

## DOMINATION GAME CRITICAL GRAPHS

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### Abstract

The domination game is played on a graph  $G$  by two players who alternately take turns by choosing a vertex such that in each turn at least one previously undominated vertex is dominated. The game is over when each vertex becomes dominated. One of the players, namely Dominator, wants to finish the game as soon as possible, while the other one wants to delay the end. The number of turns when Dominator starts the game on  $G$  and both players play optimally is the graph invariant  $\gamma_g(G)$ , named the game domination number. Here we study the  $\gamma_g$ -critical graphs which are critical with respect to vertex predomination. Besides proving some general properties, we characterize  $\gamma_g$ -critical graphs with  $\gamma_g = 2$  and with  $\gamma_g = 3$ , moreover for each  $n$  we identify the (infinite) class of all  $\gamma_g$ -critical ones among the  $n$ th powers  $C_N^n$  of cycles. Along the way we determine  $\gamma_g(C_N^n)$  for all  $n$  and  $N$ . Results of a computer search for  $\gamma_g$ -critical trees are presented and several problems and research directions are also listed.

**Keywords:** domination number, domination game, domination game critical graphs, powers of cycles, trees.

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## 1. INTRODUCTION

Critical graphs are indispensable when investigating many central graph invariants. Several different criticality concepts were investigated with respect to the chromatic number and the chromatic index. An important concept in this respect is the one of *color-critical graphs* which are the graphs  $G$  such that  $\chi(H) < \chi(G)$  holds for any proper subgraph  $H$  of  $G$  (equivalently,  $\chi(G - e) < \chi(G)$  for any edge  $e$  of  $G$ ), see the books [1, Section 14.2] and [24, Section 5.2], and the recent paper [22]. For the parallel concept on the chromatic index we refer to [20] and the references therein. A good source for critical graphs with respect to the independence number is the book [21]. Different criticality concepts were investigated also related to domination. A standard example is formed by  $\gamma$ -critical graphs [5], that is, the graphs for which  $\gamma(G - v) < \gamma(G)$  holds for any vertex  $v$  of  $G$ . For distance domination-critical graphs see [15, 23], while for domination dot-critical graphs see the seminar paper [9] and the recent developments [11, 12].

The domination game [4] is played on an arbitrary graph  $G$  by *Dominator* and *Staller*, see [2, 6, 7, 8, 10, 19] for some recent developments on the game and [13, 14] for the total version of the game. The players are taking turns choosing a vertex such that at least one previously undominated vertex becomes dominated. The game ends when no move is possible. Dominator wants to finish the game as soon as possible, while Staller wants to play as long as possible. By *Game 1* (*Game 2*) we mean a game in which Dominator (respectively, Staller) has the first move. Assuming that both players play optimally, the *game domination number*  $\gamma_g(G)$  (the *Staller-start game domination number*  $\gamma'_g(G)$ ) of a graph  $G$ , denotes the number of moves in Game 1 (respectively, Game 2). A *partially-dominated graph* is a graph together with a declaration that some vertices are already dominated, that is, they need not be dominated in the rest of the game. Note that a vertex declared to be already dominated can still be played in the course of the game provided it has an undominated neighbor. For a vertex subset  $S$  of a graph  $G$ , let  $G|S$  denote the partially dominated graph in which vertices from  $S$  are already dominated. If  $S = \{v\}$ , then we will simplify the notation  $G|\{v\}$  to  $G|v$ . The following result of Kinnersley, West, and Zamani is a fundamental tool for the domination game and will be used throughout the paper.

**Theorem 1** (Lemma 2.1, Continuation Principle [18]). *Let  $G$  be a graph and  $A, B \subseteq V(G)$ . If  $B \subseteq A$ , then  $\gamma_g(G|A) \leq \gamma_g(G|B)$  and  $\gamma'_g(G|A) \leq \gamma'_g(G|B)$ .*

In this paper we introduce critical graphs with respect to the domination game as follows. A graph  $G$  is *domination game critical* or shortly  $\gamma_g$ -critical if  $\gamma_g(G) > \gamma_g(G|v)$  holds for every  $v \in V(G)$ . We also say that  $G$  is  $k$ - $\gamma_g$ -critical provided that  $\gamma_g(G) = k$ .

One might think that in view of the standard coloring and domination critical graphs, a more natural option would be to consider vertex removed, or edge

removed, or edge added graphs. However, as it turned out, the domination game is not monotone with respect to removing/adding vertices or edges. It was demonstrated in [2] that removing an edge from a graph can change its game domination number by any value from  $\{-2, -1, 0, 1, 2\}$ . Moreover, removing a vertex from a graph can increase its game domination number by any value, and can also decrease it by 1 or 2. On the other hand, by the Continuation Principle,  $\gamma_g(G) \geq \gamma_g(G|v)$  holds for every  $v \in V(G)$ . Note also that our definition of  $\gamma_g$ -critical graphs is parallel with the chromatic-critical graphs with respect to a vertex removal because removing a vertex just means that it need not be colored.

We proceed as follows. In the next section we prove some general results that we need in later sections. One of them establishes the relation  $\gamma_g(G|v) \geq \gamma_g(G) - 2$ , valid for every graph  $G$  and every  $v \in V(G)$ . In Section 3 we concentrate on  $\gamma_g$ -critical graphs. Especially, we show that if  $G$  is  $\gamma_g$ -critical then  $\gamma'_g(G)$  equals either  $\gamma_g(G)$  or  $\gamma_g(G) - 1$ ; moreover, we characterize  $2$ - $\gamma_g$ -critical and  $3$ - $\gamma_g$ -critical graphs. In Section 4, the game domination number  $\gamma_g(C_N^n)$  and the Staller-start game domination number  $\gamma'_g(C_N^n)$  of the powers of cycles are determined for each  $n \geq 1$  and  $N \geq 2n + 1$ . As a consequence, for every  $k \geq 2$  we identify the infinite class of  $k$ - $\gamma_g$ -critical graphs which are powers of cycles. In Section 5, we present all  $\gamma_g$ -critical trees up to order 17 (obtained by computer search) and discuss possible infinite families of  $\gamma_g$ -critical trees. In the last section we describe some further typical examples of  $\gamma_g$ -critical graphs and raise some open problems.

## 2. PRELIMINARIES

A graph  $G$  realizes the pair  $(k, \ell)$  if  $\gamma_g(G) = k$  and  $\gamma'_g(G) = \ell$ . It was proved in [4, 18] that  $|\gamma_g(G) - \gamma'_g(G)| \leq 1$  holds for any graph  $G$ . Hence any graph realizes either  $(k, k+1)$ ,  $(k, k)$ , or  $(k, k-1)$  (for some integer  $k$ ) and is consequently called a PLUS graph, an EQUAL graph, or a MINUS graph, respectively. Moreover, we say that  $G$  is a *no-minus graph* [10], if for any  $S \subseteq V(G)$ ,  $\gamma_g(G|S) \leq \gamma'_g(G|S)$  holds (in other words,  $G|S$  is not a MINUS).

A variation of the domination game when Dominator (respectively, Staller) is allowed, but not obligated, to skip exactly one move in the course of the game, is called the *Dominator-pass game* (respectively, *Staller-pass game*). The number of moves in such a game, where both players are playing optimally, is denoted by  $\gamma_g^{dp}(G)$  (respectively,  $\gamma_g^{sp}(G)$ ) when Dominator starts the game (unless he decides to pass already the first move) and by  $\gamma_g^{ldp}(G)$  (respectively,  $\gamma_g^{lsp}(G)$ ) when Staller starts the game. These variants of the domination game turned out to be very useful, see [3, 4, 10, 18]. For our purposes we recall the following result.

**Proposition 2** (Lemma 2.2, Proposition 2.3 [10]). *If  $S$  is a subset of vertices of a graph  $G$ , then  $\gamma_g^{sp}(G|S) \leq \gamma_g(G|S) + 1$ . Moreover, if  $G$  is a no-minus graph, then  $\gamma_g^{sp}(G|S) = \gamma_g^{dp}(G|S) = \gamma_g(G|S)$ .*

Now we can prove:

**Theorem 3.** *If  $u$  is a vertex of a graph  $G$ , then  $\gamma_g(G|u) \geq \gamma_g(G) - 2$  holds. Moreover, if  $G$  is a no-minus graph, then  $\gamma_g(G|u) \geq \gamma_g(G) - 1$  holds.*

**Proof.** We assume that Dominator plays two games at the same time. The real game is played on  $G$ , while Dominator also imagines another game being played on  $G|u$ . The strategy of Dominator is to consider the imagined game as a Staller-pass game and to play optimally in it. Throughout the game Dominator will ensure that every vertex that is dominated in the real game is also dominated in the imagined game. Clearly this is true at the beginning. Dominator optimally plays in the imagined game and copies his moves to the real game. Since the described property is preserved, all of his moves are legal in the real game. Every move of Staller in the real game is copied to the imagined game. If her move is not legal in the imagined game, then the only new dominated vertex in the real one must be  $u$ . In this case Dominator just skips her move in the imagined game (which is fine because Dominator is playing a Staller-pass game). From this move on, the sets of dominated vertices are the same in both games, hence all the moves until the end will be legal, and the number of moves still needed to finish the game is equal in both games.

Let  $p$  and  $q$  be the number of moves played in the real game and in the imagined game, respectively. Then, since Staller plays optimally in the real game (but Dominator might not), we have  $\gamma_g(G) \leq p$ . Since in the imagined game it is possible that one move from the real game was skipped, we have  $p \leq q + 1$ . Moreover, since Dominator is playing optimally on  $G|u$  (but Staller might not), we also infer that  $q \leq \gamma_g^{sp}(G|u)$ . Putting these inequalities together we get

$$\gamma_g(G) \leq p \leq q + 1 \leq \gamma_g^{sp}(G|u) + 1,$$

that is,  $\gamma_g(G) \leq \gamma_g^{sp}(G|u) + 1$ . Therefore, by the first assertion of Proposition 2,  $\gamma_g(G) \leq \gamma_g(G|u) + 2$ . For the case when  $G$  is a no-minus graph we can analogously apply the second statement of Proposition 2 to show that  $\gamma_g(G) \leq \gamma_g(G|u) + 1$  holds. ■

Vertices  $u$  and  $v$  of a graph  $G$  are *twins* in  $G$  if their closed neighborhoods are the same,  $N[u] = N[v]$ . In particular, twins are adjacent vertices. A graph is called *twin-free* if it contains no twins.

**Lemma 4.** *If  $u$  and  $v$  are twins in  $G$ , then  $\gamma_g(G) = \gamma_g(G|u) = \gamma_g(G|v)$ .*

**Proof.** Suppose that the same game is played on  $G$  and on  $G|u$ , that is, the same vertices are selected in both games. Then we claim that a move is legal in the game played on  $G$  if and only if the move is legal in the game played on  $G|u$ . Clearly, a legal move on  $G|u$  is also legal on  $G$ . On the other hand, if at

some point a move is legal on  $G$  but not on  $G|u$ , then  $u$  would be the only newly dominated vertex in the game played on  $G$ . This would mean that the vertex  $v$  has already been dominated, but this is not possible as  $u$  and  $v$  are twins. This proves the claim, from which the lemma follows immediately. ■

### 3. PROPERTIES OF $\gamma_g$ -CRITICAL GRAPHS

The concept of  $\gamma_g$ -critical graphs is interesting only for Game 1. Indeed, suppose that  $\gamma'_g$ -critical graphs are defined analogously, that is, as graphs  $G$  for which  $\gamma'_g(G) > \gamma'_g(G|v)$  holds for every  $v \in V(G)$ . Let  $G$  be an arbitrary graph and let  $u$  be an optimal start vertex for Staller in Game 2 on  $G$ . This implies that  $\gamma'_g(G) = 1 + \gamma_g(G|N[u])$ . Assuming that  $u$  has at least one neighbor in  $G$ , it follows that Staller can play  $u$  on her first move in Game 2 on the graph  $G|u$ . While this may not be an optimal first move for Staller on  $G|u$ , we still get  $\gamma'_g(G|u) \geq 1 + \gamma_g(G|N[u]) = \gamma'_g(G)$ . It follows that there are no non-trivial  $\gamma'_g$ -critical graphs.

Consider next the graph  $G$  from Figure 1. It has 13 vertices and is 7-critical. What appears to be quite surprising is that  $\gamma_g(G|u) = \gamma_g(G|w) = 5$  holds. Hence in the definition of the  $\gamma_g$ -critical graphs, the condition  $\gamma_g(G) > \gamma_g(G|v)$  cannot be replaced with the condition  $\gamma_g(G) = \gamma_g(G|v) + 1$ . On the other hand it follows from Theorem 3 that the decrease  $\gamma_g(G) - \gamma_g(G|v) = 2$  is largest possible.

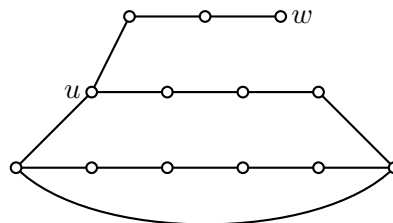


Figure 1. A critical graph on 13 vertices.

A general property of  $\gamma_g$ -critical graphs is that they cannot be PLUS.

**Proposition 5.** *If  $G$  is a  $\gamma_g$ -critical graph, then  $G$  is either a MINUS graph or an EQUAL graph.*

**Proof.** Suppose  $s$  is an optimal start vertex in Game 2 for Staller on  $G$ . Then, using the Continuation Principle,

$$\gamma'_g(G) = 1 + \gamma_g(G|N[s]) \leq 1 + \gamma_g(G|s).$$

On the other hand, since  $G$  is critical,  $\gamma_g(G|s) \leq \gamma_g(G) - 1$ . Hence

$$\gamma'_g(G) - 1 \leq \gamma_g(G|s) \leq \gamma_g(G) - 1.$$

In conclusion,  $\gamma'_g(G) \leq \gamma_g(G)$ , that is,  $G$  is not a PLUS graph. ■

The only  $1-\gamma_g$ -critical graph is  $K_1$ . As the next result asserts,  $2-\gamma_g$ -critical graphs are precisely the cocktail party graphs. Recall that the *cocktail party* graph  $K_{k \times 2}$ ,  $k \geq 1$ , is the graph obtained from  $K_{2k}$  by deleting a perfect matching. In particular,  $K_{1 \times 2}$  is the disjoint union of two vertices and  $K_{2 \times 2} = C_4$ .

**Proposition 6.** *The following conditions are equivalent for a graph  $G$ .*

- (i)  $G$  is  $2-\gamma_g$ -critical,
- (ii)  $G = K_{k \times 2}$ , for some  $k \geq 1$ ,
- (iii)  $\gamma(G) = 2$  and every pair of vertices of  $G$  forms a dominating set.

**Proof.** Let  $|V(G)| = n$ .

(i)  $\Rightarrow$  (ii) Assuming that  $G$  is a  $2-\gamma_g$ -critical graph, for every vertex  $u$  of  $G$  there exists a vertex  $u' \neq u$  which is an optimal first choice of Dominator on  $G|u$ . Since  $\gamma_g(G) = 2$ ,  $\deg(u') \leq n - 2$ . On the other hand, because  $\gamma_g(G|u) = 1$ ,  $u'$  is a dominating vertex in  $G - u$ . Therefore,  $N[u'] = V(G) \setminus \{u\}$ . Since this is true for each vertex of  $G$ , we conclude that  $G$  is isomorphic to the graph obtained from  $K_{2k}$  by deleting a perfect matching for some  $k \geq 1$ , that is,  $G = K_{k \times 2}$ .

(ii)  $\Rightarrow$  (iii) This implication is obvious.

(iii)  $\Rightarrow$  (i) Let  $x$  be an arbitrary vertex of  $G$ . Since  $\gamma(G) = 2$ ,  $\deg(x) \leq n - 2$ . On the other hand, if  $x$  is adjacent to neither  $x'$  nor to  $x''$ , then  $\{x', x''\}$  is not a dominating set. Therefore,  $\deg(x) = n - 2$ . Suppose now that  $G$  is not  $2-\gamma_g$ -critical, so that there exists a vertex  $u$  such that  $\gamma_g(G|u) = 2$ . Then any vertex  $x \neq u$  must be non-adjacent with exactly one vertex  $x' \neq u$ , which in turn implies that  $x$  is adjacent to  $u$  (since  $\deg(x) = n - 2$ ). Since  $x$  was arbitrary, we conclude that  $\deg(u) = n - 1$  and so  $\gamma(G) = 1$ , the final contradiction. ■

The equivalence between (ii) and (iii) was earlier proved in [16].

Clearly,  $\Delta(G) \leq |V(G)| - 3$  holds in a  $3-\gamma_g$ -critical graph  $G$  since otherwise  $\gamma_g(G) \leq 2$  would hold. Now we can characterize  $3-\gamma_g$ -critical graphs as follows.

**Theorem 7.** *Let  $G = (V, E)$  be a graph of order  $n$  and with  $\Delta(G) \leq n - 3$ . Then  $G$  is  $3-\gamma_g$ -critical if and only if  $G$  is twin-free, and for any  $v \in V$  there exists a vertex  $u \in V$  such that  $uv \notin E$  and  $\deg(u) = n - 3$ .*

**Proof.** Assume first that  $G$  is twin-free, and that for any  $v \in V$  there exists a vertex  $u \in V$  such that  $uv \notin E$  and  $\deg(u) = n - 3$ . Since  $\Delta(G) \leq n - 3$ , after the first move of Dominator at least two vertices remain undominated. Let  $x$  and  $y$  be such vertices. Because  $G$  is twin-free, Staller can force at least two more

moves to be made to finish the game. On the other hand, if Dominator selects a vertex of degree  $n - 3$  (which exists by our second assumption), the game will finish in the next two moves. We conclude that  $\gamma_g(G) = 3$ . Consider now any  $v \in V$  and the game on  $G|v$ . Because of the second assumption, Dominator can play a vertex  $u$  such that only one vertex remains undominated. Thus, any legal move of Staller finishes the game in the second turn. Therefore,  $G$  is  $3\text{-}\gamma_g$ -critical.

To prove the other direction, assume that  $G$  is  $3\text{-}\gamma_g$ -critical. By Lemma 4,  $G$  must be twin-free. Take any vertex  $v \in V$  and consider the domination game on  $G|v$ . Let an optimal first choice of Dominator be  $u$  and assume that  $\deg(u) < n - 3$  in  $G$ . Hence, in  $G|v$  we have two different undominated vertices, say  $x$  and  $y$ , after the choice of  $u$ . Since Staller finishes the game after her first move, there exists no vertex that dominates only one of  $x$  and  $y$ , hence  $N[x] = N[y]$ . This contradicts Lemma 4. So after Dominator's first move in  $G|v$ , there exists only one undominated vertex  $w$ . Since  $\gamma_g(G) > 2$  we get that  $v$  is not adjacent to  $u$ . From  $\gamma_g(G|v) = 2$  we conclude that  $\deg(u) = n - 3$ . Because  $u$  is not adjacent to  $v$  we are done. ■

The two conditions of Theorem 7 are independent. For instance,  $P_5$  is twin-free and does not fulfill the second condition (for the middle vertex). Of course, it is then not  $3\text{-}\gamma_g$ -critical by the theorem. Similarly, the graph from Figure 2 is not twin-free ( $u$  and  $v$  are twins), fulfills the second condition, and is not  $3\text{-}\gamma_g$ -critical. Indeed, if Dominator plays  $w$  as the first move, then the only undominated vertices left are the twins  $u$  and  $v$ , hence Staller is forced to finish the game in the next move.

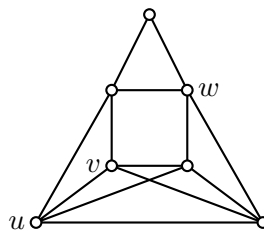


Figure 2. A graph with twins  $u$  and  $v$ .

It follows from Theorem 7 that the class of  $3\text{-}\gamma_g$ -critical graphs is quite rich (in contrast with the class of  $2\text{-}\gamma_g$ -critical graphs). For instance, the class of  $3\text{-}\gamma_g$ -critical graphs includes complements of cycles  $\overline{C}_n$ ,  $n \geq 5$ . In addition, we also get  $3\text{-}\gamma_g$ -critical graphs by removing the edges of two disjoint cycles  $C_p$  and  $C_q$  from  $K_k$ , where  $k \geq 6$ ,  $p, q \geq 3$  and  $p + q = k$ . Moreover,

**Corollary 8.** *The join of any two  $3\text{-}\gamma_g$ -critical graphs is a  $3\text{-}\gamma_g$ -critical graph.*

Proposition 6 can be reformulated in the way parallel to Theorem 7. However, as we will see in the next section (cf. Corollary 10), Theorem 7 does not extend to  $k$ -critical graphs with  $k \geq 4$ . Note also that Corollary 8 does not hold for  $k$ - $\gamma_g$ -critical graphs with  $k \geq 4$  because if  $G$  is a join of two graphs, then  $\gamma_g(G) \leq 3$ .

#### 4. POWERS OF CYCLES

For a positive integer  $n$ , the  $n$ th power  $G^n$  of a graph  $G$  is the graph with  $V(G^n) = V(G)$  and two vertices are adjacent in  $G^n$  if and only if their distance in  $G$  is at most  $n$ . In this section we consider the powers of cycles and determine their game domination number, Staller-start game domination number, and classify which are  $\gamma_g$ -critical. In this way we extend the result on cycles from [17] and also obtain infinite families of  $k$ - $\gamma_g$ -critical graphs for any  $k \geq 2$ .

**Theorem 9.** *For every  $n \geq 1$  and  $N \geq 3$ ,*

$$\gamma_g(C_N^n) = \begin{cases} \left\lceil \frac{N}{n+1} \right\rceil; & N \bmod (2n+2) \in \{0, 1, \dots, n+1\}, \\ \left\lceil \frac{N}{n+1} \right\rceil - 1; & N \bmod (2n+2) \in \{n+2, \dots, 2n+1\}. \end{cases}$$

Moreover, for every  $n \geq 1$  and  $N \geq 2n+1$ ,

$$\gamma'_g(C_N^n) = \begin{cases} \left\lceil \frac{N}{n+1} \right\rceil; & N \bmod (2n+2) \in \{0\}, \\ \left\lceil \frac{N}{n+1} \right\rceil - 1; & N \bmod (2n+2) \in \{1, \dots, n+1, 2n+1\}, \\ \left\lceil \frac{N}{n+1} \right\rceil - 2; & N \bmod (2n+2) \in \{n+2, \dots, 2n\}. \end{cases}$$

**Proof.** If  $N \leq 2n+1$ , then  $C_N^n$  is a complete graph and hence  $\gamma_g(C_N^n) = 1$  and  $\gamma'_g(C_{2n+1}^n) = 1$ , thus the statements clearly hold. Hence we assume  $N \geq 2n+2$ . After the first turn, throughout the game we always have  $2n+1$  consecutive vertices of the cycle  $C_N$ , each one being dominated. Hence, Staller may choose a vertex such that only one new vertex becomes dominated. On the other hand, Dominator cannot dominate more than  $2n+1$  vertices in a turn. Thus, Staller has a strategy which ensures that in any two consecutive turns at most  $2(n+1)$  vertices become newly dominated.

It follows for Game 1 that if Staller finishes this game, the number of turns is at least  $2 \left\lceil \frac{N}{2(n+1)} \right\rceil$ , while if the last choice is made by Dominator, it is at least



$2 \left\lceil \frac{N-2n-1}{2(n+1)} \right\rceil + 1$ . Hence, we have

$$\gamma_g(C_N^n) \geq \min \left\{ 2 \left\lceil \frac{N}{2(n+1)} \right\rceil, 2 \left\lceil \frac{N-2n-1}{2(n+1)} \right\rceil + 1 \right\}.$$

From these inequalities, checking the cases due to the residues modulo  $2n + 2$ , we obtain that the formulae given in the theorem are lower bounds on  $\gamma_g(C_N^n)$ .

In Game 2, the inequality

$$\gamma'_g(C_N^n) \geq \min \left\{ 2 \left\lceil \frac{N-2n-1}{2(n+1)} \right\rceil + 1, 2 \left\lceil \frac{N-4n-2}{2(n+1)} \right\rceil + 2 \right\}$$

must hold. Checking the cases due to the residues, we obtain that the formulae given for  $\gamma'_g(C_N^n)$  in the theorem are lower bounds.

For the upper bounds consider the following strategy of Dominator, where a *run* means a non-extendable set of consecutive dominated vertices which induce a path on  $C_N$ .

- In the first turn (if it is his turn) he is free to choose any vertex.
- In his later turns, he prefers to extend a run by exactly  $2n + 1$  vertices; if it is not possible, he selects a vertex which dominates all the vertices between the ends of two runs.

Extending an idea from [17] used there to determine the game domination number of cycles, we consider function

$$P(m) = u + (n + 1)m + 2nr,$$

where  $u$  is the number of undominated vertices and  $r$  is the number of runs after the  $m$ th turn of the game.

Now, we consider the Dominator-start game and prove the following relations by induction on  $m$ :

$$P(m) \leq \begin{cases} N + n; & m \text{ is odd,} \\ N + 2n; & m \text{ is even.} \end{cases}$$

After the first turn we have  $P(1) = N - (2n + 1) + (n + 1) + 2n = N + n$ .

- If in the  $m$ th turn, for an  $m \geq 3$  odd, Dominator extends a run with  $2n + 1$  vertices, then

$$P(m) \leq P(m - 1) - (2n + 1) + (n + 1) = P(m - 1) - n \leq N + n$$

holds. In the other case, when Dominator decreases the number of runs, the induction hypothesis implies again

$$P(m) \leq P(m - 1) - 1 + (n + 1) - 2n = P(m - 1) - n \leq N + n.$$

- If  $m$  is even, then in the  $m$ th turn Staller either does not increase the number of runs and we have

$$P(m) \leq P(m-1) - 1 + (n+1) = P(m-1) + n \leq N + 2n,$$

or a new run arises and then exactly  $2n+1$  vertices become dominated. In the latter case,

$$P(m) \leq P(m-1) - (2n+1) + (n+1) + 2n = P(m-1) + n \leq N + 2n.$$

Further, we note that if Staller finishes the game in the  $m$ th turn, then the number of runs necessarily decreases by at least 1, hence

$$P(m) \leq P(m-1) - 1 + (n+1) - 2n = P(m-1) - n \leq N$$

must be true for the number  $m$  of turns if  $m$  is even.

Consequently, if the game finishes in the  $m$ th turn, we have

$$P(m) = m(n+1) \leq \begin{cases} N+n; & m \text{ is odd,} \\ N; & m \text{ is even.} \end{cases}$$

Let us write  $N$  as  $N = 2s(n+1) + x$ , where  $s = \lfloor \frac{N}{2(n+1)} \rfloor$  and  $0 \leq x \leq 2n+1$  is the residue of  $N$  modulo  $(2n+2)$ . Checking the conditions proved for the value  $P(m)$ , we obtain the following inequalities.

- If  $x = 0$ , the condition  $(2s+1)(n+1) \leq N+n$  is not true, hence  $\gamma_g(C_N^n) \leq 2s = \lfloor \frac{N}{n+1} \rfloor$ .
- If  $x \geq 1$  then  $(2s+2)(n+1) \leq N$  cannot hold, and  $\gamma_g(C_N^n) \leq 2s+1$  is concluded. Then, for  $1 \leq x \leq n+1$  we have  $\gamma_g(C_N^n) \leq \lfloor \frac{N}{n+1} \rfloor$ , while for  $n+2 \leq x \leq 2n+1$ ,  $\gamma_g(C_N^n) \leq \lfloor \frac{N}{n+1} \rfloor - 1$  is obtained.

These bounds together prove our formulae stated for  $\gamma_g(C_N^n)$ .

Similarly, in any Staller-start game  $P(1) = N+n$  holds, and if Dominator follows the strategy described above then

$$P(m) \leq \begin{cases} N+n; & m \text{ is odd,} \\ N; & m \text{ is even} \end{cases}$$

is valid for every  $m$ . In addition, if Staller finishes the game with the  $m$ th turn,  $P(m) \leq N-n$  must be fulfilled. Again, we may consider the form  $N = 2s(n+1) + x$  and check the different cases. This yields that the formulae given for  $\gamma'_g(C_N^n)$  in the theorem are upper bounds.

These facts together prove our statements. ■

The game domination number  $\gamma_g(C_N^n|v)$ , where  $v$  is any vertex of  $C_N^n$ , can be determined in a similar way. We note that in this case

$$P(m) = m(n + 1) \leq \begin{cases} N + n - 1; & m \text{ is odd,} \\ N - 1; & m \text{ is even} \end{cases}$$

must be true, where  $P(m) = u + (n + 1)m + 2nr$  and  $m$  is the length of the game on  $C_N^n|v$ . Then, we obtain

$$\gamma_g(C_N^n|v) = \begin{cases} \left\lceil \frac{N-1}{n+1} \right\rceil; & N \bmod (2n + 2) \in \{1, \dots, n + 2\}, \\ \left\lceil \frac{N-1}{n+1} \right\rceil - 1; & N \bmod (2n + 2) \in \{0, n + 3, \dots, 2n + 1\}. \end{cases}$$

Comparing it with the result of Theorem 9, we immediately get that  $\gamma_g(C_N^n|v) < \gamma_g(C_N^n)$  holds if and only if  $N \equiv 0$  or  $1 \pmod{2n + 2}$ .

**Corollary 10.** *For every  $n \geq 1$  and  $k \geq 1$ , the graph  $C_{2(n+1)k}^n$  is  $(2k)$ - $\gamma_g$ -critical, and the graph  $C_{2(n+1)k+1}^n$  is  $(2k + 1)$ - $\gamma_g$ -critical. Further, if  $2 \leq x \leq 2k + 1$  then  $C_{2(n+1)k+x}^n$  is not  $\gamma_g$ -critical.*

Note that Corollary 10 in particular asserts that  $C_{2n+2}^n = K_{(n+1) \times 2}$  are 2- $\gamma_g$ -critical graphs and that  $C_{2n+3}^n = \overline{C}_{2n+3}$  are 3- $\gamma_g$ -critical. Moreover, by Theorem 9 the graphs  $C_{2(n+1)k}^n$  are EQUAL graphs (and  $(2k)$ - $\gamma_g$ -critical), while the graphs  $C_{2(n+1)k+1}^n$  are MINUS (and  $(2k + 1)$ - $\gamma_g$ -critical).

We also emphasize the following consequence:

**Corollary 11.** *For every  $\ell$  and  $k \geq 2$ , there exist infinitely many  $k$ - $\gamma_g$ -critical graphs of order greater than  $\ell$ .*

### 5. ON $\gamma_g$ -CRITICAL TREES

In this section we present  $\gamma_g$ -critical trees that were found by computer. The computational results indicate that the appearance of such trees is somehow random. In Figure 3 all  $\gamma_g$ -critical trees up to 17 vertices are shown. There are no  $\gamma_g$ -critical trees on up to 12 vertices, two  $\gamma_g$ -critical trees on 13 vertices, no such trees on 14 and 15 vertices, another (and only) one on 16 vertices, and ten on 17 vertices.

Let  $T_{p,q,r}$  be the graph obtained from disjoint paths  $P_{4p+1}$ ,  $P_{4q+1}$ , and  $P_{4r+1}$  by identifying three end-vertices, one from each of the paths. Hence  $|V(T_{p,q,r})| = 4(p+q+r) + 1$ . Note that  $T_{1,1,1}$  is one of the two  $\gamma_g$ -critical trees on 13 vertices and that  $T_{1,1,2}$  also appears in Figure 3. Moreover, we have verified by computer that

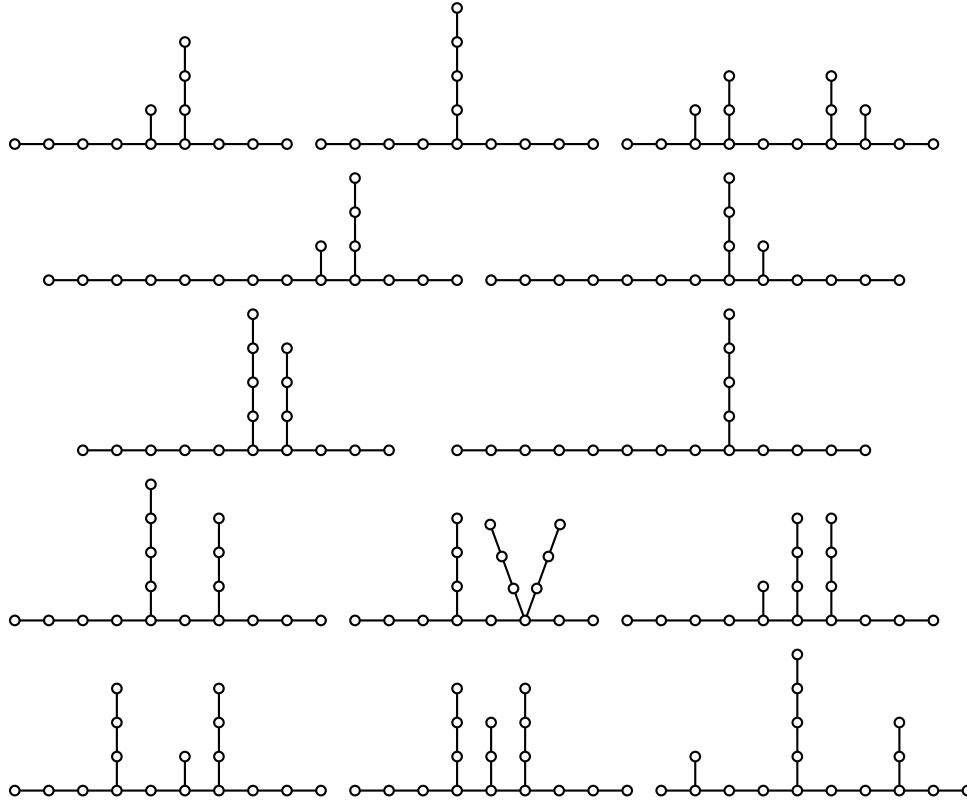


Figure 3. Critical trees on 13, 16, and 17 vertices.

$T_{p,q,r}$  is  $2((p+q+r)+1)$ -critical for  $1 \leq p, q, r \leq 3$ . These computations naturally lead to a conjecture that  $T_{p,q,r}$  is a  $\gamma_g$ -critical tree for any  $p, q, r \geq 1$ . Using the existing tools, a possible proof of this conjecture would be a technical, lengthy case analysis, hence new techniques to prove  $\gamma_g$ -criticality would be welcome.

One of the things we would need to determine in order to prove that the trees  $T_{p,q,r}$  are  $\gamma_g$ -critical, is  $\gamma_g(T_{p,q,r})$ . Note that  $T_{p,q,r}$ ,  $p, q \geq 1$ ,  $r \geq 2$ , is obtained by attaching  $P_4$  to a leaf of  $T_{p,q,r-1}$  corresponding to the parameter  $r-1$ . Because of that it would be nice if it would hold in general that if  $T'$  is obtained from a tree  $T$  by attaching  $P_4$  to one of its leaves, then  $\gamma_g(T') = \gamma_g(T) + 2$ . However, this is not true in general. There is no such example on at most 8 vertices, but there is a unique such tree  $T$  on 9 vertices shown in Figure 4. First,  $\gamma_g(T) = 5$ . On the other hand, if  $T'$  is the tree obtained from  $T$  by attaching a  $P_4$  at  $x$ , then  $\gamma_g(T') = 6$ . Moreover, if  $T''$  is a tree obtained from  $T$  by attaching a  $P_4$  at  $y$ , then  $\gamma_g(T'') = 7$ . The same holds also when a  $P_4$  is attached at  $z$ .

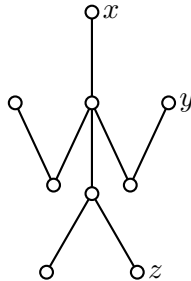


Figure 4. Tree  $T$ .

6. CONCLUDING REMARKS AND OPEN PROBLEMS

Using computer we have found more sporadic examples of  $\gamma_g$ -critical graphs. The broken ladder  $BL_k$ ,  $k \geq 0$ , is the graph obtained from the Cartesian product  $P_2 \square P_4$  by amalgamating an edge of the cycle  $C_{4k+2}$  with an edge of  $P_2 \square P_4$  whose end-points are of degree 2, see [19]. In particular,  $BL_0 = P_2 \square P_4$ . It was verified by computer that  $BL_k$  is  $(2k + 4)$ -critical for  $0 \leq k \leq 5$ . Similarly, let  $H_k$ ,  $k \geq 0$ , be the graph obtained from  $P_2 \square P_4$  by amalgamating an edge of the cycle  $C_{4k+2}$  with a middle  $P_4$ -layer edge of  $P_2 \square P_4$ . It was verified for  $k \leq 2$  that  $H_k$  is  $\gamma_g$ -critical.

If  $G$  is a vertex-transitive graph, then we only need to compare  $\gamma_g(G)$  with  $\gamma_g(G|v)$  for some vertex  $v$  of  $G$ . That is, a vertex-transitive graph  $G$  is either  $\gamma_g$ -critical or  $\gamma_g(G) = \gamma(G|v)$  holds for any vertex  $v$  of  $G$ .

Every graph contains a color-critical subgraph with the same chromatic number. Naturally extending the concept of  $\gamma_g$ -criticality to all partially dominated graphs, the same conclusion holds also for this concept. Indeed, let  $G$  be an arbitrary graph. If it is not  $\gamma_g$ -critical, then it contains a vertex  $u$  such that  $\gamma_g(G|u) = \gamma_g(G)$ . Now, if  $G|u$  is  $\gamma_g$ -critical (in the sense of the extended definition), we are done, otherwise we repeat the procedure until finally a  $\gamma_g$ -critical partially dominated graph  $G|S$  with  $\gamma_g(G|S) = \gamma_g(G)$  is found, where  $S \subseteq V(G)$ . Hence we pose:

**Problem 12.** Investigate  $\gamma_g$ -criticality of partially dominated graphs. In particular, compare it with the concept studied in this paper.

The graphs that are in a way complementary to the  $\gamma_g$ -critical graphs are the graphs  $G$  such that  $\gamma_g(G) = \gamma_g(G|v)$  holds for every  $v \in V(G)$ . For instance, among the powers of cycles, such graphs are precisely those that are not critical. Thus:

**Problem 13.** Study the graphs  $G$  for which  $\gamma_g(G) = \gamma_g(G|v)$  holds for every  $v \in V(G)$ . In particular, establish their connections with the  $\gamma_g$ -critical graphs.

Finally, in view of [10], we also pose:

**Problem 14.** Consider the behavior of the  $\gamma_g$ -criticality on the disjoint union of graphs. In particular, if  $G$  is a  $\gamma_g$ -critical graph, when is  $G \cup K_1$   $\gamma_g$ -critical?

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