On the Sensitivity Complexity of k-Uniform Hypergraph Properties[*]

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—— **Abstract** ——

In this paper we investigate the sensitivity complexity of hypergraph properties. We present a
k-uniform hypergraph property with sensitivity complexity $O(n^{\lceil k/3 \rceil})$ for any $k \geq 3$, where n
is the number of vertices. Moreover, we can do better when $k \equiv 1 \pmod 3$ by presenting a k-
uniform hypergraph property with sensitivity $O(n^{\lceil k/3 \rceil - 1/2})$. This result disproves a conjecture of
Babai [9], which conjectures that the sensitivity complexity of k-uniform hypergraph properties
is at least $\Omega(n^{k/2})$. We also investigate the sensitivity complexity of other weakly symmetric
functions and show that for many classes of transitive-invariant Boolean functions the minimum
achievable sensitivity complexity can be $O(N^{1/3})$, where N is the number of variables. Finally,
we give a lower bound for sensitivity of k-uniform hypergraph properties, which implies the
sensitivity conjecture of k-uniform hypergraph properties for any constant k.

1998 ACM Subject Classification F.1.3 Complexity Measures and Classes

Keywords and phrases Sensitivity Complexity, k-uniform Hypergraph Properties, Boolean Function, Turan's question

Digital Object Identifier 10.4230/LIPIcs.STACS.2017.51

1 Introduction

In order to understand the effect of symmetry on computational complexity, especially in
the decision tree model, Boolean functions with certain symmetry have been extensively
investigated. It is observed that symmetry usually implies high complexity or makes the
problem harder in the decision tree model. An illustrative example is the well known
evasiveness conjecture, which asserts that any non-constant monotone transitive Boolean
function is evasive, and it has attracted a lot of attention [29, 13, 25, 6]. Rivest and
Vuillemin [32] showed that any non-constant monotone graph property is weakly evasive.
Kulkarni et al. [26] showed an analogous result for 3-hypergraph properties. Black [10]
extended these results to k-uniform hypergraph properties for any fixed k.

Sensitivity complexity is an important complexity measure of Boolean functions in the
decision tree model, and sensitivity complexity of Boolean functions with certain symmetry

has also attracted a lot of attention. One of the most challenging problem here is whether *symmetry* implies high sensitivity complexity. The famous sensitivity conjecture, which asserts sensitivity complexity and block sensitivity are polynomially related, implies $s(f) = \Omega(n^\alpha)$ for transitive functions with some constant $\alpha > 0$ since it has been shown that $bs(f) = \Omega(n^{1/3})$ for transitive functions [34]. Turan [38] initiated the study of sensitivity of graph properties and proved that the sensitivity is greater than $n/4$ for any nontrivial graph property, where n is the number of vertices, and this relation is also tight up to a constant factor. He also pointed out that for symmetric functions, $s(f) \geq n/2 \geq bs(f)/2$. Recently the lower bound has been improved to $\frac{6}{17}n$ by Sun [35], and further been improved to $\lfloor \frac{n}{2} \rfloor$ for sufficient large n by Karpas [24]. Gao et al. [18] investigated the sensitivity of bipartite graph properties as well. In 2005, Chakraborty [14] constructed a minterm cyclically invariant Boolean function whose sensitivity is $\Theta(n^{1/3})$, which answers Turan's question [38] in the negative. He also showed this bound is tight for minterm transitive functions.

For hypergraph properties, Biderman et al. [9] present a sequence of k-uniform hypergraph properties with sensitivity $\Theta(\sqrt{N})$, where $N = \binom{n}{k}$ is the number of variables. Babai conjectures that this bound is tight, i.e., $s(f) = \Omega(\sqrt{N})$ for any nontrivial k-uniform hypergraph property f.

Our Results.   In this paper we disprove this conjecture by constructing k-uniform hypergraph properties with sensitivity $O(n^{\lceil k/3 \rceil})$.

▶ **Theorem 1.** *For any fixed $k \geq 3$, there exists a sequence of k-uniform hypergraph properties f such that $s(f) = O(n^{\lceil k/3 \rceil})$, where n is the number of vertices.*

Moreover, we can give better constructions when $k \equiv 1 \pmod 3$.

▶ **Theorem 2.** *For any fixed $k \geq 4$ satisfying $k \equiv 1$ (mod 3) , there exists a sequence of k-uniform hypergraph properties f such that $s(f) = O(n^{\lceil k/3 \rceil - 1/2})$, where n is the number of vertices.*

More generally, we also investigate the sensitivity of k-partite k-uniform hypergraph properties. Actually, the constructions of k-uniform hypergraph properties are inspired by the constructions of k-partite k-uniform hypergraph properties.

▶ **Theorem 3.** *For any $k \geq 3$, there exists a sequence of k-partite k-uniform hypergraph properties $f : \{0,1\}^{n^k} \to \{0,1\}$ such that $s(f) = O(n^{\lceil k/3 \rceil})$.*

▶ **Theorem 4.** *For any $k \geq 4$ satisfying $k \equiv 1$ (mod 3) , there exists a sequence of k-partite k-uniform hypergraph properties $f : \{0,1\}^{n^k} \to \{0,1\}$ such that $s(f) = O(n^{\lceil k/3 \rceil - 1/2})$.*

Let G be an Abelian group, the fundamental theorem of finite abelian groups states that $G \cong C_{m_1} \times \cdots \times C_{m_l}$, where C_m is the cyclic group of order m and $|G| = \prod_{i=1}^{l} m_i$.

▶ **Theorem 5.** *Let $G \leq S_n$ be a transitive Abelian group, then there exists a Boolean function $f : \{0,1\}^n \to \{0,1\}$ invariant under G such that $s(f) \leq \alpha n^{1/3}$, where α is a number only depending on l.*

On the other side, Chakraborty [15] observed the following lower bound on k-uniform hypergraph properties without showing the proof, which implies the sensitivity conjecture of k-uniform hypergraph properties for any constant k. For the convenience of readers, the proof will be given in the paper.

▶ **Theorem 6.** *For any fixed k and any non-trivial k-uniform hypergraph property f, $s(f) = \Omega(n)$, where n is the number of vertices.*

Similar lower bound holds for the sensitivity of k-partite k-uniform hypergraph properties.

▶ **Theorem 7.** *For any fixed k and any non-trivial k-partite k-uniform hypergraph property f, $s(f) = \Omega(n)$, where n is the number of vertices in one partition.*

The sketch of the proof is as follows: we just pretend this function is a bipartite graph property by dividing k partitions into two sets of size 1 and $k - 1$ respectively and then follow the same argument in the proof of Theorem 2 in [18]. We omit the proof in this paper.

Related Work.   Sensitivity complexity and block sensitivity are first introduced by Cook, Dwork and Reischuk [16, 17] and Nisan [30] respectively, to study the time complexity of CREW-PRAMs. Block sensitivity has been shown to be polynomially related to a number of other complexity measures [12], such as decision tree complexity, certificate complexity, polynomial degree and quantum query complexity, etc, except sensitivity. The famous sensitivity conjecture, proposed by Nisan and Szegedy [31], asserts that block sensitivity and sensitivity complexity are also polynomially related. On one side, it is easy to see $s(f) \le bs(f)$ for any Boolean function f according to the definitions. On the other side, it is much more challenging to prove or disprove that block sensitivity is polynomially bounded by sensitivity. Despite of a lot of effort, the best known upper bound is exponential: $bs(f) \le \max\{2^{s(f)-1}(s(f) - \frac{1}{3}), s(f)\}$ [3]. Recently, He, Li and Sun further improve the upper bound to $(\frac{8}{9} + o(1))s(f)2^{s(f)-1}$ [23]. The best known separation between sensitivity and block sensitivity is quadratic [4]: there exists a sequence of Boolean functions f with $bs(f) = \frac{2}{3}s(f)^2 - \frac{1}{3}s(f)$. For an excellent survey on the sensitivity conjecture, see [22]. For other recent progress, see [11, 2, 1, 5, 19, 36, 20, 21, 37, 8, 28, 33, 7].

Organization.   We present some preliminaries in Section 2, and give the proofs of Theorem 1 and Theorem 2 in Section 3. We give the constructions of k-partite k-uniform hypergraph properties (Theorem 3 and 4) and the proof of Theorem 5 in Section 4 and give the proof of Theorem 6 in Section 5. Finally, we conclude this paper with some open problems in Section 6.

2   Preliminaries

Let $f : \{0,1\}^n \to \{0,1\}$ be a Boolean function and $[n] = \{1, 2, \cdots, n\}$. For an input $x \in \{0,1\}^n$ and a subset $B \subseteq [n]$, x^B denotes the input obtained by flipping all the bit x_j such that $j \in B$.

▶ **Definition 8.** The *sensitivity* of f on input x is defined as $s(f, x) := |\{i|f(x) \ne f(x^{\{i\}})\}|$. The sensitivity, 0-sensitivity and 1-sensitivity of the function f are defined as $s(f) := \max_x s(f, x)$, $s_0(f) = \max_{x \in f^{-1}(0)} s(f, x)$ and $s_1(f) = \max_{x \in f^{-1}(1)} s(f, x)$ respectively.

▶ **Definition 9.** The *block sensitivity* $bs(f, x)$ of f on input x is the maximum number of disjoint subsets $B_1, B_2, \cdots, B_r$ of $[n]$ such that for all $j \in [r]$, $f(x) \ne f(x^{B_j})$. The block sensitivity of f is defined as $bs(f) = \max_x bs(f, x)$.

▶ **Definition 10.** A *partial assignment* is a function $p : [n] \to \{0, 1, \star\}$. We call $S = \{i|p_i \ne \star\}$ the support of this partial assignment. We define the size of p (denoted by $|p|$) to be $|S|$. We call x a (full) assignment if $x : [n] \to \{0, 1\}$. We say x is consistent with p if $x|_S = p$, i.e., $x_i = p_i$ for all $i \in S$.[1]

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[1]   The function p can be viewed as a vector, and we sometimes use p_i to represent $p(i)$.

▶ **Definition 11.** For $b \in \{0, 1\}$, a $b-certificate$ for f is a partial assignment p such that $f(x) = b$ whenever x is consistent with p.

The *certificate complexity* $C(f, x)$ of f on input x is the minimum size of $f(x)$-certificate that is consistent with x. The certificate complexity of f is $C(f) = \max_x C(f, x)$.

The *1-certificate complexity* of f is $C_1(f) = \max_{x \in f^{-1}(1)} C(f, x)$, and similarly we define $C_0(f)$.

According to the definitions, it's easy to see $s(f) \leq bs(f) \leq C(f)$, $s_0(f) \leq C_0(f)$ and $s_1(f) \leq C_1(f)$.

▶ **Definition 12.** Let p and p' be two partial assignments, the distance between p and p' is defined as $dist(p, p') = |\{i | p_i = 1 \text{ and } p'_i = 0, \text{ or } p_i = 0 \text{ and } p'_i = 1\}|$.

▶ **Definition 13.** Let $f : \{0, 1\}^n \to \{0, 1\}$ be a Boolean function and G be a subgroup of S_n, we say that f is invariant under G if $f(x_1, \cdots, x_n) = f(x_{\sigma(1)}, \cdots, x_{\sigma(n)})$ for any $x \in \{0, 1\}^n$ and any $\sigma \in G$.

A Boolean function f is called *transitive* (or *weakly symmetric*) if G is a transitive group[2]. A Boolean function f is called *symmetric* if $G = S_n$.

▶ **Definition 14.** A Boolean function f invariant under a transitive group G is called minterm-transitive if there exists a partial assignment p such that $f(x) = 1$ if and only if x is consistent with $p^\sigma := (p_{\sigma(1)}, p_{\sigma(2)}, \cdots, p_{\sigma(n)})$ for some $\sigma \in G$. We call p the minterm of f.

A Boolean string can represent a graph in the following manner: $x_{(i,j)} = 1$ means there is an edge connecting vertex i and vertex j, and $x_{i,j} = 0$ means there is no such edge. Graph properties are functions for which the value is independent with the labeling of vertices, i.e. two isomorphic graphs have the same function value.

▶ **Definition 15.** A Boolean function $f : \{0, 1\}^{\binom{n}{2}} \to \{0, 1\}$ is called a graph property if for every input $x = (x_{(1,2)}, \cdots, x_{(n-1,n)})$ and every permutation $\sigma \in S_n$,

$$f(x_{(1,2)}, \cdots, x_{(n-1,n)}) = f(x_{(\sigma(1),\sigma(2))}, \cdots, x_{(\sigma(n-1),\sigma(n))}).$$

Similarly, we define k-uniform hypergraph properties.

▶ **Definition 16.** A Boolean function $f : \{0, 1\}^{\binom{n}{k}} \to \{0, 1\}$ is called a k-uniform hypergraph property if for every input $x = (x_{(1,2,...,k)}, \cdots, x_{(n-k+1,...,n-1,n)})$ and every permutation $\sigma \in S_n$,

$$f(x_{(1,2,...,k)}, \cdots, x_{(n-k+1,...,n-1,n)}) = f(x_{(\sigma(1),\sigma(2),...,\sigma(k))}, \cdots, x_{(\sigma(n-k+1),...,\sigma(n-1),\sigma(n))}).$$

Let p be a partial assignment and $\sigma \in S_n$, we define $\sigma(p)$ the induced shift of p by σ, i.e., $\sigma(p)_S = p_{\sigma(S)}$ for any subset of $[n]$ of size k S. Here $\sigma(S) = \{\sigma(i) | i \in S\}$.

▶ **Definition 17.** A Boolean function $f : \{0, 1\}^{n^k} \to \{0, 1\}$ is called k-partite k-uniform hypergraph property, if for every input $x = (x_{(1,1,\cdots,1)}, \cdots, x_{(n,n,\cdots,n)})$ and every $\sigma = (\sigma_1, \cdots, \sigma_k) \in S_n^{\otimes k}$,

$$f(x_{(1,1,\cdots,1)}, \cdots, x_{(n,n,\cdots,n)}) = f(x_{(\sigma_1(1),\cdots,\sigma_k(1))}, \cdots, x_{(\sigma_1(n),\cdots,\sigma_k(n))}).$$

It is easy to see that any (k-partite) k-uniform hypergraph property is transitive.

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[2] A group $G \leq S_n$ is transitive if for every $i < j$, there exists a $\sigma \in G$ such that $\sigma(i) = j$.

3   k-Uniform Hypergraph Properties

In this section, we give the proofs of Theorem 1 and Theorem 2.

▶ **Theorem 1** (restated). *For any fixed $k \geq 3$, there exists a sequence of k-uniform hypergraph properties f such that $s(f) = O(n^{\lceil k/3 \rceil})$, where n is the number of vertices.*

Proof. The function we construct is a minterm function. Let p be the minterm defining f, and it is constructed as follows:

First, let k_1 and k_2 be two integers such that $k_1 + 2k_2 = k$ and $\lfloor k/3 \rfloor \leq k_1, k_2 \leq \lceil k/3 \rceil$. Let $V = \{v_1, \cdots, v_n\}$ be the set of vertices and $B = \{v_n, v_{n-1}, \cdots, v_{n-k_1+1}\}$. For each $1 \leq i \leq 6$, let $W_i = \{v_{(i-1)k_2+1}, \cdots, v_{ik_2}\}$, and $C = \bigcup_{1 \leq i \leq 6} W_i$, $D = V \setminus (C \cup B)$.

- For any $S \subseteq C$ of size $2k_2$, $p(B \cup S) = 0$, except $S = W_i \cup W_{i+1}$ for $i \in [5]$ where $p(B \cup S) = 1$.
- For any S of size $2k_2$ and $k_2 \leq |S \cap C| < 2k_2$, $p(B \cup S) = 1$, except W_3 or $W_4 \subseteq S$ where $p(B \cup S) = 0$.
- All the other variables are $\star$.

If $f(x) = 1$ then x is consistent with some $\sigma(p)$, which implies $C(f, x) \leq |p|$. Thus $s_1(f) \leq C_1(f) \leq |p| = \sum_{i=k_2}^{2k_2} \binom{6k_2}{i} \binom{n - 6k_2 - k_1}{2k_2 - i} = O(n^{k_2})$. If $f(x) = 0$, then $s(f, x)$ is at most the number of shifts of p (i.e., $\sigma(p)$s) adjacent to x, thus according to the triangle inequality, $s_0(f)$ is at most the maximum number of $\sigma(p)$s where the distance between any two of them is at most 2. We claim that for any shift $\pi(p)$, there are only $O(1)$ $\sigma(p)$s satisfying $\pi(B) = \sigma(B)$ and $dist(\pi(p), \sigma(p)) \leq 2$. It is easy to see that this claim implies $s_0(f) = O(n^{k_1})$ since there are $\binom{n}{k_1} = O(n^{k_1})$ possible choices of the $\sigma(B)$s, and this will end the whole proof.

▶ **Claim 18.** *For any $\pi(p)$, there are only $O(1)$ $\sigma(p)$s satisfying $\pi(B) = \sigma(B)$ and $dist(\pi(p), \sigma(p)) \leq 2$.*

Proof. It is easy to see that this claim is equivalent to show $|\{\sigma(p)|dist(p, \sigma(p)) \leq 2$ and $\sigma(B) = B\}| = O(1)$. The case for $k_2 = 1$ is a little special, and we discuss this case first.

Case for $k_2 = 1$.   We use Figure 1 to illustrate p. Note that the vertices in D are symmetric and $|C| = O(1)$, thus $|\{\sigma(p)|\sigma(C) = C$ and $\sigma(B) = B\}| = O(1)$. So we only need to consider the set $\{\sigma(p)|\sigma(C) \neq C$ and $\sigma(B) = B\}$, and we exclude each σ case by case:

1. $\sigma(W_3)$ or $\sigma(W_4) \in \{W_1, W_2, W_5, W_6\}$.
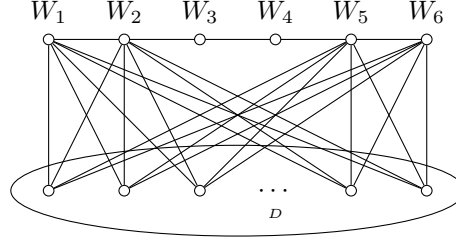   W.l.o.g, assume $\sigma(W_3) = W_1$, then

$$
\begin{aligned}
dist(p, \sigma(p)) \quad \geq \quad & |\{e \subseteq [n]|\sigma(p)(e) = 1, p(e) = 0, |e| = k, \{W_3, B\} \subseteq e\}| \\
\geq \quad & |\{e \subseteq [n]|\sigma(p)(e) = 1, |e| = k, \{W_3, B\} \subseteq e\}| \\
& -|\{e \subseteq [n]|p(e) = \{1, \star\}, |e| = k, \{W_3, B\} \subseteq e\}| \\
= \quad & |\{e \subseteq [n]|p(\sigma(e)) = 1, |e| = k, \{W_3, B\} \subseteq e\}| - O(1) \\
= \quad & |\{e \subseteq [n]|p(e) = 1, |e| = k, \{W_1, B\} \subseteq e\}| - O(1) \\
\geq \quad & n - O(1) \geq 3.
\end{aligned}
$$

2. $\sigma(W_3)$ or $\sigma(W_4) \in D$, and $\{\sigma(W_3), \sigma(W_4)\} \bigcap \{W_1, W_2, W_5, W_6\} = \emptyset$.
   W.l.o.g, assume $\sigma(W_3) \in D$, note that for any v, $p(B \cup W_3 \cup v) \neq \star$, and $|\{v \neq W_4|p(B \cup W_3 \cup v) = 1\}| = 1$. While $|\{v \neq W_4|\sigma(p)(B \cup W_3 \cup v)\}| = |\{v \neq W_4|p(B \cup \sigma(W_3) \cup \sigma(v))\}| = 4$, thus $dist(p, \sigma(p)) \geq 3$.

3. $\sigma(W_3) = W_3$ and $\sigma(W_4) = W_4$.

Figure 1 The graph to illustrate p for $k_2 = 1$.

a. $\sigma(W_5) \neq W_5$ and $\sigma(W_2) \neq W_2$.

Since $p(B \cup \sigma(W_2) \cup \sigma(W_3)) = p(B \cup \sigma(W_4) \cup \sigma(W_5)) = 0$ and $p(B \cup \sigma(W_3) \cup \sigma(S)) = p(B \cup \sigma(W_4) \cup \sigma(S')) = 1$, for some $\sigma(S) = W_2$ and $\sigma(S') = W_5$, thus $dist(p, \sigma(p)) \geq 4$.

b. $\sigma(W_5) = W_5$ and $\sigma(W_2) = W_2$.

Since $\sigma(C) \neq C$, W.l.o.g, assume $\sigma(W_1) \in D$, then $p(B \cup \sigma(W_1) \cup \sigma(W_5)) = 1$.

If $\sigma(W_6) \in D$, then $p(B \cup \sigma(W_2) \cup \sigma(W_6)) = 1$ and $p(B \cup \sigma(S) \cup \sigma(W_5)) = p(B \cup \sigma(S') \cup \sigma(W_2)) = 0$, for some $\sigma(S) = W_1$ and $\sigma(S') = W_6$.

If $\sigma(W_6) = W_6$, then $p(B \cup \sigma(W_1) \cup \sigma(W_6)) = 1$, and $p(B \cup \sigma(S) \cup \sigma(W_5)) = p(B \cup \sigma(S) \cup \sigma(W_6)) = 0$, for some $\sigma(S) = W_1$.

If $\sigma(W_6) = W_1$, then $p(B \cup \sigma(W_2) \cup \sigma(W_6)) = 1$ and $p(B \cup \sigma(W_6) \cup \sigma(W_5)) = 0$.

Thus we always have $dist(p, \sigma(p)) \geq 3$.

c. $\sigma(W_5) \neq W_5$ and $\sigma(W_2) = W_2$.

Note that $p(B \cup \sigma(W_4) \cup \sigma(W_5)) = 0$ and $p(B \cup \sigma(W_4) \cup \sigma(S)) = 1$ for some $\sigma(S) = W_5$.

If $\sigma(W_5) \in D \cup \{W_1\}$, then $p(B \cup \sigma(W_2) \cup \sigma(W_5)) = 1$.

If $\sigma(W_5) = W_6$ and $\sigma(W_6) \in D \cup \{W_1\}$, then $p(B \cup \sigma(W_2) \cup \sigma(W_6)) = 1$.

If $\sigma(W_5) = W_6$ and $\sigma(W_6) \in W_5$, since $\sigma(C) \neq C$, thus $\sigma(W_1) \in D$ and $p(B \cup \sigma(W_1) \cup \sigma(W_5)) = 1$.

Therefore we always have $dist(p, \sigma(p)) \geq 3$.

d. $\sigma(W_2) \neq W_2$ and $\sigma(W_5) = W_5$.

Similar to the above one.

4. $\sigma(W_3) = W_4$ and $\sigma(W_4) = W_3$.

Similar to the case where $\sigma(W_3) = W_3$ and $\sigma(W_4) = W_4$.

Case for $k_2 \geq 2$. Similarly, since $|\{\sigma(p)|\sigma(C) = C$ and $\sigma(B) = B\}| = O(1)$, we only need to consider the set $\{\sigma(p)|\sigma(C) \neq C$ and $\sigma(B) = B\}$, and we exclude each σ case by case:

1. $\sigma(W_3)$ or $\sigma(W_4) \notin \{W_3, W_4\}$.

Assume $\sigma(W_3) \notin \{W_3, W_4\}$, note that for any $S \cap (B \cup W_3) = \emptyset$, $p(B \cup W_3 \cup S) \neq \star$ and there are only two such Ss to make $p = 1$. While no matter what $\sigma(W_3)$ is, it's easy to see there are at least five (actually many) such Ss to make $p(B \cup \sigma(W_3) \cup \sigma(S)) = 1$, thus $dist(p, \sigma(p)) \geq 3$.

2. $\sigma(W_3), \sigma(W_4) \in \{W_3, W_4\}$

W.l.o.g, assume $\sigma(W_3) = W_3$ and $\sigma(W_4) = W_4$.

a. $\sigma(W_5) \neq W_5$ and $\sigma(W_2) \neq W_2$.

Now $p(B \cup \sigma(W_3) \cup \sigma(W_2)) = p(B \cup \sigma(W_4) \cup \sigma(W_5)) = 0$ and $p(B \cup \sigma(W_3) \cup \sigma(S)) = p(B \cup \sigma(W_4) \cup \sigma(S')) = 1$, for some $\sigma(S) = W_2$ and $\sigma(S') = W_5$. Therefore, $dist(p, \sigma(p)) \geq 4$.

b. $\sigma(W_5) = W_5$ or $\sigma(W_2) = W_2$.

Assume $\sigma(W_5) = W_5$, since $\sigma(C) \cap D \neq \emptyset$, there exists some $W \in \{W_1, W_2, W_6\}$ such that $\sigma(W) \cap D \neq \emptyset$.

Moreover, for any $S \subseteq W_3 \cup W_4 \cup W_5$ and $S \notin \{W_3, W_4, W_5\}$ with $|S| = k_2$, we have $\sigma(S) \subseteq W_3 \cup W_4 \cup W_5$ and $\sigma(S) \notin \{W_3, W_4, W_5\}$, thus $p(B \cup W \cup S) = 0 \neq p(B \cup \sigma(W) \cup \sigma(S)) = 1$, and note that there are at least $\binom{6}{2} - 3 = 12$ such Ss. Thus, $dist(p, \sigma(p)) \geq 3$.

◄

▶ **Theorem 2** (restated). *For any fixed $k \geq 4$ satisfying $k \equiv 1$ (mod 3) , there exists a sequence of k-uniform hypergraph properties f such that $s(f) = O(n^{\lceil k/3 \rceil - 1/2})$, where n is the number of vertices.*

Proof. We still use minterm functions here.

Let $k = 3l + 1$. Note that in the above construction for $(3l + 1)$-uniform hypergraph properties, $s_1(f) \leq |p| = O(n^l)$ and $s_0(f) = O(n^{l+1})$. Intuitively, we can pack $\sqrt{n}$ minterms together to get a super minterm, expecting to decrease the number of shifts of p satisfying the distance constraint (i.e., where any of two shifts of p have distance at most 2). However, just packing minterms naively doesn't work here, we need to do more.

Let p be the minterm defining f. p is constructed as follows:

The notions V, B, W_i, C and D are defined the same as in Theorem 1, where we let $k_1 = k_2 = l$. Besides that, let $D_1 = \{v_{6l+1}, v_{6l+2}, \cdots, v_{6l+\sqrt{n}}\}$ and $D_2 = D \setminus D_1$.

- For any $S \subseteq C$ of size $2l$ and any $v \in D_1$, $p(B \cup S \cup v) = 0$, except $S = W_i \cup W_{i+1}$ for $i \in [5]$ where $p(B \cup S \cup v) = 1$.
- For any $S \subseteq C$ of size $2l$ and any $v \in D_2$, $p(B \cup S \cup v) = 1$.
- For any S satisfying $l \leq |S \cap C| < 2l$, $|S| = 2l + 1$ and $S \cap D_1 \neq \emptyset$, $p(B \cup S) = 1$, except W_3 or $W_4 \subseteq S$ where $p(B \cup S) = 0$.
- All the other variables are $\star$.

It is not hard to see that $|p| = O(n^{l+1/2})$, thus $s_1(f) \leq C_1(f) \leq |p| = O(n^{l+1/2})$.

Similar to the argument in the proof of Theorem 1, we just need to show the following claim to complete the proof.

▶ **Claim 19.** *There are only $O(\sqrt{n})$ $\sigma(p)$s with the same $\pi(B) = \sigma(B)$ satisfying $dist(\pi(p), \sigma(p)) \leq 2$.*

Proof. By contradiction, suppose there are $C\sqrt{n}$ such $\sigma(p)$s where C is a sufficient large number, thus there must exist a vertex v such that $\sigma(v) \in D_1$ for at least C such $\sigma(p)$s, w.l.o.g, assume this set contains p. And we will argue that there are only $O(1)$ such $\sigma(p)$s satisfying $dist(\sigma(p), p) \leq 2$, then it's a contradiction, which completes the proof.   ◄

Since the vertices in D_1 or D_2 are symmetric, thus $|\{\sigma(p)|\sigma(C) = C \text{ and } \sigma(D_1) = D_1\}| = O(1)$.

If $\sigma(C) = C$ and $\exists v_1 \in D_1, v_2 \in D_2$ satisfying $\sigma(v_1) = v_2$, then $dist(\sigma(p), p) \geq 3$, since almost all variables which contains v_1, C and B are 0 in p, while all these variable are 1 in $\sigma(p)$.

If $\sigma(C) \neq C$, since $\sigma(v) \in D_1$, then we find that $p(S \cup v) = p'(S)$ where p' is the minterm defined in Theorem 1 for $3l$-uniform hypergraph properties. Similarly, $\sigma(p)(S \cup v) = p(\sigma(S) \cup \sigma(v)) = p'(\sigma(S))$. We only consider those Ss satisfying $v \notin \sigma(S) \cup S$ and follows the similar proof of Claim 1 in Theorem 1. Finally we can obtain $dist(p, \sigma(p)) \geq 3$.   ◄

Table 1 The table to illustrate p of k-partite k-uniform hypergraph properties.

$\vec{b} =$	$\vec{z}_1$	$\vec{z}_2$	$\vec{z}_3$	$\vec{z}_4$	$\vec{z}_5$	$\vec{z}_6$	$\vec{z}_7$	$\cdots$
$\vec{a} = \vec{z}_1$	0	0	0	1	1	1	1	1
$\vec{z}_2$	0	0	1	1	1	1	1	1
$\vec{z}_3$	1	0	1	0	0	0	0	0
$\vec{z}_4$	1	1	0	$\star$	$\star$	$\star$	$\star$	$\star$
$\vec{z}_5$	1	1	0	$\star$	$\star$	$\star$	$\star$	$\star$
$\vec{z}_6$	1	1	0	$\star$	$\star$	$\star$	$\star$	$\star$
$\vec{z}_7$	1	1	0	$\star$	$\star$	$\star$	$\star$	$\star$
$\cdots$	1	1	0	$\star$	$\star$	$\star$	$\star$	$\star$

4   k-Partite k-Uniform Hypergraph Properties and Abelian Groups

In this section, we give the constructions of k-partite k-uniform hypergraph properties first.

▶ **Theorem 3** (restated). *For any $k \geq 3$, there exists a sequence of k-partite k-uniform hypergraph properties $f : \{0,1\}^{n^k} \to \{0,1\}$ such that $s(f) = O(n^{\lceil k/3 \rceil})$.*

Proof. The function we use here is also a minterm function. Let k_1 and k_2 be the integers such that $k_1 + 2k_2 = k$ and $\lfloor k/3 \rfloor \leq k_1, k_2 \leq \lceil k/3 \rceil$. We divide the k partitions into three sets, and each of them is of size k_2, k_2 and k_1. And they are indicated by $\vec{a}, \vec{b} \in [n]^{k_2}$ and $\vec{c} \in [n]^{k_1}$ respectively. Assume t is an integer, let $\vec{z}_t$ ($\vec{z}_t^c$ respectively) be the t-th smallest vector in $[n]^{k_2}$ ($[n]^{k_1}$ respectively) in the lexicographic order. We use Table 1 to illustrate the minterm p:

- For $\vec{b} = \vec{z}_1, \vec{z}_2$ or $\vec{z}_3$, $p(\vec{z}_1, \vec{b}, \vec{z}_1^c) = 0$, otherwise $p(\vec{z}_1, \vec{b}, \vec{z}_1^c) = 1$.
- For $\vec{b} = \vec{z}_1$ or $\vec{z}_2$, $p(\vec{z}_2, \vec{b}, \vec{z}_1^c) = 0$, otherwise $p(\vec{z}_2, \vec{b}, \vec{z}_1^c) = 1$.
- For $\vec{b} = \vec{z}_1$ or $\vec{z}_3$, $p(\vec{z}_3, \vec{b}, \vec{z}_1^c) = 1$, otherwise $p(\vec{z}_3, \vec{b}, \vec{z}_1^c) = 0$.
- For $\vec{a} \notin \{\vec{z}_1, \vec{z}_2, \vec{z}_3\}$ and $\vec{b} = \vec{z}_1$ or $\vec{z}_2$, $p(\vec{a}, \vec{b}, \vec{z}_1^c)) = 1$.
- For $\vec{a} \notin \{\vec{z}_1, \vec{z}_2, \vec{z}_3\}$ and $\vec{b} = \vec{z}_3$, $p(\vec{a}, \vec{b}, \vec{z}_1^c) = 0$.
- Otherwise $p(\vec{a}, \vec{b}, \vec{c}) = \star$.

It's easy to see $s_1(f) \leq C_1(f) \leq |p| = O(n^{k_2})$. By discussing case by case, it can be verified that for any p^π there are at most $O(1)$ p^σs satisfying $\pi(\vec{c}) = \sigma(\vec{c})$ and $dist(p^\pi, p^\sigma) \leq 2$. Observe that there are n^{k_1} choices of $\pi(\vec{c})$, thus there are at most $O(n^{k_1})$ p^σs such that the distance between any two of them is at most 2. Similar to the argument in the proof of Theorem 1, we can conclude $s_0(f) = O(n^{k_1})$. The verifying procedure is straightforward but tedious, and we omit it here.   ◀

In the following, we give the proofs of Theorem 4 and Theorem 5.

▶ **Theorem 4** (restated). *For any $k \geq 4$ satisfying $k \equiv 1 \pmod 3$, there exists a sequence of k-partite k-uniform hypergraph properties $f : \{0,1\}^{n^k} \to \{0,1\}$ such that $s(f) = O(n^{\lceil k/3 \rceil - 1/2})$.*

Proof. We still use minterm functions here. Let $k = 3l + 1$ where $l \geq 1$. We divide the k partitions into four sets of size l, l, l and 1, and each set is indicated by $\vec{a}, \vec{b}, \vec{c} \in [n]^l$ and $\vec{d} \in [n]$ respectively. Assume t is an integer, let $\vec{z}_t$ be the t-th smallest vector in $[n]^l$ in the lexicographic order. The minterm p is constructed as follows:

- For any $\vec{d} \in [\lceil \sqrt{n} \rceil]$, and any $\vec{a}$ and $\vec{b}$, $p(\vec{a}, \vec{b}, \vec{z}_1, \vec{d}) = p'(\vec{a}, \vec{b}, \vec{z}_1)$. Here p' is the partial assignment defined in the proof of Theorem 3.

- For any $\vec{d} \notin [\lceil\sqrt{n}\rceil]$ and $\vec{a}, \vec{b} \in \{\vec{z}_1, \vec{z}_2, \vec{z}_3\}$, $p(\vec{a}, \vec{b}, \vec{z}_1, \vec{d}) = 1$.
- Otherwise $p(\vec{a}, \vec{b}, \vec{c}, \vec{d}) = \star$.

It's easy to see $s_1(f) \leq |p| = O(n^{l+1/2})$. It is also not hard to verify that there are at most $\lceil\sqrt{n}\rceil$ p^σs with the same $\sigma(\vec{c})$ and satisfying the condition that the distance between any two of them is at most 2, thus $s_0(f) = O(n^{l+1/2})$. ◄

▶ **Theorem 5** (restated). *Let $G \leq S_n$ be a transitive Abelian group, then there exists a Boolean function $f : \{0,1\}^n \to \{0,1\}$ invariant under G such that $s(f) \leq \alpha n^{1/3}$, where α is a number only depending on l.*

Proof. First note that the transitive action of a group G on $[n]$ is equivalent to the action of G by left multiplication on a coset space G/Stab_1, here Stab_1 is the stabilizer of the element $1 \in [n]$. Since G is an Abelian group, $\mathrm{Stab}_1 = \cdots = \mathrm{Stab}_n$, thus $\mathrm{Stab}_1 = \{e\}$. Therefore, the action of G on $[n]$ is equivalent to the action of G by multiplication on itself. So we can relabel the variables $(x_1, \cdots, x_n)$ as $(x_{(1,\cdots,1)}, \cdots, x_{(m_1,\cdots,m_l)})$ to make $(\sigma_1 \otimes \cdots \otimes \sigma_l)(x) = (x_{(\sigma_1(1),\cdots,\sigma_l(1))}, \cdots, x_{(\sigma(m_1),\cdots,\sigma_l(m_2))})$ for any $\sigma_1 \otimes \cdots \otimes \sigma_l \in C_{m_1} \times \cdots \times C_{m_l}$.

Let p_m be the minterm of $f : \{0,1\}^m \to \{0,1\}$ defined by Chakraborty in Theorem 3.1 in [14]. We define the minterm p as $p(i_1, \cdots, i_l) = \bigoplus_{j=1}^l p_{m_j}(i_j)$. Here $\star \oplus b = \star$, for $b = 0, 1$, or $\star$. It is easy to see $s_1(f) \leq |p| = \prod_{j=1}^l |p_{m_j}| \leq \gamma n^{1/3}$, where γ is a number only depending on l. Moreover, according to the construction of p_m, it is easy to see that there are at most $\beta n^{1/3}$ $\sigma(p)$s where the distance between any two of them is at most 2. Here β is another number only depending on l, thus $s_0(f) \leq \beta n^{1/3}$. This completes the proof. ◄

5 Lower bounds

In this section, we give the proof of Theorem 6.

▶ **Theorem 6** (restated). *For any fixed k and any non-trivial k-uniform hypergraph property f, $s(f) = \Omega(n)$, where n is the number of vertices.*

Proof. W.l.o.g we assume that for the empty graph $\overline{K_n}$, $f(\overline{K_n}) = 0$. Since f is non-trivial, there must exist a graph G such that $f(G) = 1$. Let's consider graphs in $f^{-1}(1) = \{G | f(G) = 1\}$ with the minimum number of edges. Define $m = \min\{|E(G)| : f(G) = 1\}$.

We claim that if $m \geq \frac{1}{k+2}n$, then $s(f) \geq \frac{1}{k+2}n$. Let G be a graph in $f^{-1}(1)$ and $|E(G)| = m$. Consider the subfunction f' where $\forall e \notin E(G)$, x_e is restricted to 0, since G has the the minimum number of edges, deleting any edges from G will change the values of $f(G)$, therefore, f' is an AND function. Thus, $s(f) \geq s(f') = m \geq \frac{1}{k+2}n$.

In the following we assume $m < \frac{1}{k+2}n$. Again let G be a graph in $f^{-1}(1)$ with $|E(G)| = m$. Let us consider the isolated vertices set I, as $\sum_{v \in V} \deg(v) = k|E(G)| < \frac{k}{k+2}n$, we have $|I| \geq n - \sum_{v \in V} \deg(v) > \frac{2}{k+2}n$. Suppose $s(f) < \frac{1}{k+2}n$, we will deduce that there exists another graph with fewer edges and the same value, against the assumption that G has the minimum number of edges in $f^{-1}(1)$, which ends the whole proof.

Pick a vertex u with $deg(u) = d > 0$. Suppose in the graph G vertex u is adjacent to $(k-1)$-edges $\{e_1^{(k-1)}, e_2^{(k-1)}, \cdots, e_d^{(k-1)}\}$ and $I = \{u_1, u_2, \cdots, u_t\}$, where $t = |I|$.

Consider the t-variable Boolean function $g_1 : \{0,1\}^t \to \{0,1\}$, where

$$g_1(x_1, \cdots, x_t) = f(G + x_1(e_1^{(k-1)}, u_1) + \cdots + x_t(e_1^{(k-1)}, u_t)).$$

It is easy to see that g_1 is a symmetric function. We claim that g_1 is a constant function: if not, we have $s(g_1) \geq \frac{1}{2}t$ [38], which implies $s(f) > \frac{1}{k+2}n$ since g_1 is a restriction of f. In particular, $g_1(1, \cdots, 1) = g_1(0, \cdots, 0)$, i.e. $f(G_1) = f(G)$, where $G_1 = G + \sum_{i=1}^t (e_1^{(k-1)}, u_i)$.

Define $G_i = G_{i-1} + \sum_{j=1}^t (e_i^{(k-1)}, u_j)$ $(i = 2, \cdots, d)$. Similarly, we can show that

$$f(G) = f(G_1) = \cdots = f(G_d).$$

Next we will delete all the edges between $\{u, u_1, \cdots, u_t\}$ and $\{e_1^{(k-1)}, e_2^{(k-1)}, \cdots, e_d^{(k-1)}\}$ from G_d by reversing the adding edge procedure of $G \to G_1 \to \cdots \to G_d$. More precisely, define $H_1 = G_d$; for $i = 2, \cdots, d$, define

$$H_i = H_{i-1} - (e_i^{(k-1)}, u) - (e_i^{(k-1)}, u_1) - \cdots - (e_i^{(k-1)}, u_t),$$

and

$$h_i(y_0, y_1, \cdots, y_t) = f(H_i + y_0(e_i^{(k-1)}, u) + y_1(e_i^{(k-1)}, u_1) + \cdots + y_t(e_i^{(k-1)}, u_t)).$$

Similarly, by the fact $s(f) < \frac{1}{k+2}n$ we can show that all the functions $h_2, \cdots, h_d$ are constant, which implies $f(H_1) = f(H_2) = \cdots = f(H_d)$. So we find another graph H_d with fewer edges than G and $f(H_d) = 1$. ◀

6 Conclusion

In this paper, we present a k-uniform hypergraph property with sensitivity complexity $O(n^{\lceil k/3 \rceil})$ for any $k \geq 3$ and we can do better when $k \equiv 1 \pmod 3$. Besides that, we also investigate the sensitivity complexity of other transitive Boolean functions with certain symmetry. All the functions we constructed in this paper are minterm transitive functions. On the other side, Chakraborty [14] proved that the sensitivity complexity of any minterm transitive Boolean function $f : \{0,1\}^N \to \{0,1\}$ is at least $\Omega(N^{1/3})$. Kulkarni et al. [27] pointed out that the existence of any transitive function $f : \{0,1\}^N \to \{0,1\}$ with $s(f) = N^\alpha$ where $\alpha < 1/3$ implies a larger than quadratic separation between block sensitivity and sensitivity. We conjecture that the examples here are almost tight.

▶ **Conjecture 20.** *For any $k \geq 3$ and for any non-trivial k-hypergraph property f, $s(f) = \Omega(n^{k/3})$, where n is the number of vertices.*

▶ **Conjecture 21.** *For any $k \geq 3$, there exists a sequence of k-uniform hypergraph properties f with $s(f) = O(n^{k/3})$, where n is the number of vertices.*

A more general question is the following variant of Turan's question proposed by Chakraborty [14]: If $f : \{0,1\}^N \to \{0,1\}$ is Boolean function invariant under a transitive group of permutations, then is it true that $s(f) = \Omega(N^c)$ for some constant $c > 0$? We conjecture that the inequality holds for $c = 1/3$, which would imply Conjecture 20 and the sensitivity conjecture of transitive functions.

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