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Abstract We show that a version of the Discontinuous Galerkin Method (DGM) can be included in the Gradient Discretisation Method (GDM) framework. We prove that it meets the main mathematical gradient discretisation properties on any kind of polytopal mesh, and that it is identical to the Symmetric Interior Penalty Galerkin (SIPG) method in the case of first order polynomials. A numerical study shows the effect of the numerical parameter included in the scheme.

Key words: Gradient Discretisation method, Discontinuous Galerkin method

MSC (2010)**:** 65M08, 65N08, 35Q30

1 Introduction

Discontinuous Galerkin (DG) methods are being more and more studied. They present the advantage to be suited to elliptic and parabolic problems, while opening the possibility to closely approximate weakly regular functions on general meshes. Although the convergence of DG methods has been proved on a variety of problems (see [2] and references therein), note that the stabilisation of the classical DG schemes has to be specified in each case. On the other hand, convergence and error estimate results for a wide variety of numerical methods applied to some elliptic, parabolic, coupled, linear and nonlinear problems are proved on the generic "gradi-

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ent scheme" issued from the Gradient Discretisation Method framework, assuming that a very small number of core properties hold true (see [3]).

The aim of this paper is to show that, from the DG setting, we can build a Gradient Discretisation which satisfies all these core properties. This is done on general polytopal meshes in any space dimension. This work immediately extends the range of problems which can be handled by Discontinuous Galerkin methods to all for which the Gradient Discretisation is shown to converge (degenerate parabolic problems, two-phase flow problems,...). Note that the gradient scheme resulting from the Discontinuous Galerkin Gradient Discretisation (DGGD) may be not identical to the corresponding stabilised DG scheme proposed in the literature, although we show in this paper that it is identical to the Symmetric Interior Penalty Galerkin (SIPG) method in the $\mathbb{P}^1(\mathbb{R}^d)$ case for the following elliptic problem:

$$\begin{aligned}&\overline{u}\in H_0^1(\Omega),\ \forall\overline{v}\in H_0^1(\Omega),\\&\int_\Omega \Lambda(\boldsymbol{x})\nabla\overline{u}(\boldsymbol{x})\cdot\nabla\overline{v}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}=\int_\Omega f(\boldsymbol{x})\overline{v}(\boldsymbol{x})\mathrm{d}\boldsymbol{x},\end{aligned}\tag{1}$$

where:

- Ω is an open bounded polytopal connected subset of $\mathbb{R}^d$ ($d\in\mathbb{N}^\star$), (2a)
- Λ is a measurable function from Ω to the set of $d\times d$ symmetric matrices and there exists $\underline{\lambda},\overline{\lambda}>0$ such that, for a.e. $\boldsymbol{x}\in\Omega$, $\Lambda(\boldsymbol{x})$ has eigenvalues in $[\underline{\lambda},\overline{\lambda}]$, (2b)
- $f\in L^2(\Omega)$. (2c)

This paper is organised as follows. In Section 2, we give a gradient discretisation version of Discontinuous Galerkin schemes. We then prove in Section 3 that this gradient discretisation satisfies the core properties which are sufficient for convergence and error estimates results. A short numerical example finally shows the role the numerical parameter used in the design of the scheme on its accuracy (Section 4).

2 Discontinuous Galerkin Gradient Discretisation (DGGD)

We consider a polytopal mesh of Ω, in the sense of [3, Definition 7.2], defined by the triplet $\mathfrak{T}=(\mathscr{M},\mathscr{F},\mathscr{P})$.

The set $\mathscr{M}$ is a finite family of non empty connected polytopal open disjoint subsets of Ω. For $K\in\mathscr{M}$, $|K|>0$ is the measure of K and h_K denotes the diameter of K.

The set $\mathscr{F}$ contains the "faces" of the mesh – "edges" in 2D. x For all $\sigma\in\mathscr{F}$, the set $\mathscr{M}_\sigma$, which contains the elements of $\mathscr{M}$ having σ as face or edge, has exactly one element if $\sigma\in\mathscr{F}_{\rm ext}$ (the exterior faces) or two elements if $\sigma\in\mathscr{F}_{\rm int}$ (the interior faces). For all $K \in \mathcal{M}$, $\mathcal{F}_K \subset \mathcal{F}$ contains the faces of K, and for any $\sigma \in \mathcal{F}_K$, we denote by $\boldsymbol{n}_{K,\sigma}$ the (constant) unit vector normal to σ outward to K.

We denote by $\mathcal{P}$ the family of points $\mathcal{P} = (\boldsymbol{x}_K)_{K\in\mathcal{M}}$, such that for all $K \in \mathcal{M}$, $K \in \mathcal{M}$ is strictly star-shaped with respect to $\boldsymbol{x}_K$ (see Figure 1). This implies that the orthogonal distance $d_{K,\sigma}$ between $\boldsymbol{x}_K$ and $\sigma \in \mathcal{F}_K$ is such that $d_{K,\sigma} > 0$. For all $K \in \mathcal{M}$ and $\sigma \in \mathcal{F}_K$, we denote by $D_{K,\sigma}$ the cone with vertex $\boldsymbol{x}_K$ and basis σ.

The size of the polytopal mesh is defined by:

$$h_{\mathcal{M}} = \sup\{h_K, K \in \mathcal{M}\}. \tag{3}$$

Finally, for a given polytopal mesh $\mathfrak{T}$ we define a number that measures the regularity properties of the mesh:

$$\eta_{\mathfrak{T}} = \max\{\frac{h_K}{h_L} + \frac{h_L}{h_K},\ \sigma \in \mathcal{F}_{\rm int}\,,\ \mathcal{M}_\sigma = \{K,L\}\} \cup \{\frac{h_K}{d_{K,\sigma}}, K \in \mathcal{M}, \sigma \in \mathcal{F}_K\}. \tag{4}$$

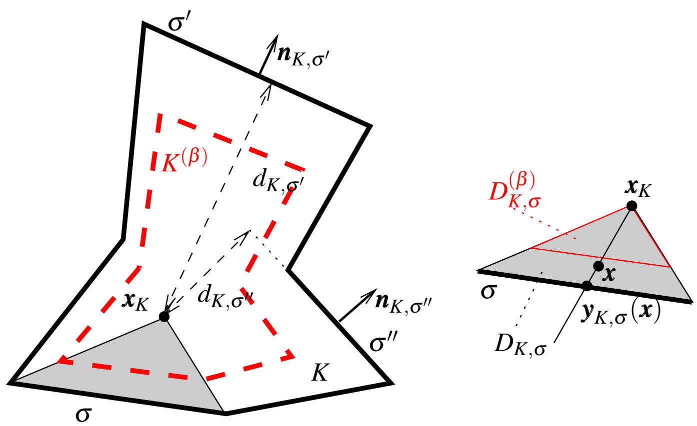

Fig. 1 A cell K of a polytopal mesh and notation on $D_{K,\sigma}$

Let us now define the Discontinuous Galerkin Gradient Discretisation (DGGD) for the approximation of (1) in the sense of [5, 3]. For a given value $k \in \mathbb{N}^\star$, and for a given $p \in]1,+\infty[$, we define the space $X_{\mathcal{D},0}$ of all functions $v \in L^p(\Omega)$ such that, for all $K \in \mathcal{M}$, $v_{|K} \in \mathbb{P}^k(\mathbb{R}^d)$, the latter denoting the space of polynomial function with degree less or equal to k (recall that the dimension of $\mathbb{P}^k(\mathbb{R}^d)$ is $\frac{(k+d)!}{k!d!}$, and therefore the dimension of $X_{\mathcal{D},0}$ is equal to $\frac{(k+d)!}{k!d!}\#\mathcal{M}$). We denote by $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$, where $\Pi_{\mathcal{D}} = \mathrm{Id}$. Let us now define $\nabla_{\mathcal{D}}$. Let $\beta \in]0,1[$ be given. For $v \in X_{\mathcal{D},0}$, for $K \in \mathcal{M}$ and for any $\sigma \in \mathcal{F}_K$, we set

$$
\begin{aligned}
&\nabla_{\mathscr{D}} v(\boldsymbol{x}) = \nabla v_{|K}(\boldsymbol{x}) && \text{for a.e. } \boldsymbol{x} \in D_{K,\sigma}^{(\beta)},\\
&\nabla_{\mathscr{D}} v(\boldsymbol{x}) = \nabla v_{|K}(\boldsymbol{x}) + \frac{d\,[v]_{K,\sigma}(\boldsymbol{y}_{K,\sigma}(\boldsymbol{x}))}{(1-\beta^d)d_{K,\sigma}}\boldsymbol{n}_{K,\sigma} && \text{for a.e. } \boldsymbol{x} \in D_{K,\sigma} \setminus D_{K,\sigma}^{(\beta)},
\end{aligned} \tag{5}
$$

where (see Figure 1):

- $$D_{K,\sigma}^{(\beta)} := \{\boldsymbol{x} \in D_{K,\sigma}, \frac{(\boldsymbol{x}-\boldsymbol{x}_K)\cdot \boldsymbol{n}_{K,\sigma}}{d_{K,\sigma}} < \beta\} \text{ and } K^{(\beta)} = \bigcup_{\sigma\in\mathscr{F}_K} D_{K,\sigma}^{(\beta)}, \tag{6}$$

  (note that we have $|D_{K,\sigma} \setminus D_{K,\sigma}^{(\beta)}| = \frac{1-\beta^d}{d} d_{K,\sigma}|\sigma|$),
- $\boldsymbol{y}_{K,\sigma}(\boldsymbol{x}) \in \sigma$ is the intersection between σ and the line joining $\boldsymbol{x}_K$ and $\boldsymbol{x}$; it satisfies

$$\boldsymbol{x} = \boldsymbol{x}_K + \frac{(\boldsymbol{x}-\boldsymbol{x}_K)\cdot \boldsymbol{n}_{K,\sigma}}{d_{K,\sigma}}(\boldsymbol{y}_{K,\sigma}(\boldsymbol{x}) - \boldsymbol{x}_K),$$

- for all $K \in \mathscr{M}$, we denote by

$$
\begin{aligned}
\forall \boldsymbol{y} \in \sigma,\ &[v]_{K,\sigma}(\boldsymbol{y}) = \tfrac{1}{2}(v_{|L}(\boldsymbol{y}) - v_{|K}(\boldsymbol{y})) && \text{if } \mathscr{M}_\sigma = \{K,L\},\\
&[v]_{K,\sigma}(\boldsymbol{y}) = 0 - v_{|K}(\boldsymbol{y}) && \text{if } \mathscr{M}_\sigma = \{K\}.
\end{aligned} \tag{7}
$$

Remark 1. It is possible to consider $\beta_{K,\sigma}$ instead of a constant β, without changing the mathematical analysis done in this paper. It is also possible to consider the more general definition for the discrete gradient

$$\nabla_{\mathscr{D}} v(\boldsymbol{x}) = \nabla v_{|K}(\boldsymbol{x}) + \psi(\frac{(\boldsymbol{x}-\boldsymbol{x}_K)\cdot \boldsymbol{n}_{K,\sigma}}{d_{K,\sigma}})\frac{[v]_{K,\sigma}(\boldsymbol{y}_{K,\sigma}(\boldsymbol{x}))}{d_{K,\sigma}}\boldsymbol{n}_{K,\sigma} \text{ for a.e. } \boldsymbol{x} \in D_{K,\sigma},$$

where $\psi\ :\ ]0,1[\to \mathbb{R}^+$ is a bounded measurable function such that $\psi(s) = 0$ on $]0,\beta[$ and

$$\int_{1-\beta}^1 \psi(s)s^{d-1}\mathrm{d}s = 1.$$

Then the following mathematical analysis holds as well.

Remark 2 (Piecewise constant reconstruction). One can for example replace $\Pi_{\mathscr{D}}$ by $\widehat{\Pi}_{\mathscr{D}}$ such that, for all $K \in \mathscr{M}$, and a.e. $\boldsymbol{x} \in K$, $\widehat{\Pi}_{\mathscr{D}} v(\boldsymbol{x}) = \frac{1}{|K|}\int_K v(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$, which provides a piecewise constant reconstruction, choosing a basis including the value at the centre of gravity of K.

Using the DGGD $\mathscr{D} = (X_{\mathscr{D},0}, \Pi_{\mathscr{D}}, \nabla_{\mathscr{D}})$, the gradient scheme for the discretisation of (1) is given by: find $u \in X_{\mathscr{D},0}$ such that

$$\int_\Omega \Lambda(\boldsymbol{x})\nabla_{\mathscr{D}} u(\boldsymbol{x}) \cdot \nabla_{\mathscr{D}} v(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \int_\Omega f(\boldsymbol{x})\Pi_{\mathscr{D}} v(\boldsymbol{x})\mathrm{d}\boldsymbol{x}, \forall v \in X_{\mathscr{D},0}. \tag{8}$$

Owing to the properties proved in Section 3, the DGGD scheme then satisfies the convergence and error estimates properties detailed in [3].

Link with the SIPG scheme

In the case $k=1$, the gradient of any element of $X_{\mathscr{D},0}$ restricted to $K\in\mathscr{M}$ is constant in K. Let us assume that Λ follows the same property. Then the left hand side of (8) can be computed in this particular case:

$$\int_\Omega \Lambda(\boldsymbol{x})\nabla_{\mathscr{D}}u(\boldsymbol{x})\cdot\nabla_{\mathscr{D}}v(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \sum_{K\in\mathscr{M}}\Bigg(\int_K \Lambda_K \nabla u_{|K}\cdot\nabla v_{|K}\mathrm{d}\boldsymbol{x}$$
$$+\sum_{\sigma\in\mathscr{F}_K}\Bigg(\int_\sigma \Lambda_K\big([u]_{K,\sigma}(\boldsymbol{y})\nabla v_{|K}+[v]_{K,\sigma}(\boldsymbol{y})\nabla u_{|K}\big)\cdot\boldsymbol{n}_{K,\sigma}\mathrm{d}\gamma(\boldsymbol{y})$$
$$+\frac{d\,\Lambda_K\boldsymbol{n}_{K,\sigma}\cdot\boldsymbol{n}_{K,\sigma}}{(1-\beta^d)d_{K,\sigma}}\int_\sigma [u]_{K,\sigma}(\boldsymbol{y})[v]_{K,\sigma}(\boldsymbol{y})\mathrm{d}\gamma(\boldsymbol{y})\Bigg)\Bigg).$$

We then recover the SIPG scheme as presented in [4] or [2], the penalty coefficient τ_σ (term $\frac{\sigma_e}{|e|^{\beta_0}}$ of [4, eqn. (11)], term $\frac{\eta}{h_F}$ of [2, eqn. (4.12)]) being equal, in the preceding relation, to

$$\tau_\sigma = \frac{d}{4(1-\beta^d)}\Big(\frac{\Lambda_K\boldsymbol{n}_{K,\sigma}\cdot\boldsymbol{n}_{K,\sigma}}{d_{K,\sigma}}+\frac{\Lambda_L\boldsymbol{n}_{L,\sigma}\cdot\boldsymbol{n}_{L,\sigma}}{d_{L,\sigma}}\Big) \text{ if } \mathscr{M}_\sigma=\{K,L\},$$

and

$$\tau_\sigma = \frac{d}{(1-\beta^d)}\frac{\Lambda_K\boldsymbol{n}_{K,\sigma}\cdot\boldsymbol{n}_{K,\sigma}}{d_{K,\sigma}} \text{ if } \mathscr{M}_\sigma=\{K\}.$$

Note that τ_σ has a minimum value letting $\beta\to 0$, which can be compared, for example, to that given by [2, Lemma 4.12]. In our setting, it does not depend on the regularity of the mesh nor on the maximum cardinal of $\mathscr{F}_K$ (in the DGGD scheme, we don't handle separately the case $d=1$ and the cases $d>1$).

3 Mathematical properties of the DGGD method

In this paper, we denote, for $\xi\in\mathbb{R}^d$ by $|\xi|=(\sum_{i=1}^d\xi_i^2)^{1/2}$ the Euclidean norm of ξ.

Lemma 1. *Let $n\in\mathbb{N}$ and $\beta\in]0,1[$ be given. Let $\mathfrak{T}$ be a polytopal mesh. Then there holds*

$$\forall v\in\mathbb{P}^n(\mathbb{R}^d),\ \forall K\in\mathscr{M},\ \forall\sigma\in\mathscr{F}_K,\ \int_{D_{K,\sigma}}|v(\boldsymbol{x})|^p\mathrm{d}\boldsymbol{x}\le\frac{(n+1)^{p-1}}{\beta^{d+pn}C_{p,n}}\int_{D^{(\beta)}_{K,\sigma}}|v(\boldsymbol{x})|^p\mathrm{d}\boldsymbol{x},$$

where $C_{p,n}$ only depends on p, n and d, and where $D^{(\beta)}_{K,\sigma}$ is defined by (6)*.*

Proof. This lemma is proved thanks to the change of variable $\boldsymbol{x}=\boldsymbol{x}_K+s(\boldsymbol{y}-\boldsymbol{x}_K)$, where $\boldsymbol{y}\in\sigma$ and $s\in]0,\beta[$ (we then have $\mathrm{d}\boldsymbol{x}=d_{K,\sigma}s^{d-1}\mathrm{d}\gamma(\boldsymbol{y})\mathrm{d}s$). ■

Lemma 2. *There exists $A > 0$, only depending on β, p, k and d, such that*

$$\forall v \in X_{\mathscr{D},0},\ \frac{1}{A}\|v\|_{\mathrm{DG},p} \leq \|\nabla_{\mathscr{D}} v\|_{L^p(\Omega)^d} \leq A\|v\|_{\mathrm{DG},p}, \tag{9}$$

where

$$\|v\|_{\mathrm{DG},p}^p = \sum_{K\in\mathscr{M}} \Big(\int_K |\nabla v_{|K}(\boldsymbol{x})|^p \mathrm{d}\boldsymbol{x} + \sum_{\sigma\in\mathscr{F}_K} \frac{1}{d_{K,\sigma}^{p-1}} \int_\sigma |[v]_{K,\sigma}(\boldsymbol{y})|^p \mathrm{d}\gamma(\boldsymbol{y}) \Big). \tag{10}$$

Remark 3 (DG norm). Note that Definition (10) for the DG norm is slightly different from [1, eqn. (5)] or [2, eqn. (5.1)], with the use of $d_{K,\sigma}$ instead that of $\mathrm{diam}(\sigma)$, and with notation (7) for the jump at the faces of the mesh. This allows the application of discrete functional analysis results without regularity hypotheses on the polytopal mesh.

Proof. We apply the inequality $|a+b|^p \leq (1+c^{p'})^{p-1}(|a|^p + |\frac{b}{c}|^p)$ with $\frac{1}{p}+\frac{1}{p'} = 1$, $a = \nabla v_{|K}(\boldsymbol{x}) + \frac{d\ [v]_{K,\sigma}(\boldsymbol{y}_{K,\sigma}(\boldsymbol{x}))}{(1-\beta^d)d_{K,\sigma}} \boldsymbol{n}_{K,\sigma}$ and $b = -\nabla v_{|K}(\boldsymbol{x})$, for some $c > 0$ chosen accounting for Lemma 1, applied to the components of $\nabla v_{|K}$, which are polynomial too. ∎

From an adaptation of the discrete functional analysis results provided in [1] to our polytopal mesh framework, we conclude on one hand that $\|\nabla_{\mathscr{D}} \cdot\|_{L^p(\Omega)^d}$ is a norm on $X_{\mathscr{D},0}$, and on the other hand the two following lemmas.

Lemma 3 (coercivity). *Let $\mathscr{D}$ be a DGGD. We define $C_{\mathscr{D}} \geq 0$ by*

$$C_{\mathscr{D}} = \max_{v\in X_{\mathscr{D},0}\setminus\{0\}} \frac{\|\Pi_{\mathscr{D}} v\|_{L^p(\Omega)}}{\|\nabla_{\mathscr{D}} v\|_{L^p(\Omega)^d}}. \tag{11}$$

Then there exists C_P only depending on $|\Omega|$, β, p, k and d such that $C_P \geq C_{\mathscr{D}}$, which means that any sequence $(\mathscr{D})_{m\in\mathbb{N}}$ is coercive in the sense of [3, Definition 2.2].

Lemma 4 (compactness). *Let $(\mathscr{D})_{m\in\mathbb{N}}$ be a sequence of DGGD. Then, for all $(v_m)_{m\in\mathbb{N}}$ such that, for all $m\in\mathbb{N}$, $v_m \in X_{\mathscr{D}_m,0}$ and such that the sequence $(\|\nabla_{\mathscr{D}_m} v_m\|_{L^p(\Omega)})_{m\in\mathbb{N}}$ is bounded, the sequence $(v_m)_{m\in\mathbb{N}}$ is relatively compact in $L^p(\Omega)$, which means that any sequence $(\mathscr{D})_{m\in\mathbb{N}}$ is compact in the sense of [3, Definition 2.8].*

Lemma 5 (GD-consistency). *Let $(\mathscr{D})_{m\in\mathbb{N}}$ be a sequence of DGGD such that $h_{\mathfrak{T}_m}$ tends to 0 as $m\to\infty$ while $\eta_{\mathfrak{T}_m}$ remains bounded. We define $S_{\mathscr{D}} : W_0^{1,p}(\Omega) \to [0,+\infty)$ by*

$$\forall \varphi \in W_0^{1,p}(\Omega),\ S_{\mathscr{D}}(\varphi) = \min_{v\in X_{\mathscr{D},0}} \Big(\|\Pi_{\mathscr{D}} v - \varphi\|_{L^p(\Omega)} + \|\nabla_{\mathscr{D}} v - \nabla\varphi\|_{L^p(\Omega)^d} \Big). \tag{12}$$

Then it holds $$\forall \varphi \in C_c^\infty(\Omega),\ \lim_{m\to\infty} S_{\mathscr{D}_m}(\varphi) = 0,$$
which is a sufficient condition for the GD-consistency of $(\mathscr{D})_{m\in\mathbb{N}}$ in the sense of [3, Definition 2.4] thanks to [3, Lemma 2.13].

Proof. Let $\varphi \in C_c^\infty(\Omega)$, and let M be an upper bound of $D_2\varphi := (\partial^2_{ij}\varphi)_{1\le i,j\le d}$ on $\overline{\Omega}$. We let $\mathscr{D} = \mathscr{D}_m$ for a given m, and we consider $v \in X_{\mathscr{D},0}$ defined by

$$\forall K \in \mathscr{M},\ \forall \boldsymbol{x} \in K,\ v_{|K}(\boldsymbol{x}) = \varphi(\boldsymbol{x}_K) + \nabla\varphi(\boldsymbol{x}_K)\cdot(\boldsymbol{x} - \boldsymbol{x}_K).$$

Indeed, $v_{|K} \in \mathbb{P}^k(\mathbb{R}^d)$ since $k \ge 1$. We then perform Taylor expansions at the second order of the function φ, which allow to conclude the existence of $C \ge 0$, increasingly depending on $\eta_{\mathfrak{T}}$, such that

$$\|\nabla\varphi - \nabla_{\mathscr{D}} v\|_{L^p(\Omega)^d} \le C M h_{\mathscr{M}} (|\Omega|)^{1/p}.$$

■

Lemma 6 (limit conformity). *Let $(\mathscr{D})_{m\in\mathbb{N}}$ be a sequence of DGGD such that $h_{\mathfrak{T}_m}$ tends to 0 as $m \to \infty$. Let $p' = \frac{p}{p-1}$ and define $W_{\mathscr{D}}$: $W^{p'}_{\mathrm{div}}(\Omega) \to [0, +\infty)$ by*

$$\forall \boldsymbol{\varphi} \in W^{p'}_{\mathrm{div}}(\Omega),\quad W_{\mathscr{D}}(\boldsymbol{\varphi}) = \max_{v \in X_{\mathscr{D},0}\setminus\{0\}} \frac{\left| \int_\Omega \left(\nabla_{\mathscr{D}} v(\boldsymbol{x}) \cdot \boldsymbol{\varphi}(\boldsymbol{x}) + \Pi_{\mathscr{D}} v(\boldsymbol{x}) \mathrm{div}\boldsymbol{\varphi}(\boldsymbol{x})\right) \mathrm{d}\boldsymbol{x} \right|}{\|\nabla_{\mathscr{D}} v\|_{L^p(\Omega)^d}}. \tag{13}$$

Then it holds $$\forall \boldsymbol{\varphi} \in C^\infty(\mathbb{R}^d)^d,\ \lim_{m\to\infty} W_{\mathscr{D}_m}(\boldsymbol{\varphi}) = 0,$$
which is a sufficient condition for the limit conformity of $(\mathscr{D})_{m\in\mathbb{N}}$ in the sense of [3, Definition 2.6] thanks to [3, Lemma 2.14] since $(\mathscr{D})_{m\in\mathbb{N}}$ is coercive.

Proof. The proof relies on the coefficient of $[v]_{K,\sigma}$, which ensures that the terms at the faces of the mesh behave as $Ch_{\mathscr{M}}$. ■

4 Numerical results

The aim of this section is to assess the influence of the parameter $\beta \in]0,1[$ on the accuracy of the gradient scheme (8) issued from the DGGD for the discretisation of (1). We consider the $1D$ case $\Omega =]0,1[$, and the polytopal mesh $\mathfrak{T}$ defined, for $N \in \mathbb{N}^\star$ and $h = \frac{1}{N}$, by $\mathscr{M} = \{](i-1)h, ih[,\ i = 1,\ldots,N\}$, $\mathscr{F} = \{\{ih\},\ i = 0,\ldots,N\}$, $\mathscr{P} = \{(i-\frac{1}{2})h,\ i = 1,\ldots,N\}$. We consider one of the test cases studied in [4], that is Problem (1) with $\Lambda = \mathrm{Id}$ and $\overline{u}(x) = \cos(8\pi x) - 1$ (hence $f(x) = (8\pi)^2\cos(8\pi x)$). Considering first degree polynomials, the set $X_{\mathscr{D},0}$ is a vector space with dimension $2N$. In the following tables (where "order" is the convergence order with respect to the size of the mesh), the columns "FE" correspond to the conforming $\mathbb{P}^1$ Finite Element solution, and we check that the results provided by "[4]" with $\sigma_n = 4.5$, which corresponds to $\beta = 1 - 1/\sigma_n$ for the interior faces, and $\beta = 1 - 2/\sigma_n$ for the exterior faces, are close to ours:

$N\backslash\beta$	0	0.5	0.9	0.99	FE	[4]
10	0.496	0.241	0.347	0.394	0.399	0.247
order	1.438	1.529	1.734	1.843	1.855	
20	0.183	0.083	0.104	0.110	0.110	0.083
order	1.092	1.706	1.909	1.959	1.964	
40	0.086	0.026	0.028	0.028	0.028	0.024
order	1.013	1.894	1.973	1.989	1.991	
80	0.043	0.007	0.007	0.007	0.007	
order	0.999	1.967	1.992	1.997	1.998	
160	0.021	0.002	0.002	0.002	0.002	

L^2 error of the solution

$N\backslash\beta$	0	0.5	0.9	0.99	FE	[4]
10	13.233	11.533	11.360	11.349	11.348	11.777
order	0.172	0.781	0.862	0.863	0.863	
20	11.743	6.714	6.251	6.240	6.240	6.421
order	0.010	1.004	0.966	0.965	0.965	
40	11.666	3.348	3.199	3.197	3.197	3.253
order	-0.008	1.034	0.992	0.991	0.991	
80	11.728	1.635	1.609	1.608	1.608	
order	-0.007	1.014	0.998	0.998	0.998	
160	11.781	0.810	0.805	0.805	0.805	

L^2 error of the broken gradient

Although we did not prove that the linear systems are invertible when $\beta = 0$, we note that in practice a solution is obtained but that the broken gradient does not seem to converge. In this very regular case, the L^2 error is the lowest for $\beta = 0.5$ but the convergence seems slightly better for β closer to 1, and it tends to the results of the finite element method as $\beta \to 1$.

5 Conclusion

The version of the DG method included in the GDM framework has the advantages to be samely defined for $d = 1$ and $d > 1$, to hold on any polytopal mesh provided that the grid block are strictly star-shaped, to involve Discrete Functional Analysis results which do not depend on the regularity of the mesh, and to apply on any problem on which the GDM is shown to converge. This version is identical to the SIPG method in the case $k = 1$. The differences with the SIPG scheme in the case $k > 1$ remain to be assessed.

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