



Finite Volume for Fusion Simulations

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Finite Volume for Fusion Simulations

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Plan

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Resistive MHD equations

- Non-conservative equations: $E^{hd} = \frac{p}{\gamma-1} + \frac{1}{2}\rho\mathbf{u}^2$,
 $\mathbf{E} = \mathbf{B} \times \mathbf{u} + \eta\mathbf{J}$

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) & = 0, \\ \partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p & = \mathbf{J} \times \mathbf{B}, \\ \partial_t E^{hd} + \nabla \cdot [(E^{hd} + p)\mathbf{u}] & = (\mathbf{J} \times \mathbf{B}) \cdot \mathbf{u} + \eta \mathbf{J}^2, \\ \partial_t \mathbf{B} + \nabla \times \mathbf{E} & = 0. \end{cases}$$

- Conservative equations: $E = E^{hd} + \frac{1}{2}\mathbf{B}^2$, $p^* = p + \frac{1}{2}\mathbf{B}^2$

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) & = 0, \\ \partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} - \mathbf{B} \otimes \mathbf{B}) + \nabla p^* & = 0, \\ \partial_t E + \nabla \cdot [(E + p^*)\mathbf{u} - (\mathbf{u} \cdot \mathbf{B})\mathbf{B}] & = -\nabla \cdot (\eta \mathbf{J} \times \mathbf{B}), \\ \partial_t \mathbf{B} + \nabla \cdot (\mathbf{u} \otimes \mathbf{B} - \mathbf{B} \otimes \mathbf{u}) & = -\nabla(\eta \mathbf{J}). \end{cases}$$

MHD system

- Hyperbolic part of the system: 3 wave types
 - 2 Alfvén waves,
 - 4 magneto-acoustic waves (2 fast and 2 slow),
 - 2 material waves (moving with \mathbf{u}).
- Involution equation:

$$\partial_t(\nabla \cdot \mathbf{B}) = 0.$$

Then

$$\nabla \cdot \mathbf{B}(t = 0) = 0 \Rightarrow \nabla \cdot \mathbf{B}(t) = 0 \quad \forall t.$$

- Numerical issues:
 - 1 Shock capturing: need a robust and accurate scheme.
 - 2 Free-divergence constraint.
 - 3 Time scale $T \gg T_{alfven} = L/v_{alfven}$: need implicit scheme.

Free-divergence constraint: Well-known methods

2 families of methods:

- Vector potential \mathbf{A} :

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

- Divergence cleaning methods:

$$\text{Enforce } \nabla \cdot \mathbf{B} = 0.$$

Vector potential \mathbf{A}

Replace \mathbf{B} by \mathbf{A} in the equation:

$$\partial_t \mathbf{A} - \mathbf{u} \times (\nabla \times \mathbf{A}) + \eta \nabla \times (\nabla \times \mathbf{A}) = -\nabla U.$$

Advantage:

- Insure $\nabla \cdot \mathbf{B} = 0$.

Drawbacks:

- One order higher in the spatial derivatives.
- More complex system of equation.

Divergence cleaning methods: Previous works

- Powell et al. (JCP, 1999): Add a source term proportionnal to $\nabla \cdot \mathbf{B}$.

$$\partial_t \mathbf{B} + \nabla \cdot (\mathbf{u} \otimes \mathbf{B} - \mathbf{B} \otimes \mathbf{u}) = -(\nabla \cdot \mathbf{B})\mathbf{u}.$$

- Munz et al. (JCP, 2000), Dedner et al. (JCP, 2001): Generalized Lagrange Multiplier (GLM) method.

$$\begin{aligned} \partial_t \mathbf{B} + \nabla \cdot (\mathbf{u} \otimes \mathbf{B} - \mathbf{B} \otimes \mathbf{u}) + \nabla \Psi &= 0, \\ \mathcal{D}(\Psi) + \nabla \cdot \mathbf{B} &= 0, \\ \mathcal{D}(\Psi) &= \frac{1}{c_h^2} \partial_t \Psi + \frac{1}{c_p^2} \Psi. \end{aligned}$$

- Balsara and Spicer (JCP, 1998): Constrained transport method.

$$\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathcal{S} = - \oint_{\partial S} \mathbf{E} \cdot d\mathbf{l}.$$

Divergence cleaning methods

Advantages:

- Easily incorporated with a Riemann type scheme.

Drawbacks:

- Divergence cleaning:
 - $\nabla \cdot \mathbf{B}$ appears in several equations.
 - Not approximated with the same discretization.
 - Divergence cleaning method insure $\nabla \cdot \mathbf{B} = 0$ for one discrete approximation.
- Constrained transport: difficult to implement for arbitrary meshes.

Proposed method

- A mixture of the two methods.
- Idea: work with a redundant system.
- Both vector potential and magnetic fields equations.
- Based on the relaxation scheme method:
 - 1 Evolution step: FV method for all the variables.
 - 2 Projection step: enforce $\mathbf{B} = \nabla \times \mathbf{A}$.

2-D implementation

- Scalar potential ψ :

$$\mathbf{B} = B_z \mathbf{e}_z + \mathbf{e}_z \times \nabla \psi.$$

- Augmented system:

$$\left\{ \begin{array}{l} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) \\ \partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} - \mathbf{B} \otimes \mathbf{B}) + \nabla p^* \\ \partial_t E + \nabla \cdot [(E + p^*) \mathbf{u} - (\mathbf{u} \cdot \mathbf{B}) \mathbf{B}] \\ \partial_t \mathbf{B} + \nabla \cdot (\mathbf{u} \otimes \mathbf{B} - \mathbf{B} \otimes \mathbf{u}) \\ \partial_t \psi + \mathbf{u} \cdot \nabla \psi \end{array} \right. \begin{array}{l} = 0, \\ = 0, \\ = -\eta \nabla \cdot (\mathbf{J} \times \mathbf{B}), \\ = \eta \nabla^2 \mathbf{B}, \\ = \eta \nabla^2 \psi. \end{array}$$

Evolution step

- System $\partial_t U + \partial_x F(U) + \partial_y G(U) = 0$:

$$U_{i,j}^{n+1} = U_{i,j}^n - \frac{\Delta t}{\Delta x} (F_{i+1/2,j} - F_{i-1/2,j}) - \frac{\Delta t}{\Delta y} (G_{i,j+1/2} - G_{i,j-1/2}).$$

- $F_{i+1/2,j}$ and $G_{i,j+1/2}$ computed with a Riemann solver.

Evolution step: HLLD scheme

- Miyoshi and Kusano (JCP, 2005).
- Approximate Riemann solver with 4 intermediate states.

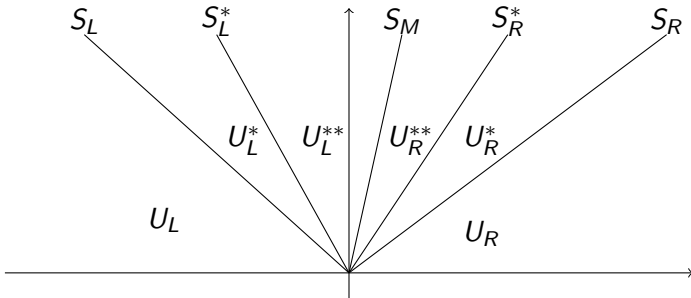


Figure : Riemann fan with four intermediate states.

Evolution step: HLLD scheme

- Intermediate fluxes:

$$\begin{cases} F_K^* &= F_K + S_K U_K^* - S_K U_K \\ F_K^{**} &= F_K + S_K^* U_K^{**} - (S_K^* - S_K) U_K^* - S_K U_K \end{cases}, K = L, R.$$

- $\rho\psi$ flux:

$$F(\rho\psi) = F(\rho) \times \begin{cases} \psi_L, & 0 \leq S_M \\ \psi_R, & 0 \geq S_M \end{cases}.$$

Scheme with projection

- 1 Evolution step: HLLD scheme

$$\left\{ \begin{array}{l} \mathbf{B}_{i,j}^{n+1/2}, \\ \psi_{i,j}^{n+1/2}. \end{array} \right.$$

- 2 Projection step:

$$\left\{ \begin{array}{l} \psi_{i,j}^{n+1} = \psi_{i,j}^{n+1/2}, \\ \mathbf{B}_{i,j}^{n+1} = B_{z,i,j}^{n+1/2} \mathbf{e}_z + \mathbf{e}_z \times (\nabla \psi)_{i,j}^{n+1}. \end{array} \right.$$

Numerical tests

Shock capturing tests:

- Brio-Wu,
- Orszag-Tang,
- Kelvin-Helmholtz.

Plasma fusion tests:

- Screw pinch equilibrium.

1-D Brio-Wu shock tube

- Magnetic field:

$$\mathbf{B} = B_x \mathbf{e}_x + \partial_x \psi \mathbf{e}_y.$$

- 1-D equation for the potential ψ :

$$\partial_t(\rho\psi) + \partial_x(\rho\psi u) = B_x \rho v.$$

- Initial conditions:

	ρ	\mathbf{u}	p	B_x	$B_y = \partial_x \psi$	ψ
$x < 0.5$	1	0	1	0.75	1	x
$x > 0.5$	0.125	0	0.1	0.75	-1	$1 - x$

- 1-D mesh: 100 points
- Final time: 0.1

1-D Brio-Wu: Magnetic component B_y

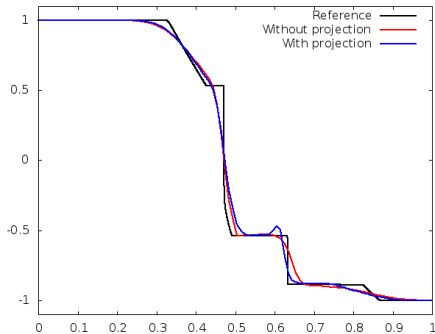


Figure : HLLD O(1).

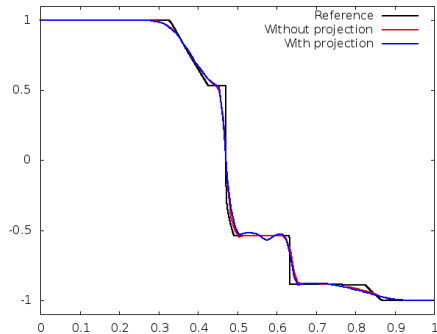


Figure : HLLD O(2).

2-D Brio-Wu shock tube

- Potential:

$$\psi(x, y) = \begin{cases} x - 0.75y, & x < 0.5 \\ 1 - x - 0.75y, & x > 0.5 \end{cases} .$$

- 2-D mesh: 100×10 cells.

2-D Brio-Wu shock tube

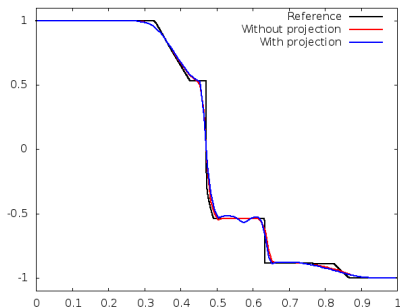


Figure : B_y HLLD O(2).

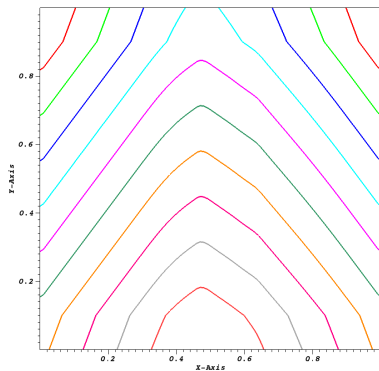


Figure : ψ , HLLD O(2).

Orszag-Tang

- Initial conditions: $\gamma = 5/3$

ρ	$u(x, y)$	$v(x, y)$	$p(x, y)$	$B_x(x, y)$	$B_y(x, y)$	$\psi(x, y)$
γ^2	$-\sin(2\pi y)$	$\sin(2\pi x)$	γ	$-\sin(2\pi y)$	$\sin(4\pi x)$	$-\frac{1}{2\pi} \cos(2\pi y) - \frac{1}{4\pi} \cos(4\pi x)$

- Mesh: 512×512 cells.
- Final time: 0.5.

Orszag-Tang: Pressure field

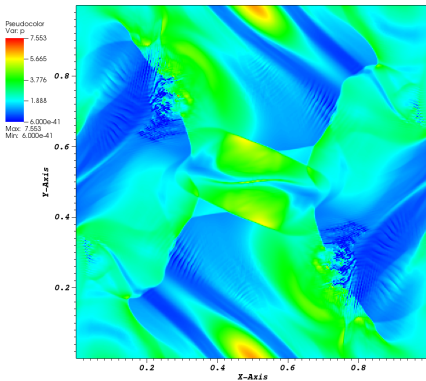


Figure : Without projection.

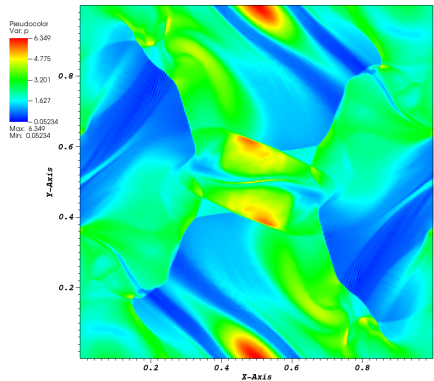


Figure : With projection.

Orszag-Tang

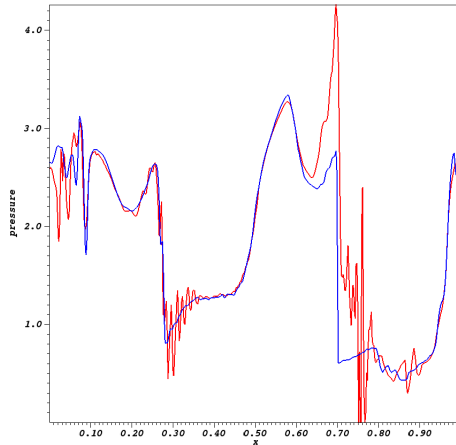


Figure : Pressure at $y = 0.3125$ Red: without projection, Blue: with projection.

Orszag-Tang

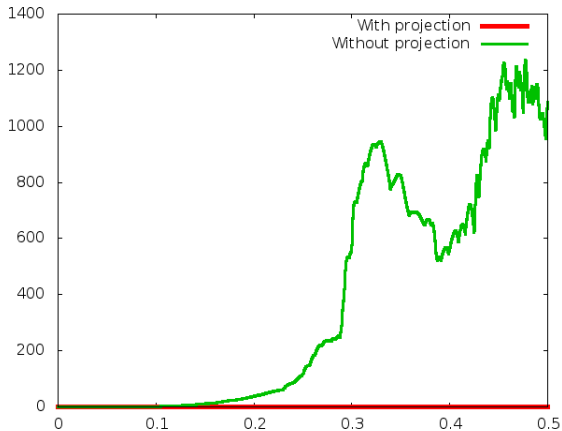


Figure : $\|\nabla \cdot \mathbf{B}\|_\infty$.

Kelvin-Helmholtz instabilities

- Initial data

ρ	p	$u(x, y)$	$v(x, y)$	$B_x(x, y)$	$B_y(x, y)$	$B_z(x, y)$	$\psi(x, y)$
1	$\frac{1}{\gamma}$	$\frac{1}{2} \tanh(\frac{y}{y_0})$	0	$0.1 \cos(\frac{\pi}{3})\sqrt{\rho}$	0	$0.1 \sin(\frac{\pi}{3})\sqrt{\rho}$	$-0.1 \cos(\frac{\pi}{3})\sqrt{\rho}y$

- Single mode perturbation

$$v(x, y) = 0.01 \sin(2\pi x) \exp(-\frac{y^2}{\sigma^2}), \quad \sigma = 0.01.$$

- Mesh: 256×512 points

- $\frac{B_{pol}}{B_{tor}} = \frac{\sqrt{B_x^2 + B_y^2}}{B_z}$.

Kelvin-Helmholtz: $t = 5.0$

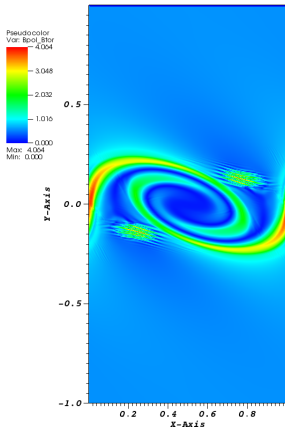


Figure : Without projection.

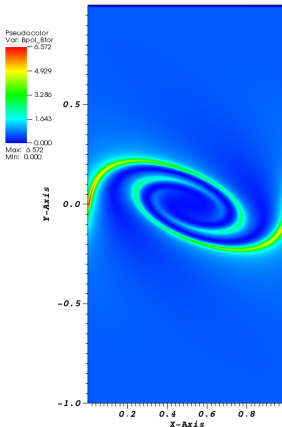


Figure : With projection.

Kelvin-Helmholtz: $t = 8.0$

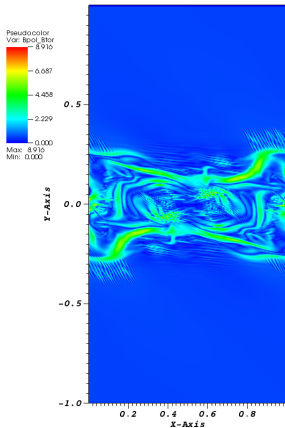


Figure : Without projection.

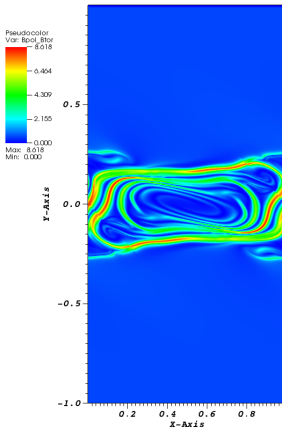


Figure : With projection.

Kelvin-Helmholtz: $t = 12.0$

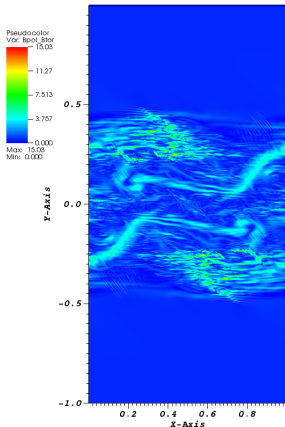


Figure : Without projection.

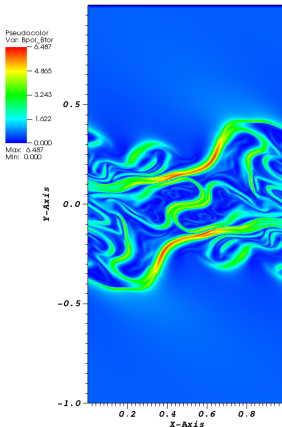


Figure : With projection.

Kelvin-Helmholtz: $t = 20.0$

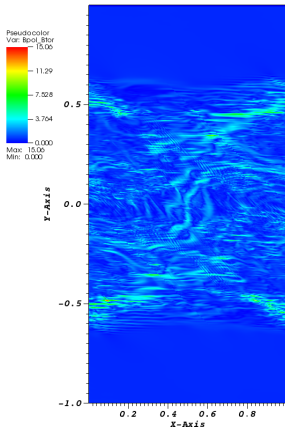


Figure : Without projection.

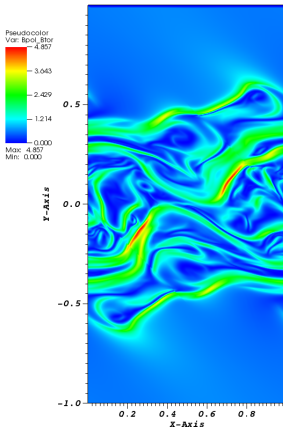


Figure : With projection.

Kelvin-Helmholtz

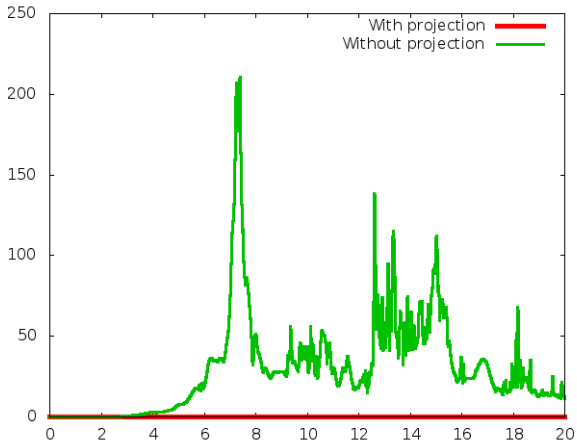


Figure : $\|\nabla \cdot \mathbf{B}\|_\infty$.

Preliminary test for fusion plasma

- Equilibrium computation: $\nabla p = \mathbf{J} \times \mathbf{B}$.
- Initial conditions:

$$\left\{ \begin{array}{ll} R_0 = 10, & B_r = 0, \\ \rho = 1, & B_\theta(r) = \frac{r}{R_0(3r^2+1)}, \\ \mathbf{u} = 0, & B_z = 1, \\ p(r) = \frac{1}{6R_0^2(3r^2+1)^2}, & \psi(r) = \frac{1}{6R_0} \ln(3r^2 + 1). \end{array} \right.$$

- Aligned mesh: cylindrical coordinates (100×10 cells).
- Non aligned mesh: Cartesian coordinates (200×200 cells).

Cylindrical coordinates

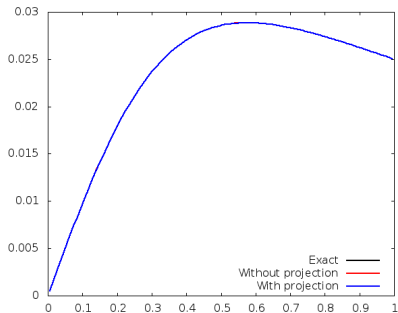


Figure : B_θ .

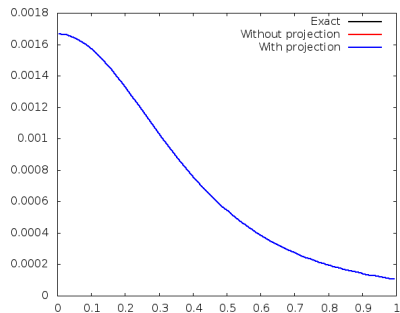


Figure : p .

Cylindrical coordinates: steady state

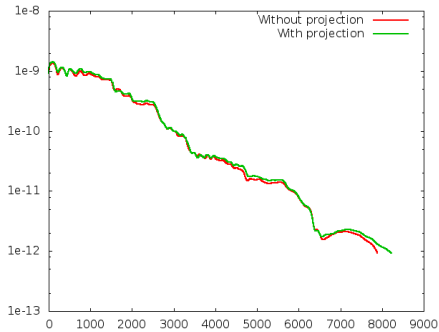


Figure : Residual on r -momentum equation.

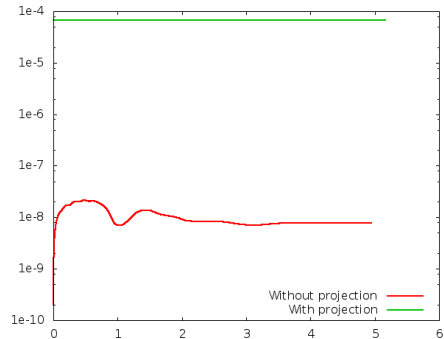


Figure : Relative error of B_θ .

Cartesian coordinates

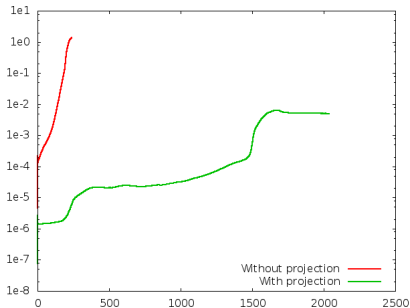


Figure : Relative error of p in function of Alfvén time.

- Non aligned meshes: with projection simulations maintains the equilibrium on 1000 Alfvén times (around 0.5ms and 10^6 time steps).
- However, we want to perform the simulation on several milliseconds.

Remedies

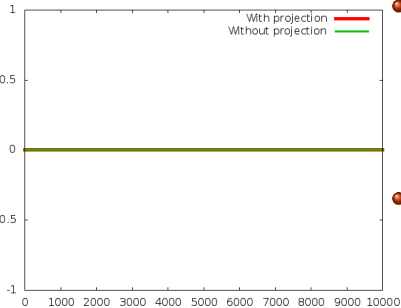


Figure : Relative error of p .

- A simple trick: remove the truncation error evaluated at time $t = 0$.

$$\begin{aligned} \partial_t(\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p \\ - \mathbf{J} \times \mathbf{B} - (\nabla_h p - \mathbf{J}_h \times \mathbf{B}_h)_{eq} = 0. \end{aligned}$$

- Well-balanced scheme: work in progress.

Conclusions and perspectives

Conclusions:

- Shock capturing tests: Scheme with projection gives satisfactory results.
- Plasma fusion test: Work in progress on well-balanced scheme.

Perspectives:

- Perform more tests for plasma fusion (kink instability).
- Adapt to 3-D geometry.
- Test on resistive problems.

Cylindrical equations

$$\begin{cases}
 \partial_t(r\rho) + \partial_r(r\rho u_r) + \partial_\theta(\rho u_\theta) & = 0, \\
 \partial_t(r\rho u_r) + \partial_r[r(\rho u_r^2 + p^* - B_r^2)] + \partial_\theta(\rho u_r u_\theta - B_r B_\theta) & = \rho u_\theta^2 + p^* - B_\theta^2, \\
 \partial_t(r^2 \rho u_\theta) + \partial_r[r^2(\rho u_r u_\theta - B_r B_\theta)] + \partial_\theta[r(\rho u_\theta^2 + p^* - B_\theta^2)] & = 0, \\
 \partial_t(r\rho u_z) + \partial_r[r(\rho u_r u_z - B_r B_z)] + \partial_\theta(\rho u_\theta u_z - B_\theta B_z) & = 0, \\
 \partial_t(rE) + \partial_r[r[(E + p^*)u_r - (\mathbf{u} \cdot \mathbf{B})B_r]] + \partial_\theta[(E + p^*)u_\theta - (\mathbf{u} \cdot \mathbf{B})B_\theta] & = 0, \\
 \partial_t(rB_r) + \partial_\theta(u_\theta B_r - u_r B_\theta) & = 0, \\
 \partial_t B_\theta + \partial_r(u_r B_\theta - u_\theta B_r) & = 0, \\
 \partial_t(rB_z) + \partial_r[u_r B_z - u_z B_r] + \partial_\theta(u_\theta B_z - u_z B_\theta) & = 0, \\
 \partial_t(r\rho\psi) + \partial_r(r\rho\psi u_r) + \partial_\theta(\rho\psi u_\theta) & = 0.
 \end{cases}$$

Scheme with projection

1 Evolution step: HLLD scheme

$$\left\{ \begin{array}{l} \rho_{i,j}^{n+1/2}, \\ \mathbf{B}_{i,j}^{n+1/2}, \\ \psi_{i,j}^{n+1/2}. \end{array} \right.$$

2 Projection step:

$$\left\{ \begin{array}{l} \psi_{i,j}^{n+1} = \psi_{i,j}^{n+1/2}, \\ \mathbf{B}_{i,j}^{n+1} = B_{z,i,j}^{n+1/2} \mathbf{e}_z + \mathbf{e}_z \times (\nabla \psi)_{i,j}^{n+1}, \\ E_{i,j}^{n+1} = \frac{p_{i,j}^{n+1/2}}{\gamma-1} + \frac{1}{2} \rho^{n+1/2} \mathbf{u}^2 + \frac{1}{2} (\mathbf{B}_{i,j}^{n+1})^2. \end{array} \right.$$