# Betti Numbers and Generalized Hamming Weights

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We can associate to each linear code $\mathscr{C}$ defined over a finite field the matroid $M[H]$ of its parity check matrix $H$. For any matroid $M$ one can define its generalized Hamming weights which are the same as those of the code $\mathscr{C}$. In [2] the authors show that the generalized Hamming weights of a matroid are determined by the $\mathbb{N}$-graded Betti numbers of the Stanley-Reisner ring of the simplicial complex whose faces are the independent set of $M$. In this talk we go a step further. Our practical results indicate that the generalized Hamming weights of a linear code $\mathscr{C}$ can be obtained from the monomial ideal associated with a test-set for $\mathscr{C}$. Moreover, recall that in [3] we use the Gröbner representation of a linear code $\mathscr{C}$ to provide a test-set for $\mathscr{C}$.

Our results are still a work in progress, but its applications to Coding Theory and Cryptography are of great value.

## 1 Notation and Prerequisites

We begin with an introduction of basic definitions and some known results. By $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{F}_q$ (where $q$ is a primer power) we denote the set of positive integers, the set of integers and the finite field with $q$ elements, respectively.

**Definition 1** *A matroid $M$ is a pair $(E,I)$ consisting of a finite set $E$ called ground set and a collection $I$ of subsets of $E$ called independent sets, satisfying the following conditions:*

1. *The empty set is independent, i.e. $\emptyset \in I$*
2. *If $A \in I$ and $B \subset A$, then $B \in I$*
3. *If $A,B \in I$ and $|A| < |B|$, then there exists $e \in B \setminus A$ such that $A \cup \{e\} \in I$*

Let $M = (E,I)$ be a matroid. A maximal independent subset of $E$ is called a *basis* of $M$. A direct consequence of the previous definition is that all bases of $M$ have the same cardinality. Thus, we define the *rank* of the matroid $M$ as the cardinality of any basis of $M$, denoted by $\text{rank}(M)$. A subset $E$ that does not belong

to $I$ is called *dependent set*. Minimal dependent subsets of $E$ are known as *circuits* of $M$. A set is said to be a *cycle* if it is a disjoint union of circuits. The collection of cycles of $M$ is denoted by $\mathscr{C}(M)$. For all $\sigma \in E$, the *nulity function* of $\sigma$ is given by $n(\sigma) := |\sigma| - \text{rank}(M_\sigma)$ with $\text{rank}(M_\sigma) = \max\{|A| \mid A \in I \text{ and } A \subset \sigma\}$, i.e. the restriction of $\text{rank}(M)$ to the subsets of $\sigma$.

Let us consider an $m \times n$ matrix $A$ in $\mathbb{F}_q$ whose columns are indexed by $E = \{1,\ldots,n\}$ and take $I$ to be the collection of subsets $J$ of $E$ for which the column vectors $\{A_j \mid j \in J\}$ are linearly independent over $\mathbb{F}_q$. Then $(E,I)$ defines a matroid denoted by $M[A]$. A matroid $M = (E,I)$ is $\mathbb{F}_q$-representable if it is isomorphic to $M[A]$ for some $A \in \mathbb{F}_q^{m\times n}$. Then the matrix $A$ is called the representation matrix of $M$. The following well known results describes the relation between the colleciton of all cycles of a matroid $M$ and its representation matrix.

**Proposition 1** *Let $M = (E,I)$ be a $\mathbb{F}_q$-representable matroid. Then $\mathscr{C}(M)$ is the null space of a representation matrix of $M$. Furthermore, the dimension of $\mathscr{C}(M)$ is $|E| - \text{rank}(M)$.*

Let $\Delta$ be a simplicial complex on the finite ground set $E$. Let $\mathbb{K}$ be a field and let $\mathbf{x}$ be the indeterminates $\mathbf{x} = \{x_e \mid e \in E\}$. The *Stanley-Reisner* ideal of $\Delta$ is, by definition,

$$I_\Delta = \langle \mathbf{x}^\sigma \mid \sigma \notin \Delta \rangle$$

The *Stanley-Reisner ring* of $I_\Delta$, denoted by $R_\Delta$, is defined to be the quotient ring $R_\Delta = \frac{\mathbb{K}[\mathbf{x}]}{I_\Delta}$. This ring has a minimal free resolution as $\mathbb{N}^E$-graded module:

$$0 \longleftarrow R_\Delta \longleftarrow P_0 \longleftarrow P_1 \longleftarrow \cdots \longleftarrow P_l \longleftarrow 0$$

where each $P_i$ is given by $P_i = \bigoplus_{\alpha\in\mathbb{N}^E} \mathbb{K}[\mathbf{x}](-\alpha)^{\beta_{i,\alpha}}$. We write $\beta_{i,\alpha}$ for the $\mathbb{N}^E$-graded Betti Numbers of $\Delta$.

## 1.1 Matroids and Simplicial complex

A matroid $M = (E,I)$ is a simplicial complex whose faces are the independent sets. Thus, $I_M := \langle \mathbf{x}^\sigma \mid \sigma \in \mathscr{C} \rangle$ where $\mathscr{C}$ is the set of all circuits of $M$. Define $N_i = \{\sigma \in N \mid n(\sigma) = d\}$.

**Theorem 1 ([2]Theorem 1)** *Let $M$ be a matroid on the ground set $E$. Let $\sigma \subset E$. Then, $\beta_{i,\sigma} \neq 0$ if and only if $\sigma$ is minimal in $N_i$.*

**Definition 2** *Let $M = (E,I)$ be a matroid, we define the generalized Hamming weights of $M$ to be $d_i = \min\{|\sigma| \mid n(\sigma) = i\}$.*

**Corollary 1** *Let $M$ be a matroid on the ground set $E$. Then,*

$$d_i = \min\{d \mid \beta_{i,d} \neq 0 \quad \textit{for all} \quad 1 \leq i \leq |E| - \text{rank}(M)\}.$$

## 1.2 Matroids and linear codes

An $[n,k]_q$ linear code $\mathscr{C}$ is a $k$-dimensional subspace of $\mathbb{F}_q^n$. We define a *generator matrix* of $\mathscr{C}$ to be a $k \times n$ matrix $G$ whose row vectors span $\mathscr{C}$, while a *parity check matrix* of $\mathscr{C}$ is an $(n-k) \times n$ matrix $H$ whose null space is $\mathscr{C}$.

Let us denote by $\mathrm{d}_H(\cdot,\cdot)$ and $\mathrm{w}_H(\cdot)$ the *Hamming distance* and the *Hamming weight* on $\mathbb{F}_q^n$, respectively. We write $d$ for the *minimum Hamming distance* of the code $\mathscr{C}$, which is equal to its minimum weight. Thus, the error correcting capability of $\mathscr{C}$ is $t = \left\lfloor \frac{d-1}{2} \right\rfloor$ where $\lfloor \cdot \rfloor$ is the greatest integer function. For every codeword $\mathbf{c} \in \mathscr{C}$ its *support*, $\mathrm{supp}(\mathbf{c})$, is defined as its support as a vector in $\mathbb{F}_q^n$, i.e. $\mathrm{supp}(\mathbf{c}) = \{i \mid c_i \neq 0\}$. We will denote by $\mathscr{M}_{\mathscr{C}}$ the set of codewords of minimal support of $\mathscr{C}$.

A *test-set* $\mathscr{T}_{\mathscr{C}}$ for $\mathscr{C}$ is a set of codewords such that for every word $\mathbf{y} \in \mathbb{F}_q^n$, either $\mathbf{y}$ belongs to the set of coset leaders, or there exists an element $\mathbf{t} \in \mathscr{T}_{\mathscr{C}}$ such that $\mathrm{w}_H(\mathbf{y}-\mathbf{t}) < \mathrm{w}_H(\mathbf{y})$.

**Definition 3** *The $r^{th}$ generalized Hamming weight of $\mathscr{C}$ denoted by $d_r(\mathscr{C})$ is the smallest support of an $r$-dimensional subcode of $\mathscr{C}$. That is,*

$$d_r(\mathscr{C}) = \min\{\mathrm{supp}(D) \mid D \subseteq \mathscr{C} \text{ and } \mathrm{rank}(D) = r\}$$

In [3] the authors associate a binomial ideal to an arbitrary linear code provided by the rows of a generator matrix and the relations given by the additive table of the defining field.

Let $\mathbf{X}$ denote $n$ vector variables $X_1,\ldots,X_n$ such that each variable $X_i$ can be decomposed into $q-1$ components $x_{i,1},\ldots,x_{i,q-1}$ with $i = 1,\ldots,n$. A monomial in $\mathbf{X}$ is a product of the form:

$$\mathbf{X}^{\mathbf{u}} = X_1^{\mathbf{u}_1} \cdots \mathbf{X}_n^{\mathbf{u}_n} = \left(x_{1,1}^{u_{1,1}} \cdots x_{1,q-1}^{u_{1,q-1}}\right) \cdots \left(x_{n,1}^{u_{n,1}} \cdots x_{n,q-1}^{u_{n,q-1}}\right)$$

where $\mathbf{u} \in \mathbb{Z}_{\geq 0}^{n(q-1)}$. The total degree of $\mathbf{X}^{\mathbf{u}}$ is the sum $\deg(\mathbf{X}^{\mathbf{u}}) = \sum_{i=1}^{n} \sum_{j=1}^{q-1} u_{i,j}$. When $\mathbf{u} = (0,\ldots,0)$, note that $\mathbf{X}^{\mathbf{u}} = 1$. Then, the polynomial ring $\mathbb{K}[\mathbf{X}]$ is the set of all polynomials in $\mathbf{X}$ with coefficients in $\mathbb{K}$.

Recall that the multiplicative group $\mathbb{F}_q^*$ of nonzero elements of $\mathbb{F}_q$ is cyclic. A generator of the cyclic group $\mathbb{F}_q^*$ is called a primitive element of $\mathbb{F}_q$, i.e. $\mathbb{F}_q$ consist of 0 and all powers from 1 to $q-1$ of that primitive element. Let $\alpha$ be a primitive element of $\mathbb{F}_q$. We define by $\mathscr{R}_{X_i}$, the set of all the binomials on the variables $X_i$ associated to the relations given by the additive table of the field $\mathbb{F}_q = \langle \alpha^j \mid j = 1,\ldots,q-1 \rangle \cup \{0\}$, i.e.

$$\mathscr{R}_{X_i} = \left\{ \ \{x_{i,u}x_{i,v} - x_{i,w} \mid \alpha^u + \alpha^v = \alpha^w\} \quad \cup \quad \{x_{i,u}x_{i,v} - 1 \mid \alpha^u + \alpha^v = 0\} \ \right\}$$

with $i = 1, \ldots, n$. Note that there are $\binom{q}{2}$ different binomials in $\mathscr{R}_{X_i}$. We define $\mathscr{R}_{\mathbf{X}}$ as the ideal generated by the union of all binomial ideals $\mathscr{R}_{X_i}$, i.e. $\mathscr{R}_{\mathbf{X}} = \left\langle \cup_{i=1}^{n} \mathscr{R}_{X_i} \right\rangle$

We will use the following characteristic crossing functions. These applications aim at describing a one-to-one correspondence between the finite field $\mathbb{F}_q$ with $q$ elements and the standard basis of $\mathbb{Z}^{q-1}$, denoted as $E_q = \{\mathbf{e}_1, \ldots, \mathbf{e}_{q-z}\}$ where $\mathbf{e}_i$ is the unit vector with a 1 in the $i$-th coordinate and 0's elsewhere.

$$\Delta: \quad \mathbb{F}_q \quad \longrightarrow \quad E_q \cup \{\mathbf{0}\} \subseteq \mathbb{Z}^{q-1} \qquad \text{and} \qquad \nabla: \quad E_q \cup \{\mathbf{0}\} \quad \longrightarrow \quad \mathbb{F}_q$$

1. The map $\Delta$ replaces the element $\mathbf{a} = \alpha^i \in \mathbb{F}_q$ by the vector $\mathbf{e}_i$ and $0 \in \mathbb{F}_q$ by the zero vector $\mathbf{0} \in \mathbb{Z}^{q-1}$.

2. The map $\nabla$ recovers the element $\alpha^j \in \mathbb{F}_q$ from the unit vector $\mathbf{e}_j$ and the zero element $0 \in \mathbb{F}_q$ from the zero vector $\mathbf{0} \in \mathbb{Z}^{q-1}$.

These maps will be used with matrices and vectors acting coordinate-wise. Although $\Delta$ is not a linear function. Note that we have:

$$\mathbf{X}^{\Delta \mathbf{a}} \cdot \mathbf{X}^{\Delta \mathbf{b}} = \mathbf{X}^{\Delta \mathbf{a} + \Delta \mathbf{b}} = \mathbf{X}^{\Delta(\mathbf{a}+\mathbf{b})} \mod \mathscr{R}_{\mathbf{X}} \text{ for all } \mathbf{a}, \mathbf{b} \in \mathbb{F}_q^n.$$

Let $\mathscr{C}$ be an $[n,k]_q$ linear code. We define the *ideal associated* to $\mathscr{C}$ as the binomial ideal:

$$I(\mathscr{C}) = \left\langle \left\{ \mathbf{X}^{\Delta \mathbf{a}} - \mathbf{X}^{\Delta \mathbf{b}} \mid \mathbf{a} - \mathbf{b} \in \mathscr{C} \right\} \right\rangle \subseteq \mathbb{K}[\mathbf{X}]$$

Given the rows of a generator matrix $\mathscr{C}$, labelled by $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\} \subseteq \mathbb{F}_q^n$, we define the following ideal:

$$I_+(\mathscr{C}) = \left\langle \quad \left\{ \mathbf{X}^{\Delta(\alpha^j \mathbf{w}_i)} - 1 \right\}_{\substack{i=1,\ldots,n \\ j=1,\ldots q-1}} \quad \cup \quad \left\{ \mathscr{R}_{X_i} \right\}_{i=1,\ldots,n} \quad \right\rangle \subseteq \mathbb{K}[\mathbf{X}]$$

**Theorem 2** *[3][Theorem 2.3]* $I(\mathscr{C}) = I_+(\mathscr{C})$

**Remark 1** *In the binary case, given a generator matrix $G \in \mathbb{F}_2^{k \times n}$ of an $[n,k]_2$-code $\mathscr{C}$ and let label its rows by $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\} \subseteq \mathbb{F}_2^n$. We define the ideal associated to $\mathscr{C}$ as the binomial ideal:*

$$I_+(\mathscr{C}) = \left\langle \quad \left\{ \mathbf{X}^{\mathbf{w}_i} - 1 \right\}_{i=1,\ldots,k} \quad \cup \quad \left\{ x_i^2 - 1 \right\}_{i=1,\ldots,n} \quad \right\rangle \subseteq \mathbb{K}[\mathbf{X}]$$

Now, let $\mathscr{G} = \{g_1, \ldots, g_s\}$ be the reduced Gröbner basis of the ideal $I_+(\mathscr{C})$ with respect to $\succ$, where we take $\succ$ to be any degree compatible ordering on $\mathbb{K}[\mathbf{X}]$ with

$X_1 \prec \ldots \prec X_n$. By Lemma [3][Lemma 3.3] we know that all elements of $\mathscr{G} \setminus \mathscr{R}_{\mathbf{X}}$ are in standard form, so for $g_i \in \mathscr{G} \setminus \mathscr{R}_{\mathbf{X}}$ with $i = 1, \ldots, s$, we define

$$g_i = \mathbf{X}^{\Delta \mathbf{g}_i^+} - \mathbf{X}^{\Delta \mathbf{g}_i^-} \quad \text{with} \quad \mathbf{X}^{\Delta \mathbf{g}_i^+} \succ \mathbf{X}^{\Delta \mathbf{g}_i^-} \quad \text{and} \quad \mathbf{g}_i^+ - \mathbf{g}_i^- \in \mathscr{C}.$$

Using [3][Proposition 4], we know that the set $\mathscr{T} = \left\{ \mathbf{g}_i^+ - \mathbf{g}_i^- \mid i = 1, \ldots, s \right\}$ is a test-set for $\mathscr{C}$.

**Example 1** *Consider the* $[6,3,2]_2$ *binary code* $\mathscr{C}$ *defined by the following generator matrix:*

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \in \mathbb{F}_2^{3 \times 6}$$

*Let us label the rows of G by* $\mathbf{w}_1$ *and* $\mathbf{w}_2$*. By the previous theorem, the ideal associated to the linear code* $\mathscr{C}$ *may be defined as the following ideal:*

$$\begin{aligned} I_+(\mathscr{C}) &= \left\langle \ \{\mathbf{X}^{\mathbf{w}_i} - 1\}_{i=1,2} \ \cup \ \{\mathscr{R}_{X_i}\}_{i=1,\ldots,6} \ \right\rangle \\ &= \left\langle \ \left\{ \begin{array}{c} x_1 x_6 - 1 \\ x_2 x_3 x_5 - 1 \\ x_4 x_5 x_6 - 1 \end{array} \right\} \ \cup \ \left\{ x_i^2 - 1 \right\}_{i=1,\ldots,6} \ \right\rangle \end{aligned}$$

*If we compute a reduced Gröbner basis* $\mathscr{G}$ *of* $I_+(\mathscr{C})$ *we obtained a test-set consisting of* 4 *codewords:*

$$\mathscr{T}_{\mathscr{C}} = \{(1,0,0,0,0,1), (0,1,1,0,1,0), (0,1,1,1,1,0,1), (0,0,0,1,1,1)\}$$

For fuller discussion of this algebraic structure see [4, 1] and the references therein.

The connection between linear codes and matroids will turn out to be fundamental for the development of the subsequent results. Thus, a brief review will be provided here.

Given an $m \times n$ matrix $H$ in $\mathbb{F}_q$, then $H$ can be seen not only as the representation matrix of the $\mathbb{F}_q$-representable matroid $M[H]$ but also as a parity check matrix of an $[n,k]$-code $\mathscr{C}$. Furthermore, there exists a one to one correspondence between $\mathbb{F}_q$-representable matroids and linear codes, since for any $H, H' \in \mathbb{F}_q^{m \times n}$, $M[H] = M[H']$ if an only if $H$ and $H'$ are parity check matrices of the same code $\mathscr{C}$. This association enables us to work with $\mathbb{F}_q$-representable matroids and linear codes as if they were the same object and thus we can conclude some properties of linear codes using tools from matroid theory and vice-versa.

## 2 Our Conjecture

Let $M = (E, I)$ be a matroid and $\mathscr{C}$ be the set of all circuits of $M$. Consider $\mathscr{T}$ a collection of cycles of $M$ with the following property: $\bigcup_{\tau\in\mathscr{C}} \tau = \bigcup_{\tau\in\mathscr{T}} \tau$. We define the ideal $I_{\mathscr{T}} = \langle \mathbf{x}^{\sigma} \mid \sigma \in \mathscr{T} \rangle$.

**Conjecture 1** *Let $\beta'_{i,\alpha}$ the $\mathbb{N}^E$-graded betti number of $I_{\mathscr{T}}$, related with the minimal free resolution of $R = \frac{\mathbb{K}[X]}{I_{\mathscr{T}}}$ as $\mathbb{N}^E$-graded module. Then, we have a similar result as Theorem 1 and Corollary 1.*

If we talk about linear codes, the conjecture allows us to compute the set of generalized Hamming weight of a linear code $\mathscr{C}$ using a Test-set for $\mathscr{C}$, in other words, by computing a Grobner basis of the ideal associated to $\mathscr{C}$.

**Corollary 2** *Let $\mathscr{T}_{\mathscr{C}}$ be a test-set for the linear code $\mathscr{C}$. Consider the monomial ideal: $I_{\mathscr{T}_{\mathscr{C}}} = \langle \mathbf{x}^{\sigma} \mid \sigma \in \mathscr{T}_{\mathscr{C}} \rangle$. Let $\beta'_{i,\alpha}$ the $\mathbb{N}^E$-graded betti numbers of $I_{\mathscr{T}_{\mathscr{C}}}$. Then,*

$$d_i(\mathscr{C}) = \min\left\{d \mid \beta'_{i,d} \neq 0\right\} \text{ for } 1 \leq i \leq n-k$$

**Example 2** *Now we use the same code of Example 1. In this case the support of a test-set $T_{\mathscr{C}}$ is given by: $\mathscr{T} = \{\{2,3,5\},\{2,3,4,6\},\{4,5,6\},\{1,6\}\}$ i.e. we consider the ideal: $I_{\mathscr{T}} = \langle x_2x_3x_5, x_2x_3x_4x_6, x_4x_5x_6, x_1x_6\rangle \subseteq \mathbb{K}[x_1,\ldots,x_6]$. We get the Betti diagram*

| | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 1 | | |
| 2 | 2 | 1 | |
| 3 | 1 | 4 | 2 |

*Thus $\beta'_{1,2}$, $\beta'_{2,4}$ and $\beta'_{3,6}$ are the minimal $\beta'_{i,d} \neq 0$ with $i = 1,2,3$. Or equivalently, $d_1 = 2$, $d_2 = 4$ and $d_3 = 6$.*

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