High order stabilized finite element method for MHD plasma modeling

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Outline

1. Introduction
2. Stabilization for Full MHD
3. Applications: Internal kink, Bohm B.C.
4. Conclusions and perspectives

Motivation and goal

Motivations

- Go from Reduced to the full MHD model
- Ensure stability of flows dominated by convection
- Boundary conditions

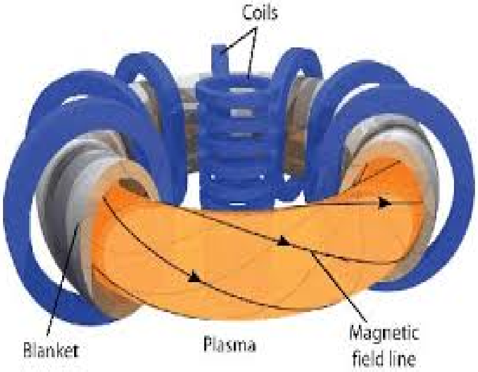
Main goal: use JOREK to simulate Edge-localized modes (ELMS) with the Full-MHD model

Tokamaks

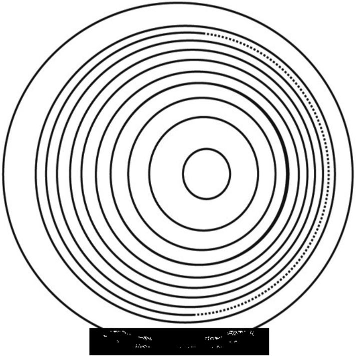
- Tokamak stands for: Toroidal chamber with magnetic coils (From Russian: Toroidal'naya Kamera s Magnitnymi Katushhami).
- Principal tokamaks at present: JET (UK), Asdex-Upgrade (Germany), Tore-Supra (France) and DIIID (USA)
- Drifts $\rightarrow$ Nested magnetic field lines $\rightarrow$ Confinement is improved
- The amplification factor of a fusion reactor is defined as

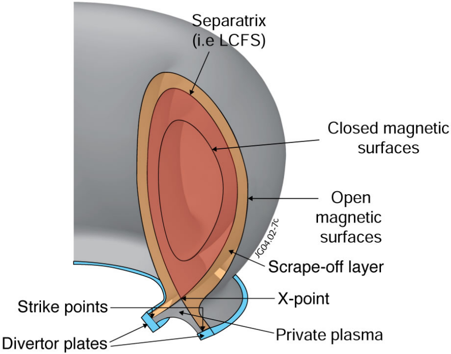
$$Q = \frac{P_{fus}}{P_{inj}}$$

- JET: $Q \approx 1$ - ITER: $Q = 10$
- Economically exploitable $Q = 50$



There are two main configurations for tokamaks





Limiter
Material structure into the plasma

Divertor
X-point configuration
Material structure placed outside

Jorek overview

- Nonlinear reduced MHD and full MHD under implementation
- Cubic C1 finite elements (quadrangles)/Fourier (modal)
- Galerkin formulation
- Linearized Crank-Nicholson and Gear: Implicit scheme
- GMRES solver, Hybrid paralelization (MPI/OpenMPI)

Jorek: Numerical method

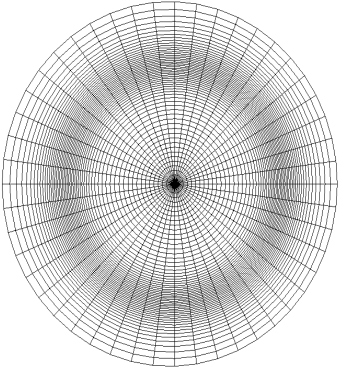
- Spatial discretization: 4^{th}-order accurate Galerkin method based on Bézier Formalism
- An arbitrary quantity is expanded on a unit square as

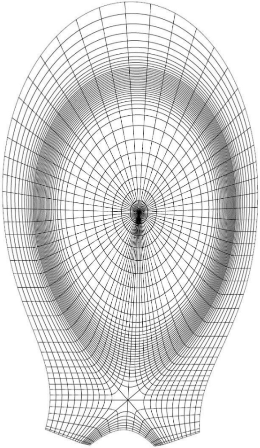
$$P(s,t) = \sum_{i=1}^{4}\sum_{j=1}^{4} p_{ij}\sigma_{ij}b_{ij}(s,t)$$

- $(s,t) \in [0,1]$, σ assure continuity of P
- Basis functions b_{ij}: Bernstein polynomials

$$\begin{aligned}
b_{11} &= (1-s)^2(1-t)^2(1+2s)(1+2t),\\
b_{12} &= 3(1-s)^2(1-t)^2(1+2t)s,\\
b_{13} &= 3(1-s)^2(1-t)^2(1+2s)t,\\
b_{14} &= 9(1-s)^2(1-t)^2 st.
\end{aligned}$$

Examples of meshes: poloidal and X-point





System of equations: Full MHD

The single-fluid full MHD system in consideration is

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{m} = \nabla \cdot (\underline{\mathcal{D}} \nabla \rho'), \\ \dfrac{\partial \boldsymbol{m}}{\partial t} + \nabla \cdot \left(\dfrac{\boldsymbol{m} \otimes \boldsymbol{m}}{\rho} - \boldsymbol{B} \otimes \boldsymbol{B} \right) + \nabla \left(p + \dfrac{\boldsymbol{B} \cdot \boldsymbol{B}}{2} \right) = \nabla \cdot \underline{\tau}, \\ \dfrac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \left(\left(\mathcal{E} + p + \dfrac{\boldsymbol{B} \cdot \boldsymbol{B}}{2} \right) \dfrac{\boldsymbol{m}}{\rho} - \dfrac{\boldsymbol{B} \cdot \boldsymbol{m}}{\rho} \boldsymbol{B} \right) = \nabla \cdot \boldsymbol{q} + \nabla \cdot \left(\underline{\tau} \dfrac{\boldsymbol{m}}{\rho} \right), \\ \dfrac{\partial \boldsymbol{B}}{\partial t} + \nabla \cdot \left(\dfrac{\boldsymbol{m} \otimes \boldsymbol{B} - \boldsymbol{B} \otimes \boldsymbol{m}}{\rho} \right) = -\nabla \times (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \end{cases}$$

with $\boldsymbol{m} \equiv \rho \boldsymbol{v}$, $\mathcal{E} = \rho \varepsilon + \frac{1}{2\rho} \boldsymbol{m} \cdot \boldsymbol{m} + \frac{\boldsymbol{B} \cdot \boldsymbol{B}}{2}$, $p = (\gamma - 1)\rho\varepsilon = \rho T$,

$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = \underline{\eta} \boldsymbol{J}$, $\boldsymbol{J} \equiv \nabla \times \boldsymbol{B}$.

Full MHD

The ideal MHD system driving the previous system can be written in the following compact form

$$\frac{\partial \boldsymbol{w}}{\partial t} + \nabla \cdot \underline{\boldsymbol{f}} = 0,$$

where

$$\boldsymbol{w} = \begin{pmatrix} \rho \\ \boldsymbol{m} \\ \mathcal{E} \\ \boldsymbol{B} \end{pmatrix}, \quad \underline{\boldsymbol{f}} = \begin{pmatrix} \boldsymbol{m}^T \\ \boldsymbol{v} \otimes \boldsymbol{m} + P\mathbb{I} - \boldsymbol{B} \otimes \boldsymbol{B} \\ \mathcal{H}\boldsymbol{m}^T - (\boldsymbol{B} \cdot \boldsymbol{v})\boldsymbol{B}^T \\ \boldsymbol{v} \otimes \boldsymbol{B} - \boldsymbol{B} \otimes \boldsymbol{v} \end{pmatrix}$$

$P = p + \frac{\boldsymbol{B} \cdot \boldsymbol{B}}{2}$, $\mathcal{H} = \frac{\mathcal{E} + P}{\rho}$

Full MHD

Quasi-linear form

$$\frac{\partial \boldsymbol{w}}{\partial t} + \underline{\boldsymbol{L}}(\boldsymbol{w}, \boldsymbol{\partial})\boldsymbol{w} = 0$$

The operator $\underline{\boldsymbol{L}}(\boldsymbol{w}, \boldsymbol{\partial})$

$$\underline{\boldsymbol{L}}(\boldsymbol{w}, \boldsymbol{\partial}) = \left(\begin{array}{c|c|c|c} 0 & \boldsymbol{\partial}^T & 0 & \boldsymbol{0}^T \\ \hline -\frac{\boldsymbol{m}}{\rho^2}\boldsymbol{m}\cdot\boldsymbol{\partial} + \frac{\partial P}{\partial \rho}\boldsymbol{\partial} & \begin{array}{c}\frac{1}{\rho}(\boldsymbol{m}\otimes\boldsymbol{\partial} + \boldsymbol{m}\cdot\boldsymbol{\partial}) \\ + \left(\frac{\partial P}{\partial \boldsymbol{m}} \otimes \boldsymbol{m}\right)^T\end{array} & \frac{\partial P}{\partial \mathcal{E}}\boldsymbol{\partial} & \begin{array}{c}(\boldsymbol{B}\otimes\boldsymbol{\partial})^T - \boldsymbol{B}\cdot\boldsymbol{\partial} \\ + \left(\frac{\partial P}{\partial \boldsymbol{B}} \otimes \boldsymbol{m}\right)^T\end{array} \\ \hline \frac{\partial \mathcal{H}}{\partial \rho}\boldsymbol{m}\cdot\boldsymbol{\partial} + \frac{\boldsymbol{B}\cdot\boldsymbol{m}}{\rho^2}\boldsymbol{B}\cdot\boldsymbol{\partial} & \begin{array}{c}\left(\frac{\partial \mathcal{H}}{\partial \boldsymbol{m}}\right)^T \boldsymbol{m}\cdot\boldsymbol{\partial} + \mathcal{H}\boldsymbol{\partial}^T \\ -\frac{\boldsymbol{B}^T}{\rho}\boldsymbol{B}\cdot\boldsymbol{\partial}\end{array} & \frac{\partial \mathcal{H}}{\partial \mathcal{E}}\boldsymbol{m}\cdot\boldsymbol{\partial} & \begin{array}{c}\left(\frac{\partial \mathcal{H}}{\partial \boldsymbol{B}}\right)^T \boldsymbol{m}\cdot\boldsymbol{\partial} - \frac{\boldsymbol{B}\cdot\boldsymbol{m}}{\rho}\boldsymbol{\partial}^T \\ -\frac{\boldsymbol{m}^T\boldsymbol{B}\cdot\boldsymbol{\partial}}{\rho}\end{array} \\ \hline -\frac{\boldsymbol{B}\boldsymbol{m}\cdot\boldsymbol{\partial} - \boldsymbol{m}\boldsymbol{B}\cdot\boldsymbol{\partial}}{\rho^2} & \frac{\boldsymbol{B}\otimes\boldsymbol{\partial} - \boldsymbol{B}\cdot\boldsymbol{\partial}}{\rho} & \boldsymbol{0} & \frac{\boldsymbol{m}\cdot\boldsymbol{\partial} - \boldsymbol{m}\boldsymbol{\partial}^T}{\rho} \end{array}\right)$$

Seven eigenvalues: 2 slow and 2 fast accoustic waves, 2 Alfven waves and 1 material wave

Weak formulation

Rewriting the system as $\boldsymbol{R}(\boldsymbol{w}) = 0$, in which

$$\boldsymbol{R}(\boldsymbol{w}) := \frac{\partial \boldsymbol{w}}{\partial t} + \nabla \cdot \underline{\boldsymbol{f}} - \nabla \cdot \underline{\boldsymbol{g}}, \qquad \underline{\boldsymbol{g}} : \text{ Transport process}$$

Weak formulation

$$\int_{\Omega_{\mathbf{x},\mathbf{h}}} \boldsymbol{R}(\boldsymbol{w}) \cdot \boldsymbol{w}^\star = 0, \quad \forall \boldsymbol{w}^\star \in \overrightarrow{\mathcal{W}}_{\boldsymbol{h}}(\Omega_{x,h})$$

Divergence free constraint
Vector potential formulation

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \quad \text{Coulomb gauge} : \nabla \cdot \boldsymbol{A} = 0$$

$(\nabla \cdot \boldsymbol{A})(\nabla \cdot \boldsymbol{A}^\star)$ added as a penalization term

VMS-stabilization

It comes to add a contribution

$$\int_{\Omega_{\mathbf{x},\mathbf{h}}} \boldsymbol{R}(\boldsymbol{w}) \cdot \boldsymbol{w}^{\star} + \int_{\Omega_{\mathbf{x},\mathbf{h}}} \boldsymbol{w}^{'} \cdot (\underline{\boldsymbol{L}}^{T}(\boldsymbol{w}, \partial))\boldsymbol{w}^{\star}) = 0, \quad \forall \boldsymbol{w}^{\star} \in \vec{\mathcal{W}}_{\boldsymbol{h}}(\Omega_{x,h})$$

where $\boldsymbol{w}^{'}$ is vector of subscales and $\underline{\boldsymbol{L}}^{T}(\boldsymbol{w}, \partial)$ is the adjoint of $\underline{\boldsymbol{L}}(\boldsymbol{w}, \partial)$
$\boldsymbol{w}^{'}$ can be approximated by

$$\boldsymbol{w}^{'} \approx \underline{\mathcal{T}}(\underline{\boldsymbol{L}}(\boldsymbol{w}, \partial)\delta \boldsymbol{w})$$

Several options for $\underline{\mathcal{T}}$, choice based on heuristic arguments

$$\underline{\mathcal{T}} = \alpha \frac{h_e}{2||a||} \Rightarrow \frac{||\frac{\partial \boldsymbol{x}}{\partial \zeta}||_2}{||\Lambda||_{\infty}} \mathbb{I}$$

Stabilized weak formulation

Set of interpolated variables

$$\boldsymbol{w} = \begin{pmatrix} \rho \\ \boldsymbol{m} & = & \rho \boldsymbol{v} \\ \mathcal{E} & = & \frac{\rho T}{\gamma - 1} + \frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v} + \frac{1}{2}\boldsymbol{B}\cdot\boldsymbol{B} \\ \boldsymbol{B} \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{Y} = \begin{pmatrix} \rho \\ \boldsymbol{v} \\ T \\ \boldsymbol{A} \end{pmatrix}$$

Stabilized weak formulation

$$\int_{\Omega_{\mathbf{x},\mathbf{h}}} (\underline{\boldsymbol{M}}(\boldsymbol{Y})\partial_t \boldsymbol{Y})\cdot\tilde{\boldsymbol{w}}^\star - \int_{\Omega_{\mathbf{x},\mathbf{h}}} \underline{\boldsymbol{F}} : (\boldsymbol{\partial}\odot\boldsymbol{w}^\star) + \int_{\Omega_{\mathbf{x},\mathbf{h}}} \underline{\boldsymbol{D}} : (\boldsymbol{\partial}\odot\boldsymbol{w}^\star) - \frac{1}{\epsilon}\int_{\Omega_{\mathbf{x},\mathbf{h}}} (\nabla\cdot\boldsymbol{A})(\nabla\cdot\boldsymbol{A}^\star)$$
$$+ \int_{\partial\Omega_{\mathbf{x},\mathbf{h}}} \underline{\boldsymbol{F}} : (\boldsymbol{n}\odot\boldsymbol{w}^\star) - \int_{\partial\Omega_{\mathbf{x},\mathbf{h}}} \underline{\boldsymbol{D}} : (\boldsymbol{n}\odot\boldsymbol{w}^\star) - \frac{1}{\epsilon}\int_{\partial\Omega_{\mathbf{x},\mathbf{h}}} (\boldsymbol{n}\times\partial_t\boldsymbol{A})\cdot\boldsymbol{B}^\star + \frac{1}{\epsilon}\int_{\partial\Omega_{\mathbf{x},\mathbf{h}}} \boldsymbol{S}(\boldsymbol{w},\boldsymbol{w}_b,\boldsymbol{w}^\star)$$
$$= -\int_{\Omega_{\mathbf{x},\mathbf{h}}} (\underline{\boldsymbol{L}}(\boldsymbol{w},\boldsymbol{\partial})\delta\boldsymbol{w})\cdot\underline{\mathcal{T}}(\underline{\boldsymbol{L}}^T(\boldsymbol{w},\boldsymbol{\partial}))\boldsymbol{w}^\star)$$

Stabilized weak form

In the previous equation

$$\underline{\boldsymbol{F}} = \begin{pmatrix} \rho \boldsymbol{v} \\ \rho \boldsymbol{v} \otimes \boldsymbol{v} + P\mathcal{I} - \boldsymbol{B} \otimes \boldsymbol{B} \\ \rho \mathcal{H} \boldsymbol{v} - \boldsymbol{B} \cdot \boldsymbol{v} \boldsymbol{B} \\ -\boldsymbol{v} \times \boldsymbol{B} \end{pmatrix}, \quad \underline{\boldsymbol{D}} = \begin{pmatrix} \underline{\mathcal{D}} \partial \rho' \\ \underline{\boldsymbol{\tau}} \\ \underline{\boldsymbol{\tau}} \boldsymbol{v} \\ -(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \end{pmatrix}$$

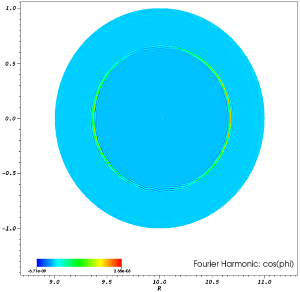
and the boundary conditions

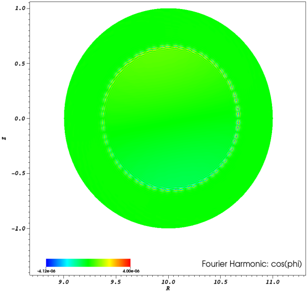
$$\boldsymbol{S}(\boldsymbol{w}, \boldsymbol{w}_b, \boldsymbol{w}^\star) = \begin{pmatrix} (\rho - \rho_b)\rho^\star \chi_\rho \\ \rho(\boldsymbol{v} - \boldsymbol{v}_b) \cdot \boldsymbol{m}^\star \chi_{\boldsymbol{v}} + (\rho \boldsymbol{v} - \boldsymbol{m}_b) \cdot \boldsymbol{m}^\star \chi_{\boldsymbol{m}} \\ \rho(T - T_b)\mathcal{E}^\star \chi_T + (p - p_b)\mathcal{E}^\star \chi_p \\ (\boldsymbol{A} - \boldsymbol{A}_b) \cdot \boldsymbol{J}^\star \chi_{\boldsymbol{A}} + (\boldsymbol{B} - \boldsymbol{B}_b) \cdot \boldsymbol{B}^\star \chi_{\boldsymbol{B}} \end{pmatrix}$$

For further details, please refer to the Research Report: hal.inria.fr/hal-01294788

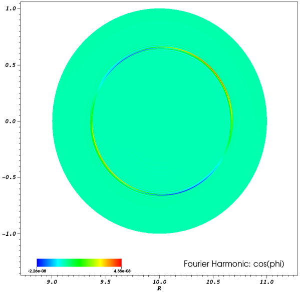
Resistive internal kink

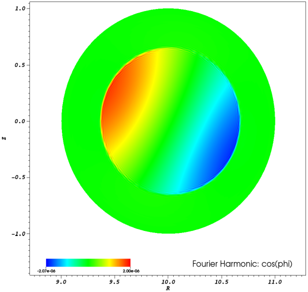
Numerical viscosity



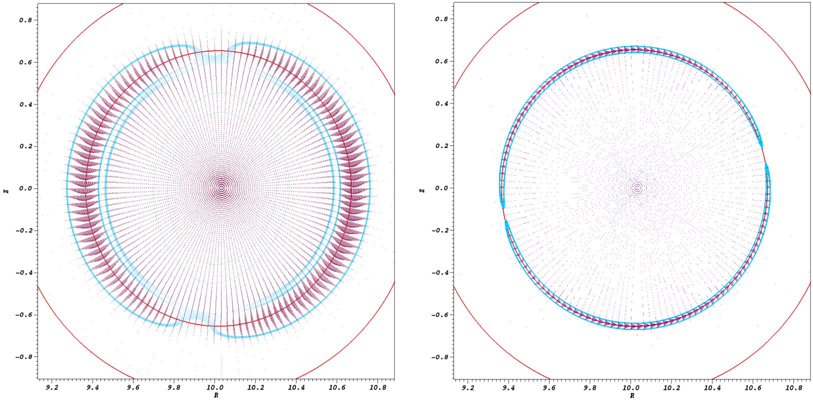


VMS-Stabilization





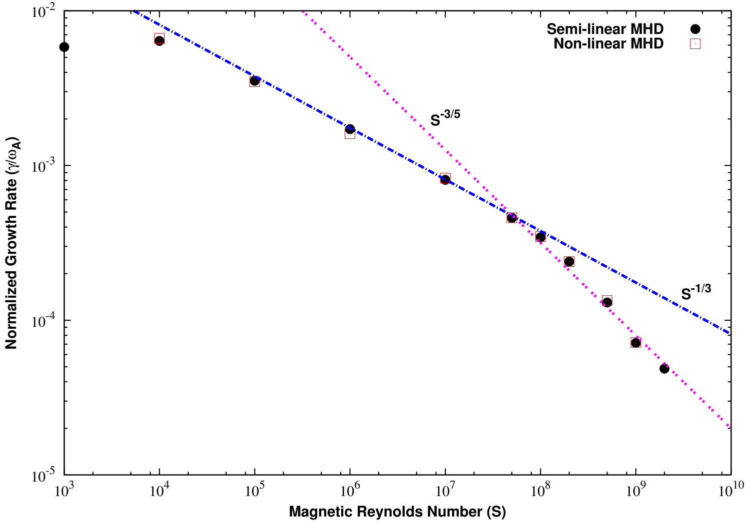
Resistive internal kink dynamics



Structure layer: $S = 10^5 \approx 10^{-1}$ and $S = 10^8 \approx 2.10^{-2}$

$S \propto \eta^{-1}$: Magnetic Reynolds number.

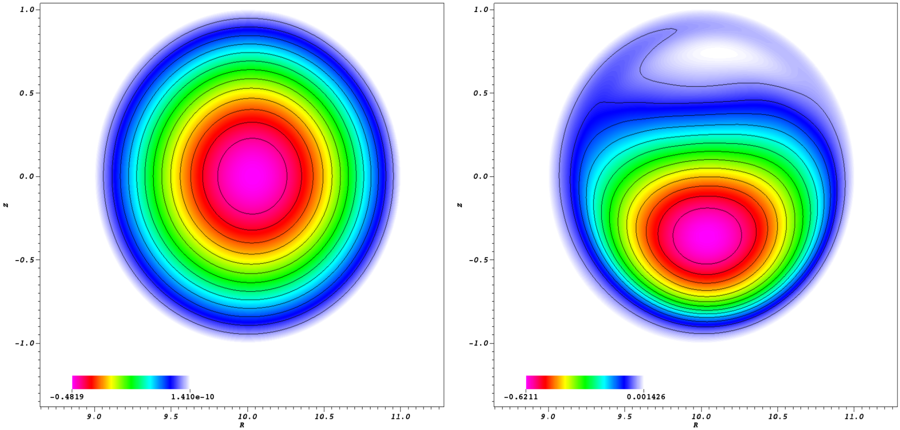
Resistive internal kink: growth rate



$\gamma \propto S^{-\frac{1}{3}} \rightarrow$ Kink $\quad \gamma \propto S^{-\frac{3}{5}} \rightarrow$ Tearing $\quad$ Treshold $\approx S = 10^8$

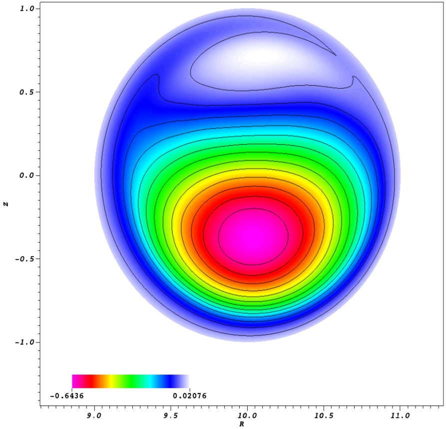
Resistive internal kink: magnetic reconnection

$t = 0\tau_A$ $t = 4400\tau_A$

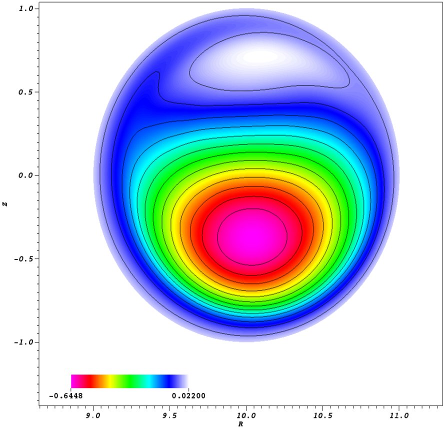


Resistive internal kink: magnetic reconnection

$t = 4460\tau_A$

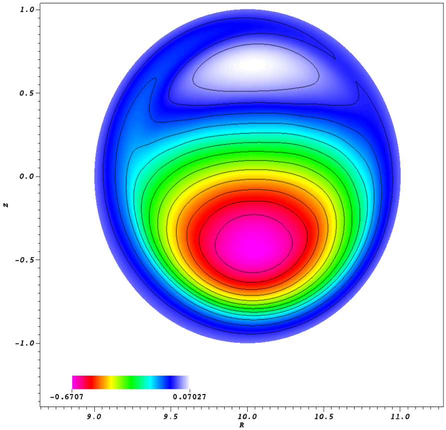


$t = 4464\tau_A$

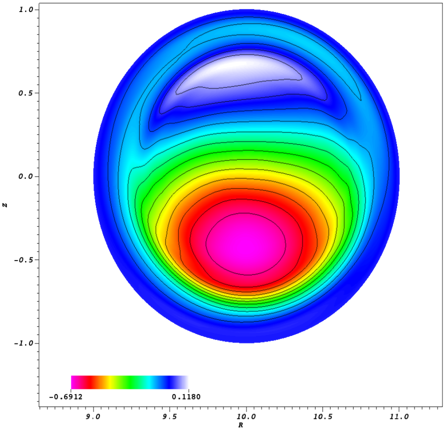


Resistive internal kink: magnetic reconnection

$t = 4648\tau_A$



$t = 5484\tau_A$



Boundary Conditions: Bohm

We need to compute the following integrals at the boundary

$$-\int_{\partial\Omega_{x,h}} \pm\rho\frac{\sqrt{\gamma T}}{||\boldsymbol{B}||}\boldsymbol{B}\cdot\boldsymbol{n}\rho^\star,$$

$$-\int_{\partial\Omega_{x,h}} \left(\pm\rho\boldsymbol{v}\frac{\sqrt{\gamma T}}{||\boldsymbol{B}||}\boldsymbol{B}\cdot\boldsymbol{n}\right)\cdot\boldsymbol{m}^\star,$$

$$-\int_{\partial\Omega_{x,h}} \pm(p\boldsymbol{n}+\pi\boldsymbol{n}-\boldsymbol{B}(\boldsymbol{B}\cdot\boldsymbol{n}))\cdot\boldsymbol{m}^\star$$

$$-\int_{\partial\Omega_{x,h}} \pm p\frac{\sqrt{\gamma T}}{||\boldsymbol{B}||}\boldsymbol{B}\cdot\boldsymbol{n}\mathcal{E}^\star$$

Where $\boldsymbol{n}$ is the outward normal and

$$\pm \equiv \frac{\boldsymbol{B}\cdot\boldsymbol{n}}{|\boldsymbol{B}\cdot\boldsymbol{n}|}$$

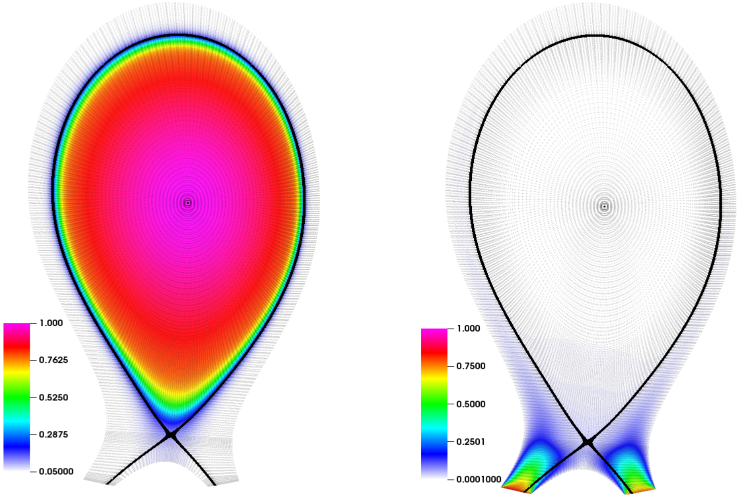
so that $\boldsymbol{v}_b\cdot\boldsymbol{n}$ is always positive

Also, a penalization term

$$\frac{1}{\epsilon}\int_{\partial\Omega_{x,h}} \left(\rho\boldsymbol{v}\mp\rho\sqrt{\gamma T}\frac{\boldsymbol{B}}{||\boldsymbol{B}||}\right)\cdot\boldsymbol{m}^\star$$

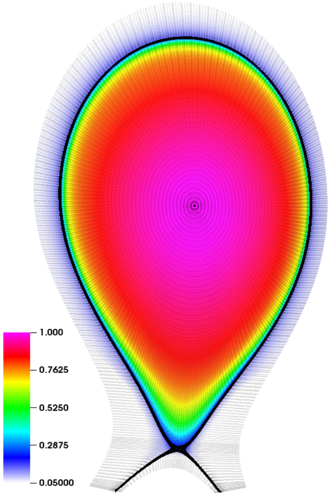
X-point results: Initial density and velocity

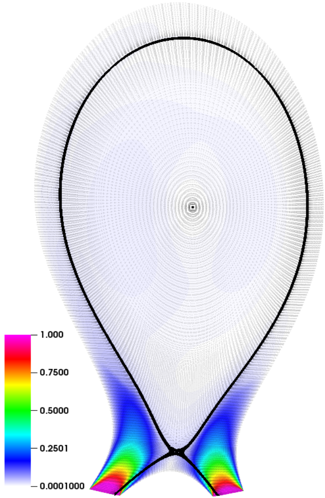
$$t = 1\tau_A$$



X-point results: Initial density and velocity

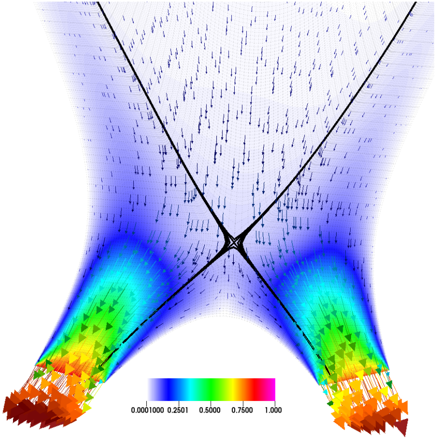
$$t = 400\tau_A$$



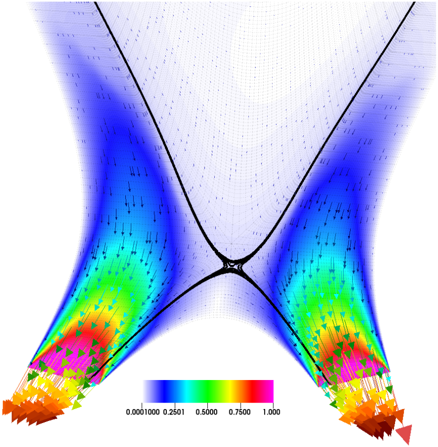


X-point results: Velocity field

$t = 1\tau_A$

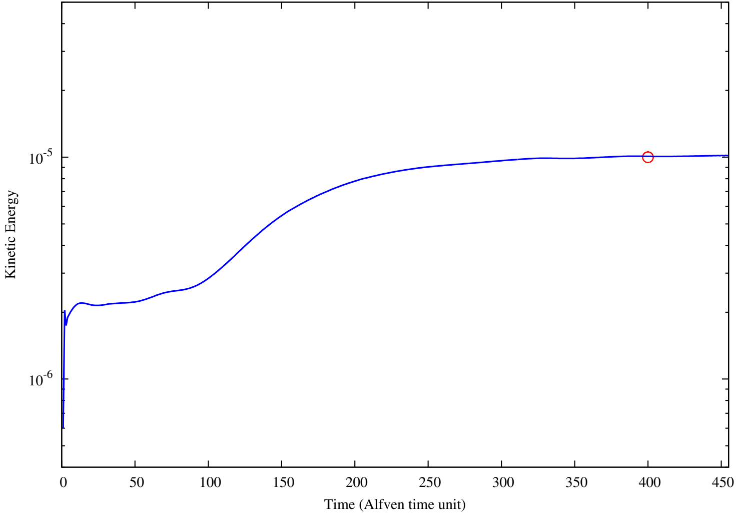


$t = 400\tau_A$



X-point results: Quasi-steady state

Evolution of the Kinetic Energy: $n = 0$



Conclusions and perspectives

Conclusions

- Full MHD equations successfully implemented for simulating **circular** plasmas
- Stabilization scheme has improved the quality of the results
- Quasi-steady state for X-point geometry with Bohm B.C.

Perspectives

- Simulation of ELMs using the full model