Structural Properties and Constant Factor-Approximation of Strong Distance-r Dominating Sets in Sparse Directed Graphs*

Stephan Kreutzer[1], Roman Rabinovich[2], Sebastian Siebertz[3], and Grischa Weberstädt[4]

1 Technische Universität Berlin, Berlin, Germany
stephan.kreutzer@tu-berlin.de
2 Technische Universität Berlin, Berlin, Germany
roman.rabinovich@tu-berlin.de
3 Technische Universität Berlin, Berlin, Germany
sebastian.siebertz@tu-berlin.de
4 Technische Universität Berlin, Berlin, Germany
grischa.weberstaedt@campus.tu-berlin.de

Abstract

Bounded expansion and nowhere dense graph classes, introduced by Nešetřil and Ossona de Mendez [26, 27], form a large variety of classes of uniformly sparse graphs which includes the class of planar graphs, actually all classes with excluded minors, and also bounded degree graphs. Since their initial definition it was shown that these graph classes can be defined in many equivalent ways: by generalised colouring numbers, neighbourhood complexity, sparse neighbourhood covers, a game known as the splitter game, and many more.

We study the corresponding concepts for directed graphs. We show that the densities of bounded depth directed minors and bounded depth topological minors relate in a similar way as in the undirected case. We provide a characterisation of bounded expansion classes by a directed version of the generalised colouring numbers. As an application we show how to construct sparse directed neighbourhood covers and how to approximate directed distance-r dominating sets on classes of bounded expansion. On the other hand, we show that linear neighbourhood complexity does not characterise directed classes of bounded expansion.

1998 ACM Subject Classification G.2.1 Combinatorics, G.2.2 Graph Theory

Keywords and phrases Directed Graph Structure Theory, Bounded Expansion, Generalised Colouring Numbers, Splitter Game, Approximation Algorithms, Dominating Set

Digital Object Identifier 10.4230/LIPIcs.STACS.2017.48

1 Introduction

Structural graph theory provides a wealth of tools and concepts that have proved to be very powerful in the study of approximation, parameterized or classical polynomial time algorithms for common NP-hard graph problems. The algorithmic properties of classes of graphs of *bounded tree width*, of *bounded genus* – especially *planar graphs* – or which exclude a fixed *(topological) minor* have been very well studied in the literature and powerful and

* Stephan Kreutzer, Roman Rabinovich and Sebastian Siebertz's research has been supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (ERC Consolidator Grant DISTRUCT, grant agreement No 648527).

very general algorithmic techniques for solving NP-hard problems on any of these classes of graphs have emerged.

Initially, many graph theoretical concepts studied in this area were based on topological aspects of graphs, such as tree width or excluded minors. In [26], Nešetřil and Ossona de Mendez introduced the concept of classes of *bounded expansion* which properly generalise classes of graphs excluding a fixed minor. Bounded expansion classes are defined in terms of edge densities of bounded depth minors. Since their initial definition in [26] it was shown that the concept of graph classes of bounded expansion as well as their generalisation to nowhere dense classes of graphs can equivalently be defined in many other ways: by *generalised colouring numbers* [38], *low tree depth colourings* [26], bounded *neighbourhood complexity* [33], *sparse neighbourhood covers* [15, 16, 28] and a game known as the *splitter game* [16]. This indicates that bounded expansion is a natural concept appearing frequently in different contexts. This intuition is supported for instance by [32] where it was shown that many types of real-world networks indeed are of bounded expansion.

One important consequence of the large number of different characterisations of bounded expansion classes is that every new characterisation developed in the literature also provides a different set of algorihmic techniques that can be used for different types of problems. This has led to a growing number of algorithmic results on bounded expansion classes for parameterized algorithms [8, 12, 16, 29], approximation algorithms [11], kernelization [9, 13] and others.

Starting with Johnson, Robertson, Seymour and Thomas' introduction of *directed tree width* [17], several attempts have been made to generalise the successful theory of algorithmic graph structure theory to the world of directed graphs. Most of these proposals were again based on generalising topological properties, especially bounded tree width, to directed graphs [1, 3, 4, 5, 25, 30, 34]. However, it was subsequently shown that many NP- or W[1]-hard computational problems for directed graphs remain intractable on classes of bounded directed tree width [21, 24]. In fact, it was even claimed in the literature that there cannot be any algorithmically useful digraph width measure [14]. One important reason for these hardness results is that these problems remain intractable even on acyclic or nearly acyclic digraphs and most of proposed width measures have small width on acyclic digraphs.

To overcome these problems, Kreutzer and Tazari [23] initiated the study of generalisations of bounded expansion and nowhere dense classes of graphs to the directed setting. In [23] they studied a concept called *nowhere crownful* classes of digraphs, proved an equivalent characterisation of these classes by *uniformly quasi-wideness* and showed that a variant of the directed dominating set problem becomes fixed-parameter tractable on nowhere crownful classes of digraphs. In the same paper, they defined the concept of *directed bounded expansion*, but did not study it any further. The main and decisive difference between these new approaches and the previous proposals of directed width measures is that nowhere crownful and bounded expansion classes of digraphs do not contain the class of acyclic digraphs as "low width" classes. Quite the contrary, they were specifically designed to distinguish between "easy" instances of acyclic digraphs and computationally hard ones.

Contributions of this paper. The focus of this paper is to study in depth classes of digraphs of bounded expansion. Intuitively, a class of digraphs has *bounded expansion* if there is a function $f:\mathbb{N} \to \mathbb{N}$ such that for all $G \in \mathcal{C}$ and all $r \geq 1$, every *depth-r minor* H of G has edge densitiy $\frac{|E(H)|}{|V(H)|} \leq f(r)$. While there is a clear and generally accepted concept of undirected minor, there are various definitions of directed minors that have been studied in the literature, among them *butterfly* minors [17] but also *directed minors* as used in [23] to define nowhere crownful classes. In Section 3 we show that the concept of bounded expansion of digraphs is irrespective of the notion of directed minor used. In particular, a class has bounded expansion with respect to the directed minors in [23] if, and only if, it has bounded expansion with respect to *topological minors*, i.e. subdivisions. As subdivisions are a canonical notion even in the directed setting, this yields a natural and undisputed definition of bounded expansion.

We show next that classes of bounded expansion can also be characterised by generalised colouring numbers (see Section 4). Generalised colouring numbers have proved to be an extremely fruitful concept on undirected graphs, both structurally and algorithmically. The characterisation of directed bounded expansion by colouring numbers allows us to employ similar algorithmic techniques now also in the directed setting. We show in Section 6 that bounded expansion classes admit constant step winning strategies in a splitter game (see Section 6) and also admit sparse neighbourhood covers (see Section 7).

The dominating set problem is one of the best studied NP-hard graph problems in the literature and has served as benchmark for many different algorithmic techniques. It remains NP-complete on many classes of graphs and is notoriously hard to approximate. In fact, Raz and Safra [31] showed that approximating the domination number within any factor better than $O(\log n)$ is already NP-hard. Computing dominating sets, or the distance-r variant of it, is a graph theoretical abstraction of many real-life problems such as distributing facilities such as routers or distribution centres to cover a given area or many other similar problems. One of the main applications of dominating sets is to choose a small number of positions in a graph or network so that every vertex in the graph can communicate with a member of the dominating set. The radius r thereby determines the range that can be covered by a single element of the dominating set.

Motivated by the application above where we want to choose vertices so that every vertex of the network can communicate with an element of the dominating set within distance r, we study the *strong distance-r dominating set* problem. Here we are asked to find a minimal set X of vertices in a digraph G such that every vertex $v \in V(G)$ is contained together with some element of X on a closed directed walk of length at most $2r$.[1] We show in Section 8 that for every $r \geq 1$ there is a constant factor approximation algorithm for the strong r dominating set problem on any class of digraphs of bounded expansion.

2 Background from graph theory

In this section we fix our notation. We refer to [2] for background on digraph theory.

Digraphs, walks and neighbourhoods. A *digraph* G consists of a set $V(G)$ of *vertices* and a set $E(G) \subseteq V(G) \times V(G)$ of *arcs*. We assume that a digraph G has no loops, i.e. no edges of the form (v, v) for $v \in V(G)$. A *walk of length* k in a digraph G is a sequence $W = v_0, \ldots, v_k$ of vertices of G such that for each $0 \leq i < k$ there is an edge $(v_i, v_{i+1}) \in E(G)$. A walk is *closed* if $v_0 = v_k$, and *open* otherwise. If W is open, then vertex v_0 is the *initial vertex* of W,

[1] A different, perhaps more natural variant would be to require that for every vertex $v \in V(G)$ there is an $x \in X$ such that G contains a directed path of length at most r from x to v and a directed path of length at most r from v to x. We have opted for our notion of strong domination as it fits more nicely with the concept of strong neighbourhoods in Section 2. But the same algorithmic techniques we use in Section 8 could also be used to design a constant factor approximation algorithm for this different definition of strong domination.

vertex v_k is its *terminal vertex*, and v_0 and v_k are *end-vertices* of W. If all vertices of W are distinct, then W is a *path from* v_0 *to* v_k.

Let G be a digraph, let $v \in V(G)$ and let $r \geq 0$. The *r-out-neighbourhood* of v, denoted by $N^+_{G,r}(v)$, or just $N^+_r(v)$ if G is understood, is defined as the set of vertices u in G such that G contains a path of length at most r from v to u. We write $N^+(v)$ for $N^+_1(v) \setminus \{v\}$.

The *r-in-neighbourhood* of v, denoted by $N^-_{G,r}(v)$, or just $N^-_r(v)$ if G is understood, is defined as the set of vertices u in G such that G contains a path of length at most r from u to v. We write $N^-(v)$ for $N^-_1(v) \setminus \{v\}$.

The *r-strong-neighbourhood* of v, denoted by $\tilde{N}_{G,r}(v)$, or just $\tilde{N}_r(v)$ if G is understood, is defined as the set of vertices u in G such that G contains a closed walk of length at most $2r$ containing u and v.

The *out-degree* of a vertex $v \in V(G)$ is $d^+(v) := |N^+(v)|$, its *in-degree* is $d^-(v) := |N^-(v)|$ and its *degree* is $d(v) := |N^+(v)| + |N^-(v)|$. The *minimum out-degree* of G is defined as $\delta^+(G) := \min\{d^+(v) \ : \ v \in V(G)\}$, *minimum in-degree* and *minimum degree* are defined analogously.

If the edge relation of a digraph G is symmetric, i.e. if $(u,v) \in E(G)$ implies $(v,u) \in E(G)$, then we speak of an *undirected graph*. If G is a digraph, we write $\bar{G}$ for the *underlying undirected graph of* G, which has the same vertices as G and for each arc $(u,v) \in E(G)$ it holds that $(u,v) \in E(\bar{G})$ and $(v,u) \in E(\bar{G})$. Note that $|E(G)| \leq \left|E(\bar{G})\right| \leq 2\,|E(G)|$.

Directed shallow minors. The theory of directed minors is by far not as established as its undirected counterpart, in particular, there are several competing notions of directed minors.

▶ **Definition 2.1.** A *butterfly contraction* is the operation of contracting an edge $e = (u,v)$ where either u has out-degree 1 or v has in-degree 1. A graph H is said to be a *butterfly minor* of a graph G, written $H \preccurlyeq^b G$, if it can be obtained from G by a series of vertex and edge deletions and butterfly contractions.

For undirected graphs, the notion of minors that are obtained by series of vertex and edge deletions and edge contractions can equivalently be defined in terms of minor models. In the directed setting these two notions are different (every butterfly minor is also a directed minor but not vice versa) [23].

▶ **Definition 2.2.** A digraph H has a *directed model* in a digraph G if there is a function δ mapping vertices $v \in V(H)$ of H to sub-graphs $\delta(v) \subseteq G$ and edges $e \in E(H)$ to edges $\delta(e) \in E(G)$ such that
- if $v \neq u$ then $\delta(v) \cap \delta(u) = \emptyset$;
- if $e = (u,v)$ and $\delta(e) = (u', v')$ then $u' \in \delta(u)$ and $v' \in \delta(v)$.

For $v \in V(H)$ let $\mathrm{in}(\delta(v)) := V(\delta(v)) \cap \bigcup_{e=(u,v)\in E(H)} V(\delta(e))$ and $\mathrm{out}(\delta(v)) := V(\delta(v)) \cap \bigcup_{e=(v,w)\in E(H)} V(\delta(e))$.

We furthermore require that for every $v \in V(H)$
- there is a directed path in $\delta(v)$ from any $u \in \mathrm{in}(\delta(v))$ to every $u' \in \mathrm{out}(\delta(v))$;
- there is at least one source vertex $s_v \in \delta(v)$ that reaches every element of $\mathrm{out}(\delta(v))$;
- there is at least one sink vertex $t_v \in \delta(v)$ that can be reached from every element of $\mathrm{in}(\delta(v))$.

We write $H \preccurlyeq^d G$ if H has a directed model in G and call H a *directed minor* of G. We call the sets $\delta(v)$ for $v \in V(H)$ the *branch-sets* of the model.

▶ **Definition 2.3.** For $r \geq 0$, a digraph H is a *depth-r minor* of a digraph G, denoted as $H \preccurlyeq^d_r G$, if there exists a directed model of H in G in which the length of all the paths in the branch-sets of the model are bounded by r.

Finally, we consider the notion of directed topological minors.

▶ **Definition 2.4.** A digraph H is a *topological minor* of a digraph G if there is a function δ mapping vertices $v \in V(H)$ to vertices of $V(G)$ and edges $e \in E(H)$ to directed paths in G such that $\delta(v) \neq \delta(u)$ for all distinct $u, v \in V(H)$, and if $e = (u, v) \in E(H)$, then $\delta(e)$ is a path from $\delta(u)$ to $\delta(v)$ in G which is internally vertex disjoint from all $\delta(e')$ with $e' \in E(H)$, $e' \neq e$. For $r \geq 0$, H is a *topological depth-r minor* of G, written $H \preccurlyeq_r^t G$, if it is a topological minor and all paths $\delta(e)$ have length at most $2r$.

▶ **Lemma 2.5.** *For all digraphs H, G and $r \geq 0$ it holds that $H \preccurlyeq_r^t G$ implies $H \preccurlyeq_r^d G$.*

The key to relating the edge density of depth-r minors and depth-r topological minors in the directed setting is based on a special requirement on directed bipartite graphs.

▶ **Definition 2.6.** A *directed bipartite graph* is a directed graph $G = (A \dot\cup B, E)$ whose vertex set is partitioned into two sets A and B and $E \subseteq A \times B$.

The reason is that the branch sets of directed bipartite graphs can be chosen to have a particularly simple form.

▶ **Definition 2.7.** An *in-branching* is an orientation of a rooted tree with all edges oriented towards the root, an *out-branching* is defined analogously as a tree with all edges oriented away from the root.

▶ **Lemma 2.8** (see [23]). *If H is a directed bipartite graph with $H \preccurlyeq^d G$, we can choose the branch-sets of the model of H in G to be in- or out-branchings. In this case $H \preccurlyeq^d G \Leftrightarrow H \preccurlyeq^b G$.*

3 Classes of bounded expansion

Following [26] (see [23] for the directed case), we define classes of digraphs of bounded expansion by bounding the density of bounded depth minors.

▶ **Definition 3.1.** Let G be a digraph and let $r \geq 0$. The *greatest reduced average degree of rank r* (short *grad*) of G, denoted $\nabla_r(G)$ is

$$\nabla_r(G) := \max\left\{\frac{|E(H)|}{|V(H)|} \ : \ H \preccurlyeq_r^d G\right\}$$

and its *topological greatest average degree of rank r* (short *top-grad*) is

$$\widetilde{\nabla}_r(G) := \max\left\{\frac{|E(H)|}{|V(H)|} \ : \ H \preccurlyeq_r^t G\right\}.$$

▶ **Definition 3.2.** A class $\mathcal{C}$ of digraphs has *bounded expansion* if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that for all $r \geq 0$ it holds that $\nabla_r(G) \leq f(r)$ for all $G \in \mathcal{C}$.

▶ **Example 3.3.** Every class of digraphs of bounded expansion has bounded edge density, hence not every class of acyclic digraphs has bounded expansion. The class of subdivided cliques with all edges oriented away from the subdivision vertices has bounded expansion. The only directed (topological) minors of a graph G from this class are the subgraphs of G, and hence it holds that $\nabla_r(G) \leq 2$ for all $r \geq 0$. The underlying undirected class is not even nowhere dense in the undirected setting.

In the undirected case, we can give an equivalent definition of bounded expansion classes in terms of densities of topological depth-r minors, due to the following theorem proved by Dvořák [10].

▶ **Theorem** (Theorem 3.9 of [10]). *Let $r, d \geq 1$ and let $p = 4(4d)^{(r+1)^2}$. Let G be an undirected graph. If $\nabla_r(G) \geq p$, then $\widetilde{\nabla}_r(G) \geq d$.*

A slightly worse bound can be given in the directed case, the only reason that we do not achieve the same bounds is that we may loose more edges when going to a bipartite subgraph than in the undirected case.

▶ **Lemma 3.4.** *Every digraph G contains a bipartite subgraph H with $d(v) \geq \frac{1}{8}\nabla_0(G)$, for all $v \in V(H)$.*

We can now follow the lines of Dvořák's proof to obtain the following theorem.

▶ **Theorem 3.5.** *Let $r, d \geq 1$ and let $p = 32 \cdot (4d)^{(r+1)^2}$. Let G be a digraph. If $\nabla_r(G) \geq p$, then $\widetilde{\nabla}_r(G) \geq d$.*

▶ **Corollary 3.6.** *A class $\mathcal{C}$ of digraphs has bounded expansion if and only if there is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that for all $r \in \mathbb{N}$ it holds that $\widetilde{\nabla}_r(G) \leq f(r)$ for all $G \in \mathcal{C}$.*

4 Generalised colouring numbers

The *colouring number* $\mathrm{col}(G)$ of an undirected graph G is the minimum integer k such that there is a linear order $<_L$ of the vertices of G for which each vertex v has *back-degree* at most $k-1$, i.e. at most $k-1$ neighbours u with $u <_L v$. It is well-known that for any graph G, the chromatic number $\chi(G)$ satisfies $\chi(G) \leq \mathrm{col}(G)$.

Some generalisations of the colouring number of a graph have been studied in the literature. These include the *arrangeability* [6] used in the study of Ramsey numbers of graphs, the *admissibility* [19], and the *rank* [18] used in the study of the game chromatic number of graphs. Three natural generalisation of the colouring number are the series adm_r, col_r and wcol_r of *generalised colouring numbers* introduced by Kierstead and Yang [20] (see Dvořák [11] for the general definition of adm_r) in the context of colouring games and marking games on graphs. As proved by Zhu [38], these invariants can be used to characterise bounded expansion classes of graphs.

In this section, we define directed versions of the above invariants and show that also directed classes of bounded expansion can be characterised by bounds on the generalised colouring numbers.

▶ **Definition 4.1.** Let G be a digraph. By $\Pi(G)$ we denote the set of all linear orders of $V(G)$. For $L \in \Pi(G)$, we write $u <_L v$ if u is smaller than v with respect to L, and $u \leq_L v$ if $u <_L v$ or $u = v$. Let $u, v \in V(G)$. For a $r \geq 0$, we say that u is *weakly r-reachable* from v with respect to L, if there is a path P of length ℓ, $0 \leq \ell \leq r$, connecting u and v (in either direction) such that u is minimum among the vertices of P (with respect to L). By $\mathrm{WReach}_r[G, L, v]$ we denote the set of vertices that are weakly r-reachable from v w.r.t. L.

Vertex u is *strongly r-reachable* from v with respect to L, if there is a path P of length ℓ, $0 \leq \ell \leq r$, connecting u and v (in either direction) such that $u \leq_L v$ and such that all internal vertices w of P satisfy $v <_L w$. Let $\mathrm{SReach}_r[G, L, v]$ be the set of vertices that are strongly r-reachable from v w.r.t. L. Note that we have $v \in \mathrm{SReach}_r[G, L, v] \subseteq \mathrm{WReach}_r[G, L, v]$.

▶ **Definition 4.2.** For a non-negative integer r, we define the *weak r-colouring number* $\mathrm{wcol}_r(G)$ of G and the *r-colouring number* $\mathrm{col}_r(G)$ of G respectively as

$$\begin{aligned} \mathrm{wcol}_r(G) &:= \min_{L\in\Pi(G)} \max_{v\in V(G)} \big|\mathrm{WReach}_r[G,L,v]\big|, \\ \mathrm{col}_r(G) &:= \min_{L\in\Pi(G)} \max_{v\in V(G)} \big|\mathrm{SReach}_r[G,L,v]\big|. \end{aligned}$$

▶ **Definition 4.3.** For a non-negative integer r, the *r-admissibility* $\mathrm{adm}_r[G,L,v]$ of v w.r.t. L is the maximum size k of a family $\{P_1,\ldots,P_k\}$ of paths of length at most r with one end v, and the other end at a vertex w with $w \le_L v$, and satisfy $V(P_i)\cap V(P_j) = \{v\}$ for all $1 \le i < j \le k$. As for $r > 0$ we can always let the paths end in the first vertex smaller than v, we can assume that the internal vertices of the paths are larger than v. Note that $\mathrm{adm}_r[G,L,v]$ is an integer, whereas $\mathrm{WReach}_r[G,L,v]$ and $\mathrm{SReach}_r[G,L,v]$ are vertex sets. The *r-admissibility* $\mathrm{adm}_r(G)$ of G is

$$\mathrm{adm}_r(G) \quad = \quad \min_{L\in\Pi(G)} \max_{v\in V(G)} \mathrm{adm}_r[G,L,v].$$

As in the undirected setting, we can show that these measures are strongly related.

▶ **Theorem 4.4.** *Let G be a digraph and let $r \ge 1$. Then*

$$\mathrm{col}_r(G) \le 2\cdot(\mathrm{adm}_r(G)-1)^r + 1 \qquad \textit{and} \qquad \mathrm{wcol}_r(G) \le 2\cdot \mathrm{adm}_r(G)^r.$$

In the following, we will prove that the above invariants can also be used to characterise bounded expansion classes of digraphs.

▶ **Definition 4.5.** Let G be a digraph, $X \subseteq V(G)$, $u \in V(G)\setminus X$ and $r \in \mathbb{N}$. The *r-projection* of u onto X is the set $M_r^G(u,X)$ of all vertices $v \in X$ such that there is a directed path between u and v in G (in either direction) of length at most r with all internal vertices in $V(G)\setminus X$.

The next lemma is proved as in the undirected case, compare e.g. to Lemma 2.9 of [9].

▶ **Lemma 4.6.** *Let G be a digraph, $r \ge 0$ and $X \subseteq V(G)$. There exists a set $\mathrm{cl}_r(X) \subseteq V(G)$, called an r-*closure *of X with the following properties. Let $\xi := \lceil 2\nabla_{r-1}(G)\rceil$.*

1. *$X \cap \mathrm{cl}_r(X) = \emptyset$;*
2. *$|\mathrm{cl}_r(X)| \le (r-1)\xi\cdot|X|$; and*
3. *$|M_r^{G-\mathrm{cl}_r(X)}(u,X)| \le \xi$ for all $u \in V(G)\setminus(X\cup \mathrm{cl}_r(X))$.*

Just as in the undirected case, we know how to find an optimal order for the r-admissibility of a digraph (see [11], Algorithm 2, for a proof). For a set $S \subseteq V(G)$ and $v \in S$, let $b_r(S,v)$ be the maximum number of directed paths from v of length at most r intersecting only in v whose internal vertices belong to $V(G)\setminus S$ and whose end-vertices belong to S.

▶ **Lemma 4.7** (see [11]). *Let G be a digraph and let L be the order of $V(G)$ obtained iteratively as follows. Let $S := V(G)$. For $i = n, n-1, \ldots, 1$, choose $v_i \in S$ minimising $p_i = b_r(S,v_i)$ and set $S := S\setminus\{v_i\}$. Then L is optimal for $\mathrm{adm}_r(G)$.*

Clearly, the adm_r-value of the computed linear order is $\max_{1\le i\le n} p_i$. Hence, according to the lemma, if $\mathrm{adm}_r(G) = c$, then we can find a set $S \subseteq V(G)$ such that every $v \in S$ satisfies $b_r(v,S) \ge c$. Based on this obstruction for small admissibility, we are going to relate the grad of G to its admissibility (compare with Lemma 3.4 of [38] and Theorem 3.1 of [15] for the undirected case).

▶ **Theorem 4.8.** *For every digraph G and every $r \in \mathbb{N}$ it holds that* $\mathrm{adm}_r(G) < 16r^2\nabla_{r-1}(G)^4$.

Proof. Let $\xi := 2\nabla_{r-1}(G)$ and assume $\mathrm{adm}_r(G) \geq c := r^2\xi^4$. According to Lemma 4.7, there exists a set $S \subseteq V(G)$ such that every $v \in S$ satisfies $b_r(v, S) \geq c$. We construct $\mathrm{cl}_r(S)$ according to Lemma 4.6, which has size at most $(r-1)\xi \cdot |S|$. We now iteratively contract short paths leading from S to $\mathrm{cl}_r(S)$. Define $\mathcal{P}_0$ as the set of paths between $v \in S$ and $w \in \mathrm{cl}_r(S) \cup S$ of length at most r with all internal vertices in $V(G) \setminus (\mathrm{cl}_r(S) \cup S)$ (if two paths have the same initial and terminal vertex, we add only one of them). We hence have $|\mathcal{P}_0| \geq \frac{c}{2} \cdot |S|$ by assumption.

As long as there exists $P \in \mathcal{P}_i$, contract P to an edge and remove from $\mathcal{P}_i$ all paths which intersect P to obtain $\mathcal{P}_{i+1}$. As every internal vertex u of P satisfies $|M_r^{G-\mathrm{cl}_r(S)}(u, S)| \leq \xi$ by assumption, P can intersect with at most $r\xi^2$ many other paths $P' \in \mathcal{P}_i$. Hence, hence after $i+1$ contractions, we have $|\mathcal{P}_{i+1}| \geq \frac{c}{2}|S| - (i+1)r\xi^2$. Note that we are constructing a graph $H \preccurlyeq^d_{r-1} G$ with vertex set $S \cup \mathrm{cl}_r(S)$, that is, with at most $((r-1)\xi + 1) \cdot |S|$ vertices, which by assumption on $\nabla_{r-1}(G)$ can have at most $\xi/2 \cdot ((r-1)\xi + 1) \cdot |S|$ many edges. This gives a contradiction for $c > 2r\xi^2((r-1)\xi + 1) \cdot \xi/2$, e.g. for $c = r^2\xi^4 = 16r^2\nabla_{r-1}(G)^4$. ◀

To complete the characterisation, we show the following.

▶ **Theorem 4.9.** *For every digraph G and every $r \in \mathbb{N}$ it holds that* $\widetilde{\nabla}_r(G) \leq 16(\mathrm{adm}_{2r}(G) + 1)$.

Proof. Let $c := \mathrm{adm}_{2r}(G) + 1$. Assume towards a contradiction that there is $H \preccurlyeq^t_r G$ of edge density $16c$. Let $H' \subseteq H$ be a bipartite graph with minimum degree at least $2c$, which exists by Lemma 3.4. Let L be an order of $V(G)$ witnessing that $\mathrm{adm}_{2r}(G) = c$. Let v be a principal vertex of H' and let $\{P_1, \ldots, P_t\}$ be the set of paths corresponding to the edges connecting v with its neighbours in H'. At most c paths among $P_1, \ldots, P_t$ contain an internal vertex that is smaller than v with respect to L, as otherwise, $\mathrm{adm}_{2r}(G) \geq c$. Remove all edges e from H' that correspond to a path P_e in G such that the principal vertex is larger than some internal vertex of P_e to obtain a graph H''. By the above argument, H'' has minimum degree at least c. Hence every vertex v reaches in G at least c vertices by paths that are internally vertex disjoint and contain no vertex smaller than c. Considering the largest vertex of H'' in G with respect to L, this is a contradiction to our assumption. ◀

▶ **Corollary 4.10.** *A class $\mathcal{C}$ of digraphs has bounded expansion if, and only if, there is a function $f : \mathbb{N} \to \mathbb{N}$ such that* $\mathrm{wcol}_r(G) \leq f(r)$ *for all $G \in \mathcal{C}$ and all $r \geq 1$.*

Finally, let us note that we can efficiently compute for every graph G from a bounded expansion class $\mathcal{C}$ an order L of $V(G)$ witnessing that the r-admissibility is small. The following lemma, which follows from a simple colour coding argument (see [7, Chapter 5.2 and 5.6]), shows that we can efficiently compute the number $b_r(S, v)$ for every $S \subseteq V(G)$ and every $v \in V(G)$ if this number, or decide that the vertex will not be the next in the order.

▶ **Lemma 4.11.** *There is an algorithm which, given a digraph G, a set $S \subseteq V(G)$, a vertex $v \in S$ and numbers r, k, decides whether there are k disjoint paths from v to vertices in S which are internally disjoint from S in time $2^{k+r} \cdot n^{\mathcal{O}(1)}$.*

▶ **Corollary 4.12.** *Let $\mathcal{C}$ be a class of digraphs of bounded expansion. There is a function g such that for all $r \geq 0$ and all $G \in \mathcal{C}$ we can compute an optimal order for* $\mathrm{adm}_r(G)$ *in time $g(r) \cdot n^{\mathcal{O}(1)}$.*

5 Neighbourhood Complexity

Recently, a measure called *distance-r neighbourhood complexity* was used to characterise classes of undirected bounded expansion [33]. Similar measures can be defined in the directed setting.

▶ **Definition 5.1.** Let G be a digraph, let $X \subseteq V(G)$ and let $r \geq 1$. The *distance-r out-neighbourhood complexity of X in G*, denoted $\nu^+(G)$, is defined by

$$\nu^+(G,X) = \max_{H \subseteq G, X \subseteq V(H)} \left|\{N_r^+(v) \cap X \ : \ v \in V(H)\}\right| .$$

Analogously, one can define the *distance-r in-neighbourhood complexity* when using $N_r^-(v)$ and the *distance-r mixed neighbourhood complexity* when using $(N_r^+(v) \cup N_r^-(v))$ in the above definition.

Closure under subgraphs in the above definition is required to characterise sparse graph classes. Classically, this closure is not part of the definition, when it is e.g. used to define classes of bounded VC-dimension [35, 36, 37].

It was proved in [33] that a class $\mathcal{C}$ of undirected graphs has bounded expansion if and only if for every $r \geq 1$ there is a constant c_r such that for all $G \in \mathcal{C}$ and all $X \subseteq V(G)$ it holds that $\nu(G,X) \leq c_r \cdot |X|$. The analogous statement for classes of directed graphs does not hold, not even for $r = 1$, due to the simple fact that a directed bipartite graph does not contain directed minors other than its subgraphs.

▶ **Theorem 5.2.** *For every $k \geq 1$ there exists a class $\mathcal{C}_k$ of digraphs such that for all $G \in \mathcal{C}_k$ and all $r \geq 0$ it holds that $\nabla_r(G) \leq k$ (hence $\mathcal{C}_k$ has bounded expansion) and for each $G \in \mathcal{C}_k$ there exists $X \subseteq V(G)$ such that $\nu_1^+(G,X) = |X|^k$.*

For every bounded expansion class of digraphs we do obtain polynomial bounds though. To prove the following lemma, we use Lemma 4.6 to show that there are only few high degree vertices in the r-neighbourhood of $X \subseteq V(G)$.

▶ **Theorem 5.3.** *Let $\mathcal{C}$ be a class of digraphs of bounded expansion. Then for all $r \geq 1$ there exists $k \geq 1$ such that for all $G \in \mathcal{C}$ and $X \subseteq V(G)$ we have $\nu_r^+(G,X) \leq |X|^k$. The same statement holds for in-neighbourhood complexity and mixed neighbourhood complexity.*

6 A Splitter Game for Classes of Digraphs of Bounded Expansion

In this section we establish a very useful property of bounded expansion classes of digraphs based on a directed version of a game, known as the *splitter game*, originally introduced as a characterisation of nowhere dense classes of undirected graphs in [16].

Let G be a digraph and let $\ell, m, r \geq 0$. The *(ℓ, m, r)-strong directed splitter game* on G is played by two players, *Connector* and *Splitter*, as follows. Let $G_0 := G$. In round $i+1$ of the game, Connector picks a vertex $v_{i+1} \in V(G_i)$. Then Splitter chooses a subset $W_{i+1} \subseteq V(G_i)$ with $|W_{i+1}| \leq m$. Define G_{i+1} as the induced subgraph of G_i with $V(G_{i+1}) = \tilde{N}_{G_i,r}(v_{i+1}) \setminus W_{i+1}$. Splitter wins if $V(G_{i+1}) = \emptyset$. Otherwise the game continues to the next round. If Splitter has not won after ℓ rounds, then Connector wins.

A *strategy* for Splitter is a function f associating to every partial play $(v_1, W_1, \ldots, v_s, W_s)$ with associated sequence $G_0, \ldots, G_s$ and every move $v_{s+1} \in A_s$ by Connector a move $W_{s+1} \subseteq V(G_s)$ with $|W_{s+1}| \leq m$ for Splitter. A strategy f is a *winning strategy* for Splitter if she wins every play in which she follows the strategy f. If such a winning strategy exists, we say that Splitter *wins* the (ℓ, m, r)-directed splitter game on G.

For undirected graphs the splitter game can be used to characterise nowhere dense classes of graphs. This is not the case for directed graphs, however, short winning strategies can be provided for bounded expansion classes (compare to [22]).

▶ **Theorem 6.1.** *Let G be a graph, let $r \in \mathbb{N}$ and let $\ell = \mathrm{wcol}_{4r}(G)$. Then splitter wins the $(\ell, 1, r)$-strong splitter game.*

Proof. Let L be a linear order that witnesses $\mathrm{wcol}_{4r}(G) = \ell$. First note the following. Let $v \in V(G)$ and let $m \in \tilde{N}_r(v)$ be the L-minimal element of $\tilde{N}_r(v)$. Then for every $w \in \tilde{N}_r(v)$, $G[\tilde{N}_r(v)]$ contains a directed path from w to m of length at most $4r$. Hence, $m \in \mathrm{WReach}(G, L, w)$ for every $w \in \tilde{N}_r(v)$.

We now describe a winning strategy for splitter in the $(\ell, 1, r)$-splitter game. Suppose in round $i + 1 \leq \ell$, connector chooses a vertex $v_{i+1} \in V(G_i)$. Let W_{i+1} (splitter's choice) be the minimum vertex of $\tilde{N}_{G_i,r}(v_{i+1})$ with respect to L. Then for each $u \in N_{G_i,r}(v_{i+1})$ there is a path between u and w_{i+1} of length at most $4r$ that uses only vertices of $N_r^{G_i}(v_{i+1})$. As w_i is L-minimal in $N_r^{G_i}(v_{i+1})$, w_{i+1} is weakly $4r$-reachable from each $u \in N_r^{G_i}(v_{i+1})$. Now let $G_{i+1} := G_i[N_r^{G_i}(v_{i+1}) \setminus \{w_{i+1}\}]$. As w_{i+1} is not part of G_{i+1}, in the next round splitter will choose another vertex which is weakly $4r$-reachable from every vertex of the remaining r-neighbourhood. As $\mathrm{wcol}_{4r}(G) = \ell$, the game must stop after at most ℓ rounds. ◀

Note that unlike the undirected case of nowhere dense classes of graphs, the strong splitter game is not a characterisation of bounded expansion classes, as splitter wins the $(1, 1, 1)$-strong splitter game on every acyclic digraph, but the class of acyclic digraphs does not have bounded expansion.

7 Neighbourhood Covers

Neighbourhood covers of small radius and small size play a key role in the design of many data structures for distributed systems. There is also a deep connection between sparse neighbourhood covers of small radius and sparse graph spanners of low stretch. In this section we will show that classes of digraphs of bounded expansion admit sparse strong neighbourhood covers that can be computed by a fixed-parameter algorithm.

▶ **Definition 7.1.** Let $r \in \mathbb{N}$. A *strong r-neighbourhood cover* $\mathcal{X}$ of a graph G is a mapping $\mathcal{X} : V(G) \to 2^{V(G)}$ such that $G[\mathcal{X}(v)]$ is strongly connected and $\tilde{N}_r(v) \subseteq \mathcal{X}(v)$. We call each $G[\mathcal{X}(v)]$ a *cluster* of $\mathcal{X}$.

The *radius* of a cluster $C := G[\mathcal{X}(v)]$ is defined as the minimal $r \in \mathbb{N}$ for which there is a vertex $w \in V(C)$ and for every $w \in V(C)$, the cluster C contains a directed path of length at most r from w to v and a directed path of length at most r from v to w. The *radius* $\mathrm{rad}(\mathcal{X})$ of a cover $\mathcal{X}$ is the maximum radius of any of its clusters.

The *degree* $d^{\mathcal{X}}(v)$ of v in $\mathcal{X}$ is the number of clusters that contain v. The *maximum degree* $\Delta(\mathcal{X})$ of $\mathcal{X}$ is $\Delta(\mathcal{X}) = \max_{v \in V(G)} d^{\mathcal{X}}(v)$.

The main result of this section is the following theorem.

▶ **Theorem 7.2.** *Let $\mathcal{C}$ be a class of digraphs of bounded expansion. There are functions $f, h : \mathbb{N} \to \mathbb{N}$ such that for all $r \in \mathbb{N}$ and all graphs $G \in \mathcal{C}$, there exists a strong r-neighbourhood cover of radius at most $4r$ and maximum degree at most $f(r)$ and this cover can be computed in time $h(r) \cdot n^{\mathcal{O}(1)}$.*

In the next lemma we use the weak colouring number to prove the existence of sparse neighbourhood covers in bounded expansion classes of digraphs.

▶ **Definition 7.3.** Let G be a digraph, let $<$ be an ordering of $V(G)$ and let $r > 0$. For a vertex $v \in V(G)$ we define $X_r[G, <, v]$ as the set of vertices $w \in V(G)$ such that $v \in \mathrm{WReach}_r[G, <, w]$.

▶ **Lemma 7.4.** *Let G be a graph such that $\mathrm{wcol}_{4r}(G) \leq s$ and let $<$ be an order witnessing this. For $v \in V(G)$, let $m(v)$ be the minimum of $\tilde{N}_r(v)$ with respect to $<$. Then $\mathcal{X} : V(G) \to 2^{V(G)}$ with $\mathcal{X}(v) = X_{4r}[G, <, m(v)]$ is a strong r-neighbourhood cover of G with radius at most $4r$ and maximum degree at most s.*

Proof. Clearly, by construction of the sets $X_{4r}[G, <, v]$ the radius of each cluster is at most $4r$. Furthermore, for $v \in V(G)$ we have $\tilde{N}_r(v) \subseteq \mathcal{X}(v)$. To see this, let $m(v)$ be the minimum of $\tilde{N}_r(v)$ with respect to $<$. Then $m(v)$ is weakly $4r$-reachable from every $w \in N_r(v) \setminus \{m(v)\}$. There is a path from w to v of length at most $2r$ (on the closed walk containing both w and v) and a path from v to $m(v)$ of length at most $2r$ (again on a closed walk containing the two). Now the concatenation of the two paths is a walk of length at most $4r$ which contains as a sub-walk which is a path of length at most $4r$. As this path uses only vertices of $\tilde{N}_r(v)$ and $m(v)$ is the minimum element, we conclude that $\tilde{N}_r(v) \subseteq X_{4r}[G, <, m(v)]$. Finally observe that for every $v \in V(G)$,

$$\begin{aligned} d^{\mathcal{X}}(v) &= |\{u \in V(G) : v \in X_{4r}[G, <, u]\}| \\ &= |\{u \in V(G) : u \in \mathrm{WReach}_{4r}[G_<, v]\}| = |\mathrm{WReach}_{4r}[G_<, v]| \leq s. \end{aligned}$$ ◀

Now to prove Theorem 7.2 it suffices to note that the clusters $X_{4r}[G, <, v]$ can be computed in the desired time. According to Corollary 4.12, we can compute an order $<$ which is optimal for $\mathrm{adm}_{4r}(G)$ in time $g(4r) \cdot n^{\mathcal{O}(1)}$. According to Theorem 4.4, this order also satisfies $|\mathrm{WReach}_{4r}[G, <, v]| \leq f(r)$ for properly defined function f. We order the vertices of G in order $<$. Now, to compute $X_{4r}[G, <, v]$ for a vertex v, we just have to perform the first $4r$ levels of a breadth-first search around v (where we follow paths either in the direction of the edges or against the direction of the edges) which stops in every branch if a vertex smaller than v is encountered. Defining g accordingly finishes the proof of the theorem.

8 Constant-Factor Approximation Algorithms for Strong Dominating Sets

In this section we prove that strong dominating sets can be approximated up to a constant factor on any class $\mathcal{C}$ of directed bounded expansion. Our approach is inspired by [11].

▶ **Definition 8.1** (Strong r-Dominating Sets).

1. Let $r \geq 1$ and let G be a digraph. A vertex $v \in V(G)$ *strongly-r-dominates* a vertex $u \in V(G)$ if there is a closed walk of length at most $2r$ in G containing u and v.
2. A *strong-r-dominating set* is a set $X \subseteq V(G)$ such that every vertex in G is strongly dominated by a vertex in X.
3. The *strong r-domination number* of G, denoted $\mathrm{sdom}_r(G)$, is the minimum size of a strong r-dominating set of G.

▶ **Theorem 8.2.** *Let $\mathcal{C}$ be a class of digraphs of directed bounded expansion. Let $r \geq 1$. There is a polynomial time constant factor approximation algorithm for strong r-dominating sets. More precisely, for every value of r, there is an algorithm running in time $g(r) \cdot n^{\mathcal{O}(1)}$ for some function g which, on input $G \in \mathcal{C}$ computes a strong-r-dominating set $D \subseteq V(G)$ of order at most $\mathrm{wcol}_{4r}(G)^2 \cdot \mathrm{sdom}_r(G)$.*

In the remainder of this section we prove Theorem 8.2. Let r be given. An *r-obstruction set* is a set $X \subseteq V(G)$ such that for any distinct $x, y \in X$, there are *no* two closed directed walks $W_1, W_2 \subseteq V(G)$, each of length at most $2r$, such that $W_1 \cap W_2 \neq \emptyset$ and $x \in W_1$ and $y \in W_2$. Note that we do not require $W_1 \neq W_2$. We call a pair u, v of vertices which form an r-obstruction set $\{u, v\}$ *r-separated*. Otherwise, we call the pair u, v *r-dependent*.

For a set $X \subseteq V(G)$, we define $\mathrm{Sdom}_r(X)$ as the set of vertices $v \in V(G)$ such that v is strongly-r-dominated by a vertex in X.

As no two distinct vertices of an obstruction set lie on a closed walk of length at most $2r$, no two vertices from the set can be dominated strongly r-dominated by a single vertex. Hence, if G contains an obstruction set of order k then $\mathrm{sdom}_r(G) \geq k$.

▶ **Lemma 8.3.** *There exists a polynomial time algorithm which, given a number $r \geq 1$ and a digraph G together with an ordering L witnessing $\mathrm{wcol}_{4r}(G)$, computes an obstruction set $X \subseteq V(G)$ of order k, for some k, and an r-dominating set of order at most $\mathrm{wcol}_{4r}(G)^2 \cdot k$.*

Proof. Let L be an ordering of G witnessing $\mathrm{wcol}_{4r}(G)$. We greedily compute sets $A, D, S \subseteq V(G)$ as follows. Start with $A_0 = D_0 := \emptyset$ and $S_0 := V(G)$. Now suppose A_i, D_i, S_i have already been defined. If $S_i = \emptyset$ then the construction stops here. Otherwise, let a be the $<_L$-minimal element of S_i and define $A_{i+1} := A_i \cup \{a\}$, $D_{i+1} := D_i \cup \mathrm{WReach}_{4r}(G, L, a)$. Finally, we define $S_{i+1} := S_i \setminus \mathrm{Sdom}_r(\mathrm{WReach}_{4r}(G, L, a))$.

Now let i be minimal such that $S_i = \emptyset$. Such an index i exists as we add a vertex to A – and remove it from S – at each iteration.

Clearly, $D := D_i$ is a strong r-dominating set of G and $|D| \leq \mathrm{wcol}_{4r}(G) \cdot |A|$. We will show next that G contains an r-obstruction set $X \subseteq A$ of order $\frac{1}{\mathrm{wcol}_{4r}(G)} \cdot |A|$. Hence, D is a factor $(\mathrm{wcol}_{4r}(G))^2$ approximation of the strong r-domination number of G.

We construct a digraph H with vertex set A and edges (u, v) if $u <_L v$ and u and v are r-dependent. We will show next that the maximal out-degree of H is $< \mathrm{wcol}_{4r}(G)$. For $a \in A$ let $a_1, \ldots, a_l$ be the elements of A which are L-smaller than a and such that a and a_i are r-dependent, for all $1 \leq i \leq l$. We claim that $l < \mathrm{wcol}_{4r}(G)$.

For every $1 \leq i \leq l$ let $W_1 := W_1(i)$ and $W_2 := W_2(i)$ be two closed directed walks in G of length at most $2r$ which intersect each other and such that $a \in V(W_1)$ and $a_i \in V(W_2)$. Such walks exist as a_i and a are r-dependent. Let $z = z(i)$ be the L-minimal vertex in $V(W_1 \cup W_2)$. Note that $z \in \mathrm{WReach}_{4r}(G, L, a)$. For, $W_1 \cup W_2$ form a strongly connected subgraph on at most $4r$ vertices containing z and a and hence there is a directed path from a to z of length at most $4r$. As z is the L-minimal element of $W_1 \cup W_2$, this implies that $z \in \mathrm{WReach}_{4r}(G, L, a)$.

We claim first that $z \notin V(W_1)$. For otherwise, z would strongly r-dominate a and hence in the i-th iteration of the algorithm, a would have been removed from S_i, contradicting the fact that $a \in A$. Thus, $z(i) \in W_2(i)$ for all $1 \leq i \leq l$.

Now suppose $z(i) = z(j)$ for some $1 \leq i < j \leq s$. But then again a_j, which is contained in $W_2(j)$, is strongly r-dominated by $z(i)$ and hence a_j would have been removed from S_i at step i. Hence, $z(i) \neq z(j)$for all $1 \leq i \neq j \leq s$. It follows that $l \leq |\mathrm{WReach}_{4r}(G, L, a)| - 1 < \mathrm{wcol}_{4r}(G)$.

This shows that the maximum outdegree of any vertex in H is $< \mathrm{wcol}_{4r}(G)$. Hence, H is $\mathrm{wcol}_{4r}(G) - 1$-degenerate and therefore $\mathrm{wcol}_{4r}(G)$-colourable. Thus, the colour class C of maximal size contains at least $\frac{1}{\mathrm{wcol}_{4r}(G)} \cdot |A|$ elements of A and forms an r-obstruction set, witnessing that the strong r-domination number of G is at least $\frac{1}{\mathrm{wcol}_{4r}(G)^2}|A|$. This completes the proof of the lemma. ◀

Theorem 8.2 now follows immediately from the previous lemma together with Corollary 4.12. Note, however, that our algorithm runs in polynomial time for any fixed r but it depends exponentially on $\mathrm{wcol}_{4r}(\mathcal{C})$. The reason is that we currently do not know how to compute a good approximation of the wcol-ordering in polynomial time.

References

1 Saeed Akhoondian Amiri, Lukasz Kaiser, Stephan Kreutzer, Roman Rabinovich, and Sebastian Siebertz. Graph searching games and width measures for directed graphs. In *32nd International Symposium on Theoretical Aspects of Computer Science, STACS 2015, March 4-7, 2015, Garching, Germany*, pages 34–47, 2015.

2 Jørgen Bang-Jensen and Gregory Z. Gutin. *Digraphs: theory, algorithms and applications*. Springer Science & Business Media, 2008.

3 Dietmar Berwanger, Anuj Dawar, Paul Hunter, and Stephan Kreutzer. Dag-width and parity games. In *Symp. on Theoretical Aspects of Computer Science (STACS)*, 2006.

4 Dietmar Berwanger, Anuj Dawar, Paul Hunter, Stephan Kreutzer, and Jan Obdrzálek. The dag-width of directed graphs. *J. Comb. Theory, Ser. B*, 102(4):900–923, 2012. doi:10.1016/j.jctb.2012.04.004.

5 Dietmar Berwanger, Erich Grädel, Lukasz Kaiser, and Roman Rabinovich. Entanglement and the complexity of directed graphs. *Theor. Comput. Sci.*, 463:2–25, 2012. doi:10.1016/j.tcs.2012.07.010.

6 Guantao Chen and Richard H. Schelp. Graphs with linearly bounded ramsey numbers. *J. Combin. Theory Ser. B*, 57:138–149, 1993.

7 Marek Cygan, Fedor V. Fomin, Ł ukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michał Pilipczuk, and Saket Saurabh. *Parameterized Algorithms*. Springer, 2015.

8 Anuj Dawar and Stephan Kreutzer. Domination problems in nowhere-dense classes of graphs. In *Foundations of software technology and theoretical computer science (FSTTCS)*, pages 157–168, 2009.

9 Pål Grønås Drange, Markus S. Dregi, Fedor V Fomin, Stephan Kreutzer, Daniel Lokshtanov, Marcin Pilipczuk, Michał Pilipczuk, Felix Reidl, Saket Saurabh, Fernando Sánchez Villaamil, Sebastian Siebertz, et al. Kernelization and sparseness: the case of dominating set. *arXiv preprint arXiv:1411.4575*, 2014.

10 Zdenek Dvorák. Asymptotical structure of combinatorial objects. *Charles University, Faculty of Mathematics and Physics*, 51:357–422, 2007.

11 Zdeněk Dvořák. Constant-factor approximation of the domination number in sparse graphs. *European Journal of Combinatorics*, 34(5):833–840, 2013.

12 Zdenek Dvorak, Daniel Král, and Robin Thomas. Testing first-order properties for subclasses of sparse graphs. *J. ACM*, 60(5):36, 2013.

13 Jakub Gajarskỳ, Petr Hliněnỳ, Jan Obdržálek, Sebastian Ordyniak, Felix Reidl, Peter Rossmanith, Fernando Sanchez Villaamil, and Somnath Sikdar. Kernelization using structural parameters on sparse graph classes. In *European Symposium on Algorithms*, pages 529–540. Springer, 2013.

14 Robert Ganian, Petr Hliněný, Joachim Kneis, Daniel Meister, Jan Obdržálek, Peter Rossmanith, and Somnath Sikdar. Are there any good digraph width measures? In Venkatesh Raman and Saket Saurabh, editors, *Parameterized and Exact Computation*, volume 6478 of *Lecture Notes in Computer Science*, pages 135–146. Springer Berlin Heidelberg, 2010. doi:10.1007/978-3-642-17493-3_14.

15 Martin Grohe, Stephan Kreutzer, Roman Rabinovich, Sebastian Siebertz, and Konstantinos Stavropoulos. Colouring and covering nowhere dense graphs. In *Graph-Theoretic Concepts in Computer Science – 41st International Workshop, WG 2015, Garching, Germany, June 17-19, 2015, Revised Papers*, pages 325–338, 2015.

16 Martin Grohe, Stephan Kreutzer, and Sebastian Siebertz. Deciding first-order properties of nowhere dense graphs. In *Proceedings of the 46th Annual ACM Symposium on Theory of Computing*, pages 89–98. ACM, 2014.

17 Thor Johnson, Neil Robertson, Paul D. Seymour, and Robin Thomas. Directed tree-width. *Journal of Combinatorial Theory, Series B*, 82(1):138–154, 2001.

18 Hal A. Kierstead. A simple competitive graph coloring algorithm. *J. Combin. Theory Ser. B*, 78:57–68, 2000.

19 Hal A. Kierstead and William T. Trotter. Planar graph coloring with an uncooperative partner. In W.T. Trotter, editor, *Planar Graphs*, volume 9 of *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.*, pages 85–93. AMS, 1993.

20 Hal A. Kierstead and Daqing Yang. Orderings on graphs and game coloring number. *Order*, 20:255–264, 2003.

21 Stephan Kreutzer and Sebastian Ordyniak. Digraph decompositions and monotonicity in digraph searching. In *Graph-Theoretic Concepts in Computer Science, 34th International Workshop, WG 2008, Durham, UK, June 30 – July 2, 2008. Revised Papers*, pages 336–347, 2008.

22 Stephan Kreutzer, MichałPilipczuk, Roman Rabinovich, and Sebastian Siebertz. The generalised colouring numbers on classes of bounded expansion. In *41st International Symposium on Mathematical Foundations of Computer Science, MFCS 2016, August 22-26, 2016 – Kraków, Poland*, pages 85:1–85:13, 2016.

23 Stephan Kreutzer and Siamak Tazari. Directed nowhere dense classes of graphs. In *Proceedings of the twenty-third annual ACM-SIAM symposium on Discrete Algorithms*, pages 1552–1562. SIAM, 2012.

24 Michael Lampis, Georgia Kaouri, and Valia Mitsou. On the algorithmic effectiveness of digraph decompositions and complexity measures. *Discrete Optimization*, 8(1):129–138, 2011.

25 Daniel Meister, Jan Arne Telle, and Martin Vatshelle. Characterization and recognition of digraphs of bounded kelly-width. In *Workshop on Graph Theoretical Aspects of Computer Science (WG)*, pages 270–279, 2007. doi:10.1007/978-3-540-74839-7_26.

26 Jaroslav Nešetřil and Patrice Ossona de Mendez. Grad and classes with bounded expansion I. decompositions. *European Journal of Combinatorics*, 29(3):760–776, 2008.

27 Jaroslav Nešetřil and Patrice Ossona de Mendez. On nowhere dense graphs. *European Journal of Combinatorics*, 32(4):600–617, 2011.

28 Jaroslav Nešetřil and Patrice Ossona De Mendez. On low tree-depth decompositions. *Graphs and Combinatorics*, 31(6):1941–1963, 2015.

29 Jaroslav Nešetřil and Patrice Ossona de Mendez. Grad and classes with bounded expansion I–III. *European Journal of Combinatorics*, 29, 2008. Series of 3 papers appearing in volumes (3) and (4).

30 Jan Obdrzálek. Dag-width: connectivity measure for directed graphs. In *Symp. on Discrete Algorithms (SODA)*, pages 814–821, 2006.

31 Ran Raz and Shmuel Safra. A sub-constant error-probability low-degree test, and a sub-constant error-probability PCP characterization of NP. In *Proceedings of the twenty-ninth annual ACM Symposium on Theory of Computing*, page 475–484, 1997.

32 Felix Reidl. *Structural sparseness and complex networks*. Dr., Aachen, Techn. Hochsch., Aachen, 2016.

33 Felix Reidl, Fernando Sánchez Villaamil, and Konstantinos Stavropoulos. Characterising bounded expansion by neighbourhood complexity. *arXiv preprint arXiv:1603.09532*, 2016.

34 Mohammad Ali Safari. D-width: A more natural measure for directed tree width. In *Proc. of Mathematical Foundations of Computer Science (MFCS)*, number 3618 in Lecture Notes in Computer Science, pages 745–756, 2005.

35 Norbert Sauer. On the density of families of sets. *Journal of Combinatorial Theory, Series A*, 13(1):145–147, 1972.

36 Saharon Shelah. A combinatorial problem; stability and order for models and theories in infinitary languages. *Pacific Journal of Mathematics*, 41(1):247–261, 1972.

37 Vladimir N. Vapnik and Alexey Y. Chervonenkis. On the uniform convergence of relative frequencies of events to their probabilities. In *Measures of Complexity*, pages 11–30. Springer, 2015.

38 Xuding Zhu. Colouring graphs with bounded generalized colouring number. *Discrete Mathematics*, 309(18):5562–5568, 2009.