Minkowski Games*

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Abstract

We introduce and study Minkowski games. In these games, two players take turns to choose positions in $\mathbb{R}^d$ based on some rules. Variants include boundedness games, where one player wants to keep the positions bounded (while the other wants to escape to infinity), and safety games, where one player wants to stay within a given set (while the other wants to leave it).

We provide some general characterizations of which player can win such games, and explore the computational complexity of the associated decision problems. A natural representation of boundedness games yields coNP-completeness, whereas the safety games are undecidable.

1998 ACM Subject Classification F.3.1 Specifying and Verifying and Reasoning about Programs

Keywords and phrases Control in $\mathbb{R}^d$, determinacy, polytopic/arbitrary, coNP-complete, undecidable

Digital Object Identifier 10.4230/LIPIcs.STACS.2017.50

1 Introduction

Minkowski games. In this paper we define and study *Minkowski games*. A *Minkowski play* is an infinite duration interaction between two players, called Player A and Player B, in the space $\mathbb{R}^d$. A *move* in a Minkowski play is a subset of $\mathbb{R}^d$. Player A has a set of moves $\mathcal{A}$ and Player B has a set of moves $\mathcal{B}$. The play starts in a position $a_0 \in \mathbb{R}^d$ and is played for an infinite number of rounds as follows. For a round starting in position a, Player A chooses $A \in \mathcal{A}$ and Player B chooses a vector b in $a + A$, where $+$ denotes the Minkowski sum. Next, Player B chooses $B \in \mathcal{B}$ and Player A chooses a vector a' in $b + B$. Then a new round starts in the position a'. The *outcome* of a Minkowski play is thus an infinite sequence of vectors $a_0b_0a_1b_1 \dots a_nb_n \dots$ obtained during this infinite interaction. Each outcome is either winning for Player A or for Player B, and this is specified by a winning condition.

We consider two types of winning conditions. First, we consider *boundedness*: an outcome $a_0b_0a_1b_1 \dots a_nb_n \dots$ is winning for Player A in the boundedness game if there is a bounded subset Safe $\subseteq \mathbb{R}^d$ such that the outcome stays in Safe, *i.e.* for all $i \geq 0$, $a_i \in$ Safe and $b_i \in$ Safe, otherwise it is winning for Player B. Second, we consider *safety*: given a subset Safe $\subseteq \mathbb{R}^d$, an outcome is winning for Player A if it stays in Safe, otherwise Player B wins.

In this paper $\mathcal{A}$ and $\mathcal{B}$ are finite sets $\{A_1, A_2, \dots, A_{n_A}\}$ and $\{B_1, B_2, \dots, B_{n_B}\}$, and both the moves and Safe in the safety Minkowski games are bounded. A long version [16] of this

* This work was supported by the ERC inVEST (279499) project.

34th Symposium on Theoretical Aspects of Computer Science (STACS 2017).
Editors: Heribert Vollmer and Brigitte Vallée; Article No. 50; pp. 50:1–50:13
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

paper, providing comprehensive lemmas and proofs, relaxes these assumptions when possible and gives counterexamples otherwise.

The Minkowski games are a natural mathematical abstraction to model the interaction between two agents taking actions, modeled by moves, with imprecision as the adversary resolves nondeterminism by picking a vector in the move chosen by the other player.[1] Perhaps more importantly, the appeal of Minkowski games comes also from their simple and natural definition. We provide in this paper general results about these games and study several of their incarnations in which moves are given as (*i*) bounded polyhedral sets, (*ii*) sets defined using the first-order theory of the reals, or (*iii*) represented as compact or overt sets as defined in computable analysis [17]. Our results are as follows.

Results. We establish a necessary and sufficient condition for Player A to have a winning strategy in a *boundedness Minkowski game* with finitely many moves and give a simple proof (in comparison with Borel determinacy) that these games are determined (Theorem 5). We then turn our attention to computational complexity aspects of determining the winner of a game, *i.e.* who has a winning strategy. The necessary and sufficient condition that we have identified leads to a coNP solution when the moves are given as bounded rational polyhedral sets and we provide matching lower bounds (Theorem 6). These results hold both for moves represented by sets of linear inequalities and moves represented as the convex hulls of a finite set of points. Additionally, we show that for every fixed dimension d, determining the winner can be done in polynomial time (Corollary 11). When the moves are defined using the first-order theory of the reals, determining the winner of a boundedness game is shown to be 2ExpTime-complete (Proposition 12). Finally, in the computable analysis setting the problem is semi-decidable (Proposition 14), and this is the best that we can hope for.

We characterize the set of winning positions in a *safety Minkowski game* as the greatest fixed point of an operator that removes the points where Player B can provably win (in finitely many rounds). We show that this greatest fixed point can be characterized by an approximation sequence of at most ω steps (Proposition 17). This leads to semi-decidability in the general setting of computable analysis (Proposition 18). Then we show that identifying the winner in a safety Minkowski game is undecidable even for finite sets of moves that are given as bounded rational polyhedral sets (Theorem 19).

Motivations and related works. Infinite duration games are commonly used as mathematical framework for modeling the controller synthesis problem for reactive systems [15]. For reactive systems embedded in some physical environment, games played on hybrid automata have been considered, see e.g. [9] and references therein. In such a model, one controller interacts with an environment whose physical properties are modeled by valuations of d real-valued variables (vectors in $\mathbb{R}^d$). Most of the problems related to the synthesis of controllers for hybrid automata are undecidable [9]. Restricted subclasses with decidable properties, such as timed automata and initialized rectangular automata have been considered [11, 8]. Most of the undecidability properties of those models rely on the coexistence of continuous and discrete evolutions of the configurations of hybrid automata. The one-sided version of the model of Minkowski games (where $\mathcal{B} = \{\{0\}\}$) can be seen as a restricted form of an hybrid games in which each continuous evolution is of a unique fixed duration and space independent (such as in linear and rectangular hybrid automata). It is usually called discrete

[1] See further discussions on the practical appeal of these games for modeling systems evolving in multi-dimensional spaces when we report on related works.

time control in this setting. To the best of our knowledge, the closest models to Minkowski games that have been considered in the literature so far are Robot games defined by Doyen et al. in [7] and Bounded-Rate Multi-Mode Systems introduced by Alur et al. in [2, 1]. Minkowski games generalize robot games: there the moves are always singletons given as integer vectors. While we show that the Safety problem is undecidable for bounded safety objectives, it is easy to show that this problem is actually decidable for robot games. However, in [7] they investigate reachability of a specific position rather than safety conditions as we do here. Reachability was proven undecidable in [13] even for two-dimensional robot games. Boundedness objectives have not been studied for Robot games.

Bounded-Rate Multi-Mode Systems (BRMMS) are a restricted form of hybrid systems that can switch freely among a finite number of modes. The dynamics in each mode is specified by a bounded set of possible rates. The possible rates are either given by convex polytopes or as finite set of vectors. There are several differences with Minkowski games. First BRMMS are asymmetric and are thus closer to the special case of one-sided Minkowski games. Second, the actions in BRMMS are given by a mode and a time delay $\delta \in \mathbb{R}$ while the time elapsing in our model can be seen as uniform and fixed. The ability to choose delays that are as small as needed makes the safety control problem for BRMMS with modes given as polytopes decidable while we show that the safety Minkowski games with moves defined by polytopes are undecidable. The discrete time control of BRMMS, which is more similar to the safety Minkowski games, has been solved only for modes given as finite sets of vectors and left open for modes given as polytopes. Our undecidability results can be easily adapted to the discrete time control of BRMMS and thus solves the open question left in that paper. Boundedness objectives have not been studied for BRMMS.

2 Preliminaries

Linear constraints. Let $d \in \mathbb{N}_{>0}$, and $X = \{x_1, x_2, \ldots, x_d\}$ be a set of variables. A *linear term* on X is a term of the form $\alpha_1 x_1 + \cdots + \alpha_d x_d$ where $x_i \in X$, $\alpha_i \in \mathbb{R}$ for all i, $1 \leq i \leq n$. A *linear constraint* is a formula $\alpha_1 x_1 + \cdots + \alpha_d x_d \sim c$, where $\sim \in \{<, \leq, =, \geq, >\}$, that compares a linear term with a constant $c \in \mathbb{R}$. Given a valuation $v : X \to \mathbb{R}$, amounting to a vector in $\mathbb{R}^d$, we write $v \models \alpha_1 x_1 + \cdots + \alpha_d x_d \sim c$ iff $\alpha_1 v(x_1) + \cdots + \alpha_d v(x_d) \sim c$. Given a linear constraint $\phi \equiv \alpha_1 x_1 + \cdots + \alpha_d x_d \sim c$, we write $[\![\phi]\!] = \{v \in \mathbb{R}^d \mid v \models \alpha_1 x_1 + \cdots + \alpha_d x_d \sim c\}$. The constraint is called rational, if c and all the α_i are in $\mathbb{Q}$.

Polyhedra, polytopes, convex hull. Given a finite set $\mathcal{H} = \{\phi_1, \phi_2, \ldots, \phi_n\}$ of linear constraints, we note $[\![\mathcal{H}]\!] = \{v \in \mathbb{R}^d \mid \forall \phi \in \mathcal{H} : v \models \phi\}$ the set of vectors that satisfies all the linear constraints in $\mathcal{H}$. Such a set is a convex set and is usually called a *polyhedra*. In the special case that is bounded, then it is called a *polytope*. When a polytope is *closed*, then it is well-known that it can not only be represented by a finite set of linear inequalities that are all non-strict but also as the *convex hull* of a finite set of (extremal) vectors. The convex hull of a subset of a $\mathbb{R}$-vector space is noted and defined as follows:

$$\mathsf{CH}(\mathcal{V}) := \left\{ \sum_{i=0}^{n} \alpha_i x_i \;\middle|\; n \in \mathbb{N} \wedge \sum_{i=0}^{n} \alpha_i = 1 \wedge \forall i (x_i \in \mathcal{V} \wedge \alpha_i \geq 0) \right\}.$$

Carathéodory's theorem says that for all $\mathcal{V} \subseteq \mathbb{R}^d$, every point in $\mathsf{CH}(\mathcal{V})$ is a convex combination of at most $d+1$ points from $\mathcal{V}$. As a consequence, the n ranging over $\mathbb{N}$ in the definition of the convex hull can safely be replaced with fixed d.

Let P be a closed polytope. P has two families of representations: its H-representations are the finite sets of linear inequalities $\mathcal{H}$ such that $[\![\mathcal{H}]\!] = P$, and its V-representations are the finite sets of vectors $\mathcal{V}$ such that $\mathsf{CH}(\mathcal{V}) = P$. Some algorithmic operations are easier to perform on one representation or on the other. In general there is no polynomial time translation from one representation to the other unless P = NP. Nevertheless, such a polynomial time translation exists for fixed dimension. (We denote by $\mathsf{Ver(P)}$ the extremal points, *i.e.*, the vertices of a polytope P. It is the minimal set whose convex hull equals P.)

▶ **Theorem 1** ([4]). *Let $d \in \mathbb{N}$. There exists a polynomial time algorithm that given a H-representation of a rational closed polytope P of dimension d, computes a V-presentation of P, and conversely, there exists a polynomial time algorithm that given a V-representation of P, computes a H-presentation of P.*

Minkowski sum. For subsets $A, B \in \mathbb{R}^d$ their Minkowski sum $A + B$ is defined as $\{a + b \mid a \in A \wedge b \in B\}$. The Minkowski sum inherits commutativity and associativity from the usual sum of vectors. The set $\{0\}$ is the neutral element, but there are no inverse elements in general. If $A = \{a\}$ then $A + B$ (resp. $B + A$) is written $a + B$ (resp. $B + a$) in a slight abuse of notation. It is straightforward to prove that $\mathsf{CH}(A) + \mathsf{CH}(B) = \mathsf{CH}(A + B)$. Especially, if A and B are convex, so is $A + B$. While $A + A$ may be a strict superset of $2A := \{2a \mid a \in A\}$ in general, for convex A we find $A + A = 2A$.

Minkowski games – Strategies. We have described in the introduction how the players interact in Minkowski games by choosing in each round a move and by resolving nondeterminism among the moves chosen by the other player. We now formally define the notions of strategies for each player, and the associated outcomes. When playing Minkowski games, players are applying *strategies*. In a game with moves $\mathcal{A}$ and $\mathcal{B}$, strategies for the two players are defined as follows. A strategy for Player A is a function

$$\lambda_A : (\mathbb{R}^d)^* \to (\mathcal{A} \cup (\mathbb{R}^d)^*) \times \mathcal{B} \to \mathbb{R}^d$$

that respects the following consistency constraint: for all finite sequences of positions $\rho \in (\mathbb{R}^d)^*$ that ends in $v \in \mathbb{R}^d$, and moves $B \in \mathcal{B}$, $\lambda_A(\rho, B) \in v + B$. Symmetrically, a strategy for Player B is a function $\lambda_B : (\mathbb{R}^d)^* \to (\mathcal{B} \cup (\mathbb{R}^d)^*) \times \mathcal{A} \to \mathbb{R}^d$ with the symmetric consistency constraint. The play $a_0 b_0 a_1 b_1 \ldots a_n b_n \ldots$ induced by the strategies λ_A and λ_B is defined inductively by (for all $i \geq 0$)

$$b_i := \lambda_B(a_0 b_0 a_1 b_1 \ldots a_i, \lambda_A(a_0 b_0 a_1 b_1 \ldots a_i)) \text{ and } a_{i+1} := \lambda_A(a_0 b_0 \ldots a_i b_i, \lambda_B(a_0 b_0 \ldots a_i b_i)).$$

Winning conditions and variants of Minkowski games. By fixing the rule that determines who wins a Minkowski play, we obtain *Minkowski games*. Here we consider three types of Minkowski games, as below.

▶ **Definition 2.** A *boundedness Minkowski game* is a pair $\langle \mathcal{A}, \mathcal{B} \rangle$ of sets of moves in $\mathbb{R}^d$ for Player A and Player B. A play in a boundedness Minkowski game starts in some irrelevant $a_0 \in \mathbb{R}^d$, and the resulting play $a_0 b_0 a_1 b_1 \ldots a_n b_n \ldots$ is winning for Player A if there exists a bounded subset Safe of $\mathbb{R}^d$ such that $a_i, b_i \in \mathsf{Safe}$ for all $i \in \mathbb{N}$, otherwise Player B wins the game. The associated decision problem asks if Player A has a strategy λ_A which is winning against all the strategies λ_B of Player B.

▶ **Definition 3.** A *safety Minkowski game* is defined by $\langle \mathcal{A}, \mathcal{B}, \mathsf{Safe}, a_0 \rangle$, where $\mathcal{A}$ and $\mathcal{B}$ are sets of moves for Player A and Player B, $\mathsf{Safe} \subseteq \mathbb{R}^d$ (bounded unless stated otherwise), and $a_0 \in \mathsf{Safe}$ is the initial position. A play in a safety Minkowski game starts in a_0, and the resulting play $a_0 b_0 a_1 b_1 \dots a_n b_n \dots$ is winning for Player A if $a_i, b_i \in \mathsf{Safe}$ for all $i \in \mathbb{N}$, otherwise Player B wins the game. The associated decision problem asks if Player A has a strategy λ_A which is winning against all the strategies λ_B of Player B.

A game is *single-sided* if $\mathcal{B} = \{\{0\}\}$, i.e. if Player B has only one trivial move. We use single-sided games to show that several of our lower-bounds hold for this subclass of games.

3 General results on the boundness problem

We first consider the special case of one-sided boundedness Minkowski games and provide a sufficient (and necessary) condition for Player A to win. The proof showcases some ideas that we use to fully characterize the general case. The general characterization in particular implies that the condition for the one-sided case is necessary.

▶ **Proposition 4.** We consider a one-sided boundedness Minkowski game $\langle \mathcal{A}, \{0\} \rangle$ where $\mathcal{A} = \{A_1, \dots, A_n\}$ and such that $0 \in \mathsf{CH}((x_i)_{1 \leq i \leq n})$ for all tuples $(x_i)_{1 \leq i \leq n}$ in $A_1 \times \dots \times A_n$. Then Player A wins the boundedness game.

Proof. We describe the current state by some list of pairs $(x_i, \alpha_i)_{i \leq n}$ such that $x_i \in A_i$ and $\alpha_i \in [0,1]$. We keep two invariants satisfied throughout the play: First, it will always be the case that the current position is equal to $\sum_{i \leq n} \alpha_i x_i$, which by boundedness of each A_i implies that Player A wins. Second, we maintain the invariant that there is some $k \leq n$ with $\alpha_k = 0$. Initially, the choice of the x_i is arbitrary, and all α_i are 0. This ensures that the strategy we describe for Player A is well-defined.

On his turn, Player A plays some A_k for k with $\alpha_k = 0$. Player B reacts with some $x'_k \in A_k$, and we set $x_k := x'_k$ and $\alpha_k := 1$.

If immediately after the move, no α_i is currently 0, we write a convex combination $0 = \sum_{i \leq n} \beta_i x_i$, which is possible by assumption. Let $r := \max_{i \leq n} \frac{\beta_i}{\alpha_i}$, and then update $\alpha_i = \alpha_i - r^{-1}\beta_i$. By the choice of the β_i, this leaves $\sum_{i \leq n} \alpha_i x_i$ unchanged. The choice of r ensures that $\alpha_i \in [0,1]$ remains true, and more over, after the updating process, there is some $k \leq n$ with $\alpha_k = 0$. Thus, the invariant is true again after the updating process. ◀

If $0 \notin \mathsf{CH}((x_i)_{i \leq m})$ for some $x_i \in A_i$, then Player B could win by playing x_i as response to a move A_i by Player A. If w is the shortest vector from 0 to $\mathsf{CH}((x_i)_{i \leq m})$, then that strategy ensures that after any round, the position has moved by at least $|w|$ in direction w – thus, the positions will leave any bounded region eventually.

We introduce some notation to state the general case. For a set of moves $\mathcal{B}$ let $\mathsf{CH}(\mathcal{B}) := \{\mathsf{CH}(B) \mid B \in \mathcal{B}\}$ and $\overline{\mathcal{B}} := \{\overline{B} \mid B \in \mathcal{B}\}$, where $\overline{B}$ is the topological closure of B. We say that a strategy for Player B in a Minkowski game is *simple*, if it prescribes choosing always the same $B \in \mathcal{B}$, and if the choice $a_i \in A_i$ depends only on the choice of $A_i \in \mathcal{A}$ by Player A.

▶ **Theorem 5.**

- *Boundedness Minkowski games are determined;*
- *the winner is the same for* $\langle \mathcal{A}, \mathcal{B} \rangle$ *and* $\langle \mathsf{CH}(\overline{\mathcal{A}}), \mathsf{CH}(\overline{\mathcal{B}}) \rangle$*;*
- *if Player B has a winning strategy, she has a simple one;*
- *Player A wins iff* $0 \in (\mathsf{CH}\{a_i \mid i \leq |\mathcal{A}|\}) + \mathsf{CH}(\overline{B})$ *for all* $(a_i)_{i \leq |\mathcal{A}|}$ *with* $a_i \in A_i$ *and* $B \in \mathcal{B}$.

4 Computational complexity of the boundedness problem

In Section 3, we have provided general results on boundedness Minkowski games. Here we study the computational complexity of the associated decision problem [2]. To formulate complexity results, we need to consider classes of games that are defined in some effective way. Here we consider three ways to represent the sets of moves: by finite sets of linear constraints (or convex hulls of finite sets of vectors), by formulas in the first-order theory of the reals (that strictly extend the expressive power of linear constraints), and as compact sets or overt sets (closed sets with positive information) in the sense of computable analysis.

Moves defined by linear constraints or as convex hulls. Here we prove the following by reducing the 3-SAT problem to the complement of the boundedness Minkowski game problem.

▶ **Theorem 6.** *Given a boundedness Minkowski game $\langle \mathcal{A}, \mathcal{B} \rangle$ where moves in the sets of moves $\mathcal{A}$ and $\mathcal{B}$ are defined by finite sets of rational linear constraints or as convex hulls of a finite sets of rational vectors, deciding the winner is* coNP-complete*. The hardness already holds for one-sided boundedness games.*

First note that Theorem 5 implies the useful Corollary 7, where a simple strategy λ_B for Player B is called a *vertex strategy*, if the $a_i \in A_i$ chosen by λ_B are always vertices of A_i.

▶ **Corollary 7.** *If Player B has a winning strategy in a boundedness Minkowski game $\langle \mathcal{A}, \mathcal{B} \rangle$ with closed moves in $\mathcal{A}$ then she has a winning vertex strategy.*

Proof. By Theorem 5 he has a simple strategy, and even a vertex one since $\mathsf{CH}(\mathsf{Ver}(P)) = P$ by definition. ◀

An important ingredient of the reduction below is a consequence of Corollary 7 and the determinacy of boundedness Minkowski games (Theorem 5): to show that Player A has a winning strategy, it suffices to show that he can spoil all the vertex strategies of Player B.

▶ **Lemma 8.** *There is a polynomial time reduction from the 3SAT problem to the complement of the boundedness problem for one-sided Minkowski games with closed, polytopic moves.*

Proof. Before going into the details of the proof, let us point out that the proof that we provide below works for both the H-representation and the V-representation. This is because the moves that we need to construct are all the convex hull of exactly three vectors. So the H-representation of such a convex hull can be obtained in polynomial time.

Let $\Psi = \{C_1, C_2, \dots, C_n\}$ be a set of clauses with 3 literals defined on the set of Boolean variables $X = \{x_1, x_2, \dots, x_m\}$. Each C_i is of the form $\ell_{i1} \vee \ell_{i2} \vee \ell_{i3}$ where each ℓ_{ij} is either x or $\neg x$ with $x \in X$.

To define the set of moves $\mathcal{A}$ for Player A, we associate a move A_i with each clause C_i. The move is a subset of $\mathbb{R}^d$, where $d = 2 \cdot |X| = 2 \cdot m$, defined from C_i as follows. We associate with each variable $x_k \in X$ two dimensions of $\mathbb{R}^d$: $2k-1$ and $2k$, and to each literals ℓ_{ij} a vector noted $\mathsf{Vect}(\ell_{ij})$ defined as follows. If the literal $\ell_{ij} = x_k$, then the vector $\mathsf{Vect}(\ell_{ij})$ has zeros everywhere but in dimension $2k-1$ and $2k$ where it is respectively equal to 1 and -1. If the literal $\ell_{ij} = \neg x_k$, then the vector $\mathsf{Vect}(\ell_{ij})$ has zeros everywhere but in dimension $2k-1$ and $2k$ where it is respectively equal to -1 and 1. So, it is the case that for all literals

[2] For all our complexity results, all the encoding of numbers and vectors that we use are the natural ones, i.e. integer or rational numbers are encoded succinctly in binary.

ℓ_1 and ℓ_2, $\mathsf{Vect}(\ell_1) + \mathsf{Vect}(\ell_2) = \mathbf{0}$ if and only if $\ell_1 \equiv \neg\ell_2$. Finally, the move associated with the clause $C_i = \ell_{i1} \vee \ell_{i2} \vee \ell_{i3}$ is

$$A_i = \mathsf{CH}(\mathsf{Vect}(\ell_{i1}), \mathsf{Vect}(\ell_{i2}), \mathsf{Vect}(\ell_{i3})).$$

We now establish the correctness of our reduction. **First**, we show that if Ψ is satisfiable then Player B has a winning strategy in the boundedness game.

Let $f : X \to \{0, 1\}$ be a valuation for the variables in X such that $f \models \Psi$. We associate with f a vertex strategy λ_B^f of Player B as follows. Because $f \models \Psi$, we know that in each clause C_i, there is a literal ℓ_{ij} such that $f \models \ell_{ij}$. Then whenever Player A chooses moves A_i, λ_B^f instructs Player B to choose $\mathsf{Vect}(\ell_{ij})$. We now claim that this strategy is winning for Player B. The argument is as follows.

Because $f \models \Psi$ and by definition of λ_B^f, it is the case that all the vertices chosen by λ_B^f are not associated with opposite literals. As a consequence, none of the vectors that will be played by Player B are opposite and so after k rounds in the game, at least one of the dimensions in the current position (assuming we started in position $\mathbf{0}$) has absolute value larger than $\frac{k}{m}$ and so there is no bounded set that can contain all the visited positions and Player B wins the boundedness game.

Second, we show that if Ψ is not satisfiable then Player B has no winning strategy. By Corollary 7, it is equivalent to show that Player B has no winning vertex strategies. We next prove that all the vertex strategies of Player B can be spoiled by Player A.

Remember that moves $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ have been defined starting from the clauses C_1, C_2, ..., C_n. Let λ_B^v be a vertex strategy of Player B, i.e. for all i, $1 \le i \le n$, $\lambda_B^v(A_i) \in \mathsf{Ver}(A_i)$. We claim that for all such vertex strategy λ_B^v, there exists i_1 and i_2, $1 \le i_1 < i_2 \le n$ such that $\lambda_B^v(A_{i_1}) = -\lambda_B^v(A_{i_2})$. If this is the case, then if Player A always alternates between move A_{i_1} and move A_{i_2}, then clearly the play remains in a bounded zone that contains $\mathbf{0}$ as the value of all the dimensions is within the interval $[-1, 1]$.

For the sake of contradiction, let us assume that it is not the case that there exists i_1 and i_2, $1 \le i_1 < i_2 \le n$ such that $\lambda_B^v(A_{i_1}) = -\lambda_B^v(A_{i_2})$ and let us derive a contradiction. To obtain the contradiction, we note that the choice of one vertex per move corresponds to the choice of one literal per clause in the SAT problem. But if there is no i_1 and i_2, $1 \le i_1 < i_2 \le n$ such that $\lambda_B^v(A_{i_1}) = -\lambda_B^v(A_{i_2})$, this means that we can choose one literal per clause so that we never choose two literals that are opposite. But this is only possible when Ψ is satisfiable, so we obtain our contradiction. ◀

▶ **Lemma 9.** *Negative instances of the boundedness Minkowski games expressed with moves defined as sets of linear inequalities or convex hull of finite sets of vectors can be recognized by a nondeterministic polynomial time Turing machine.*

Proof Sketch. To show that Player A has no winning strategy, by Theorem 5, it suffices to exhibit $a_1 \in \mathsf{Ver}(A_1)$, $a_2 \in \mathsf{Ver}(A_2)$, ..., $a_{n_A} \in \mathsf{Ver}(A_{n_A})$, and one $B \in \mathcal{B}$, such that

$$\mathbf{0} \notin \mathsf{CH}(a_1, a_2, \ldots, a_{n_A}) + \overline{B}.$$

Checking this can be done in polynomial time by reducing the testing to the satisfiability of a set of linear constraints. ◀

Fixed dimension and polytopic moves. Here we show that given $d \in \mathbb{N}$, deciding the winner of a boundedness Minkowski game with polytopic moves in $\mathbb{R}^d$ can be done in deterministic polynomial time. It relies on quick (for a fixed d) translations from V-representations of (closed) polytopes into H-representations, and vice-versa (see Theorem 1), so w.l.o.g. we focus here on the V-representation of polytopes. In this setting, by Theorem 5 it suffices to consider games with finite moves, since the extremal points of a polytope are finitely many. The degree of the polynomial (upper-)bounding the algorithmic complexity will be $2d+2$ in general, and $d+1$ for single-sided games. By Theorem 5 again, Player B has a winning strategy iff there exist $a_1 \in A_1, \ldots, a_n \in A_n$ (the moves in $\mathcal{A}$) and $B \in \mathcal{B}$ such that $0 \notin \mathsf{CH}(a_1, \ldots, a_n) + \mathsf{CH}(B)$. Trying out all the tuples $(a_1, a_2, \ldots, a_n)$ is not tractable. Instead we use a separation result.

▶ **Theorem 10.** *Let $\langle \mathcal{A}, \mathcal{B} \rangle$ be a Minkowski game with finite moves in $\mathbb{R}^d$, and let $C = \{e_1, \ldots, e_d\}$ be the canonical basis of $\mathbb{R}^d$. The game is won by Player B iff there exist $B \in \mathcal{B}$ and affinely independent $x_1, \ldots, x_d \in (\cup\mathcal{A} + B) \cup C$ s.t. for all $A \in \mathcal{A}$ there exists $a \in A$ s.t. the affine hull of $x_1, \ldots, x_d$ separates $a + B$ from 0.*

▶ **Corollary 11.** *Deciding the winner of a boundedness Minkowski game with rational polytopic moves in $\mathbb{R}^d$ that involve n vertices can be done in time $O(n^{2d+2})$.*

Moves defined in the first-order theory of the reals. Here we show that if the moves are definable in the first-order theory of the reals, then so is the condition

$$0 \in (\mathsf{CH}\{a_i \mid i \leq n\}) + \mathsf{CH}(\overline{B}) \text{ for all } (a_i)_{i \leq n} \text{ with } a_i \in \mathsf{CH}(\overline{A_i}) \text{ and } B \in \mathcal{B} \tag{1}$$

from Theorem 5. As the first-order theory is decidable, so is the question of who is winning a given boundedness Minkowski game with such moves.

We consider first-order formulas with binary function symbols $+$ and $\cdot$, constants 0 and 1 and binary relation symbol $<$. A move $A \subseteq \mathbb{R}^d$ is defined by some formula ϕ_A with d free variables $x_1, \ldots, x_k$ iff $A = \{(x_1, \ldots, x_k) \in \mathbb{R}^d \mid \phi(x_1, \ldots, x_n)\}$. If ϕ defines A, then

$$\begin{aligned}\phi_{\text{conv}} = \exists a_1^1, \ldots, a_{d+1}^1, \ldots, a_{d+1}^d, \alpha_1, \ldots, \alpha_{d+1} \quad & \bigwedge_{i=1}^{d+1} \phi(a_i^1, \ldots, a_i^d) \\ \wedge \sum_{i=1}^{d+1} \alpha_i = 1 \wedge \bigwedge_{i=1}^{d+1} 0 \leq \alpha_i \wedge \bigwedge_{j=1}^{d} \left(x_j = \sum_{i=1}^{d+1} \alpha_i a_i^j \right)&\end{aligned}$$

defines $\mathsf{CH}(A)$. Also, the formula

$$\phi_{\text{cl}} = \forall \varepsilon \quad \varepsilon > 0 \Rightarrow \left(\exists a_1, \ldots, a_d \quad \phi(a_1, \ldots, a_d) \wedge \bigwedge_{i=1}^{d} a_i < x_i + \varepsilon \wedge x_i < a_i + \varepsilon \right)$$

defines $\overline{A}$. It then follows that the condition (1) above is expressible as some formula ϕ_{win} obtained from the formulas ϕ_A, ϕ_B defining the moves in $\mathcal{A}$ and $\mathcal{B}$. Moreover, the length of the formula ϕ_{win} is polynomially bounded in the sum of the length of the ϕ_A, ϕ_B.

▶ **Proposition 12.** Deciding the winner of a boundedness Minkowski game with moves defined in the first-order theory of the reals is 2EXPTIME-COMPLETE.

Proof Sketch. Since by [3, 5] deciding truth in the first-order theory of the reals is 2EXPTIME-COMPLETE. ◀

The computable analysis perspective. If we represent the sets involved in the boundedness Minkowski games via polyhedra or first order formula, we have only restricted expressivity available to us. Using notions from computable analysis [17], we can however consider computability for all boundedness Minkowski games with closed moves – which is not a problematic restriction. As the involved spaces are all connected, we cannot expect decidability, and instead turn our attention to semidecidability, i.e. truth values in the Sierpinski space $\mathbb{S}$.

We do have to decide on a representation for the sets, though. Pointing to [14] for definitions and explanations, we have the spaces $\mathcal{A}(\mathbb{R}^d)$ of closed subsets, $\mathcal{K}(\mathbb{R}^d)$ of compact subsets and $\mathcal{V}(\mathbb{R}^d)$ of overt subsets available. In $\mathcal{A}(\mathbb{R}^d)$, a closed subset can be seen as being represented by an enumeration of rational balls exhausting its complement. The space $\mathcal{K}(\mathbb{R}^d)$ adds to that some $K \in \mathbb{N}$ such that the set is contained in $[-K, K]^d$. In $\mathcal{V}(\mathbb{R}^d)$, a closed subset is instead represented by listing all rational balls intersecting it.

A relevant property is that universal quantification over compact sets from $\mathcal{K}(\mathbb{R}^d)$ and existential quantification over overt sets from $\mathcal{V}(\mathbb{R}^d)$ preserve open predicates. We can use the former to find that:

▶ **Proposition 13.** The Minkowski sum $+ : \mathcal{A}(\mathbb{R}^d) + \mathcal{K}(\mathbb{R}^d) \to \mathcal{A}(\mathbb{R}^d)$ is computable.

It was already shown in [10, Proposition 1.5] (also [18]) that convex hull is a computable operation on compact sets, but not on closed sets. Put together, we find that:

▶ **Proposition 14.** Consider boundedness Minkowski games, where moves $A \in \mathcal{A}$ are given as overt sets (i.e. in $\mathcal{V}(\mathbb{R}^d)$) and moves $B \in \mathcal{B}$ are given as compact sets (i.e. in $\mathcal{K}(\mathbb{R}^d)$). The set of games won by Player B constitutes a computable open subset.

5 The safety problem

We now turn our attention to the safety Minkowski games. We want to understand which initial positions $a_0 \in \mathsf{Safe}$ give Player A a winning strategy in the safety game $\langle \mathcal{A}, \mathcal{B}, \mathsf{Safe}, a_0 \rangle$. In a minor abuse of notation, we speak of **the** safety Minkowski game $\langle \mathcal{A}, \mathcal{B}, \mathsf{Safe} \rangle$, and call the set of a_0 such that Player A has a winning strategy the *winning region* W.

We give two general results below: first, the winning region is the greatest fixed point of an operator that removes the points where Player B can provably win (in finitely many rounds); second, for finite $\mathcal{A}$ this fixed point is approximated in a Kleene fixed-point style.

Let $\langle \mathcal{A}, \mathcal{B}, \mathsf{Safe} \rangle$ be a safety game. Given E a target set, $f(E)$ is defined below as the positions from where Player A can ensure to fall in E after one round of the game.

▶ **Definition 15.** For all $E \subseteq \mathbb{R}^d$ let $g(E) := f(E) \cap \mathsf{Safe}$, where

$$f(E) := \{x \in \mathbb{R}^d \mid \exists A \in \mathcal{A}, \forall a \in A, \forall B \in \mathcal{B}, \exists b \in B, x + a + b \in E\}.$$

▶ **Lemma 16.** *The winning region W is the greatest fixed point of g.*

▶ **Proposition 17.** Let $S_0 := \mathbb{R}^d$, let $S_{n+1} := g(S_n)$ for all n, and let $S_\omega := \cap_{n \in \mathbb{N}} S_n$. S_ω is the greatest fixed point of g.

Computable analysis perspective. We start our investigation of the computational complexity of determining the winner in safety Minkowski games by considering the general setting of computable analysis, as we did in the end of Section 4 for the boundedness games. We point again to [14] for notation and definition, and in particular make use of the characterizations of $\mathcal{V}(\mathbb{R}^d)$ and $\mathcal{K}(\mathbb{R}^d)$ via the preservation of open predicates under quantification. We obtain:

▶ **Proposition 18.** Given a safety Minkowski game $\langle \mathcal{A}, \mathcal{B}, \mathsf{Safe}, a_0 \rangle$ with finite $\mathcal{A}$ of overt sets (i.e. from $\mathcal{V}(\mathbb{R}^d)$), finite $\mathcal{B}$ of compact sets (i.e. from $\mathcal{K}(\mathbb{R}^d)$), and closed Safe (i.e. from $\mathcal{A}(\mathbb{R}^d)$, we can semidecide (recognize) if Player B has a winning strategy.

Undecidability for polytopic sets

▶ **Theorem 19.** *There is $d \in \mathbb{N}$, a convex rational polytope Safe and a finite family $\mathcal{A}$ of closed convex rational polytopes all in $\mathbb{R}^d$ such that it is undecidable, whether Player A has a winning strategy in the one-sided safety Minkowski game $\langle \mathcal{A}, \mathsf{Safe}, a_0 \rangle$, given a_0 as a rational vector.*

To prove this theorem, we provide a reduction from the control state reachability problem for two counter machines to the problem of deciding if Player B has a winning strategy in a safety Minkowski game. As the first step, we introduce a slightly more general version of one-sided Minkowski games, and show a reduction to one-sided safety Minkowski games:

▶ **Definition 20.** A safety-reachability one-sided Minkowski game is given by a tuple $\langle \mathcal{A}, \mathsf{Safe}, \mathsf{Goal}, a_0 \rangle$, where $\langle \mathcal{A}, \mathsf{Safe}, a_0 \rangle$ is some d-dimensional safety one-sided Minkowski game, and $\mathsf{Goal} \subseteq \mathsf{Safe}$. It is played like the safety Minkowski game, and if Player A wins $\langle \mathcal{A}, \mathsf{Safe}, a_0 \rangle$, then he wins $\langle \mathcal{A}, \mathsf{Safe}, \mathsf{Goal}, a_0 \rangle$. If the play enters Goal prior to leaving Safe for the first time, also Player A wins. Else Player B wins.

▶ **Proposition 21.** Given a d-dimensional safety-reachability one-sided Minkowski game $\langle \mathcal{A}, \mathsf{Safe}, \mathsf{Goal}, a_0 \rangle$, we define the associated $d+1$-dimensional safety one-sided Minkowski game $\langle \mathcal{A}', \mathsf{Safe}', a_0' \rangle$ as follows:

1. $a_0' := \langle a_0, 0 \rangle$
2. $\mathsf{Safe}' := \mathsf{CH}\left((\mathsf{Safe} \times \{0\}) \cup (\mathsf{Goal} \times \{1\})\right)$
3. $\mathcal{A}' := \{A \times \{0\} \mid A \in \mathcal{A}\} \cup \{\{(0, \ldots, 0, 1)\}, \{(0, \ldots, 0, -1)\}\}$

Now Player A (resp. Player B) has a winning strategy in the original game iff he (resp. she) has one in the associated game.

2CM and the control state reachability problem. A two-counter machine, 2CM for short, is defined by a finite directed graph (Q, E) with labeled edges. Vertices have out-degree 0, 1 or 2. If the out-degree is 1, the corresponding edge is labeled with one of INC^i, DEC^i for $i \in \{0, 1\}$. If it is 2, one outgoing edge is labeled with $\mathrm{isZero?}^i$ and the other with $\mathrm{isNotZero?}^i$ for some $i \in \{0, 1\}$. There is a designated starting vertex $q_0 \in Q$.

A finite or infinite path through the graph is a *valid computation starting from n_0 and n_1* if the following is true: the path starts at q_0. If one starts with $c_0 := n_0$ and $c_1 := n_1$ and increments (decrements) c_i by 1 whenever encountering a label INC^i (DEC^i), then at the moment an edge labeled with $\mathrm{isZero?}^i$ ($\mathrm{isNotZero?}^i$) is passed, we find that $c_i = 0$ ($c_i \neq 0$).

▶ **Theorem 22** ([12, Theorem Ia]). *There is a 2CM such that it is undecidable whether there exists an infinite valid computation starting from n_0 and n_1 (where n_0 and n_1 are the input).*

We will slightly modify the *2CM* to simplify the construction. We subdivide every edge by adding another vertex on it. If the original edge was labeled INC^i (DEC^i), then the two new edges will be labeled INCa^i and INCb^i (DECa^i and DECb^i). If the original edge was labeled $\mathrm{isZero?}^i$ or $\mathrm{isNotZero?}^i$, we move the label to the newly-added vertex.

Now we are ready to reduce the non-halting problem of modified 2CM's to the existence of a winning strategy for Player A in a safety-reachability one-sided Minkowski game. The general idea of the reduction is as follows. First, Player A is forced to simulate the computation of the 2CM in order to avoid violating the safety condition of the safety Minkowski game.

The value of each counter c_i, $i \in \{1,2\}$, is coded in some dimension y_i such that when the counter c_i is equal to $k \in \mathbb{N}$ then the value of $y_i = \frac{1}{2^k}$. The role of Player B is restricted to assist Player A to multiply or divide the x_i by 2. His failure to operate as intended will let the play reach Goal. Additionally, each vertex Q is associated with one dimension that will be non-zero iff the computation is currently in that vertex.

All the moves and invariants that we use are definable by finite sets of linear constraints.

Defining the reduction. We are given a modified 2CM with vertex set Q (called control states) and edges E. The associated safety-reachability Minkowski game will be played in $\mathbb{R}^{4+|Q|}$. The first 4 dimensions are (x_0, y_0, x_1, y_1), where the y_i encode the counter values, and the x_i are auxiliary values. The remaining $|Q|$ dimensions are indexed with the states q.

Every instruction $e \in E$ corresponds to some move A_e for Player A. The move A_e will always decompose as $A_e = A_e^{xy} \times \{a_e^Q\}$. If e is an edge from q_i to q_f, then $a_e^Q \in \mathbb{R}^{|Q|}$ will have -1 at component q_i, $+1$ at component q_f and 0 elsewhere.

Label of e	Value of A_e^{xy}	Label of e	Value of A_e^{xy}
-	$\{(0,0,0,0)\}$		
INCa^0	$\mathsf{CH}\{(0,0),(1,-1)\} \times \{(0,0)\}$	DECa^0	$\mathsf{CH}\{(0,0),(1,0)\} \times \{(0,0)\}$
INCa^1	$\{(0,0)\} \times \mathsf{CH}\{(0,0),(1,-1)\}$	DECa^1	$\{(0,0)\} \times \mathsf{CH}\{(0,0),(1,0)\}$
INCb^0	$\mathsf{CH}\{(0,0),(-1,0)\} \times \{0,0\}$	DECb^0	$\mathsf{CH}\{(0,0),(-1,1)\} \times \{(0,0)\}$
INCb^1	$\{0,0\} \times \mathsf{CH}\{(0,0),(-1,0)\}$	DECb^1	$\{(0,0)\} \times \mathsf{CH}\{(0,0),(-1,1)\}$

It remains to define the sets Safe and Goal. For that, let Q_z^i be the set of states labeled with isZero?^{i}, and let Q_n^i be the set of states labeled with isNotZero?^{i}. Let Q_o be the set of unlabeled states with non-zero outdegree. Let e_q be the $|Q|$-dimensional vector having 1 in component q and 0 elsewhere.

$$\begin{aligned}\mathsf{Safe} := \mathsf{CH}[&\left(\bigcup_{q\in Q_o}[0,1]^4\times\{e_q\}\right)\cup\left(\bigcup_{q\in Q_n^0}[0,1]\times[0,0.7]\times[0,1]^2\times\{e_q\}\right)\\&\cup\left(\bigcup_{q\in Q_n^1}[0,1]^3\times[0,0.7]\times\{e_q\}\right)\cup\left(\bigcup_{q\in Q_z^0}[0,1]\times\{1\}\times[0,1]^2\times\{e_q\}\right)\\&\cup\left(\bigcup_{q\in Q_z^1}[0,1]^3\times\{1\}\times\{e_q\}\right)]\end{aligned}$$

$$\mathsf{Goal} := \mathsf{Safe}\cap(\{(x,y)\in\mathbb{R}^2 \mid y\neq x\neq 0\}\times\mathbb{R}^{2+|Q|}\cup\mathbb{R}^2\times\{(x,y)\in\mathbb{R}^2 \mid y\neq x\neq 0\}\times\mathbb{R}^{|Q|})$$

The starting position of the game is as follows: $(0, 2^{-n_0}, 0, 2^{-n_1}, 0, \ldots, 0, 1, 0, \ldots,)$, where n_0 and n_1 are the starting values for the counters, and the unique 1 in the latter part is found at the index corresponding to the starting state of the *2CM*.

Correctness of the reduction. We claim that Player A has a winning strategy in the constructed game, iff the (modified) *2CM* has a valid infinite computation path. As moves correspond to edges, every sequence of moves chosen by Player A in the game can be seen as a sequence of edges for the *2CM*.

First we argue that every sequence of edges which is not a path induces a losing strategy in the game. As the values of the components associated with the control states must remain between 0 and 1, and every move has components -1, $+1$ somewhere and 0 elsewhere it follows that every non-losing sequences of moves ensure that exactly one state-component q_i of the position is 1, and the others are 0. Every move coming from an edge not having the initial state q_i will lose immediately.

Next, we shall explain how the moves for INCa^i and INCb^i together cause the desired effect. If the current relevant part of the position is $(0, 2^{-k})$, then after the move INCa^i Player B may pick any $(x, y) \in (0, 2^{-k}) + \mathsf{CH}\{(0,0),(1,-1)\}$, in other words, Player B picks some $t \in [0,1]$ and sets the position to $(t, 2^{-k} - t)$. If Player B picks $t = 0$, then Player A can repeat the same move. By the definition of Goal, the only other safe choice for Player B is to pick $t = 2^{-k-1}$, i.e. to set the position to $(2^{-k-1}, 2^{-k-1})$. The move associated with INCb^i follows, which means that Player B gets to pick some $(2^{-k-1} - t, 2^{-k-1})$. Again, choosing $t = 0$ lets Player A repeat her move, and the only other choice compatible with avoiding Goal is to move to $(0, 2^{-k-1})$.

The construction for DECa^i, DECb^i is similar: starting at $(0, 2^{-k})$, Player B can only remain, enter Goal or move to $(2^{-k}, 2^{-k})$ if Player A plays a move corresponding to DECa^i. The subsequent DECb^i move allows Player B to remain, enter Goal or to move to $(0, 2^{-k+1})$.

Finally, we need to discuss (conditional) halting: by the construction of Safe, if a vertex with out-degree 0 is reached, or a vertex labeled with an unsatisfied condition, then the play is loosing for Player A. Thus, winning strategies of Player A correspond exactly to infinite non-halting computations of the 2CM.

Structural safety games. The above undecidability result for safety game with polytopic sets motivates the study of *structural safety Minkowski games*. In a (one-sided) structural safety game, there is no designated initial state and Player A is asked to be able to maintain the system safe starting from any point in the safe region. It is not difficult to see that this stronger requirement makes the game equivalent to a "one round" game. Indeed, if Player A can maintain safety from all positions within Safe, then it means that after one round of the game, the game is again within Safe, from which Player A can win for one more round, etc.

The complexity of the structural safety games for polytopic moves and Safe is below.

▶ **Theorem 23.** *Given a one-sided structural safety Minkowski game $\langle \mathcal{A}, \mathcal{B}, \mathsf{Safe} \rangle$ where moves and the set Safe are rational polytopic, it is* coNP-Complete *to decide if Player A has a winning strategy from all positions in Safe.*

6 Open questions

By comparing the results from Section 4 on linear constraints and fixed dimension, we see that while deciding the winner in a boundedness Minkowski game is coNP-complete in general, it becomes polynomial-time if the dimension of the ambient space is fixed. Thus, it makes a good candidate for an investigation in the setting of parameterized complexity [6]. Is the problem fixed-parameter tractable? Is it hard for some W[n]-class?

We showed that from some dimension onwards, it is undecidable to determine the winner in a safety Minkowski game defined via sets of linear constraints defining open and closed convex polytopes. This immediately raises two questions: first, what happens for small dimensions? Given that our construction uses essentially two dimensions per instruction, and two per counter, an optimal value is presumably obtained by using universal machine having more than 2 counters. Second, what happens if we restrict our attention to games defined via sets of linear constraints that are all non strict (defining closed convex polytopes only)?

Acknowledgements. We would like to thank Dr. Pritha Mahata and Prof. Jean Cardinal for helpful and stimulating discussions related to a.o. convex polyhedra and linear algebra.

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