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Abstract. A graph has strong convex dimension 2 if it admits a straight-line drawing in the plane such that its vertices form a convex set and the midpoints of its edges also constitute a convex set. Halman, Onn, and Rothblum conjectured that graphs of strong convex dimension 2 are planar and therefore have at most $3n-6$ edges. We prove that all such graphs have indeed at most $2n-3$ edges, while on the other hand we present an infinite family of non-planar graphs of strong convex dimension 2. We give lower bounds on the maximum number of edges a graph of strong convex dimension 2 can have and discuss several natural variants of this graph class. Furthermore, we apply our methods to obtain new results about large convex sets in Minkowski sums of planar point sets – a topic that has been of interest in recent years.

1 Introduction

A point set $X \subseteq \mathbb{R}^2$ is *(strictly) convex* if every point in X is a vertex of the convex hull of X. A point set X is said to be *weakly convex* if X lies on the boundary of its convex hull. A *drawing* of a graph G is a mapping $f : V(G) \to \mathbb{R}^2$ such that edges are straight line segments connecting vertices and neither midpoints of edges, nor vertices, nor midpoints and vertices coincide. Through most of the paper we will not distinguish between (the elements of) a graph and their drawings.

For $i, j \in \{s, w, a\}$ we define $\mathcal{G}_i^j$ as the class of graphs admitting a drawing such that the set of vertices is $\begin{cases} \text{strictly convex} & \text{if } i = s \\ \text{weakly convex} & \text{if } i = w \\ \text{arbitrary} & \text{if } i = a \end{cases}$ and the midpoints of edges constitute a $\begin{cases} \text{strictly convex} & \text{if } j = s \\ \text{weakly convex} & \text{if } j = w \\ \text{arbitrary} & \text{if } j = a \end{cases}$ set. Further, we define $g_i^j(n)$ to be the maximum number of edges an n-vertex graph in $\mathcal{G}_i^j$ can have.

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Clearly, all $\mathcal{G}_i^j$ are closed under taking subgraphs and $\mathcal{G}_s^a = \mathcal{G}_w^a = \mathcal{G}_a^a$ is the class of all graphs.

Previous results and related problems: Motivated by a special class of convex optimization problems [5], Halman, Onn, and Rothblum [4] studied drawings of graphs in $\mathbb{R}^d$ with similar constraints as described above. In particular, in their language a graph has convex dimension 2 if and only if it is in $\mathcal{G}_a^s$ and strong convex dimension 2 if and only if it is in $\mathcal{G}_s^s$. They show that all trees and cycles are in $\mathcal{G}_s^s$, while $K_4 \in \mathcal{G}_a^s \setminus \mathcal{G}_s^s$ and $K_{2,3} \notin \mathcal{G}_a^s$. Moreover, they show that $n \leq g_s^s(n) \leq 5n-8$. Finally, they conjecture that all graphs in $\mathcal{G}_s^s$ are planar and thus $g_s^s(n) \leq 3n-6$.

The problem of computing or bounding $g_a^s(n)$ and $g_s^s(n)$ was rephrased and generalized in the setting of convex subsets of Minkowski sums of planar point sets by Eisenbrand et al. [2] and then regarded as a problem of computational geometry in its own right. We introduce this setting and give an overview of known results before explaining its relation to the original graph drawing problem.

Given two point sets $A, B \subseteq \mathbb{R}^d$ their *Minkowski sum* $A + B$ is defined as $\{a + b \mid a \in A, b \in B\} \subseteq \mathbb{R}^d$. We define $M(m,n)$ as the largest cardinality of a convex set $X \subseteq A + B$, for A and B planar point sets with $|A| = m$ and $|B| = n$. In [2] it was shown that $M(m,n) \in O(m^{2/3}n^{2/3}+m+n)$. This upper bound was complemented by Bílka et al. [1] with an asymptotically matching lower bound, even under the assumption that A itself is convex, i.e., $M(m,n) \in \Theta(m^{2/3}n^{2/3} + m + n)$. Notably, the lower bound works also for the case $A = B$ non-convex, as shown by Swanepoel and Valtr [6, Proposition 4]. In [7] Tiwary gives an upper bound of $O((m+n)\log(m+n))$ for the largest cardinality of a convex set $X \subseteq A + B$, for A and B planar convex point sets with $|A| = m$ and $|B| = n$. Determining the asymptotics in this case remains an open question.

As first observed in [2], the graph drawing problem of Halman et al. is related to the largest cardinality of a convex set $X \subset A + A$, for A some planar point set. In fact, from X and A one can deduce a graph $G \in \mathcal{G}_a^s$ on vertex set A, with an edge aa' for all $a \neq a'$ with $a + a' \in X$. The midpoint of the edge aa' then just is $\frac{1}{2}(a + a') \in \frac{1}{2}X \subset \frac{1}{2}A + \frac{1}{2}A$. Conversely, from any $G \in \mathcal{G}_a^s$ one can construct X and A as desired. The only trade-off in this translation are the pairs of the form aa, which are not taken into account by the graph-model, because they correspond to vertices. Hence, they do not play a role from the purely asymptotic point of view. Thus, the results of [1, 2, 6] yield $g_a^s(n) = \Theta(n^{4/3})$. Conversely, the bounds for $g_s^s(n)$ obtained in [4] give that the largest cardinality of a convex set $X \subseteq A + A$, for A a planar convex point set with $|A| = n$ is in $\Theta(n)$.

Our results: In this paper we study the set of graph classes defined in the introduction. We extend the list of properties of point sets considered in earlier works with *weak* convexity. We completely determine the inclusion relations on the resulting classes. We prove that $\mathcal{G}_s^s$ contains non-planar graphs, which disproves a conjecture of Halman et al. [4], and that $\mathcal{G}_s^w$ contains cubic graphs, while we believe is false for $\mathcal{G}_s^s$. We give new bounds for the parameters $g_i^j(n)$: we show that $g_s^w(n) = 2n - 3$, which is an upper bound for $g_s^s(n)$ and therefore improves the upper bound of $3n-6$ conjectured by Halman et al. [4]. Furthermore we show that $\lfloor \frac{3}{2}(n-1) \rfloor \le g_s^s(n)$.

For the relation with Minkowski sums we show that the largest cardinality of a weakly convex set $X \subseteq A + A$, for A some convex planar point set of $|A| = n$, is $2n$ and of a strictly convex set is between $\frac{3}{2}n$ and $2n - 2$.

The results for weak convexity are the first non-trivial precise formulas in this area.

A preliminary version of this paper has been published in conference proceedings [3].

2 Graph drawings

Given a graph G drawn in the plane with straight line segments as edges, we denote by P_V the convex hull of its set of vertices and by P_E the convex hull of the set of midpoints of its edges. Clearly, unless $V = \emptyset$, P_E is strictly contained in P_V.

2.1 Inclusions of classes

We show that most of the classes defined in the introduction coincide and determine the exact set of inclusions among the remaining classes.

Theorem 1. *We have* $\mathcal{G}_s^s = \mathcal{G}_w^s \subsetneq \mathcal{G}_s^w \subsetneq \mathcal{G}_w^w = \mathcal{G}_a^w = \mathcal{G}_s^a = \mathcal{G}_w^a = \mathcal{G}_a^a$ *and* $\mathcal{G}_s^s \subsetneq \mathcal{G}_a^s \subsetneq \mathcal{G}_w^w$. *Moreover, there is no inclusion relationship between* $\mathcal{G}_a^s$ *and* $\mathcal{G}_s^w$. *See Figure 1 for an illustration.*

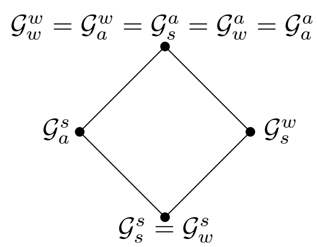

Fig. 1. Inclusions and identities among the classes $\mathcal{G}_i^j$.

Proof. Let us begin by proving that $\mathcal{G}_s^s = \mathcal{G}_w^s$, the inclusion $\mathcal{G}_s^s \subseteq \mathcal{G}_w^s$ is obvious. Take $G \in \mathcal{G}_w^s$ drawn in the required way. Since the midpoints of the edges form a convex set, there exists $\delta > 0$ such that moving every vertex by at most $< \delta$

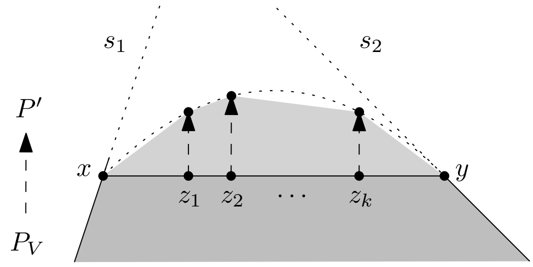

Fig. 2. The local modifications to prove $\mathcal{G}_s^s \supseteq \mathcal{G}_w^s$.

in any direction, the set of midpoints of the edges remains strictly convex. More precisely, whenever there are vertices $z_1, \ldots, z_k$ in the interior of the segment connecting two vertices x, y, we perform the following steps, see Figure 2:

We assume without loss of generality that x is drawn at $(0,0)$, y is drawn at $(1,0)$ and that P_V is entirely contained in the closed halfplane $\{(a,b) \mid b \leq 0\}$. We consider the two adjacent edges to xy in the boundary of P_V and denote by $s_1, s_2 \in \mathbb{R} \cup \{\pm\infty\}$ their slopes. Now we take $\epsilon : 0 < \epsilon < \min\{\delta, |s_1|, |s_2|\}$, we observe that $P' := P_V \cup \{(a,b) \mid 0 \leq a \leq 1 \text{ and } 0 \leq b \leq \epsilon a(1-a)\}$ is a convex set. Then, for all $i \in \{1, \ldots, k\}$, if z_i is drawn at $(\lambda_i, 0)$ with $0 < \lambda_i < 1$, we translate z_i to the point $(\lambda_i, \epsilon\lambda_i(1-\lambda_i))$. We observe that the point z_i has been moved a distance $< \epsilon/4 < \delta$ and, then, the set of midpoints of edges is still convex. Moreover, now $z_1, \ldots, z_k$ are vertices of P'. Repeating this argument when necessary we get that $G \in \mathcal{G}_s^s$.

To prove the strict inclusion $\mathcal{G}_s^s \subsetneq \mathcal{G}_s^w$ we show that the graph $K_4 - e$, i.e., the graph obtained from removing an edge e from the complete graph K_4 belongs to $\mathcal{G}_s^w$ but not to $\mathcal{G}_s^s$. Indeed, if we take x_0, x_1, x_2, x_3 the 4 vertices of $K_4 - e$ and assume that $e = x_2x_3$, it suffices to draw $x_0 = (1,0)$, $x_1 = (0,0)$, $x_2 = (0,1)$ and $x_3 = (2,1)$ to get that $K_4 - e \in \mathcal{G}_s^w$. See Figure 3 for an illustration.

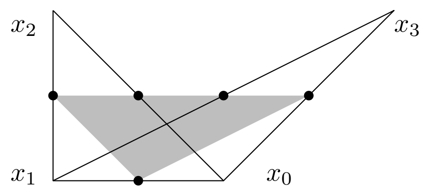

Fig. 3. A drawing proving $K_4 - e \in \mathcal{G}_s^w$.

Let us now prove that $K_4 - e \notin \mathcal{G}_s^s$. To that end, we assume that the set of vertices $\{x_0, x_1, x_2, x_3\}$ is in convex position. By means of an affine transformation we may assume that x_0, x_1, x_2, x_3 are drawn at the points $(1,0), (0,0), (0,1)$ and (a,b), with $a, b > 0$ respectively. The fact that $\{x_0, x_1, x_2, x_3\}$ is in convex position implies that $a + b > 1$. If $x_i x_{i+1 \bmod 4}$ is an edge for all $i \in \{0,1,2,3\}$, then clearly the set of midpoints is not convex because the midpoints of x_0x_2 and x_1x_3 are in the convex hull of the midpoints of the other 4 edges. So, assume that x_2x_3 is not an edge, , i.e., the drawing is like in Figure 3. So the midpoints of the edges are in positions $m_{01} = (0, 1/2), m_{12} = (1/2, 0)$, $m_{02} = (1/2, 1/2)$, $m_{13} = (a/2, b/2)$, $m_{03} = (a/2, (b + 1)/2)$. (We will generally denote midpoints in this fashion.) If $m_{01}, m_{12}, m_{02}, m_{13}$ are in convex position, then we deduce that $a < 1$ or $b < 1$ but not both, since otherwise x_3 would be in the convex hull of x_0, x_1, x_2. However, if $a < 1$, then m_{03} belongs to the convex hull of $\{m_{01}, m_{12}, m_{02}, m_{13}\}$, and if $b < 1$, then m_{13} belongs to the convex hull of $\{m_{01}, m_{12}, m_{02}, m_{03}\}$. Hence, we again have that the set of midpoints is not convex and we conclude that $K_4 - e \notin \mathcal{G}_s^s$.

The strict inclusion $\mathcal{G}_s^w \subsetneq \mathcal{G}_a^a$ comes as a direct consequence of Theorem 2.

Let us see that every graph belongs to $\mathcal{G}_w^w$, for this purpose it suffices to show that $K_n \in \mathcal{G}_w^w$. Drawing the vertices in the points with coordinates $(0, 0)$, and $(1, 2^i)$ for $i \in \{1, \ldots, n-1\}$ gives the result. Indeed, the midpoints of all the edges lie either in the vertical line $x = 1/2$ or in $x = 1$. The choice of y-coordinates ensures that in the line $x = 1$, no midpoints coincide with other midpoints nor vertices. Hence, we clearly have that $\mathcal{G}_w^w = \mathcal{G}_a^w = \mathcal{G}_s^a = \mathcal{G}_w^a = \mathcal{G}_a^a$.

The strictness in the inclusions $\mathcal{G}_s^s \subsetneq \mathcal{G}_a^s \subsetneq \mathcal{G}_w^w$ comes from the fact that $g_a^s = \Theta(n^{4/3})$ [1,2,6] and that, $g_s^s(n) \le g_s^w(n) \le 2n - 3$ by Theorem 2. This also proves that $\mathcal{G}_a^s \not\subset \mathcal{G}_s^w$.

To prove that $\mathcal{G}_s^w \not\subset \mathcal{G}_a^s$ it suffices to consider the complete bipartite graph $K_{2,3}$. Indeed, if $\{x_1, x_2, x_3\}$, $\{y_1, y_2\}$ is the vertex partition, it suffices to draw x_1, x_2, x_3 in $(0, 0)$, $(4, 0)$, $(3, 2)$, respectively, and y_1, y_2 in $(1, 1)$, $(4, 1)$, respectively, to get that $K_{2,3} \in \mathcal{G}_s^w$. See Figure 4 for an illustration.

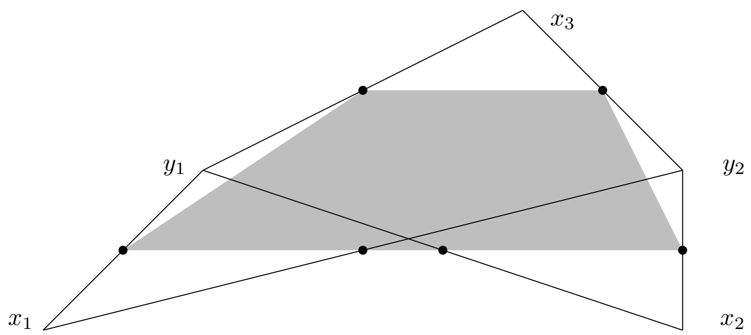

Fig. 4. A drawing proving $K_{2,3} \in \mathcal{G}_s^w$.

Finally, $K_{2,3} \notin \mathcal{G}_a^s$ was already shown in [4].

□

2.2 Bounds on numbers of edges

In this section, we show that $\lfloor \frac{3}{2}(n-1) \rfloor \leq g_s^s(n) \leq g_s^w(n) = 2n-3$.

Whenever V is weakly convex, for every vertex x, one can order the neighbors of x according to their clockwise appearance around the border of P_V starting at x. If in this order the neighbors of x are $y_1, \ldots, y_k$, then we say that $xy_2, \ldots, xy_{k-1}$ are the *interior edges of* x. Non-interior edges of x are called *exterior edges of* x. Clearly, any vertex has at most two exterior edges. A *vertex* v *sees an edge* e if the straight-line segment connecting v and the midpoint m_e of e does not intersect the interior of P_E, recall that P_E is the convex hull of the midpoints.

Lemma 1. *If $G \in \mathcal{G}_s^w$, then no vertex sees its interior edges. In particular, any vertex sees at most 2 incident edges.*

Proof. Assume that there exists a vertex x seeing an interior edge xu_i. Take u_1, u_k such that xu_1, xu_k are the exterior edges of x. We consider the induced graph G' with vertex set $V' = \{v, u_1, u_i, u_k\}$ and denote by E' its corresponding edge set. Clearly $P_{V'} \subset P_V$ and $P_{E'} \subset P_E$, so x sees xu_i in $P_{E'}$. Moreover, xu_i is still an interior edge of x in G'. Denote by m_j the midpoint of the edge vu_j, for $j \in \{1, i, k\}$. Since x sees xu_i, the closed halfplane supported by the line passing through m_1, m_k containing x also contains m_i.

However, since $P_{V'}$ is strictly convex u_i and x are separated by the line passing through u_1, u_k. This is a contradiction because $m_j = (u_j + x)/2$. See Figure 5.

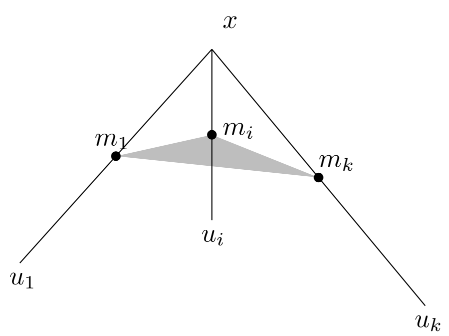

Fig. 5. The construction in Lemma 1

$\square$

Theorem 2. *If a graph $G \in \mathcal{G}_s^w$ has n vertices, then it has at most $2n-3$ edges, i.e., $g_s^w(n) \leq 2n-3$.*

Proof. Take $G \in \mathcal{G}_s^w$. Since the midpoints of the edges form a weakly convex set, every edge has to be seen by at least one of its vertices. Lemma 1 guarantees that interior edges cannot be seen. Hence, no edge can be interior to both endpoints. This proves that G has at most $2n$ edges.

We improve this bound by showing that at least three edges are exterior to both of their endpoints, i.e., are counted twice in the above estimate. During the proof let us call such edges *doubly exterior*.

Since deleting leafs only decreases the ratio of vertices and edges, we can assume that G has no leafs. Since $2 > 2n-3$ implies $n \leq 1$ and in this case our statement is clearly true, we can also assume that G has at least three edges. For an edge e, we denote by H_e^+ and H_e^- the open halfplanes supported by the line containing e. We claim that whenever an edge $e = xy$ is an interior edge of x, then $H_e^+ \cup \{x\}$ contains a doubly exterior edge. This follows by induction on the number of vertices in $H_e^+ \cap P_V$. If there is a single vertex $z \in H_e^+ \cap P_V$, then xz is an exterior edge of x because xy is interior to x. Moreover, by convexity of V and since z is the only vertex in H_e^+ the edge xz is also exterior to z, so it is doubly exterior. We assume now that there is more than one vertex in $H_e^+ \cap P_V$. Since e is interior to x, there is an edge $f = xz$ contained in $H_e^+ \cup \{x\}$ and exterior of x. If f is doubly exterior we are done. Otherwise, we set H_f^+ the halfplane supported by the line containing f and not containing y. We claim that $(H_f^+ \cup \{z\}) \cap V \subset (H_e^+ \cup \{x\}) \cap V$. Indeed, if there is a point $v \in (H_f^+ \cup \{z\}) \cap V$ but not in $H_e^+ \cup \{x\}$, then x is in the interior of the triangle with vertices $v, y, z \in V$, a contradiction. Thus, $(H_f^+ \cup \{z\}) \cap V$ is contained in $(H_e^+ \cup \{x\}) \cap V$, in particular, since $(H_f^+ \cup \{z\}) \cap V$ does not contain x the inclusion is strict. By induction, we can guarantee that $(H_e^+ \cup \{x\}) \cap P_V$ contains a doubly exterior edge.

Note that an analogous argument yields that $H_e^- \cup \{x\}$, contains a doubly exterior edge if e is an interior edge of x.

Applying this argument to any edge e which is not doubly exterior gives already two doubly exterior edges f, g contained in $H_e^+ \cup \{x\}$ and $H_e^- \cup \{x\}$, respectively. Choose an endpoint z of f, which is not an endpoint of g, which is possible since we have minimum degree at least two. Let $h = zw$ be the other exterior edge of z. If h is doubly exterior we are done. Otherwise, none of $H_h^+ \cup \{w\}$ and $H_h^- \cup \{w\}$ contains f because $z \notin H_h^+$ and $z \notin H_h^-$; moreover one of $H_h^+ \cup \{w\}$ and $H_h^- \cup \{w\}$ does not contain g. Thus, there must be a third doubly exterior edge. $\square$

Definition 1. *For every $n \geq 2$, we denote by L_n the graph consisting of two paths $P = (u_1, \ldots, u_{\lfloor \frac{n}{2} \rfloor})$ and $Q = (v_1, \ldots, v_{\lceil \frac{n}{2} \rceil})$ and the edges u_1v_1 and u_iv_{i-1} and $u_{j-1}v_j$ for $1 < i \leq \lfloor \frac{n}{2} \rfloor$ and $1 < j \leq \lceil \frac{n}{2} \rceil$. We observe that L_n has $2n-3$ edges.*

Theorem 3. *For all $n \geq 2$ we have $L_n \in \mathcal{G}_s^w$, i.e., $g_s^w(n) \geq 2n-3$.*

Proof. For every $k \geq 1$ we construct a drawing showing $L_{4k+2} \in \mathcal{G}_s^w$ (the result for other values of n follows by suppressing degree 2 vertices). We take $0 < \epsilon_0 < \epsilon_1 < \cdots < \epsilon_{2k}$ and set $\delta_j := \sum_{i=j}^{2k} \epsilon_i$ for all $j \in \{0, \ldots, 2k\}$. We consider the graph G with vertices $r_i = (i, \delta_{2i}), r'_i = (i, -\delta_{2i})$ for $i \in \{0, \ldots, k\}$ and $\ell_i = (-i, \delta_{2i-1}), \ell'_i = (-i, -\delta_{2i-1})$ for $i \in \{1, \ldots, k\}$; and edge set

$$\{r_0 r'_0\} \cup \{r_i \ell_i, r_i \ell'_i, r'_i \ell_i, r'_i \ell'_i \mid 1 \leq i \leq k\} \cup \{r_{i-1} \ell_i, r_{i-1} \ell'_i, r'_{i-1} \ell_i, r'_{i-1} \ell'_i \mid 1 \leq i \leq k\}.$$

See Figure 6 for an illustration of the final drawing. The choice of ϵ_i forming

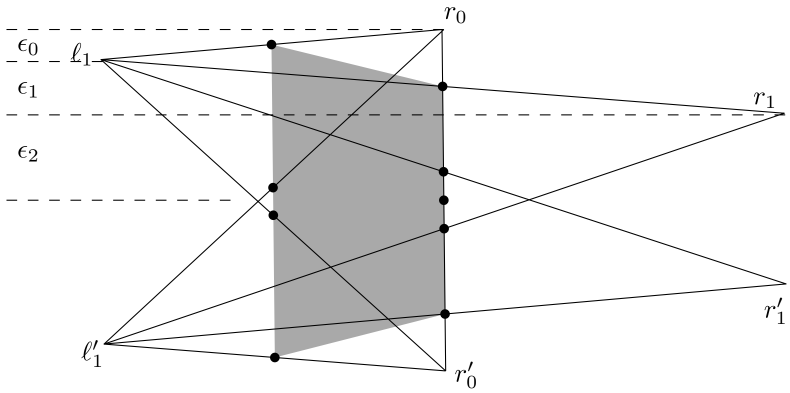

Fig. 6. The graph L_6 is in $\mathcal{G}_s^w$.

an increasing sequence yields directly that the set of vertices is strictly convex. Moreover, the midpoints of the edges all lie on the vertical lines $x = 0$ and $x = -1/2$; thus they form a weakly convex set. It is straightforward to verify that the constructed graph is L_{4k+2}.

□

We observe that $L_4 = K_4 - e$ and that L_4 is a subgraph of L_n for all $n \geq 4$. As we proved in Theorem 1, $K_4 - e$ does not belong to $\mathcal{G}_s^s$. Hence, for all $n \geq 4$ we have that $L_n \notin \mathcal{G}_s^s$.

Theorem 2 together with Theorem 3 yield the exact value of $g_s^w(n) = 2n - 3$. Moreover, since $\mathcal{G}_s^s \subset \mathcal{G}_s^w$, from Theorem 2 we also deduce the upper bound $g_s^s(n) \leq 2n - 3$. The rest of this section is devoted to provide a lower bound for $g_s^s(n)$.

Definition 2. *For every odd $n \geq 3$, we denote by B_n the graph obtained from identifying a C_3 and $\frac{n-3}{2}$ copies of C_4 altogether identified along a single edge uv. We observe that B_n has $\frac{3}{2}(n-1)$ edges and deleting a degree 2 vertex from B_n one obtains an $(n-1)$-vertex graph with $\frac{3}{2}(n-2) - \frac{1}{2}$ edges.*

Theorem 4. *For all odd $n \geq 3$ we have $B_n \in \mathcal{G}_s^s$, i.e., $g_s^s(n) \geq \lfloor \frac{3}{2}(n-1) \rfloor$.*

Proof. Let $n \geq 3$ be such that $n - 3$ is divisible by 4 (if $n - 3$ is not divisible by 4, then B_n is an induced subgraph of B_{n+1}). We will first draw B_n in an unfeasible way and then transform it into another one proving $B_n \in \mathcal{G}_s^s$.

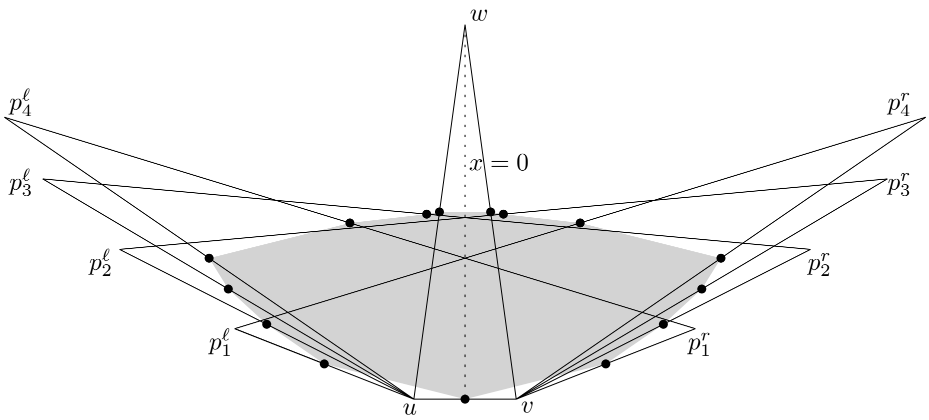

Fig. 7. The graph B_{11} is in $\mathcal{G}_s^s$.

See Figure 7 for an illustration of the final drawing.

We draw the $C_3 = (uvw)$ as an isosceles triangle with horizontal base uv. Let $u = (-1, 0)$, $v = (1, 0)$, and $w = (0, \frac{n-1}{2})$. There are $n - 3$ remaining points. Draw one half of them on coordinates $p_i^\ell = (-1 - i, i)$ for $1 \leq i \leq \frac{n-3}{2}$ and the other half mirrored along the y-axis, i.e., $p_i^r = (1 + i, i)$ for $1 \leq i \leq \frac{n-3}{2}$.

Now we add all edges $p_i^\ell u$ (left edges), $p_i^r v$ (right edges), for $1 \leq i \leq \frac{n-3}{2}$ and edges of the form $p_i^\ell p^r_{\frac{n-3}{2}+1-i}$ (diagonal edges) for all $1 \leq i \leq \frac{n-3}{2}$.

We observe that the points p_i^ℓ and u lie on the line $x + y = -1$, the points p_i^r and v lie on the line $x - y = 1$ and all midpoints of diagonal edges have y-coordinate $\frac{n-1}{4}$. In order to bring the set of vertices and the set of midpoints of edges into convex sets, we simultaneously decrease the y-coordinates of points $p^\ell_{\frac{n-3}{2}+1-i}, p^r_{\frac{n-3}{2}+1-i}$ by $2^i\epsilon$ for $i \in \{1, \dots, \frac{n-3}{2}\}$ for a sufficiently small value $\epsilon > 0$. Finally, we conveniently decrease the y-coordinate of w to get a drawing witnessing that $B_n \in \mathcal{G}_s^s$.

□

2.3 Further members of $\mathcal{G}_s^s$ and $\mathcal{G}_s^w$

We show that there are non-planar graphs in $\mathcal{G}_s^s$ and cubic graphs in $\mathcal{G}_s^w$.

Definition 3. *For all $k \geq 2$, we denote by H_k the graph consisting of a $2k$-gon with vertices $v_1, \dots, v_{2k}$ and a singly subdivided edge from v_i to $v_{i+3 \bmod 2k}$ for all i even, i.e., adjacent to the v_i there are k additional degree 2 vertices $u_1, \dots, u_k$ and edges $u_i v_{2i}$ for all $i \in \{1, \dots, k\}$, $u_i v_{2i+3}$ for all $i \in \{1, \dots, k-2\}$, $u_{k-1} v_1$ and $u_k v_3$. We observe that H_k is planar if and only if k is even, see Figure 8 for a drawing of H_3.*

Theorem 5. *For every $k \geq 2$, $H_k \in \mathcal{G}_s^s$. In particular, for every $n \geq 9$ there is a non-planar n-vertex graph in $\mathcal{G}_s^s$.*

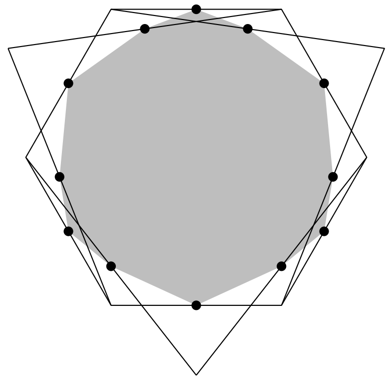

Fig. 8. The graph H_3 drawn as in Theorem 5.

Proof. We start by drawing C_{2k} as a regular $2k$-gon. Take an edge $e = xy$ and denote by x', y' the neighbors of x and y, respectively. For convenience consider e to be of horizontal slope with the $2k$-gon below it. Our goal is to place v_e a new vertex and edges $v_e x'$, $v_e y'$ preserving the convexity of vertices and midpoints of edges. We consider the upward ray r based at the midpoint m_e of e and the upward ray s of points whose x-coordinate is the average between the x-coordinates of m_e and x'. We denote by Δ the triangle with vertices the midpoint $m_{x'x}$ of the edge $x'x$, the point x and m_e. Since $s \cap \Delta$ is nonempty, we place v_e such that the midpoint of $v_e x'$ is in $s \cap \Delta$. Clearly v_e is on r and lies in the triangle defined by xy and the lines supporting edges $x'x$ and $y'y$. Hence, the middle point of $v_e y'$ is in the corresponding triangle Δ' and the convexity of vertices and midpoints of edges is preserved. See Figure 9 for an illustration. Since we only have to add a vertex on every other edge of C_{2k}, these choices are independent of each other. It is easy to verify that the constructed graph is H_k.

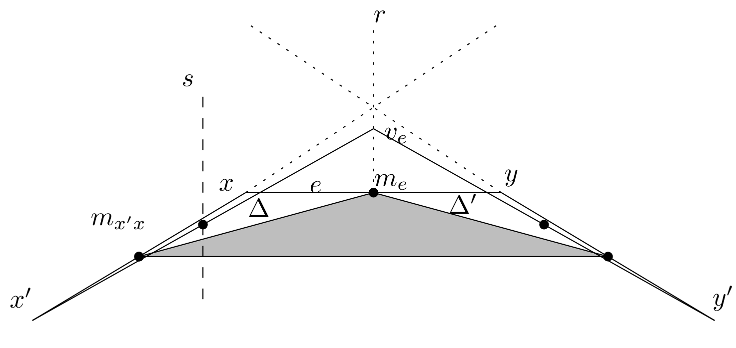

Fig. 9. The construction in Theorem 5

□

Definition 4. *For all $k \geq 3$, we denote by P_k the graph consisting of a prism over a k-cycle. We observe that P_k is a 3-regular graph.*

Theorem 6. *For every $k \geq 3$, $P_k \in \mathcal{G}_s^w$. In particular, for every even $n \geq 6$ there is a 3-regular n-vertex graph in $\mathcal{G}_s^w$.*

Proof. Let $k \geq 3$. In order to draw P_k, place $2k$ vertices $v_0, \ldots, v_{2k-1}$ as the vertices of a regular $2k$-gon in the plane. Add all *inner* edges of the form $v_i v_{i+2 \bmod 2k}$ for all i and *outer* edges $v_i v_{i+1 \bmod 2k}$ for i even. Clearly, the midpoints of outer edges form a strictly convex set and their convex hull is a regular k-gon. Now, consider four consecutive vertices in the boundary of the $2k$ gon, say $v_0, \ldots, v_3$. They induce two outer edges, v_0v_1 and v_2v_3 and two inner edges v_0v_2 and v_1v_3. Now, the triangles $v_0v_1v_2$ and $v_1v_2v_3$ share the base segment v_1v_2. Hence, the segments $m_{v_2v_3}m_{v_1v_3}$ and $m_{v_2v_0}m_{v_1v_0}$ share the slope of v_1v_2. Now, since the angle between v_1v_2 and v_2v_3 equals the angle between v_1v_2 and v_0v_1 and v_0v_1 and v_2v_3 are of equal length, the segment $m_{v_2v_3}m_{v_1v_0}$ also has the same slope. Thus, all the midpoint lie on a line and all midpoints lie on the boundary of the midpoints of outer edges. See Figure 10 for an illustration.

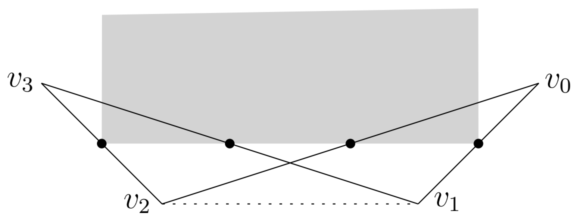

Fig. 10. The construction in Theorem 6

□

We do not know of any 3-regular graphs in $\mathcal{G}_s^s$. More generally we believe that:

Conjecture 1. If $G \in \mathcal{G}_s^s$ then G is 2-degenerate, i.e., every non-empty induced subgraph has a vertex of degree at most 2.

2.4 Structural questions

One can see, although it is tedious, that adding a leaf at the vertex r_1 of L_8 (see Definition 1) produces a graph not in $\mathcal{G}_s^w$. Under some conditions it is possible to add leafs to graphs in $\mathcal{G}_s^s$. We say that an edge is V*-crossing* if it intersects the interior of P_V.

Proposition 1. *Let $G \in \mathcal{G}_s^s$ be drawn in the required way. If uv is not V-crossing, then attaching a new vertex w to v yields a graph in $\mathcal{G}_s^s$.*

Proof. Let $G \in \mathcal{G}_s^s$ with at least 3 vertices and let $e = uv$ be the edge of G from the statement. For convenience consider that uv come in clockwise order on the boundary of P_V. Consider the supporting line H of P_E through the midpoint m_e of e, whose side containing P_E contains v. A new midpoint can go inside the triangle Δ defined by H, the two clockwisely consecutive supporting lines of P_E, both intersecting in a midpoint m'. Since P_E is contained in P_V a part of Δ lies outside P_V. Choosing the midpoint of a new edge attached to v inside this region very close to e preserves strict convexity of vertices and midpoints. See Figure 11 for an illustration.

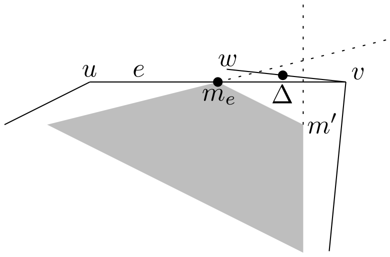

Fig. 11. The construction in Proposition 1

□

We pose the following

Question 1. Is the class $\mathcal{G}_s^s$ is closed under adding leafs?

Despite the fact that $K_{2,n} \notin \mathcal{G}_s^s$ for all $n \geq 3$, we have found in Theorem 4 a subdivision of $K_{2,n}$ which belongs to $\mathcal{G}_s^s$. Similarly, Theorem 5 gives that a subdivision of $K_{3,3}$ is in $\mathcal{G}_s^s$, while $K_{3,3}$ is not. We have the impression that subdividing edges facilitates drawings in $\mathcal{G}_s^s$. Even more, we believe that:

Conjecture 2. The edges of every graph can be (multiply) subdivided such that the resulting graph is in $\mathcal{G}_s^s$.

3 Minkowski sums

We show that the largest cardinality of a weakly convex set X, which is a subset of the Minkowski sum of a convex planar n-point set A with itself is $2n$. If X is required to be strictly convex, then the largest size of such a set lies between $\frac{3}{2}n$ and $2n - 2$.

As mentioned in the introduction there is a slight trade-off when translating the graph drawing problem to the Minkowski sum problem. Since earlier works have been considering only asymptotic bounds this was neglected. Here we are fighting for constants, so we deal with it. Recall that a point $x \in X \subseteq A + A$

is not captured by the graph model if $x = a + a$ for some $a \in A$. Indeed, the point x corresponds to a vertex in the drawing of the graph. In order to capture the trade-off, for every $i, j \in \{s, w, a\}$, we define $\widetilde{g}_i^j(n)$ as the maximum value of $n' + m$, where m is the number of edges of an n-vertex graph in $\mathcal{G}_i^j$ and n' of its vertices can be added to the set of midpoints in such a way that the resulting set is $\begin{cases} \text{strictly convex} & \text{if } j = s \\ \text{weakly convex} & \text{if } j = w \\ \text{arbitrary} & \text{if } j = a \end{cases}$.

We recall that a *vertex v sees an edge e* if the straight-line segment connecting v and the midpoint m_e of e does not intersect the interior of P_E.

Lemma 2. *Let $G \in \mathcal{G}_s^w$ be drawn in the required way and $v \in G$. If v can be added to the drawing of G such that v together with the midpoints of G is weakly convex, then every edge $vw \in G$ is seen by w.*

Proof. Otherwise the midpoint of vw will be in the convex hull of v together with parts of P_E to the left and to the right of vw, see Figure 12. □

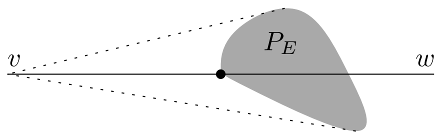

Fig. 12. The contradiction in Lemma 2

We say that an edge is *good* if it can be seen by both of its endpoints.

Theorem 7. *For every $n \geq 3$ we have $\widetilde{g}_s^w(n) = 2n$. That is, the largest cardinality of a weakly convex set $X \subseteq A + A$, for A a convex set of n points in the plane, is $2n$.*

Proof. The lower bound comes from drawing C_n as the vertices and edges of a convex polygon. The set of vertices and midpoints is weakly convex.

For the upper bound let $G \in \mathcal{G}_s^w$ with n vertices and m edges, we denote by n_i the number of vertices of G that see i of its incident edges for $i \in \{0, 1, 2\}$. Since every edge is seen by at least one of its endpoints and every vertex sees at most 2 of its incident edges (Lemma 1), we know that $m = n_1 + 2n_2 - m_g$, where m_g is the number of good edges.

Let n' be the number of vertices of G that can be added to the drawing such that together with the midpoints they are in weakly convex position. Denote by n'_i the number of these vertices that see i of its incident edges for $i \in \{0, 1, 2\}$. By Lemma 2 the edges seen by an added vertex have to be good. Thus, $m_g \geq \frac{1}{2}(n'_1 + 2n'_2)$. This yields

$$m + n' \leq n_1 + 2n_2 - \frac{1}{2}(n'_1 + 2n'_2) + n'_0 + n'_1 + n'_2 \leq n_0 + \frac{3}{2}n_1 + 2n_2 \leq 2n.$$

$\square$

Theorem 8. *For every $n \geq 3$ we have $\lfloor \frac{3}{2}n \rfloor \leq \widetilde{g}_s^s(n) \leq 2n-2$. That is, the largest cardinality of a convex set $X \subseteq A+A$, for A a convex set of n points in the plane, lies within the above bounds.*

Proof. The lower bound comes from drawing C_n as the vertices and edges of a convex polygon. The set formed by an independent set of vertices and all midpoints is in convex position.

Take $G \in \mathcal{G}_s^s$ with n vertices and m edges. The upper bound is very similar to Theorem 7. Indeed, following the same notations we also get that $m = n_1 + 2n_2 - m_g$. Again, the edges seen by an added vertex have to be good. Since now moreover the set of addable vertices has to be independent, we have $m_g \geq n'_1 + 2n'_2$. This yields

$$m + n' \leq n_1 + 2n_2 - n'_1 - 2n'_2 + n'_0 + n'_1 + n'_2 \leq n + n_2 - n'_2.$$

If $n+n_2-n'_2 > 2n-2$ then either $n_2 = n$ and $n'_2 < 2$, or $n_2 = n-1$ and $n'_2 = 0$. In both cases we get that $n' \leq 1$. By Theorem 2 we have $m \leq 2n-3$, then it follows that $m + n' \leq 2n-2$.

$\square$

4 Conclusions

We have improved the known bounds on $g_s^s(n)$, the number of edges an n-vertex graph of strong convex dimension 2 can have. Still describing this function exactly is an open problem. We believe that graphs in $\mathcal{G}_s^s$ have degeneracy 2. However, confirming our conjecture would not improve our bounds. Similarly, the exact largest cardinality $\widetilde{g}_s^s(n)$ of a convex set $X \subseteq A+A$ for A a convex planar n-point set, remains to be determined. Curiously, in both cases we have shown that the correct answer lies between $\frac{3}{2}n$ and $2n$. The more general family $\mathcal{G}_s^w$ seems to be easier to handle, in particular we have provided the exact value for both g_s^w and $\widetilde{g}_s^w$.

From a more structural point of view we wonder what graph theoretical measures can ensure that a graph belongs to $\mathcal{G}_s^s$ or $\mathcal{G}_s^w$. None of these classes is contained in the class of planar graphs. The class $\mathcal{G}_s^w$ is not closed under adding leafs. We do not know if the same holds for $\mathcal{G}_s^s$. Finally, we believe that subdividing a graph often enough ensures that it can be drawn in $\mathcal{G}_s^s$.

References

1. Ondřej Bílka, Kevin Buchin, Radoslav Fulek, Masashi Kiyomi, Yoshio Okamoto, Shin-ichi Tanigawa, and Csaba D. Tóth, *A tight lower bound for convexly independent subsets of the Minkowski sums of planar point sets*, Electron. J. Combin. **17** (2010), no. 1, Note 35, 4.

2. Friedrich Eisenbrand, János Pach, Thomas Rothvoß, and Nir B. Sopher, *Convexly independent subsets of the Minkowski sum of planar point sets*, Electron. J. Combin. **15** (2008), no. 1, Note 8, 4.
3. Ignacio García-Marco and Kolja Knauer, *Drawing graphs with vertices and edges in convex position*, Graph Drawing and Network Visualization - 23rd International Symposium, GD 2015, Los Angeles, CA, USA, September 24-26, 2015, Revised Selected Papers, 2015, pp. 348–359.
4. Nir Halman, Shmuel Onn, and Uriel G. Rothblum, *The convex dimension of a graph.*, Discrete Appl. Math. **155** (2007), no. 11, 1373–1383.
5. Shmuel Onn and Uriel G. Rothblum, *Convex combinatorial optimization.*, Discrete Comput. Geom. **32** (2004), no. 4, 549–566.
6. Konrad J. Swanepoel and Pavel Valtr, *Large convexly independent subsets of Minkowski sums*, Electron. J. Combin. **17** (2010), no. 1, Research Paper 146, 7.
7. Hans Raj Tiwary, *On the largest convex subsets in Minkowski sums.*, Inf. Process. Lett. **114** (2014), no. 8, 405–407.