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Jihene Lagha, Faouzi Triki, Habib Zribi

Small Perturbations of an Interface for Elastostatic Problems

Jihene Lagha * Faouzi Triki† Habib Zribi ‡

Abstract

We consider the Lamé system for an elastic medium consisting of an inclusion embedded in a homogeneous background medium. Based on the field expansion method (FE) and layer potential techniques, we rigorously derived the asymptotic expansion of the perturbed displacement field due to small perturbations in the interface of the inclusion. We extend these techniques to determine a relationship between traction-displacement measurements and the shape of the object and derive an asymptotic expansion for the perturbation in the elastic moments tensors (EMTs) due to the presence of small changes in the interface of the inclusion.

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1 Introduction and statement of the main results

Consider a homogeneous isotropic elastic inclusion D embedded in the background region $\mathbb{R}^2$, which is occupied by a homogeneous isotropic elastic material. The boundary ∂D of the inclusion is assumed to be of class $\mathcal{C}^2$. In this case, ∂D can be parametrized by a vector-valued function $t \to X(t)$, that is, $\partial D := \{x = X(t), t \in [a, b] \text{ with } a < b\}$, where X is a $\mathcal{C}^2$-function satisfying $|X'(t)| = 1$ for all $t \in [a, b]$, and $X(a) = X(b)$.

Let (λ_0, μ_0) denote the background Lamé constants, that are the elastic parameters in the absence of any inclusions. Assume that the Lamé constants in the inclusion D are given by (λ_1, μ_1) where $(\lambda_1, \mu_1) \neq (\lambda_0, \mu_0)$. We further assume that $\mu_j > 0$, $\lambda_j + \mu_j > 0$ for $j = 0, 1$, $(\lambda_0 - \lambda_1)(\mu_0 - \mu_1) \geq 0$. As in [22], we needed the last assumption in order to guarantee the well-posedeness of boundary integral equation representation of the displacement field (see for instance Theorem 2 in [14]).

Let $\mathbb{C}_0$ and $\mathbb{C}_1$ be the elasticity tensors for $\mathbb{R}^2\backslash\overline{D}$ and D, respectively, which are given by

$$(\mathbb{C}_m)_{ijkl} = \lambda_m \delta_{ij}\delta_{kl} + \mu_m(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \quad \text{for } i, j, k, l = 1, 2, \quad m = 0, 1.$$

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There is another way of expressing the isotropic elastic tensor which will be used later. Let $\mathbb{I}$ be the identity 4-tensor and $\mathbf{I}$ be the identity 2-tensor (the 2×2 identity matrix). Then $\mathbb{C}_m$ can be rewritten as

$$\mathbb{C}_m = \lambda_m \mathbf{I} \otimes \mathbf{I} + 2\mu_m \mathbb{I}, \qquad m = 0, 1. \tag{1.1}$$

Then, the elasticity tensor for $\mathbb{R}^2$ in the presence of the inclusion D is then given by

$$\mathbb{C} = \mathbb{C}_0 \chi_{\mathbb{R}^2 \setminus \overline{D}} + \mathbb{C}_1 \chi_D,$$

where χ_D is the indicator function of D.

In this paper, we consider the following transmission problem

$$\begin{cases} \nabla \cdot \big(\mathbb{C} \widehat{\nabla} \mathbf{u} \big) = 0 & \text{in } \mathbb{R}^2, \\ \mathbf{u}(x) - \mathbf{H}(x) = O(|x|^{-1}) & \text{as } |x| \to \infty, \end{cases} \tag{1.2}$$

where $\mathbf{H}$ is a vector-valued function satisfying $\nabla \cdot \big(\mathbb{C}_0 \widehat{\nabla} \mathbf{H} \big) = 0$ in $\mathbb{R}^2$, and $\widehat{\nabla} \mathbf{u} = \frac{1}{2} \big(\nabla \mathbf{u} + (\nabla \mathbf{u})^T \big)$ is the symmetric strain tensor. Here and throughout the paper $\mathbf{M}^T$ denotes the transpose of the matrix $\mathbf{M}$.

The elastostatic operator corresponding to the Lamé constants (λ_0, μ_0) is defined by

$$\mathcal{L}_{\lambda_0, \mu_0} \mathbf{u} := \mu_0 \Delta \mathbf{u} + (\lambda_0 + \mu_0) \nabla \nabla \cdot \mathbf{u}, \tag{1.3}$$

and the corresponding conormal derivative $\dfrac{\partial \mathbf{u}}{\partial \nu}$ on ∂D is defined to be

$$\frac{\partial \mathbf{u}}{\partial \nu} := \lambda_0 (\nabla \cdot \mathbf{u}) \mathbf{n} + \mu_0 \big(\nabla \mathbf{u} + (\nabla \mathbf{u})^T \big) \mathbf{n}, \tag{1.4}$$

where $\mathbf{n}$ is the outward unit normal to ∂D.

Similarly, we denote by $\mathcal{L}_{\lambda_1, \mu_1}$ and $\dfrac{\partial \mathbf{u}}{\partial \widetilde{\nu}}$ the Lamé operator and the conormal derivative, respectively, associated to the Lamé constants (λ_1, μ_1).

The problem (1.2) is equivalent to the following problem (see for instance [4, 5, 8])

$$\begin{cases} \mathcal{L}_{\lambda_0, \mu_0} \mathbf{u} = 0 & \text{in } \mathbb{R}^2 \setminus \overline{D}, \\ \mathcal{L}_{\lambda_1, \mu_1} \mathbf{u} = 0 & \text{in } D, \\ \mathbf{u}|_- = \mathbf{u}|_+ & \text{on } \partial D, \\ \dfrac{\partial \mathbf{u}}{\partial \widetilde{\nu}}\Big|_- = \dfrac{\partial \mathbf{u}}{\partial \nu}\Big|_+ & \text{on } \partial D, \\ \mathbf{u}(x) - \mathbf{H}(x) = O(|x|^{-1}) & \text{as } |x| \to \infty. \end{cases} \tag{1.5}$$

The quantities $\mathbf{u}|_\pm$ on ∂D denote the limits from outside and inside of D, respectively. We will also sometimes use $\mathbf{u}^e$ for $\mathbf{u}|_+$ and $\mathbf{u}^i$ for $\mathbf{u}|_-$.

Let now D_ϵ be an $\epsilon-$perturbation of $D, i.e.$, there is $h \in \mathcal{C}^1(\partial D)$ such that ∂D_ϵ is given by

$$\partial D_\epsilon := \Big\{\tilde{x} : \tilde{x} = x + \epsilon h(x)\mathbf{n}(x), x \in \partial D\Big\}. \tag{1.6}$$

Let $\mathbf{u}_\epsilon$ be the displacement field in the presence of D_ϵ. Then $\mathbf{u}_\epsilon$ is the solution to

$$\begin{cases} \mathcal{L}_{\lambda_0,\mu_0}\mathbf{u}_\epsilon = 0 & \text{in } \mathbb{R}^2\backslash\overline{D}_\epsilon, \\ \mathcal{L}_{\lambda_1,\mu_1}\mathbf{u}_\epsilon = 0 & \text{in } D_\epsilon, \\ \mathbf{u}_\epsilon|_- = \mathbf{u}_\epsilon|_+ & \text{on } \partial D_\epsilon, \\ \dfrac{\partial \mathbf{u}_\epsilon}{\partial \widetilde{\nu}}\Big|_- = \dfrac{\partial \mathbf{u}_\epsilon}{\partial \nu}\Big|_+ & \text{on } \partial D_\epsilon, \\ \mathbf{u}_\epsilon(x) - \mathbf{H}(x) = O(|x|^{-1}) & \text{as } |x| \to \infty. \end{cases} \tag{1.7}$$

The first main result of this paper is the following derivation of the leading-order term in the asymptotic expansion of $(\mathbf{u}_\epsilon - \mathbf{u})|_\Omega$ as $\epsilon \to 0$, where Ω is a bounded region outside the inclusion D, and away from ∂D.

Theorem 1.1 *Let $\mathbf{u}$ and $\mathbf{u}_\epsilon$ be the solutions to* (1.5) *and* (1.7)*, respectively. Let Ω be a bounded region outside the inclusion D, and away from ∂D. For $x \in \Omega$, the following pointwise asymptotic expansion holds:*

$$\mathbf{u}_\epsilon(x) = \mathbf{u}(x) + \epsilon \mathbf{u}_1(x) + O(\epsilon^2), \tag{1.8}$$

where the remainder $O(\epsilon^2)$ depends only on $\lambda_0, \lambda_1, \mu_0, \mu_1$, the $\mathcal{C}^2$-norm of X, the $\mathcal{C}^1$-norm of h, dist$(\Omega, \partial D)$, and $\mathbf{u}_1$ is the unique solution to

$$\begin{cases} \mathcal{L}_{\lambda_0,\mu_0}\mathbf{u}_1 = 0 & \textit{in } \mathbb{R}^2\backslash\overline{D}, \\ \mathcal{L}_{\lambda_1,\mu_1}\mathbf{u}_1 = 0 & \textit{in } D, \\ \mathbf{u}_1|_- - \mathbf{u}_1|_+ = h(\mathbb{K}_{0,1}\widehat{\nabla}\mathbf{u}^i)\mathbf{n} & \textit{on } \partial D, \\ \dfrac{\partial \mathbf{u}_1}{\partial \widetilde{\nu}}\Big|_- - \dfrac{\partial \mathbf{u}_1}{\partial \nu}\Big|_+ = \dfrac{\partial}{\partial \boldsymbol{\tau}}\Big(h\big([\mathbb{C}_1 - \mathbb{M}_{0,1}]\widehat{\nabla}\mathbf{u}^i\big)\boldsymbol{\tau}\Big) & \textit{on } \partial D, \\ \mathbf{u}_1(x) = O(|x|^{-1}) & \textit{as } |x| \to \infty, \end{cases} \tag{1.9}$$

with $\boldsymbol{\tau}$ is the tangential vector to ∂D,

$$\mathbb{M}_{0,1} := \frac{\lambda_0(\lambda_1 + 2\mu_1)}{\lambda_0 + 2\mu_0}\mathbf{I}\otimes\mathbf{I} + 2\mu_1\mathbb{I} + \frac{4(\mu_0 - \mu_1)(\lambda_0 + \mu_0)}{\lambda_0 + 2\mu_0}\mathbf{I}\otimes(\boldsymbol{\tau}\otimes\boldsymbol{\tau}), \tag{1.10}$$

$$\begin{aligned} \mathbb{K}_{0,1} := & \frac{\mu_0(\lambda_1 - \lambda_0) + 2(\mu_0 - \mu_1)(\lambda_0 + \mu_0)}{\mu_0(\lambda_0 + 2\mu_0)}\mathbf{I}\otimes\mathbf{I} + 2\big(\frac{\mu_1}{\mu_0} - 1\big)\mathbb{I} \\ & + \frac{2(\mu_1 - \mu_0)(\lambda_0 + \mu_0)}{\mu_0(\lambda_0 + 2\mu_0)}\mathbf{I}\otimes(\boldsymbol{\tau}\otimes\boldsymbol{\tau}). \end{aligned} \tag{1.11}$$

Our asymptotic expansion is also valid in the case of an elastic inclusion with high contrast parameters, for more details on the behavior of the leading and first order terms $\mathbf{u}$

and $\mathbf{u}_1$ in the asymptotic expansion of the displacement field $\mathbf{u}_\epsilon$, we refer the reader to [2, Chapter 2].

We should notice that similar asymptotic results have been obtained in the context of interface problems in elastostatics [6, 19, 20, 24], the authors derive asymptotic expansions for boundary displacement field in both cases of isotropic and anisotropic thin elastic inclusions and perturbations in the eigenvalues and elastic moments tensors (EMTs) caused by small perturbations of the shape of an elastic inclusion, the approach they use, based on energy estimates, variational approach, and fine regularity estimates for solutions of elliptic systems with discontinuous coefficients obtained by Li and Nirenberg [23]. Unfortunately, this method does not seem to work in our case.

As a consequence of the results of Theorem 1.1, we obtain the following relationship between traction-displacement measurements and the deformation h. The scalar product in $\mathbb{R}^2$, will be denoted by the dot, and sometimes to ease the notation, by $\langle , \rangle$.

Theorem 1.2 *Let S be a Lipschitz closed curve enclosing D, and away from ∂D. Let $\mathbf{u}$ and $\mathbf{u}_\epsilon$ be the solutions to (1.5) and (1.7), respectively, and $\mathbf{v}$ be the solution of the following system:*

$$\begin{cases} \mathcal{L}_{\lambda_0,\mu_0}\mathbf{v} = 0 & \text{in } \mathbb{R}^2\backslash\overline{D}, \\ \mathcal{L}_{\lambda_1,\mu_1}\mathbf{v} = 0 & \text{in } D, \\ \mathbf{v}|_- = \mathbf{v}|_+ & \text{on } \partial D, \\ \dfrac{\partial \mathbf{v}}{\partial \widetilde{\nu}}\Big|_- = \dfrac{\partial \mathbf{v}}{\partial \nu}\Big|_+ & \text{on } \partial D, \\ \mathbf{v}(x) - \mathbf{F}(x) = O(|x|^{-1}) & \text{as } |x| \to \infty. \end{cases} \tag{1.12}$$

Then, the following asymptotic expansion holds:

$$\int_S \big(\mathbf{u}_\epsilon - \mathbf{u}\big)\cdot \frac{\partial \mathbf{F}}{\partial \nu} d\sigma - \int_S \big(\frac{\partial \mathbf{u}_\epsilon}{\partial \nu} - \frac{\partial \mathbf{u}}{\partial \nu}\big)\cdot \mathbf{F} d\sigma$$
$$= \epsilon \int_{\partial D} h\bigg(\big(\big[\mathbb{M}_{0,1} - \mathbb{C}_1\big]\widehat{\nabla}\mathbf{u}^i\big)\boldsymbol{\tau}\cdot\widehat{\nabla}\mathbf{v}^i\boldsymbol{\tau} - \big(\mathbb{K}_{0,1}\widehat{\nabla}\mathbf{u}^i\big)\mathbf{n}\cdot(\mathbb{C}_1\widehat{\nabla}\mathbf{v}^i)\mathbf{n}\bigg)d\sigma + O(\epsilon^2), \tag{1.13}$$

where the remainder $O(\epsilon^2)$ depends only on $\lambda_0, \lambda_1, \mu_0, \mu_1$, the $\mathcal{C}^2$-norm of X, the $\mathcal{C}^1$-norm of h, and $dist(S, \partial D)$.

The asymptotic expansion in (1.13) can be used to design new algorithms in the identification of the shape of an elastic inclusion based on traction-displacement measurements (see for instance [1, 3, 6, 7, 17, 21, 24]).

The concept of EMTs has been studied particularly in the context of imaging of small elastic inclusions [4, 8]. Recall that EMTs $M^j_{\alpha\beta} := (m^j_{\alpha\beta 1}, m^j_{\alpha\beta 2})$ for $\alpha, \beta \in \mathbb{N}^2$ and $j = 1, 2$, associated to the inclusion D with Lamé constants (λ_1, μ_1), and the background medium with Lamé constants (λ_0, μ_0) can be described in the following manner: consider $\mathbf{H}$ to be a vector-valued function satisfying $\mathcal{L}_{\lambda_0,\mu_0}\mathbf{H} = 0$ in $\mathbb{R}^2$. Then, the displacement field $\mathbf{u}$ solution to (1.2), resulting from the perturbation of $\mathbf{H}$ due to the presence of D, has the following expansion [5, Theorem 10.2]

$$\mathbf{u}(x) = \mathbf{H}(x) + \sum_{j=1}^{2} \sum_{|\alpha|\geq 1} \sum_{|\beta|\geq 1} \frac{1}{\alpha!\beta!} \partial^\alpha \mathbf{H}_j(0) \partial^\beta \mathbf{\Gamma}(x) M^j_{\alpha\beta} \quad \forall x \text{ with } |x| > R, \tag{1.14}$$

where $D \subset B_R(0)$ and $\mathbf{\Gamma}$ is the fundamental solution to $\mathcal{L}_{\lambda_0,\mu_0}$. An alternative definition of EMTs will be given in Section 6.

The asymptotic expansion of the EMTs has been first obtained in [22, Theorem 3.1] with a remainder of the order of $O(\epsilon^{1+\gamma})$ with $0 < \gamma < 1$. The authors have used an approach based on that method proposed in [1]. In this paper we give an alternative method to prove the asymptotic behavior of EMTs resulting from small perturbations of the shape of an elastic inclusion with $\mathcal{C}^2$-boundary. Its main particularity is the fact that it is based on integral equations and layer potentials rather than variational techniques, avoiding the use (and the adaptation to our context) of the nontrivial regularity results of Li and Nirenberg [23]. Our approach gives a better estimate of the remainder (of order $O(\epsilon^2)$).

Theorem 1.3 *Let* (a^α_j) *and* (b^β_k) *be fixed constants such that* $\mathbf{H}(x) = \sum_{j=1}^{2} \sum_{\alpha\in\mathbb{N}^2} a^\alpha_j x^\alpha e_j$ *and* $\mathbf{F}(x) = \sum_{k=1}^{2} \sum_{\beta\in\mathbb{N}^2} b^\beta_k x^\beta e_k$ *are satisfy* $\nabla \cdot \big(\mathbb{C}_0 \widehat{\nabla} \cdot\big) = 0$ *in* $\mathbb{R}^2$. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be the solutions to* (1.5) *and* (1.12), *respectively. Then, the following asymptotic expansion holds:*

$$\begin{aligned}&\sum_{\alpha\beta jk} a^\alpha_j b^\beta_k m^j_{\alpha\beta k}(D_\epsilon) - \sum_{\alpha\beta jk} a^\alpha_j b^\beta_k m^j_{\alpha\beta k}(D) \\ &\qquad = \epsilon \int_{\partial D} h \bigg(\big([\mathbb{C}_1 - \mathbb{M}_{0,1}] \widehat{\nabla} \mathbf{u}^i\big) \boldsymbol{\tau} \cdot \widehat{\nabla} \mathbf{v}^i \boldsymbol{\tau} + \big(\mathbb{K}_{0,1} \widehat{\nabla} \mathbf{u}^i\big) \mathbf{n} \cdot (\mathbb{C}_1 \widehat{\nabla} \mathbf{v}^i) \mathbf{n} \bigg) d\sigma + O(\epsilon^2),\end{aligned} \tag{1.15}$$

where the remainder $O(\epsilon^2)$ *depends only on* $\lambda_0, \lambda_1, \mu_0, \mu_1$, *the* $\mathcal{C}^2$*-norm of* X, *and the* $\mathcal{C}^1$*-norm of* h.

Based on the asymptotic expansion in (1.15), we can conceive numerical algorithms in the spirit of [22] to recover fine shape details from the higher order EMTs.

The approach and techniques developed in this paper can be generalized to higher dimension interface problems and extended to other PDE systems, such as, Stokes and Maxwell.

This paper is organized as follows. In Section 2, we review some preliminary results related to small perturbations of a $\mathcal{C}^2$-interface, differentiation of tensors, and introduce a representation of the Lamé system in local coordinates. In Section 3, we formally derive the asymptotic expansion of the displacement by using the field expansion method (Theorem 1.1). In Section 4, we derive the asymptotic expansions of layer potentials. In Section 5, based on layer potentials techniques, we first justify the formal expansions, and then find the relationship between traction-displacement measurements and the deformation h (Theorem 1.1 & Theorem 1.2). In Section 6, we rigorously derive the asymptotic formula for the perturbation of the EMTs (Theorem 1.3). Finally, in the appendix, we provide some useful integral representations of quantities related to layer potentials.

2 Definitions and preliminary results

2.1 Small perturbation of a $\mathcal{C}^2$-interface

Let $a, b \in \mathbb{R}$, with $a < b$, and let $X(t) : [a, b] \to \mathbb{R}^2$ be the arclength parametrization of ∂D, namely, X is a $\mathcal{C}^2$-function satisfying $|X'(t)| = 1$ for all $t \in [a, b]$, $X(a) = X(b)$, and

$$\partial D := \{x = X(t), t \in [a, b]\}.$$

We assume that X is a positive arclength, *i.e.*, it rotates in the anticlockwise direction. Then the outward unit normal at $x \in \partial D$, $\mathbf{n}(x)$, is given by $\mathbf{n}(x) = R_{-\frac{\pi}{2}} X'(t)$, where $R_{-\frac{\pi}{2}}$ is the rotation by $-\pi/2$, the tangential vector at x, $\boldsymbol{\tau}(x) = X'(t)$, and $X'(t) \perp X''(t)$. Set the curvature $\kappa(x)$ to be defined by

$$X''(t) = \kappa(x)\mathbf{n}(x).$$

We will sometimes use $h(t)$ for $h(X(t))$ and $h'(t)$ for the tangential derivative of $h(x)$.

Then, $\tilde{x} = \tilde{X}(t) = X(t) + \epsilon h(t)\mathbf{n}(x) = X(t) + \epsilon h(t) R_{-\frac{\pi}{2}} X'(t)$ is a parametrization of ∂D_ϵ. By $\mathbf{n}(\tilde{x})$ we denote the outward unit normal to ∂D_ϵ at $\tilde{x}$. It is proved in [6] that

$$\mathbf{n}(\tilde{x}) = \frac{R_{-\frac{\pi}{2}} \tilde{X}'(t)}{|\tilde{X}'(t)|} = \frac{\Big(1 - \epsilon h(t)\kappa(x)\Big)\mathbf{n}(x) - \epsilon h'(t) X'(t)}{\sqrt{\Big(1 - \epsilon h(t)\kappa(x)\Big)^2 + \epsilon^2 h'(t)^2}} := \frac{\boldsymbol{\eta}(x)}{|\boldsymbol{\eta}(x)|}, \tag{2.1}$$

and hence $\mathbf{n}(\tilde{x})$ can be expanded uniformly as

$$\mathbf{n}(\tilde{x}) = \sum_{m=0}^{\infty} \epsilon^m \mathbf{n}_m(x), \quad x \in \partial D,$$

where the vector-valued functions $\mathbf{n}_m$ are uniformly bounded regardless of m. In particular,

$$\mathbf{n}_0(x) = \mathbf{n}(x), \quad \mathbf{n}_1(x) = -h'(t)\boldsymbol{\tau}(x), \quad x \in \partial D. \tag{2.2}$$

Likewise, denote by $d\sigma_\epsilon(\tilde{x})$ the length element to ∂D_ϵ at $\tilde{x}$ which has an uniformly expansion [6]

$$d\sigma_\epsilon(\tilde{x}) = |\tilde{X}'(t)|dt = \sqrt{(1 - \epsilon\kappa(t)h(t))^2 + \epsilon^2 h'^2(t)}dt = \sum_{m=0}^{\infty} \epsilon^m \sigma_m(x) d\sigma(x), \quad x \in \partial D, \tag{2.3}$$

where σ_m are functions bounded regardless of m, with

$$\sigma_0(x) = 1, \quad \sigma_1(x) = -\kappa(x)h(x), \quad x \in \partial D. \tag{2.4}$$

2.2 Differentiation of tensors

In this subsection, we will use the Einstein convention for the summation notation. Let $(\mathbf{e}_1, \mathbf{e}_2)$ be an orthonormal base of $\mathbb{R}^2$. Let ϕ be a differentiable scalar function. Then

$$\nabla\phi = \frac{\partial\phi}{\partial x_i}\mathbf{e}_i. \tag{2.5}$$

Let $\mathbf{u} = \mathbf{u}_i \mathbf{e}_i$ be a differentiable vector-valued function. Then

$$\nabla \mathbf{u} = \frac{\partial \mathbf{u}}{\partial x_j} \otimes \mathbf{e}_j = \frac{\partial (\mathbf{u}_i \mathbf{e}_i)}{\partial x_j} \otimes \mathbf{e}_j = \frac{\partial \mathbf{u}_i}{\partial x_j} \mathbf{e}_i \otimes \mathbf{e}_j. \tag{2.6}$$

Let $\mathbf{M} = \mathbf{M}_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$ be a differentiable matrix-valued function. Then

$$\nabla \cdot \mathbf{M} = \frac{\partial \mathbf{M}}{\partial x_k} \mathbf{e}_k = \frac{\partial (\mathbf{M}_{ij} \mathbf{e}_i \otimes \mathbf{e}_j)}{\partial x_k} \mathbf{e}_k = \frac{\partial \mathbf{M}_{ij}}{\partial x_k} (\mathbf{e}_i \otimes \mathbf{e}_j) \mathbf{e}_k = \frac{\partial \mathbf{M}_{ij}}{\partial x_k} \mathbf{e}_i (\mathbf{e}_j \cdot \mathbf{e}_k) = \frac{\partial \mathbf{M}_{ij}}{\partial x_j} \mathbf{e}_i. \tag{2.7}$$

Also, we have

$$\nabla \mathbf{M} = \frac{\partial \mathbf{M}}{\partial x_k} \otimes \mathbf{e}_k = \frac{\partial (\mathbf{M}_{ij} \mathbf{e}_i \otimes \mathbf{e}_j)}{\partial x_k} \otimes \mathbf{e}_k = \frac{\partial \mathbf{M}_{ij}}{\partial x_k} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k. \tag{2.8}$$

By $tr(\mathbf{A})$ we mean the trace of the matrix $\mathbf{A}$. Let $\mathbf{v}$ be a differentiable vector-valued function. We have the following properties

$$\nabla(\phi\ \mathbf{u}) = \phi \nabla \mathbf{u} + \mathbf{u} \otimes \nabla \phi, \tag{2.9}$$

$$\nabla(\phi\ \mathbf{M}) = \phi \nabla \mathbf{M} + \mathbf{M} \otimes \nabla \phi, \tag{2.10}$$

$$\nabla(\mathbf{u} \cdot \mathbf{v}) = (\nabla \mathbf{u})^T \mathbf{v} + (\nabla \mathbf{v})^T \mathbf{u}, \tag{2.11}$$

$$\nabla \cdot (\mathbf{u} \otimes \mathbf{v}) = \nabla \mathbf{u}\ \mathbf{v} + \nabla \cdot \mathbf{v}\ \mathbf{u}, \tag{2.12}$$

$$\nabla \cdot (\phi\ \mathbf{M}) = \mathbf{M} \nabla \phi + \phi \nabla \cdot \mathbf{M}, \tag{2.13}$$

$$\nabla \cdot (\mathbf{M}\ \mathbf{u}) = \mathbf{u} \cdot \nabla \cdot (\mathbf{M}^T) + tr(\mathbf{M} \nabla \mathbf{u}), \tag{2.14}$$

2.3 Lamé system in local coordinates

We begin with a review of some basic properties of tensor products. Let $\mathbf{A}$ and $\mathbf{B}$ be two matrices, and let $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ be 3 vectors. We have

$$(\mathbf{u} \otimes \mathbf{v}) \mathbf{w} = (\mathbf{v} \cdot \mathbf{w}) \mathbf{u}, \tag{2.15}$$

$$(\mathbf{u} \otimes \mathbf{v})^T = \mathbf{v} \otimes \mathbf{u}, \tag{2.16}$$

$$(\mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w})^T = \mathbf{v} \otimes \mathbf{w} \otimes \mathbf{u}. \tag{2.17}$$

$$\big(\mathbf{A} \otimes (\mathbf{u} \otimes \mathbf{u})\big) \mathbf{B} = \big((\mathbf{u} \otimes \mathbf{u}) : \mathbf{B}\big) \mathbf{A} = \langle \mathbf{B} \mathbf{u}, \mathbf{u} \rangle \mathbf{A}. \tag{2.18}$$

Let $\mathbf{w}$ be a twice differentiable vector-valued function on ∂D and $(\mathbf{n}, \boldsymbol{\tau})$ be the orthonormal base at each point $x \in \partial D$. Then, the gradient of $\mathbf{w}$ in local coordinates is given by

$$\nabla \mathbf{w} = \frac{\partial \mathbf{w}}{\partial \mathbf{n}} \otimes \mathbf{n} + \frac{\partial \mathbf{w}}{\partial \boldsymbol{\tau}} \otimes \boldsymbol{\tau}. \tag{2.19}$$

We obtain from (2.8) and (2.19) that

$$\nabla \nabla \mathbf{w} = \frac{\partial^2 \mathbf{w}}{\partial \mathbf{n}^2} \otimes \mathbf{n} \otimes \mathbf{n} + \frac{\partial^2 \mathbf{w}}{\partial \mathbf{n} \partial \boldsymbol{\tau}} \otimes \boldsymbol{\tau} \otimes \mathbf{n} + \frac{\partial^2 \mathbf{w}}{\partial \boldsymbol{\tau} \partial \mathbf{n}} \otimes \mathbf{n} \otimes \boldsymbol{\tau} + \frac{\partial^2 \mathbf{w}}{\partial \boldsymbol{\tau}^2} \otimes \boldsymbol{\tau} \otimes \boldsymbol{\tau}.$$

Taking the divergence of (2.19), we get from (2.7) that

$$\Delta \mathbf{w} = \nabla \cdot \nabla \mathbf{w} = \frac{\partial^2 \mathbf{w}}{\partial \mathbf{n}^2} + \frac{\partial^2 \mathbf{w}}{\partial \boldsymbol{\tau}^2} = \nabla\nabla \mathbf{w}\, \mathbf{n}\, \mathbf{n} + \nabla\nabla \mathbf{w}\, \boldsymbol{\tau}\, \boldsymbol{\tau}. \tag{2.20}$$

By using (2.5), we find

$$\begin{aligned}\nabla\nabla \cdot \mathbf{w} =& \nabla\Big(\frac{\partial \langle \mathbf{w}, \mathbf{n}\rangle}{\partial \mathbf{n}} + \frac{\partial \langle \mathbf{w}, \boldsymbol{\tau}\rangle}{\partial \boldsymbol{\tau}}\Big) \\ =& \frac{\partial^2 \langle \mathbf{w}, \mathbf{n}\rangle}{\partial \mathbf{n}^2}\mathbf{n} + \frac{\partial^2 \langle \mathbf{w}, \boldsymbol{\tau}\rangle}{\partial \mathbf{n}\partial \boldsymbol{\tau}}\mathbf{n} + \frac{\partial^2 \langle \mathbf{w}, \mathbf{n}\rangle}{\partial \boldsymbol{\tau}\partial \mathbf{n}}\boldsymbol{\tau} + \frac{\partial^2 \langle \mathbf{w}, \boldsymbol{\tau}\rangle}{\partial \boldsymbol{\tau}^2}\boldsymbol{\tau}.\end{aligned} \tag{2.21}$$

From(2.16) and (2.19), we deduce that

$$\begin{aligned}(\nabla \mathbf{w})^T &= \mathbf{n} \otimes \frac{\partial \mathbf{w}}{\partial \mathbf{n}} + \boldsymbol{\tau} \otimes \frac{\partial \mathbf{w}}{\partial \boldsymbol{\tau}} \\ &= \frac{\partial \langle \mathbf{w}, \mathbf{n}\rangle}{\partial \mathbf{n}}\mathbf{n} \otimes \mathbf{n} + \frac{\partial \langle \mathbf{w}, \boldsymbol{\tau}\rangle}{\partial \mathbf{n}}\mathbf{n} \otimes \boldsymbol{\tau} + \frac{\partial \langle \mathbf{w}, \mathbf{n}\rangle}{\partial \boldsymbol{\tau}}\boldsymbol{\tau} \otimes \mathbf{n} + \frac{\partial \langle \mathbf{w}, \boldsymbol{\tau}\rangle}{\partial \boldsymbol{\tau}}\boldsymbol{\tau} \otimes \boldsymbol{\tau},\end{aligned} \tag{2.22}$$

and then it follows from (2.7) and (2.15) that

$$\nabla \cdot (\nabla \mathbf{w})^T = \frac{\partial^2 \langle \mathbf{w}, \mathbf{n}\rangle}{\partial \mathbf{n}^2}\mathbf{n} + \frac{\partial^2 \langle \mathbf{w}, \boldsymbol{\tau}\rangle}{\partial \boldsymbol{\tau}\partial \mathbf{n}}\mathbf{n} + \frac{\partial^2 \langle \mathbf{w}, \mathbf{n}\rangle}{\partial \mathbf{n}\partial \boldsymbol{\tau}}\boldsymbol{\tau} + \frac{\partial^2 \langle \mathbf{w}, \boldsymbol{\tau}\rangle}{\partial \boldsymbol{\tau}^2}\boldsymbol{\tau}. \tag{2.23}$$

It is known that $\nabla \cdot (\nabla \mathbf{w})^T = \nabla\nabla \cdot \mathbf{w}$, which combined with (2.21), and (2.23) imply

$$\frac{\partial^2 \langle \mathbf{w}, \boldsymbol{\tau}\rangle}{\partial \boldsymbol{\tau}\partial \mathbf{n}} = \frac{\partial^2 \langle \mathbf{w}, \boldsymbol{\tau}\rangle}{\partial \mathbf{n}\partial \boldsymbol{\tau}}, \qquad \frac{\partial^2 \langle \mathbf{w}, \mathbf{n}\rangle}{\partial \mathbf{n}\partial \boldsymbol{\tau}} = \frac{\partial^2 \langle \mathbf{w}, \mathbf{n}\rangle}{\partial \boldsymbol{\tau}\partial \mathbf{n}}. \tag{2.24}$$

Using (2.8) and (2.22), we get

$$\nabla(\nabla \mathbf{w})^T = \mathbf{n} \otimes \frac{\partial^2 \mathbf{w}}{\partial \mathbf{n}^2} \otimes \mathbf{n} + \boldsymbol{\tau} \otimes \frac{\partial^2 \mathbf{w}}{\partial \mathbf{n}\partial \boldsymbol{\tau}} \otimes \mathbf{n} + \mathbf{n} \otimes \frac{\partial^2 \mathbf{w}}{\partial \boldsymbol{\tau}\partial \mathbf{n}} \otimes \boldsymbol{\tau} + \boldsymbol{\tau} \otimes \frac{\partial^2 \mathbf{w}}{\partial \boldsymbol{\tau}^2} \otimes \boldsymbol{\tau},$$

which gives

$$\nabla(\nabla \mathbf{w})^T\, \mathbf{n}\, \mathbf{n} = \frac{\partial^2 \langle \mathbf{w}, \mathbf{n}\rangle}{\partial \mathbf{n}^2}\mathbf{n} + \frac{\partial^2 \langle \mathbf{w}, \mathbf{n}\rangle}{\partial \mathbf{n}\partial \boldsymbol{\tau}}\boldsymbol{\tau}, \qquad \nabla(\nabla \mathbf{w})^T\, \boldsymbol{\tau}\, \boldsymbol{\tau} = \frac{\partial^2 \langle \mathbf{w}, \boldsymbol{\tau}\rangle}{\partial \boldsymbol{\tau}\partial \mathbf{n}}\mathbf{n} + \frac{\partial^2 \langle \mathbf{w}, \boldsymbol{\tau}\rangle}{\partial \boldsymbol{\tau}^2}\boldsymbol{\tau}. \tag{2.25}$$

Therefore, by (2.21), (2.24), and (2.25), we obtain

$$\nabla\nabla \cdot \mathbf{w} = \langle \nabla\nabla \cdot \mathbf{w}, \mathbf{n}\rangle \mathbf{n} + \langle \nabla\nabla \cdot \mathbf{w}, \boldsymbol{\tau}\rangle \boldsymbol{\tau} = \nabla(\nabla \mathbf{w})^T\, \mathbf{n}\, \mathbf{n} + \nabla(\nabla \mathbf{w})^T\, \boldsymbol{\tau}\, \boldsymbol{\tau}. \tag{2.26}$$

Let $\boldsymbol{\phi}(x)$ and $\phi(x)$ be respectively a vector and a scalar functions, which belong to $\mathcal{C}^1([a,b])$ for $x = X(\cdot) \in \partial D$. By d/dt, we denote the tangential derivative in the direction of $\boldsymbol{\tau}(x) = X'(t)$. We have

$$\frac{d}{dt}\big(\boldsymbol{\phi}(x)\big) = \nabla \boldsymbol{\phi}(x) X'(t) = \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{\tau}}(x), \qquad \frac{d}{dt}\big(\phi(x)\big) = \nabla \phi(x) \cdot X'(t) = \frac{\partial \phi}{\partial \boldsymbol{\tau}}(x).$$

The following lemma holds.

Lemma 2.1 *The restriction of the Lamé system $\mathcal{L}_{\lambda_0,\mu_0}$ in D to a neighborhood of ∂D can be expressed as follows:*

$$\begin{aligned}\mathcal{L}_{\lambda_0,\mu_0}\boldsymbol{\phi}(x) =&\mu_0\frac{\partial^2\boldsymbol{\phi}}{\partial\mathbf{n}^2}(x)+\lambda_0\nabla\nabla\cdot\boldsymbol{\phi}(x)\cdot\mathbf{n}(x)\mathbf{n}(x)+\mu_0\nabla(\nabla\boldsymbol{\phi})^T(x)\mathbf{n}(x)\mathbf{n}(x)\\&-\kappa(x)\frac{\partial\boldsymbol{\phi}}{\partial\nu}(x)+\frac{d}{dt}\Big(\big(\mathbb{C}_0\widehat{\nabla}\boldsymbol{\phi}(x)\big)\boldsymbol{\tau}(x)\Big),\quad x\in\partial D.\end{aligned}\tag{2.27}$$

Proof. According to (2.20) and (2.26). For $x\in\partial D$, we have

$$\begin{aligned}\mathcal{L}_{\lambda_0,\mu_0}\boldsymbol{\phi}(x) =&\mu_0\Delta\boldsymbol{\phi}(x)+(\lambda_0+\mu_0)\nabla\nabla\cdot\boldsymbol{\phi}(x)\\=&\mu_0\nabla\nabla\boldsymbol{\phi}(x)\mathbf{n}(x)\mathbf{n}(x)+\lambda_0\nabla\nabla\cdot\boldsymbol{\phi}(x)\cdot\mathbf{n}(x)\mathbf{n}(x)+\mu_0\nabla(\nabla\boldsymbol{\phi})^T(x)\mathbf{n}(x)\mathbf{n}(x)\\&+\mu_0\nabla\nabla\boldsymbol{\phi}(x)\boldsymbol{\tau}(x)\boldsymbol{\tau}(x)+\lambda_0\nabla\nabla\cdot\boldsymbol{\phi}(x)\cdot\boldsymbol{\tau}(x)\boldsymbol{\tau}(x)+\mu_0\nabla(\nabla\boldsymbol{\phi})^T(x)\boldsymbol{\tau}(x)\boldsymbol{\tau}(x).\end{aligned}$$

Since

$$\begin{aligned}&\mu_0\nabla\nabla\boldsymbol{\phi}(x)\boldsymbol{\tau}(x)\boldsymbol{\tau}(x)+\lambda_0\nabla\nabla\cdot\boldsymbol{\phi}(x)\cdot\boldsymbol{\tau}(x)\boldsymbol{\tau}(x)+\mu_0\nabla(\nabla\boldsymbol{\phi})^T(x)\boldsymbol{\tau}(x)\boldsymbol{\tau}(x)\\&=\frac{d}{dt}\Big(\mu_0\nabla\boldsymbol{\phi}(x)+\lambda_0\nabla\cdot\boldsymbol{\phi}(x)+\mu_0(\nabla\boldsymbol{\phi})^T(x)\Big)\boldsymbol{\tau}(x)\\&=\frac{d}{dt}\Big(\mathbb{C}_0\widehat{\nabla}\boldsymbol{\phi}(x)\Big)\boldsymbol{\tau}(x)\\&=-\kappa(x)\big(\mathbb{C}_0\widehat{\nabla}\boldsymbol{\phi}(x)\big)\mathbf{n}(x)+\frac{d}{dt}\Big(\big(\mathbb{C}_0\widehat{\nabla}\boldsymbol{\phi}(x)\big)\boldsymbol{\tau}(x)\Big),\quad x\in\partial D,\end{aligned}$$

then (2.27) holds. This completes the proof.

3 Formal derivations: the FE method

The following observations are useful.

Proposition 3.1 *Let $\mathbf{u}$ be the solution to (1.2). Then the following identities hold:*

$$\big(\mathbb{C}_0\widehat{\nabla}\mathbf{u}^e\big)\boldsymbol{\tau}=\big(\mathbb{M}_{0,1}\widehat{\nabla}\mathbf{u}^i\big)\boldsymbol{\tau},\tag{3.1}$$

$$\big(\mathbb{C}_1\widehat{\nabla}\mathbf{u}^i\big)\boldsymbol{\tau}=\big(\mathbb{M}_{1,0}\widehat{\nabla}\mathbf{u}^e\big)\boldsymbol{\tau},\tag{3.2}$$

$$\nabla\mathbf{u}^e\mathbf{n}-\nabla\mathbf{u}^i\mathbf{n}=\big(\mathbb{K}_{0,1}\widehat{\nabla}\mathbf{u}^i\big)\mathbf{n}=-\big(\mathbb{K}_{1,0}\widehat{\nabla}\mathbf{u}^e\big)\mathbf{n},\tag{3.3}$$

where the 4-tensors $\mathbb{M}_{l,k}$ and $\mathbb{K}_{l,k}$ for $l,k=0,1$, are defined by:

$$\mathbb{M}_{l,k}:=\frac{\lambda_l(\lambda_k+2\mu_k)}{\lambda_l+2\mu_l}\mathbf{I}\otimes\mathbf{I}+2\mu_k\mathbb{I}+\frac{4(\mu_l-\mu_k)(\lambda_l+\mu_l)}{\lambda_l+2\mu_l}\mathbf{I}\otimes(\boldsymbol{\tau}\otimes\boldsymbol{\tau}),$$

$$\begin{aligned}\mathbb{K}_{l,k}:=&\frac{\mu_l(\lambda_k-\lambda_l)+2(\mu_l-\mu_k)(\lambda_l+\mu_l)}{\mu_l(\lambda_l+2\mu_l)}\mathbf{I}\otimes\mathbf{I}+2\big(\frac{\mu_k}{\mu_l}-1\big)\mathbb{I}\\&+\frac{2(\mu_k-\mu_l)(\lambda_l+\mu_l)}{\mu_l(\lambda_l+2\mu_l)}\mathbf{I}\otimes(\boldsymbol{\tau}\otimes\boldsymbol{\tau}).\end{aligned}$$

Proof. The solution $\mathbf{u}$ of (1.2) satisfies the following transmission conditions along the interface ∂D:

$$\mathbf{u}^i = \mathbf{u}^e, \tag{3.4}$$

$$\nabla \mathbf{u}^i \boldsymbol{\tau} = \nabla \mathbf{u}^e \boldsymbol{\tau}, \tag{3.5}$$

$$\langle \widehat{\nabla} \mathbf{u}^i \boldsymbol{\tau}, \boldsymbol{\tau} \rangle = \langle \widehat{\nabla} \mathbf{u}^e \boldsymbol{\tau}, \boldsymbol{\tau} \rangle, \tag{3.6}$$

$$\lambda_1 \nabla \cdot \mathbf{u}^i + 2\mu_1 \langle \widehat{\nabla} \mathbf{u}^i \mathbf{n}, \mathbf{n} \rangle = \lambda_0 \nabla \cdot \mathbf{u}^e + 2\mu_0 \langle \widehat{\nabla} \mathbf{u}^e \mathbf{n}, \mathbf{n} \rangle, \tag{3.7}$$

$$\mu_1 \langle \widehat{\nabla} \mathbf{u}^i \mathbf{n}, \boldsymbol{\tau} \rangle = \mu_0 \langle \widehat{\nabla} \mathbf{u}^e \mathbf{n}, \boldsymbol{\tau} \rangle. \tag{3.8}$$

Recalling that

$$\nabla \cdot \mathbf{u}^e = \widehat{\nabla} \mathbf{u}^e : \mathbf{I} = tr(\widehat{\nabla} \mathbf{u}^e) = \langle \widehat{\nabla} \mathbf{u}^e \mathbf{n}, \mathbf{n} \rangle + \langle \widehat{\nabla} \mathbf{u}^e \boldsymbol{\tau}, \boldsymbol{\tau} \rangle. \tag{3.9}$$

From (3.6), (3.7), and (3.9), one can easily see that

$$\nabla \cdot \mathbf{u}^e = \frac{\lambda_1 + 2\mu_1}{\lambda_0 + 2\mu_0} \nabla \cdot \mathbf{u}^i + \frac{2(\mu_0 - \mu_1)}{\lambda_0 + 2\mu_0} \langle \widehat{\nabla} \mathbf{u}^i \boldsymbol{\tau}, \boldsymbol{\tau} \rangle. \tag{3.10}$$

We have

$$\begin{aligned}
\nabla \mathbf{u}^e \mathbf{n} &= \langle \nabla \mathbf{u}^e \mathbf{n}, \mathbf{n} \rangle \mathbf{n} + \langle \nabla \mathbf{u}^e \mathbf{n}, \boldsymbol{\tau} \rangle \boldsymbol{\tau} \\
&= \langle \widehat{\nabla} \mathbf{u}^e \mathbf{n}, \mathbf{n} \rangle \mathbf{n} + 2 \langle \widehat{\nabla} \mathbf{u}^e \mathbf{n}, \boldsymbol{\tau} \rangle \boldsymbol{\tau} - \langle (\nabla \mathbf{u}^e)^T \mathbf{n}, \boldsymbol{\tau} \rangle \boldsymbol{\tau} \\
&= \langle \widehat{\nabla} \mathbf{u}^e \mathbf{n}, \mathbf{n} \rangle \mathbf{n} + 2 \langle \widehat{\nabla} \mathbf{u}^e \mathbf{n}, \boldsymbol{\tau} \rangle \boldsymbol{\tau} - \langle \nabla \mathbf{u}^e \boldsymbol{\tau}, \mathbf{n} \rangle \boldsymbol{\tau}.
\end{aligned}$$

Using (3.9), we obtain

$$\nabla \mathbf{u}^e \mathbf{n} = (\nabla \cdot \mathbf{u}^e) \mathbf{n} - \langle \widehat{\nabla} \mathbf{u}^e \boldsymbol{\tau}, \boldsymbol{\tau} \rangle \mathbf{n} + 2 \langle \widehat{\nabla} \mathbf{u}^e \mathbf{n}, \boldsymbol{\tau} \rangle \boldsymbol{\tau} - \langle \nabla \mathbf{u}^e \boldsymbol{\tau}, \mathbf{n} \rangle \boldsymbol{\tau}.$$

In a similar way, we write

$$\nabla \mathbf{u}^i \mathbf{n} = (\nabla \cdot \mathbf{u}^i) \mathbf{n} - \langle \widehat{\nabla} \mathbf{u}^i \boldsymbol{\tau}, \boldsymbol{\tau} \rangle \mathbf{n} + 2 \langle \widehat{\nabla} \mathbf{u}^i \mathbf{n}, \boldsymbol{\tau} \rangle \boldsymbol{\tau} - \langle \nabla \mathbf{u}^i \boldsymbol{\tau}, \mathbf{n} \rangle \boldsymbol{\tau}.$$

It then follows from (2.18), (3.5), (3.6), (3.8), (3.9), and (3.10), that

$$\begin{aligned}
\nabla \mathbf{u}^e \mathbf{n} - \nabla \mathbf{u}^i \mathbf{n} &= (\nabla \cdot \mathbf{u}^e - \nabla \cdot \mathbf{u}^i) \mathbf{n} + 2 \langle \widehat{\nabla} \mathbf{u}^e \mathbf{n}, \boldsymbol{\tau} \rangle \boldsymbol{\tau} - 2 \langle \widehat{\nabla} \mathbf{u}^i \mathbf{n}, \boldsymbol{\tau} \rangle \boldsymbol{\tau} \\
&= \Big(\frac{\lambda_1 + 2\mu_1}{\lambda_0 + 2\mu_0} - 1 \Big) (\nabla \cdot \mathbf{u}^i) \mathbf{n} + 2 \big(\frac{\mu_1}{\mu_0} - 1 \big) \langle \widehat{\nabla} \mathbf{u}^i \mathbf{n}, \boldsymbol{\tau} \rangle \boldsymbol{\tau} \\
&\quad + \frac{2(\mu_0 - \mu_1)}{\lambda_0 + 2\mu_0} \langle \widehat{\nabla} \mathbf{u}^i \boldsymbol{\tau}, \boldsymbol{\tau} \rangle \mathbf{n} \\
&= \frac{\mu_0(\lambda_1 - \lambda_0) + 2(\mu_0 - \mu_1)(\lambda_0 + \mu_0)}{\mu_0(\lambda_0 + 2\mu_0)} (\nabla \cdot \mathbf{u}^i) \mathbf{n} + 2 \big(\frac{\mu_1}{\mu_0} - 1 \big) \widehat{\nabla} \mathbf{u}^i \mathbf{n} \\
&\quad + \frac{2(\mu_1 - \mu_0)(\lambda_0 + \mu_0)}{\mu_0(\lambda_0 + 2\mu_0)} \langle \widehat{\nabla} \mathbf{u}^i \boldsymbol{\tau}, \boldsymbol{\tau} \rangle \mathbf{n} \\
&= \big(\mathbb{K}_{0,1} \widehat{\nabla} \mathbf{u}^i \big) \mathbf{n} \quad \text{on } \partial D.
\end{aligned}$$

We obtain from (2.18), (3.6), (3.7), (3.8), and (3.10), that

$$
\begin{aligned}
\big(\mathbb{C}_0\widehat{\nabla}\mathbf{u}^e\big)\boldsymbol{\tau} &= \lambda_0(\nabla\cdot\mathbf{u}^e)\boldsymbol{\tau} + 2\mu_0(\widehat{\nabla}\mathbf{u}^e)\boldsymbol{\tau} \\
&= \frac{\lambda_0(\lambda_1+2\mu_1)}{\lambda_0+2\mu_0}(\nabla\cdot\mathbf{u}^i)\boldsymbol{\tau} + \frac{2\lambda_0(\mu_0-\mu_1)}{\lambda_0+2\mu_0}\langle\widehat{\nabla}\mathbf{u}^i\boldsymbol{\tau},\boldsymbol{\tau}\rangle\boldsymbol{\tau} \\
&\quad + 2\mu_0\langle\widehat{\nabla}\mathbf{u}^i\boldsymbol{\tau},\boldsymbol{\tau}\rangle\boldsymbol{\tau} + 2\mu_1\langle\widehat{\nabla}\mathbf{u}^i\boldsymbol{\tau},\mathbf{n}\rangle\mathbf{n} \\
&= \frac{\lambda_0(\lambda_1+2\mu_1)}{\lambda_0+2\mu_0}(\nabla\cdot\mathbf{u}^i)\boldsymbol{\tau} + \frac{2\lambda_0(\mu_0-\mu_1)}{\lambda_0+2\mu_0}\langle\widehat{\nabla}\mathbf{u}^i\boldsymbol{\tau},\boldsymbol{\tau}\rangle\boldsymbol{\tau} \\
&\quad + 2\mu_1\widehat{\nabla}\mathbf{u}^i\boldsymbol{\tau} + 2\mu_0\langle\widehat{\nabla}\mathbf{u}^i\boldsymbol{\tau},\boldsymbol{\tau}\rangle\boldsymbol{\tau} - 2\mu_1\langle\widehat{\nabla}\mathbf{u}^i\boldsymbol{\tau},\boldsymbol{\tau}\rangle\boldsymbol{\tau} \\
&= \frac{\lambda_0(\lambda_1+2\mu_1)}{\lambda_0+2\mu_0}(\nabla\cdot\mathbf{u}^i)\boldsymbol{\tau} + 2\mu_1\widehat{\nabla}\mathbf{u}^i\boldsymbol{\tau} + \frac{4(\mu_0-\mu_1)(\lambda_0+\mu_0)}{\lambda_0+2\mu_0}\langle\widehat{\nabla}\mathbf{u}^i\boldsymbol{\tau},\boldsymbol{\tau}\rangle\boldsymbol{\tau} \\
&= \big(\mathbb{M}_{0,1}\widehat{\nabla}\mathbf{u}^i\big)\boldsymbol{\tau} \quad \text{on } \partial D.
\end{aligned}
$$

The identities $\big(\mathbb{C}_1\widehat{\nabla}\mathbf{u}^i\big)\boldsymbol{\tau} = \big(\mathbb{M}_{1,0}\widehat{\nabla}\mathbf{u}^e\big)\boldsymbol{\tau}$ and $\nabla\mathbf{u}^e\mathbf{n} - \nabla\mathbf{u}^i\mathbf{n} = -\big(\mathbb{K}_{1,0}\widehat{\nabla}\mathbf{u}^e\big)\mathbf{n}$ can be obtained in exactly the same manner as above. The proof of the proposition is then achieved.

We now derive, based on the FE method [11], formally the asymptotic expansion of $\mathbf{u}_\epsilon$, solution to (1.7), as ϵ goes to zero. We start by expanding $\mathbf{u}_\epsilon$ in powers of ϵ, *i.e.*

$$
\mathbf{u}_\epsilon(x) = \mathbf{u}_0(x) + \epsilon\mathbf{u}_1(x) + O(\epsilon^2), \quad x\in\Omega,
$$

where $\mathbf{u}_n, n=0,1$, are well defined in $\mathbb{R}^2\backslash\partial D$, and satisfy

$$
\begin{cases}
\mathcal{L}_{\lambda_0,\mu_0}\mathbf{u}_n = 0 & \text{in } \mathbb{R}^2\backslash\overline{D}, \\
\mathcal{L}_{\lambda_1,\mu_1}\mathbf{u}_n = 0 & \text{in } D, \\
\mathbf{u}_n(x) - \mathbf{H}(x)\delta_{0n} = O(|x|^{-1}) & \text{as } |x|\to\infty.
\end{cases}
$$

Here δ_{0n} is the Kronecker symbol.

Let $\tilde{x} = x + \epsilon h(x)\mathbf{n}(x) \in \partial D_\epsilon$ for $x\in\partial D$. The conormal derivative $\dfrac{\partial\mathbf{u}_\epsilon^e}{\partial\nu}(\tilde{x})$ on ∂D_ϵ is given by

$$
\frac{\partial\mathbf{u}_\epsilon^e}{\partial\nu}(\tilde{x}) = \lambda_0\nabla\cdot\mathbf{u}_\epsilon^e(\tilde{x})\mathbf{n}(\tilde{x}) + \mu_0\Big(\nabla\mathbf{u}_\epsilon^e(\tilde{x}) + (\nabla\mathbf{u}_\epsilon^e)^T(\tilde{x})\Big)\mathbf{n}(\tilde{x}), \tag{3.11}
$$

where $\mathbf{n}(\tilde{x})$ is the outward unit normal to ∂D_ϵ at $\tilde{x}$ defined by (2.1). By the Taylor expansion, we write

$$
\begin{aligned}
\nabla\cdot\mathbf{u}_\epsilon^e(\tilde{x}) &= \nabla\cdot\mathbf{u}_0^e\big(x+\epsilon h(x)\mathbf{n}(x)\big) + \epsilon\nabla\cdot\mathbf{u}_1^e\big(x+\epsilon h(x)\mathbf{n}(x)\big) + O(\epsilon^2) \\
&= \nabla\cdot\mathbf{u}_0^e(x) + \epsilon h(x)\nabla\nabla\cdot\mathbf{u}_0^e(x)\cdot\mathbf{n}(x) + \epsilon\nabla\cdot\mathbf{u}_1^e(x) + O(\epsilon^2), \quad x\in\partial D.
\end{aligned} \tag{3.12}
$$

In a similar way, we get

$$
\begin{aligned}
\nabla\mathbf{u}_\epsilon^e(\tilde{x}) + (\nabla\mathbf{u}_\epsilon^e)^T(\tilde{x}) =& \big[\nabla\mathbf{u}_0^e(x) + (\nabla\mathbf{u}_0^e)^T(x)\big] + \epsilon\big[\nabla\mathbf{u}_1^e(x) + (\nabla\mathbf{u}_1^e)^T(x)\big] \\
&+ \epsilon h(x)\big[\nabla\nabla\mathbf{u}_0^e(x)\mathbf{n}(x) + \nabla(\nabla\mathbf{u}_0^e)^T(x)\mathbf{n}(x)\big] + O(\epsilon^2), \quad x\in\partial D.
\end{aligned} \tag{3.13}
$$

It then follows from (2.2), (3.11), (3.12) and (3.13) that

$$
\begin{aligned}
\frac{\partial \mathbf{u}_\epsilon^e}{\partial \nu}(\tilde{x}) =& \frac{\partial \mathbf{u}_0^e}{\partial \nu}(x) + \epsilon \frac{\partial \mathbf{u}_1^e}{\partial \nu}(x) - \epsilon h'(t)\big(\mathbb{C}_0 \widehat{\nabla} \mathbf{u}_0^e(x)\big)\boldsymbol{\tau}(x) \\
&+ \epsilon h(x) \Bigg(\lambda_0 \nabla\nabla \cdot \mathbf{u}_0^e(x) \cdot \mathbf{n}(x)\mathbf{n}(x) + \mu_0 \nabla\nabla \mathbf{u}_0^e(x)\mathbf{n}(x)\mathbf{n}(x) \\
&+ \mu_0 \nabla(\nabla \mathbf{u}_0^e)^T(x)\mathbf{n}(x)\mathbf{n}(x) \Bigg) + O(\epsilon^2), \quad x \in \partial D. \qquad (3.14)
\end{aligned}
$$

Since $\mathbf{u}_0^e$ satisfies $\mathcal{L}_{\lambda_0,\mu_0}\mathbf{u}_0^e = 0$ in $\mathbb{R}^2 \backslash \overline{D}$, then, by (2.27), we obtain

$$
\mu_0[\nabla\nabla \mathbf{u}_0^e]\,\mathbf{n}\,\mathbf{n} + \lambda_0[\nabla\nabla \cdot \mathbf{u}_0^e] \cdot \mathbf{n}\,\mathbf{n} + \mu_0[\nabla(\nabla \mathbf{u}_0^e)^T]\,\mathbf{n}\,\mathbf{n} = \kappa \frac{\partial \mathbf{u}_0^e}{\partial \nu} - \frac{\partial}{\partial \boldsymbol{\tau}}\Big(\big(\mathbb{C}_0 \widehat{\nabla} \mathbf{u}_0^e\big)\boldsymbol{\tau} \Big) \quad \text{on } \partial D,
$$

and hence, we derive from (3.14) the following formal asymptotic expansion

$$
\begin{aligned}
\frac{\partial \mathbf{u}_\epsilon^e}{\partial \nu}(\tilde{x}) =& \frac{\partial \mathbf{u}_0^e}{\partial \nu}(x) + \epsilon \frac{\partial \mathbf{u}_1^e}{\partial \nu}(x) + \epsilon \kappa(x) h(x) \frac{\partial \mathbf{u}_0^e}{\partial \nu}(x) - \epsilon \frac{d}{dt}\Big(h(x)\big[\mathbb{C}_0 \widehat{\nabla} \mathbf{u}_0^e(x)\big]\boldsymbol{\tau}(x) \Big) \\
&+ O(\epsilon^2), \quad x \in \partial D. \qquad (3.15)
\end{aligned}
$$

Similarly to (3.15) , we have

$$
\begin{aligned}
\frac{\partial \mathbf{u}_\epsilon^i}{\partial \widetilde{\nu}}(\tilde{x}) =& \frac{\partial \mathbf{u}_0^i}{\partial \widetilde{\nu}}(x) + \epsilon \frac{\partial \mathbf{u}_1^i}{\partial \widetilde{\nu}}(x) + \epsilon \kappa(x) h(x) \frac{\partial \mathbf{u}_0^i}{\partial \widetilde{\nu}}(x) - \epsilon \frac{d}{dt}\Big(h(x)\big[\mathbb{C}_1 \widehat{\nabla} \mathbf{u}_0^i(x)\big]\boldsymbol{\tau}(x) \Big) \\
&+ O(\epsilon^2), \quad x \in \partial D. \qquad (3.16)
\end{aligned}
$$

By using $\dfrac{\partial \mathbf{u}_\epsilon^i}{\partial \widetilde{\nu}} = \dfrac{\partial \mathbf{u}_\epsilon^e}{\partial \nu}$ on ∂D_ϵ, we deduce from (3.15) and (3.16) that

$$
\begin{aligned}
&\frac{\partial \mathbf{u}_0^i}{\partial \widetilde{\nu}} = \frac{\partial \mathbf{u}_0^e}{\partial \nu} \quad \text{on } \partial D, \\
&\frac{\partial \mathbf{u}_1^i}{\partial \widetilde{\nu}} - \frac{\partial \mathbf{u}_1^e}{\partial \nu} = \frac{\partial}{\partial \boldsymbol{\tau}}\Big(h\big(\mathbb{C}_1 \widehat{\nabla} \mathbf{u}_0^i\big)\boldsymbol{\tau} \Big) - \frac{\partial}{\partial \boldsymbol{\tau}}\Big(h\big(\mathbb{C}_0 \widehat{\nabla} \mathbf{u}_0^e\big)\boldsymbol{\tau} \Big) \quad \text{on } \partial D. \qquad (3.17)
\end{aligned}
$$

For $\tilde{x} = x + \epsilon h(x)\mathbf{n}(x) \in \partial D_\epsilon$. We have the following Taylor expansion

$$
\begin{aligned}
\mathbf{u}_\epsilon^e(\tilde{x}) &= \mathbf{u}_0^e(\tilde{x}) + \epsilon \mathbf{u}_1^e(\tilde{x}) + O(\epsilon^2) \\
&= \mathbf{u}_0^e(x) + \epsilon h(x) \nabla \mathbf{u}_0^e(x)\mathbf{n}(x) + \epsilon \mathbf{u}_1^e(x) + O(\epsilon^2), \quad x \in \partial D.
\end{aligned}
$$

Likewise, we obtain

$$
\mathbf{u}_\epsilon^i(\tilde{x}) = \mathbf{u}_0^i(x) + \epsilon h(x) \nabla \mathbf{u}_0^i(x)\mathbf{n}(x) + \epsilon \mathbf{u}_1^i(x) + O(\epsilon^2), \quad x \in \partial D.
$$

The transmission condition $\mathbf{u}_\epsilon^i = \mathbf{u}_\epsilon^e$ on ∂D_ϵ, immediately yields

$$
\mathbf{u}_0^i = \mathbf{u}_0^e \quad \text{on } \partial D,
$$

and

$$\mathbf{u}_1^i - \mathbf{u}_1^e = h\big(\nabla \mathbf{u}_0^e \mathbf{n} - \nabla \mathbf{u}_0^i \mathbf{n}\big) \quad \text{on } \partial D. \tag{3.18}$$

Note that $\mathbf{u}_0 = \mathbf{u}$ which is the solution to (1.5). It then follows from (3.17), (3.18), and Lemma 3.1 that

$$\mathbf{u}_1^i - \mathbf{u}_1^e = h\big(\mathbb{K}_{0,1}\widehat{\nabla}\mathbf{u}^i\big)\mathbf{n} \quad \text{on } \partial D, \tag{3.19}$$

$$\frac{\partial \mathbf{u}_1^i}{\partial \widetilde{\nu}} - \frac{\partial \mathbf{u}_1^e}{\partial \nu} = \frac{\partial}{\partial \boldsymbol{\tau}}\Big(h\big([\mathbb{C}_1 - \mathbb{M}_{0,1}]\widehat{\nabla}\mathbf{u}^i\big)\boldsymbol{\tau}\Big) \quad \text{on } \partial D. \tag{3.20}$$

Thus we formally obtain Theorem 1.1, as desired. For a proof, see Subsection 5.2.

4 Asymptotic formulae of layer potentials

4.1 Layer potentials

Let us review some well-known properties of the layer potentials on a Lipschitz domain for the elastostatics.

Let

$$\Psi := \Big\{\boldsymbol{\psi} : \partial_i \boldsymbol{\psi}_j + \partial_j \boldsymbol{\psi}_i = 0, \quad 1 \le i, j \le 2\Big\}.$$

or equivalently,

$$\Psi = \operatorname{span}\left\{\theta_1(x) := \begin{bmatrix}1\\0\end{bmatrix}, \theta_2(x) := \begin{bmatrix}0\\1\end{bmatrix}, \theta_3(x) := \begin{bmatrix}x_2\\-x_1\end{bmatrix}\right\}.$$

Introduce the space

$$L^2_\Psi(\partial D) := \Big\{\mathbf{f} \in L^2(\partial D) : \int_{\partial D} \mathbf{f}\cdot\boldsymbol{\psi}\, d\sigma = 0 \text{ for all } \boldsymbol{\psi} \in \Psi\Big\}.$$

In particular, since Ψ contains constant functions, we get

$$\int_{\partial D} \mathbf{f} d\sigma = 0$$

for any $\mathbf{f} \in L^2_\Psi(\partial D)$. The following fact is useful later.

$$\text{If} \quad \mathbf{w} \in W^{1,\frac{3}{2}}(D) \quad \text{satisfies} \quad \mathcal{L}_{\lambda_0,\mu_0}\mathbf{w} = 0 \text{ in } D, \quad \text{then} \quad \frac{\partial \mathbf{w}}{\partial \nu}\Big|_{\partial D} \in L^2_\Psi(\partial D). \tag{4.1}$$

The Kelvin matrix of fundamental solution $\mathbf{\Gamma}$ for the Lamé system $\mathcal{L}_{\lambda_0,\mu_0}$ in $\mathbb{R}^2$, is known to be

$$\mathbf{\Gamma}(x) = \frac{A}{2\pi}\log|x|\mathbf{I} - \frac{B}{2\pi}\frac{x\otimes x}{|x|^2}, \quad x \neq 0, \tag{4.2}$$

where

$$A = \frac{1}{2}\Big(\frac{1}{\mu_0} + \frac{1}{2\mu_0+\lambda_0}\Big) \qquad \text{and} \qquad B = \frac{1}{2}\Big(\frac{1}{\mu_0} - \frac{1}{2\mu_0+\lambda_0}\Big).$$

The single and double layer potentials of the density function $\boldsymbol{\phi}$ on $L^2(\partial D)$ associated with the Lamé parameters (λ_0, μ_0) are defined by

$$\boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}](x) = \int_{\partial D} \mathbf{\Gamma}(x-y)\boldsymbol{\phi}(y)d\sigma(y), \quad x \in \mathbb{R}^2, \tag{4.3}$$

$$\begin{aligned}\boldsymbol{\mathcal{D}}_D[\boldsymbol{\phi}](x) = &\int_{\partial D} \Big(\lambda_0 \nabla_y \cdot \mathbf{\Gamma}(x-y) \otimes \mathbf{n}(y) \\ &\qquad + \mu_0\Big(\big[\nabla_y \mathbf{\Gamma}(x-y)\mathbf{n}(y)\big]^T + (\nabla_y \mathbf{\Gamma})^T(x-y)\mathbf{n}(y)\Big)\Big)\boldsymbol{\phi}(y)d\sigma(y) \\ := &\int_{\partial D} \mathbf{K}(x-y)\boldsymbol{\phi}(y)d\sigma(y), \quad x \in \mathbb{R}^2 \setminus \partial D.\end{aligned} \tag{4.4}$$

The followings are well-known properties of the single and double layer potentials due to Dahlberg, Keing, and Verchota [13]. Let D be a Lipschitz bounded domain in $\mathbb{R}^2$. Then, we have

$$\frac{\partial \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}]}{\partial \nu}\Big|_{\pm}(x) = \Big(\pm\frac{1}{2}\mathbf{I} + \boldsymbol{\mathcal{K}}_D^*\Big)[\boldsymbol{\phi}](x) \quad \text{a.e. } x \in \partial D, \tag{4.5}$$

$$\boldsymbol{\mathcal{D}}_D[\boldsymbol{\phi}]\big|_{\pm}(x) = \Big(\mp\frac{1}{2}\mathbf{I} + \boldsymbol{\mathcal{K}}_D\Big)[\boldsymbol{\phi}](x) \quad \text{a.e. } x \in \partial D, \tag{4.6}$$

where $\boldsymbol{\mathcal{K}}_D$ is defined by

$$\boldsymbol{\mathcal{K}}_D[\boldsymbol{\phi}](x) = p.v.\int_{\partial D} \mathbf{K}(x-y)\boldsymbol{\phi}(y)d\sigma(y) \quad \text{a.e. } x \in \partial D,$$

and $\boldsymbol{\mathcal{K}}_D^*$ is the adjoint operator of $\boldsymbol{\mathcal{K}}_D$, that is,

$$\begin{aligned}\boldsymbol{\mathcal{K}}_D^*[\boldsymbol{\phi}](x) =& p.v.\int_{\partial D} \mathbf{K}^T(x-y)\boldsymbol{\phi}(y)d\sigma(y) \\ =& p.v.\int_{\partial D} \Big(\lambda_0 \mathbf{n}(x) \otimes \nabla_x \cdot \mathbf{\Gamma}(x-y) \\ &+ \mu_0\Big(\nabla_x \mathbf{\Gamma}(x-y)\mathbf{n}(x) + \big[(\nabla_x \mathbf{\Gamma})^T(x-y)\mathbf{n}(x)\big]^T\Big)\Big)\boldsymbol{\phi}(y)d\sigma(y) \quad \text{a.e. } x \in \partial D,\end{aligned} \tag{4.7}$$

with

$$\begin{aligned}\mathbf{K}^T(x-y) =& \frac{1}{2\pi}\frac{(A-B)}{(A+B)}\frac{\langle x-y, \mathbf{n}(x)\rangle}{|x-y|^2}\mathbf{I} + \frac{1}{2\pi}\frac{(A-B)}{(A+B)}\frac{(x-y)\otimes\mathbf{n}(x) - \mathbf{n}(x)\otimes(x-y)}{|x-y|^2} \\ &+ \frac{2}{\pi}\frac{B}{(A+B)}\frac{\langle x-y, \mathbf{n}(x)\rangle}{|x-y|^2}\frac{(x-y)\otimes(x-y)}{|x-y|^2} \qquad \text{for } x, y \in \partial D, \quad x \neq y.\end{aligned} \tag{4.8}$$

Here $p.v.$ denotes the Cauchy principal value. The operators $\mathcal{K}_D$ and $\mathcal{K}_D^*$ are singular integral operators and bounded on $L^2(\partial D)$.

Even though the derivation of the kernel $\mathbf{K}^T(x-y)$ is easy, we give its proof for the reader's convenience. Denote by $\mathbf{x} := x - y$, one can easily see from (2.12) and (2.13) that

$$\nabla_x \cdot \mathbf{\Gamma}(\mathbf{x}) = \frac{A-B}{2\pi}\frac{\mathbf{x}}{|\mathbf{x}|^2},$$

and hence

$$\mathbf{n}(x) \otimes \nabla_x \cdot \mathbf{\Gamma}(\mathbf{x}) = \frac{(A-B)}{2\pi}\frac{\mathbf{n}(x)\otimes\mathbf{x}}{|\mathbf{x}|^2}. \tag{4.9}$$

It follows from (2.8) and (2.17) that

$$\begin{aligned}\nabla_x(\mathbf{x}\otimes\mathbf{x}) = \nabla_x(\mathbf{x}_i\mathbf{x}_j e_i\otimes e_j) = \frac{\partial(\mathbf{x}_i\mathbf{x}_j)}{\partial x_k} e_i\otimes e_j\otimes e_k &= (\mathbf{x}_j\delta_{ik} + \mathbf{x}_i\delta_{jk})e_i\otimes e_j\otimes e_k \\ &= \mathbf{x}_j e_k\otimes e_j\otimes e_k + \mathbf{x}_i e_i\otimes e_k\otimes e_k \\ &= (\mathbf{I}\otimes\mathbf{x})^T + (\mathbf{x}\otimes\mathbf{I}). \qquad (4.10)\end{aligned}$$

Here we used the Einstein convention for the summation notation.

From (2.9), (2.10), (2.11), and (4.10), we get

$$\nabla_x\mathbf{\Gamma}(\mathbf{x}) = \frac{A}{2\pi}\frac{\mathbf{I}\otimes\mathbf{x}}{|\mathbf{x}|^2} + \frac{B}{\pi}\frac{\mathbf{x}\otimes\mathbf{x}\otimes\mathbf{x}}{|\mathbf{x}|^4} - \frac{B}{2\pi}\frac{(\mathbf{I}\otimes\mathbf{x})^T + (\mathbf{x}\otimes\mathbf{I})}{|\mathbf{x}|^2}, \tag{4.11}$$

and thus

$$\nabla_x\mathbf{\Gamma}(\mathbf{x})\mathbf{n}(x) = \frac{A}{2\pi}\frac{\langle\mathbf{x},\mathbf{n}(x)\rangle}{|\mathbf{x}|^2}\mathbf{I} + \frac{B}{\pi}\frac{\langle\mathbf{x},\mathbf{n}(x)\rangle}{|\mathbf{x}|^4}(\mathbf{x}\otimes\mathbf{x}) - \frac{B}{2\pi}\frac{\mathbf{x}\otimes\mathbf{n}(x) + \mathbf{n}(x)\otimes\mathbf{x}}{|\mathbf{x}|^2}.$$

Using (2.17), the transpose of $\nabla_x\mathbf{\Gamma}(\mathbf{x})$ is given by

$$(\nabla_x\mathbf{\Gamma})^T(\mathbf{x}) = \frac{A}{2\pi}\frac{(\mathbf{I}\otimes\mathbf{x})^T}{|\mathbf{x}|^2} + \frac{B}{\pi}\frac{\mathbf{x}\otimes\mathbf{x}\otimes\mathbf{x}}{|\mathbf{x}|^4} - \frac{B}{2\pi}\frac{(\mathbf{x}\otimes\mathbf{I}) + (\mathbf{I}\otimes\mathbf{x})}{|\mathbf{x}|^2},$$

and hence we obtain

$$(\nabla_x\mathbf{\Gamma})^T(\mathbf{x})\mathbf{n}(x) = \frac{A}{2\pi}\frac{\mathbf{n}(x)\otimes\mathbf{x}}{|\mathbf{x}|^2} + \frac{B}{\pi}\frac{\langle\mathbf{x},\mathbf{n}(x)\rangle}{|\mathbf{x}|^4}\mathbf{x}\otimes\mathbf{x} - \frac{B}{2\pi}\frac{\mathbf{x}\otimes\mathbf{n}(x)}{|\mathbf{x}|^2} - \frac{B}{2\pi}\frac{\langle\mathbf{x},\mathbf{n}(x)\rangle}{|\mathbf{x}|^2}\mathbf{I}.$$

Therefore

$$\begin{aligned}\nabla_x\mathbf{\Gamma}(\mathbf{x})\mathbf{n}(x) + [(\nabla_x\mathbf{\Gamma})^T(\mathbf{x})\mathbf{n}(x)]^T =& \frac{(A-B)}{2\pi}\frac{\langle\mathbf{x},\mathbf{n}(x)\rangle}{|\mathbf{x}|^2}\mathbf{I} + \frac{(A-B)}{2\pi}\frac{\mathbf{x}\otimes\mathbf{n}(x)}{|\mathbf{x}|^2} \\ & - \frac{B}{\pi}\frac{\mathbf{n}(x)\otimes\mathbf{x}}{|\mathbf{x}|^2} + \frac{2B}{\pi}\frac{\langle\mathbf{x},\mathbf{n}(x)\rangle}{|\mathbf{x}|^4}(\mathbf{x}\otimes\mathbf{x}). \qquad (4.12)\end{aligned}$$

We finally get $\mathbf{K}^T(x-y)$ in (4.8) from (4.9) and (4.12), as desired.

Let $\boldsymbol{\mathcal{D}}_D^\sharp$ be the standard double layer potential which is defined for any $\boldsymbol{\phi} \in L^2(\partial D)$ by

$$\boldsymbol{\mathcal{D}}_D^\sharp[\boldsymbol{\phi}](x) = \int_{\partial D} \frac{\partial \boldsymbol{\Gamma}(x-y)}{\partial \mathbf{n}(y)} \boldsymbol{\phi}(y) d\sigma(y), \quad x \in \mathbb{R}^2 \backslash \partial D. \tag{4.13}$$

One can easily see that

$$\begin{aligned}
\frac{\partial \boldsymbol{\Gamma}(x-y)}{\partial \mathbf{n}(y)} \boldsymbol{\phi}(y) =& \left[-\frac{A}{2\pi} \frac{\langle \mathbf{x}, \mathbf{n}(y) \rangle}{|\mathbf{x}|^2} \mathbf{I} - \frac{B}{\pi} \frac{\langle \mathbf{x}, \mathbf{n}(y) \rangle}{|\mathbf{x}|^4} (\mathbf{x} \otimes \mathbf{x}) + \frac{B}{2\pi} \frac{\mathbf{x} \otimes \mathbf{n}(y) + \mathbf{n}(y) \otimes \mathbf{x}}{|\mathbf{x}|^2} \right] \boldsymbol{\phi}(y) \\
=& -\frac{A}{2\pi} \frac{\langle \mathbf{x}, \mathbf{n}(y) \rangle}{|\mathbf{x}|^2} \boldsymbol{\phi}(y) - \frac{B}{\pi} \frac{\langle \mathbf{x}, \mathbf{n}(y) \rangle \langle \mathbf{x}, \boldsymbol{\phi}(y) \rangle}{|\mathbf{x}|^4} \mathbf{x} \\
&+ \frac{B}{2\pi} \frac{\langle \mathbf{n}(y), \boldsymbol{\phi}(y) \rangle \mathbf{x} + \langle \mathbf{x}, \boldsymbol{\phi}(y) \rangle \mathbf{n}(y)}{|\mathbf{x}|^2} \\
:=& \Lambda_1(x,y) + \Lambda_2(x,y) + \Lambda_3(x,y) \quad \text{for } x \neq y.
\end{aligned} \tag{4.14}$$

For $i = 1, 2, 3$, it follows from (2.9)-(2.14) that

$$\begin{aligned}
\mathcal{L}_{\lambda_0,\mu_0} \big(\Lambda_i(\cdot, y) \big)(x) = C_i \Bigg(& \frac{\langle \boldsymbol{\phi}(y), \mathbf{n}(y) \rangle}{|\mathbf{x}|^4} \mathbf{x} + \frac{\langle \mathbf{x}, \mathbf{n}(y) \rangle}{|\mathbf{x}|^4} \boldsymbol{\phi}(y) + \frac{\langle \mathbf{x}, \boldsymbol{\phi}(y) \rangle}{|\mathbf{x}|^4} \mathbf{n}(y) \\
& - 4 \frac{\langle \mathbf{x}, \mathbf{n}(y) \rangle \langle \mathbf{x}, \boldsymbol{\phi}(y) \rangle}{|\mathbf{x}|^6} \mathbf{x} \Bigg) \quad \text{for } x \neq y,
\end{aligned}$$

with

$$C_1 = \frac{(\lambda_0 + \mu_0) A}{\pi}, \qquad C_2 = -\frac{2\mu_0 B}{\pi}, \qquad C_3 = -\frac{(\lambda_0 + \mu_0) B}{\pi}.$$

Since $C_1 + C_2 + C_3 = 0$, then $\boldsymbol{\mathcal{D}}_D^\sharp(\boldsymbol{\phi})$ satisfies

$$\mathcal{L}_{\lambda_0,\mu_0} \big(\boldsymbol{\mathcal{D}}_D^\sharp[\boldsymbol{\phi}] \big) = 0 \quad \text{in } \mathbb{R}^2 \backslash \partial D. \tag{4.15}$$

The following proposition holds.

Proposition 4.1 *Let D be a bounded Lipschitz domain in $\mathbb{R}^2$. For $\boldsymbol{\phi} \in L^2(\partial D)$, we have*

$$\boldsymbol{\mathcal{D}}_D^\sharp[\boldsymbol{\phi}]\big|_\pm(x) = \Big(\mp \frac{1}{2\mu_0} \mathbf{I} \pm B \ \mathbf{n} \otimes \mathbf{n} + \boldsymbol{\mathcal{K}}_D^\sharp \Big) \boldsymbol{\phi}(x) \quad a.e. \ x \in \partial D, \tag{4.16}$$

$$\frac{\partial \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}]}{\partial \mathbf{n}}\Big|_\pm(x) = \Big(\pm \frac{1}{2\mu_0} \mathbf{I} \mp B \ \mathbf{n} \otimes \mathbf{n} + \big(\boldsymbol{\mathcal{K}}_D^\sharp\big)^* \Big) \boldsymbol{\phi}(x) \quad a.e. \ x \in \partial D, \tag{4.17}$$

where $\boldsymbol{\mathcal{K}}_D^\sharp$ is defined by

$$\boldsymbol{\mathcal{K}}_D^\sharp[\boldsymbol{\phi}](x) = p.v. \int_{\partial D} \frac{\partial}{\partial \mathbf{n}(y)} \boldsymbol{\Gamma}(x-y) \boldsymbol{\phi}(y) d\sigma(y) \quad a.e. \ x \in \partial D,$$

and $\big(\boldsymbol{\mathcal{K}}_D^\sharp\big)^*$ is the adjoint operator of $\boldsymbol{\mathcal{K}}_D^\sharp$, that is,

$$\big(\boldsymbol{\mathcal{K}}_D^\sharp\big)^*[\boldsymbol{\phi}](x) = p.v. \int_{\partial D} \frac{\partial}{\partial \mathbf{n}(x)} \boldsymbol{\Gamma}(x-y) \boldsymbol{\phi}(y) d\sigma(y) \quad a.e. \ x \in \partial D. \tag{4.18}$$

The operators $\boldsymbol{\mathcal{K}}_D^\sharp$ and $(\boldsymbol{\mathcal{K}}_D^\sharp)^*$ are singular integral operators and bounded on $L^2(\partial D)$.

Proof. Standard arguments yield the trace formulas [15]

$$\partial_i \big(\boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}]\big)_j(x)\big|_\pm = \pm\Big\{\frac{1}{2\mu_0}\mathbf{n}_i(x)\boldsymbol{\phi}_j(x) - B\langle\boldsymbol{\phi},\mathbf{n}\rangle\mathbf{n}_i(x)\mathbf{n}_j(x)\Big\}$$
$$+ p.v.\int_{\partial D}\partial_i\boldsymbol{\Gamma}_{jk}(x-y)\boldsymbol{\phi}_k(y)d\sigma(y), \quad x\in\partial D, \tag{4.19}$$

namely,

$$\nabla\boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}](x)\big|_\pm = \pm\Big\{\frac{1}{2\mu_0}\boldsymbol{\phi}(x)\otimes\mathbf{n}(x) - B\langle\mathbf{n}(x),\boldsymbol{\phi}(x)\rangle\mathbf{n}(x)\otimes\mathbf{n}(x)\Big\}$$
$$+ p.v.\int_{\partial D}\nabla_x\big[\boldsymbol{\Gamma}(x-y)\boldsymbol{\phi}(y)\big]d\sigma(y), \quad x\in\partial D. \tag{4.20}$$

Clearly the jump relation of the normal derivative of the single layer potential in (4.17) follows from (2.15) and (4.20). The jump formula in (4.16) can be proved by using standard arguments from the proof of the theorem 3.28 in [16]. The operators $\boldsymbol{\mathcal{K}}_D^\sharp$ and $\big(\boldsymbol{\mathcal{K}}_D^\sharp\big)^*$ are bounded on $L^2(\partial D)$ by the theorem of Coifman-McIntosh-Meyer [12].

The operators $\boldsymbol{\mathcal{D}}_D^\sharp$ and $\partial\boldsymbol{\mathcal{S}}_D/\partial\mathbf{n}$ can be viewed as unfamiliar layer potentials for the system of elastostatics.

Note that we will drop the $p.v.$ in the below; this is because ∂D is $\mathcal{C}^2$ and throughout this paper we will denote by $\widetilde{\boldsymbol{\mathcal{S}}}_D, \widetilde{\boldsymbol{\mathcal{D}}}_D$, $\widetilde{\boldsymbol{\mathcal{K}}}_D^*$, $\widetilde{\boldsymbol{\mathcal{D}}}_D^\sharp$, and $(\widetilde{\boldsymbol{\mathcal{K}}}_D^\sharp)^*$ the layer potentials corresponding to the Lamé constants (λ_1,μ_1).

Let us note simple, but important relations.

Lemma 4.2 *1. If $\mathbf{f}\in W^{1,2}(D)$ and $\mathcal{L}_{\lambda_0,\mu_0}\mathbf{f}=0$ in D, then for all $\mathbf{g}\in W^{1,2}(D)$,*

$$\int_{\partial D}\mathbf{g}\cdot\frac{\partial\mathbf{f}}{\partial\nu}\,d\sigma = \int_D \lambda_0(\nabla\cdot\mathbf{f})(\nabla\cdot\mathbf{g}) + \frac{\mu_0}{2}\big(\nabla\mathbf{f}+(\nabla\mathbf{f})^T\big):\big(\nabla\mathbf{g}+(\nabla\mathbf{g})^T\big)d\sigma. \tag{4.21}$$

2. If $\mathbf{f}\in W^{1,2}(\mathbb{R}^2\backslash\overline{D})$ and $\mathcal{L}_{\lambda_0,\mu_0}\mathbf{f}=0$ in $\mathbb{R}^2\backslash\overline{D}$, $\mathbf{f}(x)=O(|x|^{-1})$ as $|x|\to\infty$. Then for all $\mathbf{g}\in W^{1,2}(\mathbb{R}^2\backslash\overline{D})$, $\mathbf{g}(x)=O(|x|^{-1})$ as $|x|\to\infty$, we have

$$-\int_{\partial D}\mathbf{g}\cdot\frac{\partial\mathbf{f}}{\partial\nu}\,d\sigma = \int_{\mathbb{R}^2\backslash\overline{D}} \lambda_0(\nabla\cdot\mathbf{f})(\nabla\cdot\mathbf{g}) + \frac{\mu_0}{2}\big(\nabla\mathbf{f}+(\nabla\mathbf{f})^T\big):\big(\nabla\mathbf{g}+(\nabla\mathbf{g})^T\big)d\sigma. \tag{4.22}$$

Here, for 2×2 matrices $\mathbf{M}$ and $\mathbf{N}$, $\mathbf{M}:\mathbf{N}=\sum_{ij}\mathbf{M}_{ij}\mathbf{N}_{ij}$.

4.2 Asymptotic formula of $\boldsymbol{\mathcal{K}}_{D_\epsilon}^*$

Let $\tilde{x}$, $\tilde{y}\in\partial D_\epsilon$, that is,

$$\tilde{x} = x+\epsilon h(x)\mathbf{n}(x), \qquad \tilde{y} = y+\epsilon h(y)\mathbf{n}(y), \quad x,y\in\partial D. \tag{4.23}$$

Denote by

$$E(x,y) := h(x)\mathbf{n}(x) - h(y)\mathbf{n}(y).$$

It follows from (4.23) that

$$|\tilde{x}-\tilde{y}|^2 = |x-y|^2\Big(1+2\epsilon F(x,y)+\epsilon^2 G(x,y)\Big), \tag{4.24}$$

where

$$F(x,y)=\frac{\langle x-y, E(x,y)\rangle}{|x-y|^2}, \qquad G(x,y)=\frac{|E(x,y)|^2}{|x-y|^2}.$$

Since ∂D is of class $\mathcal{C}^2$, then

$$\frac{\langle x-y,\mathbf{n}(x)\rangle}{|x-y|^2}, \frac{\langle x-y,\mathbf{n}(y)\rangle}{|x-y|^2} \le C \quad \text{ for } x,y\in\partial D.$$

We have $h\nu\in\mathcal{C}^1(\partial D)$. Then, one can easily see that

$$|F(x,y)|+|G(x,y)|^{\frac{1}{2}} \le C\|X\|_{\mathcal{C}^2}\|h\|_{\mathcal{C}^1} \quad \text{ for } x,y\in\partial D.$$

We denote by $|\cdot|_\infty$ the matrix infinity norm. For $x,y\in\partial D$, we have

$$\left|\frac{(x-y)\otimes(x-y)}{|x-y|^2}\right|_\infty \le 1,$$

and

$$\left|\frac{E(x,y)\otimes(x-y)}{|x-y|^2}\right|_\infty, \left|\frac{(x-y)\otimes E(x,y)}{|x-y|^2}\right|_\infty, \left|\frac{E(x,y)\otimes E(x,y)}{|x-y|^2}\right|_\infty^{\frac{1}{2}} \le C\|X\|_{\mathcal{C}^2}\|h\|_{\mathcal{C}^1}.$$

For $\widetilde{\boldsymbol{\phi}}\in L^2(\partial D_\epsilon)$, the operator $\boldsymbol{\mathcal{K}}^*_{D_\epsilon}$ is defined by

$$\boldsymbol{\mathcal{K}}^*_{D_\epsilon}[\widetilde{\boldsymbol{\phi}}](\tilde{x}) = \int_{\partial D_\epsilon}\mathbf{K}^T(\tilde{x}-\tilde{y})\widetilde{\boldsymbol{\phi}}(\tilde{y})d\sigma_\epsilon(\tilde{y}),$$

where

$$\begin{aligned}\mathbf{K}^T(\tilde{x}-\tilde{y}) =& \frac{1}{2\pi}\frac{(A-B)}{A+B}\ \frac{\langle\tilde{x}-\tilde{y},\mathbf{n}(\tilde{x})\rangle}{|\tilde{x}-\tilde{y}|^2}\mathbf{I} + \frac{1}{2\pi}\frac{(A-B)}{(A+B)}\frac{(\tilde{x}-\tilde{y})\otimes\mathbf{n}(\tilde{x})-\mathbf{n}(\tilde{x})\otimes(\tilde{x}-\tilde{y})}{|\tilde{x}-\tilde{y}|^2}\\ &+\frac{2}{\pi}\frac{B}{A+B}\frac{\langle\tilde{x}-\tilde{y},\mathbf{n}(\tilde{x})\rangle}{|\tilde{x}-\tilde{y}|^2}\frac{(\tilde{x}-\tilde{y})\otimes(\tilde{x}-\tilde{y})}{|\tilde{x}-\tilde{y}|^2} \quad \text{ for } \tilde{x},\tilde{y}\in\partial D_\epsilon,\ \tilde{x}\neq\tilde{y}.\end{aligned}$$

It follows from (2.1), (2.3), and (4.24) that

$$\begin{aligned}&\frac{(\tilde{x}-\tilde{y})\otimes\mathbf{n}(\tilde{x})-\mathbf{n}(\tilde{x})\otimes(\tilde{x}-\tilde{y})}{|\tilde{x}-\tilde{y}|^2}d\sigma_\epsilon(\tilde{y})\\ &=\frac{(\tilde{x}-\tilde{y})\otimes\boldsymbol{\eta}(x)-\boldsymbol{\eta}(x)\otimes(\tilde{x}-\tilde{y})}{|x-y|^2}\\ &\qquad\times\frac{1}{1+2\epsilon F(x,y)+\epsilon^2G(x,y)}\frac{\sqrt{\Big(1-\epsilon h(y)\kappa(y)\Big)^2+\epsilon^2h'(s)^2}}{\sqrt{\Big(1-\epsilon h(x)\kappa(x)\Big)^2+\epsilon^2h'(t)^2}}d\sigma(y).\end{aligned}$$

We have

$$\frac{1}{1+2\epsilon F(x,y)+\epsilon^2 G(x,y)} \times \frac{\sqrt{\Big(1-\epsilon h(y)\kappa(y)\Big)^2+\epsilon^2 h'(s)^2}}{\sqrt{\Big(1-\epsilon h(x)\kappa(x)\Big)^2+\epsilon^2 h'(t)^2}} d\sigma(y)$$

$$= \Big[1-2\epsilon\frac{\langle x-y, h(x)\mathbf{n}(x)-h(y)\mathbf{n}(y)\rangle}{|x-y|^2}+\epsilon\Big(\kappa(x)h(x)-\kappa(y)h(y)\Big)\Big]d\sigma(y)+O(\epsilon^2), \quad (4.25)$$

where the remainder $O(\epsilon^2)$ depends only on the $\mathcal{C}^2$-norm of X and $\mathcal{C}^1$-norm of h.

According to (2.1) and (4.23), we write

$$\begin{aligned}\frac{(\tilde{x}-\tilde{y})\otimes\boldsymbol{\eta}(x)-\boldsymbol{\eta}(x)\otimes(\tilde{x}-\tilde{y})}{|x-y|^2} &= \big(1-\epsilon\kappa(x)h(x)\big)\frac{(x-y)\otimes\mathbf{n}(x)-\mathbf{n}(x)\otimes(x-y)}{|x-y|^2}\\ &\quad+\epsilon\frac{E(x,y)\otimes\mathbf{n}(x)-\mathbf{n}(x)\otimes E(x,y)}{|x-y|^2}\\ &\quad-\epsilon h'(t)\frac{(x-y)\otimes\boldsymbol{\tau}(x)-\boldsymbol{\tau}(x)\otimes(x-y)}{|x-y|^2}+O(\epsilon^2).\end{aligned} \quad (4.26)$$

Therefore, by (4.25) and (4.26), we get

$$\begin{aligned}&\frac{(\tilde{x}-\tilde{y})\otimes\mathbf{n}(\tilde{x})-\mathbf{n}(\tilde{x})\otimes(\tilde{x}-\tilde{y})}{|\tilde{x}-\tilde{y}|^2}d\sigma_\epsilon(\tilde{y})\\ &= \frac{\mathbf{x}\otimes\mathbf{n}(x)-\mathbf{n}(x)\otimes\mathbf{x}}{|\mathbf{x}|^2}d\sigma(y)+\epsilon\Big(\kappa(x)h(x)-\kappa(y)h(y)\Big)\frac{\mathbf{x}\otimes\mathbf{n}(x)-\mathbf{n}(x)\otimes\mathbf{x}}{|\mathbf{x}|^2}d\sigma(y)\\ &\quad-\epsilon\kappa(x)h(x)\frac{\mathbf{x}\otimes\mathbf{n}(x)-\mathbf{n}(x)\otimes\mathbf{x}}{|\mathbf{x}|^2}d\sigma(y)-\epsilon h'(t)\frac{\mathbf{x}\otimes\boldsymbol{\tau}(x)-\boldsymbol{\tau}(x)\otimes\mathbf{x}}{|\mathbf{x}|^2}d\sigma(y)\\ &\quad+\epsilon h(y)\Big[2\frac{\langle\mathbf{x},\mathbf{n}(y)\rangle}{|\mathbf{x}|^4}\Big(\mathbf{x}\otimes\mathbf{n}(x)-\mathbf{n}(x)\otimes\mathbf{x}\Big)-\frac{\mathbf{n}(y)\otimes\mathbf{n}(x)-\mathbf{n}(x)\otimes\mathbf{n}(y)}{|\mathbf{x}|^2}\Big]d\sigma(y)\\ &\quad-2\epsilon h(x)\frac{\langle\mathbf{x},\mathbf{n}(x)\rangle}{|\mathbf{x}|^4}\Big(\mathbf{x}\otimes\mathbf{n}(x)-\mathbf{n}(x)\otimes\mathbf{x}\Big)d\sigma(y)+O(\epsilon^2).\end{aligned} \quad (4.27)$$

It is proved in [6] that

$$\begin{aligned}\frac{\langle\tilde{x}-\tilde{y},\mathbf{n}(\tilde{x})\rangle}{|\tilde{x}-\tilde{y}|^2}d\sigma_\epsilon(\tilde{y}) =&\frac{\langle\mathbf{x},\mathbf{n}(x)\rangle}{|\mathbf{x}|^2}d\sigma(y)+\epsilon\Big(\kappa(x)h(x)-\kappa(y)h(y)\Big)\frac{\langle\mathbf{x},\mathbf{n}(x)\rangle}{|\mathbf{x}|^2}d\sigma(y)\\ &-\epsilon\kappa(x)h(x)\frac{\langle\mathbf{x},\mathbf{n}(x)\rangle}{|\mathbf{x}|^2}d\sigma(y)-\epsilon h'(t)\frac{\langle\mathbf{x},\boldsymbol{\tau}(x)\rangle}{|\mathbf{x}|^2}d\sigma(y)\\ &+\epsilon h(y)\Big(2\frac{\langle\mathbf{x},\mathbf{n}(y)\rangle\langle\mathbf{x},\mathbf{n}(x)\rangle}{|\mathbf{x}|^4}-\frac{\langle\mathbf{n}(x),\mathbf{n}(y)\rangle}{|\mathbf{x}|^2}\Big)d\sigma(y)\\ &+\epsilon h(x)\Big(-2\Big(\frac{\langle\mathbf{x},\mathbf{n}(x)\rangle}{|\mathbf{x}|^2}\Big)^2+\frac{1}{|\mathbf{x}|^2}\Big)d\sigma(y)+O(\epsilon^2).\end{aligned} \quad (4.28)$$

Using (4.23) and (4.24), we obtain

$$
\begin{aligned}
\frac{(\tilde{x}-\tilde{y})\otimes(\tilde{x}-\tilde{y})}{|\tilde{x}-\tilde{y}|^2} =& \frac{(\tilde{x}-\tilde{y})\otimes(\tilde{x}-\tilde{y})}{|x-y|^2} \times \frac{1}{1+2\epsilon F(x,y)+\epsilon^2 G(x,y)} \\
=& \left(1-2\epsilon\frac{\langle x-y, h(x)\mathbf{n}(x)-h(y)\mathbf{n}(y)\rangle}{|x-y|^2}\right)\frac{(x-y)\otimes(x-y)}{|x-y|^2} \\
&+\epsilon\frac{(x-y)\otimes\Big(h(x)\mathbf{n}(x)-h(y)\mathbf{n}(y)\Big)}{|x-y|^2} \\
&+\epsilon\frac{\Big(h(x)\mathbf{n}(x)-h(y)\mathbf{n}(y)\Big)\otimes(x-y)}{|x-y|^2}+O(\epsilon^2).
\end{aligned} \tag{4.29}
$$

It follows from (4.28) and (4.29) that

$$
\begin{aligned}
&\frac{\langle \tilde{x}-\tilde{y}, \mathbf{n}(\tilde{x})\rangle}{|\tilde{x}-\tilde{y}|^2}\frac{(\tilde{x}-\tilde{y})\otimes(\tilde{x}-\tilde{y})}{|\tilde{x}-\tilde{y}|^2}d\sigma_\epsilon(\tilde{y}) \\
&= \Bigg[\frac{\langle \mathbf{x}, \mathbf{n}(x)\rangle}{|\mathbf{x}|^4}(\mathbf{x}\otimes\mathbf{x})+\epsilon\Big(\kappa(x)h(x)-\kappa(y)h(y)\Big)\frac{\langle \mathbf{x}, \mathbf{n}(x)\rangle}{|\mathbf{x}|^4}(\mathbf{x}\otimes\mathbf{x}) \\
&\quad -\epsilon\kappa(x)h(x)\frac{\langle \mathbf{x}, \mathbf{n}(x)\rangle}{|\mathbf{x}|^4}(\mathbf{x}\otimes\mathbf{x})-\epsilon h'(t)\frac{\langle \mathbf{x}, \boldsymbol{\tau}(x)\rangle}{|\mathbf{x}|^4}(\mathbf{x}\otimes\mathbf{x}) \\
&\quad +\epsilon h(x)\left(-4\frac{(\langle \mathbf{x}, \mathbf{n}(x)\rangle)^2}{|\mathbf{x}|^6}(\mathbf{x}\otimes\mathbf{x})+\frac{(\mathbf{x}\otimes\mathbf{x})}{|\mathbf{x}|^4}+\frac{\langle \mathbf{x}, \mathbf{n}(x)\rangle}{|\mathbf{x}|^4}\Big(\mathbf{n}(x)\otimes\mathbf{x}+\mathbf{x}\otimes\mathbf{n}(x)\Big)\right) \\
&\quad +\epsilon h(y)\Bigg(4\frac{\langle \mathbf{x}, \mathbf{n}(y)\rangle\langle \mathbf{x}, \mathbf{n}(x)\rangle}{|\mathbf{x}|^6}(\mathbf{x}\otimes\mathbf{x})-\frac{\langle \mathbf{n}(x), \mathbf{n}(y)\rangle}{|\mathbf{x}|^4}(\mathbf{x}\otimes\mathbf{x}) \\
&\qquad -\frac{\langle \mathbf{x}, \mathbf{n}(x)\rangle}{|\mathbf{x}|^4}\Big(\mathbf{n}(y)\otimes\mathbf{x}+\mathbf{x}\otimes\mathbf{n}(y)\Big)\Bigg)\Bigg]d\sigma(y)+O(\epsilon^2).
\end{aligned} \tag{4.30}
$$

From (4.27), (4.28), and (4.30), we write

$$
\mathbf{K}^T(\tilde{x}-\tilde{y}) = \mathbf{K}^T(x-y)+\epsilon\mathbf{K}_1(x-y)+O(\epsilon^2) \quad \text{for } x,y\in\partial D, x\neq y.
$$

Introduce the integral operator $\mathcal{K}_D^{(1)}$, defined for any $\boldsymbol{\phi}\in L^2(\partial D)$ by

$$
\boldsymbol{\mathcal{K}}_D^{(1)}[\boldsymbol{\phi}](x) := \int_{\partial D}\mathbf{K}_1(x-y)\boldsymbol{\phi}(y)d\sigma(y), \quad x\in\partial D. \tag{4.31}
$$

The operator $\boldsymbol{\mathcal{K}}_D^{(1)}$ is bounded on $L^2(\partial D)$. In fact, this is an immediate consequence of the celebrated theorem of Coifman-McIntosh-Meyer [12].

Let Φ_ϵ be the diffeomorphism from ∂D onto ∂D_ϵ given by $\Phi_\epsilon(x) = x+\epsilon h(t)\mathbf{n}(x)$, where $x = X(t)\in\partial D$. The following theorem holds.

Theorem 4.3 *There exists $C > 0$ depending only on $\lambda_0, \lambda_1, \mu_0, \mu_1$, $\|X\|_{\mathcal{C}^2}$, and $\|h\|_{\mathcal{C}^1}$ such that for any $\widetilde{\boldsymbol{\phi}} \in L^2(\partial D_\epsilon)$, we have*

$$\left\| \boldsymbol{\mathcal{K}}^*_{D_\epsilon}[\widetilde{\boldsymbol{\phi}}] \circ \Phi_\epsilon - \boldsymbol{\mathcal{K}}^*_D[\boldsymbol{\phi}] - \epsilon \boldsymbol{\mathcal{K}}^{(1)}_D[\boldsymbol{\phi}] \right\|_{L^2(\partial D)} \leq C\epsilon^2 \big\|\boldsymbol{\phi}\big\|_{L^2(\partial D)}, \tag{4.32}$$

where $\boldsymbol{\phi} = \widetilde{\boldsymbol{\phi}} \circ \Phi_\epsilon$ and $\boldsymbol{\mathcal{K}}^{(1)}_D$ is defined in (4.31).

The following theorem is of particular importance to us in order to establish our asymptotic expansions.

Theorem 4.4 *Let $\boldsymbol{\phi} \in \mathcal{C}^{1,k}(\partial D)$, for some $0 < k < 1$. Then*

$$\frac{\partial \boldsymbol{\mathcal{D}}^\sharp_D[\boldsymbol{\phi}]}{\partial \nu}\Big|_+ - \frac{\partial \boldsymbol{\mathcal{D}}^\sharp_D[\boldsymbol{\phi}]}{\partial \nu}\Big|_- = \frac{\partial}{\partial \boldsymbol{\tau}}\Big(\langle \boldsymbol{\phi}, \boldsymbol{\tau}\rangle \mathbf{n} + \frac{\lambda_0}{2\mu_0 + \lambda_0} \langle \boldsymbol{\phi}, \mathbf{n}\rangle \boldsymbol{\tau} \Big) \quad \textit{on } \partial D. \tag{4.33}$$

Proof. For a function $\mathbf{w}$ defined on $\mathbb{R}^2 \backslash \partial D$, we denote

$$\mathbf{w}(x)|_\pm = \lim_{t\neq 0, t \to 0^\pm} \mathbf{w}(x_t) \quad \text{for } x \in \partial D, \quad x_t := x + t\mathbf{n}(x).$$

Let $\widetilde{\boldsymbol{\phi}} \in L^2(\partial D_\epsilon)$ and $\boldsymbol{\phi} = \widetilde{\boldsymbol{\phi}} \circ \Phi_\epsilon$. Following the same arguments as in the case of $\boldsymbol{\mathcal{K}}_{D_\epsilon}$ (taking $h = 1$) and using the integral representations in the appendix, we can prove that

$$\begin{aligned}
&\Big(\frac{\partial \boldsymbol{\mathcal{S}}_{D_\epsilon}}{\partial \nu}[\widetilde{\boldsymbol{\phi}}] \circ \Phi_\epsilon - \frac{\partial \boldsymbol{\mathcal{S}}_D}{\partial \nu}[\boldsymbol{\phi}] \Big)(x_t) \\
&= \epsilon \Bigg(\kappa(x) \frac{\partial \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}]}{\partial \nu}(x_t) - \frac{\partial \boldsymbol{\mathcal{S}}_D[\kappa\boldsymbol{\phi}]}{\partial \nu}(x_t) + \frac{\partial \boldsymbol{\mathcal{D}}^\sharp_D[\boldsymbol{\phi}]}{\partial \nu}(x_t) - \kappa(x)\frac{\partial \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}]}{\partial \nu}(x_t) \\
&\qquad + \lambda_0 \nabla\nabla \cdot \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}](x_t) \cdot \mathbf{n}(x_t)\mathbf{n}(x_t) + \mu_0 \nabla\Big(\nabla \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}](x_t) + \big(\nabla \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}](x_t)\big)^T \Big)\mathbf{n}(x_t)\mathbf{n}(x_t) \Bigg) \\
&\quad + O(\epsilon^2).
\end{aligned}$$

If $\boldsymbol{\phi} \in \mathcal{C}^{1,k}(\partial D)$, then $\boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}]$ is $\mathcal{C}^{2,k}$ and $\boldsymbol{\mathcal{D}}^\sharp_D[\boldsymbol{\phi}]$ is $\mathcal{C}^{1,k}$ on $\overline{D}$ and $\mathbb{R}^2 \backslash D$. Thus

$$\begin{aligned}
&\Big(\frac{\partial \boldsymbol{\mathcal{S}}_{D_\epsilon}}{\partial \nu}[\widetilde{\boldsymbol{\phi}}] \circ \Phi_\epsilon - \frac{\partial \boldsymbol{\mathcal{S}}_D}{\partial \nu}[\boldsymbol{\phi}] \Big)\Big|_\pm \\
&\quad = \epsilon \Bigg(\kappa \frac{\partial \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}]}{\partial \nu} - \frac{\partial \boldsymbol{\mathcal{S}}_D[\kappa\boldsymbol{\phi}]}{\partial \nu} + \frac{\partial \boldsymbol{\mathcal{D}}^\sharp_D[\boldsymbol{\phi}]}{\partial \nu} - \kappa \frac{\partial \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}]}{\partial \nu} + \lambda_0 \nabla\nabla \cdot \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}] \cdot \mathbf{n}\mathbf{n} \\
&\qquad\qquad + \mu_0 \nabla \Big(\nabla \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}] + \big(\nabla \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}]\big)^T \Big)\mathbf{n}\mathbf{n} \Bigg)\Bigg|_\pm + O(|t|^k\epsilon) + O(\epsilon^2) \quad \text{on } \partial D.
\end{aligned}$$

Since $\mathcal{L}_{\lambda_0,\mu_0}\boldsymbol{\mathcal{S}}_D[\cdot] = 0$ in $\mathbb{R}^2 \backslash \partial D$, it follows from the representation of the Lamé system on ∂D in (2.27) that

$$\begin{aligned}
&\frac{\partial}{\partial \boldsymbol{\tau}}\Big((\mathbb{C}_0 \widehat{\nabla} \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}])\boldsymbol{\tau} \Big)\Big|_\pm \\
&= \Bigg(\kappa \frac{\partial \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}]}{\partial \nu} - \lambda_0 \nabla\nabla \cdot \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}] \cdot \mathbf{n}\mathbf{n} - \mu_0 \nabla \Big(\nabla \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}] + \big(\nabla \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}]\big)^T \Big)\mathbf{n}\mathbf{n} \Bigg)\Bigg|_\pm \quad \text{on } \partial D,
\end{aligned}$$

and hence

$$\Big(\frac{\partial \boldsymbol{\mathcal{S}}_{D_\epsilon}}{\partial \nu}[\widetilde{\boldsymbol{\phi}}] \circ \Phi_\epsilon - \frac{\partial \boldsymbol{\mathcal{S}}_D}{\partial \nu}[\boldsymbol{\phi}]\Big)\Big|_\pm = \epsilon\bigg(\kappa \frac{\partial \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}]}{\partial \nu} - \frac{\partial \boldsymbol{\mathcal{S}}_D[\kappa\boldsymbol{\phi}]}{\partial \nu} + \frac{\partial \boldsymbol{\mathcal{D}}_D^\sharp[\boldsymbol{\phi}]}{\partial \nu} - \frac{\partial}{\partial \boldsymbol{\tau}}\Big(\big(\mathbb{C}_0 \widehat{\nabla} \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}]\big)\boldsymbol{\tau}\Big)\bigg)\Big|_\pm + O(|t|^k\epsilon) + O(\epsilon^2) \ \text{ on } \partial D.$$

According to (4.5), we have

$$\Big(\frac{\partial \boldsymbol{\mathcal{S}}_{D_\epsilon}[\widetilde{\boldsymbol{\phi}}]}{\partial \nu} \circ \Phi_\epsilon - \frac{\partial \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}]}{\partial \nu}\Big)\Big|_+ = \Big(\frac{\partial \boldsymbol{\mathcal{S}}_{D_\epsilon}[\widetilde{\boldsymbol{\phi}}]}{\partial \nu} \circ \Phi_\epsilon - \frac{\partial \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}]}{\partial \nu}\Big)\Big|_- \quad \text{on } \partial D,$$

which gives

$$\begin{aligned} &\kappa \frac{\partial \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}]}{\partial \nu}\Big|_+ - \frac{\partial \boldsymbol{\mathcal{S}}_D[\kappa\boldsymbol{\phi}]}{\partial \nu}\Big|_+ + \frac{\partial \boldsymbol{\mathcal{D}}_D^\sharp[\boldsymbol{\phi}]}{\partial \nu}\Big|_+ - \frac{\partial}{\partial \boldsymbol{\tau}}\Big(\big(\mathbb{C}_0 \widehat{\nabla} \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}]\big)\boldsymbol{\tau}\Big)\Big|_+ \\ &= \kappa \frac{\partial \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}]}{\partial \nu}\Big|_- - \frac{\partial \boldsymbol{\mathcal{S}}_D[\kappa\boldsymbol{\phi}]}{\partial \nu}\Big|_- + \frac{\partial \boldsymbol{\mathcal{D}}_D^\sharp[\boldsymbol{\phi}]}{\partial \nu}\Big|_- - \frac{\partial}{\partial \boldsymbol{\tau}}\Big(\big(\mathbb{C}_0 \widehat{\nabla} \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}]\big)\boldsymbol{\tau}\Big)\Big|_- \quad \text{on } \partial D. \end{aligned} \tag{4.34}$$

By (4.5) again, we have

$$\Big(\kappa \frac{\partial \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}]}{\partial \nu} - \frac{\partial \boldsymbol{\mathcal{S}}_D[\kappa\boldsymbol{\phi}]}{\partial \nu}\Big)\Big|_+ = \Big(\kappa \frac{\partial \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}]}{\partial \nu} - \frac{\partial \boldsymbol{\mathcal{S}}_D[\kappa\boldsymbol{\phi}]}{\partial \nu}\Big)\Big|_- \quad \text{on } \partial D.$$

It then follows from (4.34) that

$$\frac{\partial \boldsymbol{\mathcal{D}}_D^\sharp[\boldsymbol{\phi}]}{\partial \nu}\Big|_+ - \frac{\partial}{\partial \boldsymbol{\tau}}\Big(\big(\mathbb{C}_0 \widehat{\nabla} \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}]\big)\boldsymbol{\tau}\Big)\Big|_+ = \frac{\partial \boldsymbol{\mathcal{D}}_D^\sharp[\boldsymbol{\phi}]}{\partial \nu}\Big|_- - \frac{\partial}{\partial \boldsymbol{\tau}}\Big(\big(\mathbb{C}_0 \widehat{\nabla} \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}]\big)\boldsymbol{\tau}\Big)\Big|_- \quad \text{on } \partial D, \tag{4.35}$$

that is, $\frac{\partial \boldsymbol{\mathcal{D}}_D^\sharp[\boldsymbol{\phi}]}{\partial \nu} - \frac{\partial}{\partial \boldsymbol{\tau}}\Big(\big(\mathbb{C}_0 \widehat{\nabla} \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}]\big)\boldsymbol{\tau}\Big)$ is continuous on ∂D, but $\frac{\partial \boldsymbol{\mathcal{D}}_D^\sharp[\boldsymbol{\phi}]}{\partial \nu}$ and $\frac{\partial}{\partial \boldsymbol{\tau}}\Big(\big(\mathbb{C}_0 \widehat{\nabla} \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}]\big)\boldsymbol{\tau}\Big)$ are discontinuous on ∂D, and we have the following relationship

$$\frac{\partial \boldsymbol{\mathcal{D}}_D^\sharp[\boldsymbol{\phi}]}{\partial \nu}\Big|_+ - \frac{\partial \boldsymbol{\mathcal{D}}_D^\sharp[\boldsymbol{\phi}]}{\partial \nu}\Big|_- = \frac{\partial}{\partial \boldsymbol{\tau}}\Big(\big(\mathbb{C}_0 \widehat{\nabla} \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}]\big)\boldsymbol{\tau}\big|_+ - \big(\mathbb{C}_0 \widehat{\nabla} \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}]\big)\boldsymbol{\tau}\big|_-\Big) \quad \text{on } \partial D.$$

It follows from (4.19) that

$$\big(\mathbb{C}_0 \widehat{\nabla} \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}]\big)\boldsymbol{\tau}\big|_+ - \big(\mathbb{C}_0 \widehat{\nabla} \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}]\big)\boldsymbol{\tau}\big|_- = \langle \boldsymbol{\phi}, \boldsymbol{\tau}\rangle \mathbf{n} + \frac{\lambda_0}{2\mu_0 + \lambda_0}\langle \boldsymbol{\phi}, \mathbf{n}\rangle \boldsymbol{\tau} \quad \text{on } \partial D.$$

Thus (4.4) is proved, as desired. This finishes the proof of the theorem.

As a direct consequence of (4.5), (4.35), and the expansions in the appendix, the integral representation of $\boldsymbol{\mathcal{K}}_D^{(1)}$ in (4.31), can be rewritten as

$$\begin{aligned} \boldsymbol{\mathcal{K}}_D^{(1)}[\boldsymbol{\phi}](x) =& \Big(\kappa h(x) \frac{\partial \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}]}{\partial \nu}(x) - \frac{\partial \boldsymbol{\mathcal{S}}_D[\kappa h\boldsymbol{\phi}]}{\partial \nu}(x)\Big)\Big|_\pm \\ &+ \bigg(\frac{\partial \boldsymbol{\mathcal{D}}_D^\sharp[h\boldsymbol{\phi}]}{\partial \nu}(x) - \frac{d}{dt}\Big(h(x)\big(\mathbb{C}_0 \widehat{\nabla} \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}](x)\big)\boldsymbol{\tau}(x)\Big)\bigg)\Big|_\pm, \quad x \in \partial D. \end{aligned} \tag{4.36}$$

4.3 Asymptotic expansion of $\boldsymbol{\mathcal{S}}_{D_\epsilon}$

For $\widetilde{\boldsymbol{\phi}} \in L^2(\partial D_\epsilon)$, we have

$$\boldsymbol{\mathcal{S}}_{D_\epsilon}[\widetilde{\boldsymbol{\phi}}](\tilde{x}) = \int_{\partial D_\epsilon} \left(\frac{A}{2\pi} \log|\tilde{x}-\tilde{y}| - \frac{B}{2\pi} \frac{(\tilde{x}-\tilde{y}) \otimes (\tilde{x}-\tilde{y})}{|\tilde{x}-\tilde{y}|^2} \right) \widetilde{\boldsymbol{\phi}}(\tilde{y}) d\sigma_\epsilon(\tilde{y}), \quad \tilde{x} \in \partial D_\epsilon.$$

It follows from (2.3) and (4.24) that

$$\begin{aligned}
&\log|\tilde{x}-\tilde{y}| d\sigma_\epsilon(\tilde{y}) \\
&= \frac{1}{2} \log \Big(|x-y|^2 \big(1 + 2\epsilon F(x,y) + \epsilon^2 G(x,y) \big) \Big) d\sigma_\epsilon(\tilde{y}) \\
&= \Big(\log|x-y| + \epsilon F(x,y) + O(\epsilon^2) \Big) \times \Big(d\sigma(y) - \epsilon \kappa(y) h(y) d\sigma(y) + O(\epsilon^2) \Big) \\
&= \left[\log|x-y| + \epsilon \left(-\kappa(y) h(y) \log|x-y| + h(x) \frac{\langle x-y, \mathbf{n}(x) \rangle}{|x-y|^2} - h(y) \frac{\langle x-y, \mathbf{n}(y) \rangle}{|x-y|^2} \right) \right] d\sigma(y) \\
&\quad + O(\epsilon^2) \Big(\log|x-y| + 1 \Big).
\end{aligned} \tag{4.37}$$

According to (2.3), (4.29), and (4.37), we obtain

$$\begin{aligned}
&\boldsymbol{\Gamma}(\tilde{x}-\tilde{y}) d\sigma(\tilde{y}) \\
&= \Bigg[\boldsymbol{\Gamma}(\mathbf{x}) - \epsilon \kappa(y) h(y) \boldsymbol{\Gamma}(\mathbf{x}) \\
&\quad + \epsilon h(x) \left(\frac{A}{2\pi} \frac{\langle \mathbf{x}, \mathbf{n}(x) \rangle}{|\mathbf{x}|^2} \mathbf{I} + \frac{B}{\pi} \frac{\langle \mathbf{x}, \mathbf{n}(x) \rangle}{|\mathbf{x}|^2} \frac{\mathbf{x} \otimes \mathbf{x}}{|\mathbf{x}|^2} - \frac{B}{2\pi} \frac{\mathbf{x} \otimes \mathbf{n}(x) + \mathbf{n}(x) \otimes \mathbf{x}}{|\mathbf{x}|^2} \right) \\
&\quad + \epsilon h(y) \left(-\frac{A}{2\pi} \frac{\langle \mathbf{x}, \mathbf{n}(y) \rangle}{|\mathbf{x}|^2} \mathbf{I} - \frac{B}{\pi} \frac{\langle \mathbf{x}, \mathbf{n}(y) \rangle}{|\mathbf{x}|^2} \frac{\mathbf{x} \otimes \mathbf{x}}{|\mathbf{x}|^2} + \frac{B}{2\pi} \frac{\mathbf{x} \otimes \mathbf{n}(y) + \mathbf{n}(y) \otimes \mathbf{x}}{|\mathbf{x}|^2} \right) \Bigg] d\sigma(y) \\
&\quad + O(\epsilon^2) \Big(\log|\mathbf{x}| + 1).
\end{aligned} \tag{4.38}$$

Introduce an integral operator $\boldsymbol{\mathcal{S}}_D^{(1)}$, defined for any $\boldsymbol{\phi} \in L^2(\partial D)$ by

$$\boldsymbol{\mathcal{S}}_D^{(1)}[\boldsymbol{\phi}](x) = -\boldsymbol{\mathcal{S}}_D[\kappa h \boldsymbol{\phi}](x) + \left(h(x) \frac{\partial \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}]}{\partial \mathbf{n}}(x) + \boldsymbol{\mathcal{D}}_D^\sharp[h\boldsymbol{\phi}](x) \right) \Big|_\pm, \quad x \in \partial D. \tag{4.39}$$

The operators $\boldsymbol{\mathcal{S}}_D^{(1)}$ and $\frac{\partial \boldsymbol{\mathcal{S}}_D^{(1)}}{\partial \boldsymbol{\tau}}$ are bounded on $L^2(\partial D)$ by the theorem of Coifman, McIntosh, and Meyer [12]. Therefore, we get from (4.38)

$$\left\| \boldsymbol{\mathcal{S}}_{D_\epsilon}[\widetilde{\boldsymbol{\phi}}] \circ \Phi_\epsilon - \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}] - \epsilon \boldsymbol{\mathcal{S}}_D^{(1)}[\boldsymbol{\phi}] \right\|_{L^2(\partial D)} \leq C\epsilon^2 \|\boldsymbol{\phi}\|_{L^2(\partial D)}, \tag{4.40}$$

where $\boldsymbol{\phi} = \widetilde{\boldsymbol{\phi}} \circ \Phi_\epsilon$.

We have

$$\frac{\partial \boldsymbol{\mathcal{S}}_{D_\epsilon}[\widetilde{\boldsymbol{\phi}}]}{\partial \boldsymbol{\tau}}(\tilde{x}) = \int_{\partial D} \nabla \boldsymbol{\Gamma}\big(\tilde{x} - \Phi_\epsilon(y)\big) R_{\frac{\pi}{2}} \boldsymbol{\eta}(x) \boldsymbol{\phi}(y) \times \frac{\sqrt{\Big(1 - \epsilon h(y)\kappa(y)\Big)^2 + \epsilon^2 h'(s)^2}}{\sqrt{\Big(1 - \epsilon h(x)\kappa(x)\Big)^2 + \epsilon^2 h'(t)^2}} d\sigma(y),$$

where $\nabla \boldsymbol{\Gamma}$ and $\boldsymbol{\eta}$ are defined in (4.11) and (2.1), respectively. Following the same argument as in the case of $\boldsymbol{\mathcal{K}}^*_{D_\epsilon}$, we can prove that

$$\left\| \frac{\partial \boldsymbol{\mathcal{S}}_{D_\epsilon}[\widetilde{\boldsymbol{\phi}}]}{\partial \boldsymbol{\tau}} \circ \Phi_\epsilon - \frac{\partial \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}]}{\partial \boldsymbol{\tau}} - \epsilon \frac{\partial \boldsymbol{\mathcal{S}}^{(1)}_D[\boldsymbol{\phi}]}{\partial \boldsymbol{\tau}} \right\|_{L^2(\partial D)} \leq C\epsilon^2 \big\|\boldsymbol{\phi}\big\|_{L^2(\partial D)}. \tag{4.41}$$

Throughout this paper $W^2_1(\partial D)$ denotes the first L^2-Sobolev of space of order 1 on ∂D. From (4.40) and (4.41), we obtain the following theorem.

Theorem 4.5 *There exists $C > 0$ depending only on $\lambda_0, \lambda_1, \mu_0, \mu_1$, $\|X\|_{\mathcal{C}^2}$, and $\|h\|_{\mathcal{C}^1}$ such that for any $\widetilde{\boldsymbol{\phi}} \in L^2(\partial D_\epsilon)$,*

$$\left\| \boldsymbol{\mathcal{S}}_{D_\epsilon}[\widetilde{\boldsymbol{\phi}}] \circ \Phi_\epsilon - \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}] - \epsilon \boldsymbol{\mathcal{S}}^{(1)}_D[\boldsymbol{\phi}] \right\|_{W^2_1(\partial D)} \leq C\epsilon^2 \big\|\boldsymbol{\phi}\big\|_{L^2(\partial D)}, \tag{4.42}$$

where $\boldsymbol{\phi} = \widetilde{\boldsymbol{\phi}} \circ \Phi_\epsilon$ and $\boldsymbol{\mathcal{S}}^{(1)}_D$ is defined in (4.39).

5 Asymptotic of the displacement field

The following solvability result done by Escauriaza and Seo [14].

Theorem 5.1 *Suppose that $(\lambda_0 - \lambda_1)(\mu_0 - \mu_1) \geq 0$ and $0 < \lambda_1, \mu_1 < +\infty$. For any given $(\mathbf{F}, \mathbf{G}) \in W^2_1(\partial D) \times L^2(\partial D)$, there exists a unique pair $(\mathbf{f}, \mathbf{g}) \in L^2(\partial D) \times L^2(\partial D)$ such that*

$$\begin{cases} \widetilde{\boldsymbol{\mathcal{S}}}_D[\mathbf{f}]\big|_- - \boldsymbol{\mathcal{S}}_D[\mathbf{g}]\big|_+ = \mathbf{F} & \text{on } \partial D, \\ \big(-\frac{1}{2}\mathbf{I} + \widetilde{\boldsymbol{\mathcal{K}}}^*_D\big)[\mathbf{f}] - \big(\frac{1}{2}\mathbf{I} + \boldsymbol{\mathcal{K}}^*_D\big)[\mathbf{g}] = \mathbf{G} & \text{on } \partial D, \end{cases} \tag{5.1}$$

and there exists a constant $C > 0$ depending only on $\lambda_0, \mu_0, \lambda_1, \mu_1$, and the Lipschitz character of D such that

$$\|\mathbf{f}\|_{L^2(\partial D)} + \|\mathbf{g}\|_{L^2(\partial D)} \leq C\Big(\|\mathbf{F}\|_{W^2_1(\partial D)} + \|\mathbf{G}\|_{L^2(\partial D)}\Big). \tag{5.2}$$

Moreover, if $\mathbf{G} \in L^2_\Psi(\partial D)$, then $\mathbf{g} \in L^2_\Psi(\partial D)$.

The following proposition is of particular importance to us.

Proposition 5.2 *Suppose that $(\lambda_0 - \lambda_1)(\mu_0 - \mu_1) \geq 0$ and $0 < \lambda_1, \mu_1 < +\infty$. For any given $(\mathbf{F}, \mathbf{G}) \in W^2_1(\partial D) \times L^2(\partial D)$, there exists a unique pair $(\mathbf{f}, \mathbf{g}) \in L^2(\partial D) \times L^2(\partial D)$ such that*

$$\begin{cases} \Big(\widetilde{\boldsymbol{\mathcal{S}}}_D + \epsilon \widetilde{\boldsymbol{\mathcal{S}}}^{(1)}_D\Big)[\mathbf{f}] - \Big(\boldsymbol{\mathcal{S}}_D + \epsilon \boldsymbol{\mathcal{S}}^{(1)}_D\Big)[\mathbf{g}] = \mathbf{F} & \text{on } \partial D, \\ \Big(-\frac{1}{2}\mathbf{I} + \widetilde{\boldsymbol{\mathcal{K}}}^*_D + \epsilon \widetilde{\boldsymbol{\mathcal{K}}}^{(1)}_D\Big)[\mathbf{f}] - \Big(\frac{1}{2}\mathbf{I} + \boldsymbol{\mathcal{K}}^*_D + \epsilon \boldsymbol{\mathcal{K}}^{(1)}_D\Big)[\mathbf{g}] = \mathbf{G} & \text{on } \partial D. \end{cases} \tag{5.3}$$

Furthermore, there exists a constant $C > 0$ depending only on $\lambda_0, \mu_0, \lambda_1, \mu_1$, and the Lipschitz character of D such that

$$\|\mathbf{f}\|_{L^2(\partial D)} + \|\mathbf{g}\|_{L^2(\partial D)} \le C\Big(\|\mathbf{F}\|_{W_1^2(\partial D)} + \|\mathbf{G}\|_{L^2(\partial D)}\Big). \tag{5.4}$$

Proof. Let $\mathcal{X} := L^2(\partial D) \times L^2(\partial D)$ and $\mathcal{Y} := W_1^2(\partial D) \times L^2(\partial D)$. For $n = 0, 1$, define the operator $\mathcal{T}_n : \mathcal{X} \to \mathcal{Y}$ by

$$\mathcal{T}_0(\mathbf{f}, \mathbf{g}) := \Big(\widetilde{\boldsymbol{\mathcal{S}}}_D[\mathbf{f}]\big|_- - \boldsymbol{\mathcal{S}}_D[\mathbf{g}]\big|_+, \Big(-\frac{1}{2}\mathbf{I} + \widetilde{\boldsymbol{\mathcal{K}}}_D^*\Big)[\mathbf{f}] - \Big(\frac{1}{2}\mathbf{I} + \boldsymbol{\mathcal{K}}_D^*\Big)[\mathbf{g}]\Big),$$

and

$$\mathcal{T}_1(\mathbf{f}, \mathbf{g}) := \Big(\widetilde{\boldsymbol{\mathcal{S}}}_D^{(1)}[\mathbf{f}]\big|_- - \boldsymbol{\mathcal{S}}_D^{(1)}[\mathbf{g}]\big|_+, \widetilde{\boldsymbol{\mathcal{K}}}_D^{(1)}[\mathbf{f}] - \boldsymbol{\mathcal{K}}_D^{(1)}[\mathbf{g}]\Big).$$

The operator $\mathcal{T}_1$ is bounded on $\mathcal{X}$ because it is a linear combination of bounded integral operators. According to Theorem 5.1, the operator $\mathcal{T}_0$ is invertible. For ϵ small enough, it follows from Theorem 1.16, section 4 of [18], that the operator $\mathcal{T}_0 + \epsilon\mathcal{T}_1$ is invertible. This completes the proof of solvability of (5.3). The estimate (5.4) is a consequence of solvability and the closed graph theorem.

5.1 Representation of solutions

For more details on the following representation formulae, we refer to [4, 5, 8]. The solution $\mathbf{u}_\epsilon$ to (1.7) can be represented as

$$\mathbf{u}_\epsilon(x) = \begin{cases} \mathbf{H}(x) + \boldsymbol{\mathcal{S}}_{D_\epsilon}[\boldsymbol{\varphi}_\epsilon](x), & x \in \mathbb{R}^2 \backslash \overline{D}_\epsilon, \\ \widetilde{\boldsymbol{\mathcal{S}}}_{D_\epsilon}[\boldsymbol{\psi}_\epsilon](x), & x \in D_\epsilon, \end{cases} \tag{5.5}$$

where the pair $(\boldsymbol{\psi}_\epsilon, \boldsymbol{\varphi}_\epsilon)$ is the unique solution in $L^2(\partial D_\epsilon) \times L^2_\Psi(\partial D_\epsilon)$ of

$$\begin{cases} \widetilde{\boldsymbol{\mathcal{S}}}_{D_\epsilon}[\boldsymbol{\psi}_\epsilon]\big|_- - \boldsymbol{\mathcal{S}}_{D_\epsilon}[\boldsymbol{\varphi}_\epsilon]\big|_+ = \mathbf{H} \quad \text{on } \partial D_\epsilon, \\ \big(-\frac{1}{2}\mathbf{I} + \widetilde{\boldsymbol{\mathcal{K}}}_{D_\epsilon}^*\big)[\boldsymbol{\psi}_\epsilon] - \big(\frac{1}{2}\mathbf{I} + \boldsymbol{\mathcal{K}}_{D_\epsilon}^*\big)[\boldsymbol{\varphi}_\epsilon] = \dfrac{\partial \mathbf{H}}{\partial \nu} \quad \text{on } \partial D_\epsilon. \end{cases} \tag{5.6}$$

Similarly, the solution to (1.5) has the following representation

$$\mathbf{u}(x) = \begin{cases} \mathbf{H}(x) + \boldsymbol{\mathcal{S}}_D[\boldsymbol{\varphi}](x), & x \in \mathbb{R}^2 \backslash \overline{D}, \\ \widetilde{\boldsymbol{\mathcal{S}}}_D[\boldsymbol{\psi}](x), & x \in D, \end{cases} \tag{5.7}$$

where the pair $(\boldsymbol{\psi}, \boldsymbol{\varphi})$ is the unique solution in $L^2(\partial D) \times L^2_\Psi(\partial D)$ of

$$\begin{cases} \widetilde{\boldsymbol{\mathcal{S}}}_D[\boldsymbol{\psi}]\big|_- - \boldsymbol{\mathcal{S}}_D[\boldsymbol{\varphi}]\big|_+ = \mathbf{H} \quad \text{on } \partial D, \\ \big(-\frac{1}{2}\mathbf{I} + \widetilde{\boldsymbol{\mathcal{K}}}_D^*\big)[\boldsymbol{\psi}] - \big(\frac{1}{2}\mathbf{I} + \boldsymbol{\mathcal{K}}_D^*\big)[\boldsymbol{\varphi}] = \dfrac{\partial \mathbf{H}}{\partial \nu} \quad \text{on } \partial D. \end{cases} \tag{5.8}$$

Let Ω be a bounded region outside the inclusion D, and away from ∂D. It then follows from (5.5) and (5.7) that

$$\mathbf{u}_\epsilon(x) - \mathbf{u}(x) = \boldsymbol{\mathcal{S}}_{D_\epsilon}[\boldsymbol{\varphi}_\epsilon](x) - \boldsymbol{\mathcal{S}}_D[\boldsymbol{\varphi}](x), \quad x \in \Omega. \tag{5.9}$$

In order to prove the asymptotic expansion for $(\mathbf{u}_\epsilon - \mathbf{u})|_\Omega$ as ϵ tends to 0, we next investigate the asymptotic behavior of $\boldsymbol{\mathcal{S}}_{D_\epsilon}[\boldsymbol{\varphi}_\epsilon]$ as $\epsilon \to 0$.

5.2 Proof of the theorem 1.1

For $\tilde{x} = x + \epsilon h(x)\mathbf{n}(x) \in \partial D_\epsilon$. We have the following Taylor expansion

$$\mathbf{H}\big(x + \epsilon h(x)\mathbf{n}(x)\big) = \mathbf{H}(x) + \epsilon h(x)\frac{\partial \mathbf{H}}{\partial \mathbf{n}}(x) + O(\epsilon^2), \quad x \in \partial D, \tag{5.10}$$

where the remainder $O(\epsilon^2)$ depends only on $\|h\|_{\mathcal{C}^0(\partial D)}$ and $\|X\|_{\mathcal{C}^1(\partial D)}$.

Similarly, by the Taylor expansion, (2.2), and (2.27), we obtain that

$$\begin{aligned}
\frac{\partial \mathbf{H}}{\partial \nu}(\tilde{x}) =& \lambda_0 \nabla \cdot \mathbf{H}(\tilde{x})\mathbf{n}(\tilde{x}) + \mu_0\Big(\nabla \mathbf{H}(\tilde{x}) + (\nabla \mathbf{H})^T(\tilde{x})\Big)\mathbf{n}(\tilde{x}) \\
=& \lambda_0 \nabla \cdot \mathbf{H}(x)\mathbf{n}(x) + \mu_0\Big(\nabla \mathbf{H}(x) + (\nabla \mathbf{H})^T(x)\Big)\mathbf{n}(x) \\
&+ \epsilon h(x)\Big[\lambda_0 \nabla\nabla \cdot \mathbf{H}(x)\cdot \mathbf{n}(x) + \mu_0 \nabla\nabla \mathbf{H}(x)\mathbf{n}(x) + \mu_0 \nabla(\nabla \mathbf{H})^T(x)\mathbf{n}(x)\Big]\mathbf{n}(x) \\
&- \epsilon h'(t)\Big[\lambda_0 \nabla \cdot \mathbf{H}(x)\boldsymbol{\tau}(x) + \mu_0\Big(\nabla \mathbf{H}(x) + (\nabla \mathbf{H})^T(x)\Big)\boldsymbol{\tau}(x)\Big] + O(\epsilon^2) \\
=& \frac{\partial \mathbf{H}}{\partial \nu}(x) + \epsilon \kappa(x) h(x)\frac{\partial \mathbf{H}}{\partial \nu}(x) - \epsilon \frac{d}{dt}\Big(h(x)\big(\mathbb{C}_0 \widehat{\nabla} \mathbf{H}\big)\boldsymbol{\tau}(x)\Big) + O(\epsilon^2), \quad x \in \partial D.
\end{aligned} \tag{5.11}$$

Now, we introduce $(\boldsymbol{\psi}^{(1)}, \boldsymbol{\varphi}^{(1)})$ as a solution to the following system

$$\begin{cases}
\widetilde{\boldsymbol{\mathcal{S}}}_D[\boldsymbol{\psi}^{(1)}]\big|_- - \boldsymbol{\mathcal{S}}_D[\boldsymbol{\varphi}^{(1)}]\big|_+ = h\dfrac{\partial \mathbf{H}}{\partial \mathbf{n}} - \big(\widetilde{\boldsymbol{\mathcal{S}}}_D^{(1)}[\boldsymbol{\psi}] - \boldsymbol{\mathcal{S}}_D^{(1)}[\boldsymbol{\varphi}]\big) \quad \text{on } \partial D, \\
\Big(-\dfrac{1}{2}\mathbf{I} + \widetilde{\boldsymbol{\mathcal{K}}}_D^*\Big)[\boldsymbol{\psi}^{(1)}] - \Big(\dfrac{1}{2}\mathbf{I} + \boldsymbol{\mathcal{K}}_D^*\Big)[\boldsymbol{\varphi}^{(1)}] = \kappa h \dfrac{\partial \mathbf{H}}{\partial \nu} - \dfrac{\partial}{\partial \boldsymbol{\tau}}\Big(h\big(\mathbb{C}_0 \widehat{\nabla}\mathbf{H}\big)\boldsymbol{\tau}\Big) \\
\qquad\qquad -\big(\widetilde{\boldsymbol{\mathcal{K}}}_D^{(1)}[\boldsymbol{\psi}] - \boldsymbol{\mathcal{K}}_D^{(1)}[\boldsymbol{\varphi}]\big) \quad \text{on } \partial D,
\end{cases} \tag{5.12}$$

where $(\boldsymbol{\psi}, \boldsymbol{\varphi})$ is the solution to (5.8). One can easily check the existence and uniqueness of $(\boldsymbol{\psi}^{(1)}, \boldsymbol{\varphi}^{(1)})$ by using the theorem 5.1.

It follows from (5.6), (5.12), and Theorems 4.3 and 4.5 that

$$\begin{cases}
\Big(\widetilde{\boldsymbol{\mathcal{S}}}_D + \epsilon \widetilde{\boldsymbol{\mathcal{S}}}_D^{(1)}\Big)[\widetilde{\boldsymbol{\psi}} - \boldsymbol{\psi} - \epsilon \boldsymbol{\psi}^{(1)}]\big|_- - \Big(\boldsymbol{\mathcal{S}}_D + \epsilon \boldsymbol{\mathcal{S}}_D^{(1)}\Big)[\widetilde{\boldsymbol{\varphi}} - \boldsymbol{\varphi} - \epsilon \boldsymbol{\varphi}^{(1)}]\big|_+ \\
\qquad = \mathbf{H}\circ \Phi_\epsilon - \mathbf{H} - \epsilon h \dfrac{\partial \mathbf{H}}{\partial \mathbf{n}} + O_1(\epsilon^2) \quad \text{on } \partial D, \\
\Big(-\dfrac{1}{2}\mathbf{I} + \widetilde{\boldsymbol{\mathcal{K}}}_D^* + \epsilon \widetilde{\boldsymbol{\mathcal{K}}}_D^{(1)}\Big)[\widetilde{\boldsymbol{\psi}} - \boldsymbol{\psi} - \epsilon \boldsymbol{\psi}^{(1)}] - \Big(\dfrac{1}{2}\mathbf{I} + \boldsymbol{\mathcal{K}}_D^* + \epsilon \boldsymbol{\mathcal{K}}_D^{(1)}\Big)[\widetilde{\boldsymbol{\varphi}} - \boldsymbol{\varphi} - \epsilon \boldsymbol{\varphi}^{(1)}] \\
\qquad = \dfrac{\partial \mathbf{H}}{\partial \nu}\circ \Phi_\epsilon - \dfrac{\partial \mathbf{H}}{\partial \nu} - \epsilon \kappa h \dfrac{\partial \mathbf{H}}{\partial \nu} + \epsilon \dfrac{\partial}{\partial \boldsymbol{\tau}}\Big(h\big(\mathbb{C}_0 \widehat{\nabla}\mathbf{H}\big)\boldsymbol{\tau}\Big) + O_2(\epsilon^2) \quad \text{on } \partial D,
\end{cases} \tag{5.13}$$

with $\widetilde{\boldsymbol{\varphi}} := \boldsymbol{\varphi}_\epsilon \circ \Phi_\epsilon$, $\widetilde{\boldsymbol{\psi}} := \boldsymbol{\psi}_\epsilon \circ \Phi_\epsilon$, and $\|O_1(\epsilon^2)\|_{W_1^2(\partial D)}, \|O_2(\epsilon^2)\|_{L^2(\partial D)} \le C\epsilon^2$, where the constant C depends only on $\lambda_0, \lambda_1, \mu_0, \mu_1$, the $\mathcal{C}^2$-norm of X, and the $\mathcal{C}^1$-norm of h.

The following lemma follows immediately from (5.10), (5.11), (5.13), and the estimate in (5.4).

Lemma 5.3 *For ϵ small enough, there exists C depending only on $\lambda_0, \lambda_1, \mu_0, \mu_1$, the $\mathcal{C}^2$-norm of X, and the $\mathcal{C}^1$-norm of h such that*

$$\Big\|\boldsymbol{\psi}_\epsilon \circ \Phi_\epsilon - \boldsymbol{\psi} - \epsilon \boldsymbol{\psi}^{(1)}\Big\|_{L^2(\partial D)} + \Big\|\boldsymbol{\varphi}_\epsilon \circ \Phi_\epsilon - \boldsymbol{\varphi} - \epsilon \boldsymbol{\varphi}^{(1)}\Big\|_{L^2(\partial D)} \le C\epsilon^2, \tag{5.14}$$

where $(\boldsymbol{\psi}_\epsilon, \boldsymbol{\varphi}_\epsilon), (\boldsymbol{\psi}, \boldsymbol{\varphi})$, *and* $(\boldsymbol{\psi}^{(1)}, \boldsymbol{\varphi}^{(1)})$ *are the solutions to* (5.6), (5.8), *and* (5.12), *respectively.*

Recall that the domain D is separated apart from Ω, then

$$\sup_{x\in\Omega, y\in\partial D} \left|\partial^i \mathbf{\Gamma}(x-y)\right| \le C, \quad i \in \mathbb{N}^2,$$

for some constant $C > 0$ depending on $dist(D, \Omega)$. After the change of variables $\tilde{y} = \Phi_\epsilon(y)$, we get from (2.3), (5.14), and the Taylor expansion of $\mathbf{\Gamma}(x-\tilde{y})$ for $y \in \partial D$, and $x \in \Omega$ fixed that

$$\begin{aligned}
\boldsymbol{\mathcal{S}}_{D_\epsilon}[\boldsymbol{\varphi}_\epsilon](x) &= \int_{\partial D_\epsilon} \mathbf{\Gamma}(x-\tilde{y})\boldsymbol{\varphi}_\epsilon(\tilde{y})d\sigma(\tilde{y}) \\
&= \int_{\partial D} \Big(\mathbf{\Gamma}(x-y) + \epsilon h(y)\nabla\mathbf{\Gamma}(x-y)\mathbf{n}(y)\Big)\Big(\boldsymbol{\varphi}(y) + \epsilon\boldsymbol{\varphi}^{(1)}(y)\Big) \\
&\qquad\qquad \times \Big(1 - \epsilon\kappa(y)h(y)\Big)d\sigma(y) + O(\epsilon^2) \\
&= \boldsymbol{\mathcal{S}}_D[\boldsymbol{\varphi}](x) + \epsilon\Big(\boldsymbol{\mathcal{S}}_D[\boldsymbol{\varphi}^{(1)}](x) - \boldsymbol{\mathcal{S}}_D[\kappa h\boldsymbol{\varphi}](x) + \boldsymbol{\mathcal{D}}^\sharp_D[h\boldsymbol{\varphi}](x)\Big) + O(\epsilon^2). \qquad (5.15)
\end{aligned}$$

The following theorem follows immediately from (5.9) and (5.15).

Theorem 5.4 *Let* ϵ *be small enough. The following pointwise expansion holds for* $x \in \Omega$

$$\mathbf{u}_\epsilon(x) = \mathbf{u}(x) + \epsilon\Big(\boldsymbol{\mathcal{S}}_D[\boldsymbol{\varphi}^{(1)}](x) - \boldsymbol{\mathcal{S}}_D[\kappa h\boldsymbol{\varphi}](x) + \boldsymbol{\mathcal{D}}^\sharp_D[h\boldsymbol{\varphi}](x)\Big) + O(\epsilon^2), \qquad (5.16)$$

where $\boldsymbol{\varphi}$ *and* $\boldsymbol{\varphi}^{(1)}$ *are defined by* (5.8) *and* (5.12), *respectively. The remainder* $O(\epsilon^2)$ *depends only on* $\lambda_0, \lambda_1, \mu_0, \mu_1$, *the* $\mathcal{C}^2$*-norm of* X, *the* $\mathcal{C}^1$*-norm of* h, *and* $dist(\Omega, D)$.

We now prove the following representation theorem for the solution of the transmission problem (1.9) which will be very helpful in the proof of theorem 1.1.

Theorem 5.5 *The solution* $\mathbf{u}_1$ *of* (1.9) *is represented by*

$$\mathbf{u}_1(x) = \begin{cases} \boldsymbol{\mathcal{S}}_D[\boldsymbol{\varphi}^{(1)}](x) - \boldsymbol{\mathcal{S}}_D[\kappa h\boldsymbol{\varphi}](x) + \boldsymbol{\mathcal{D}}^\sharp_D[h\boldsymbol{\varphi}](x), & x \in \mathbb{R}^2\backslash\overline{D}, \\ \widetilde{\boldsymbol{\mathcal{S}}}_D[\boldsymbol{\psi}^{(1)}](x) - \widetilde{\boldsymbol{\mathcal{S}}}_D[\kappa h\boldsymbol{\psi}](x) + \widetilde{\boldsymbol{\mathcal{D}}}^\sharp_D[h\boldsymbol{\psi}](x), & x \in D, \end{cases} \qquad (5.17)$$

where $(\boldsymbol{\psi}, \boldsymbol{\varphi})$ *and* $(\boldsymbol{\psi}^{(1)}, \boldsymbol{\varphi}^{(1)})$ *are defined by* (5.8) *and* (5.12), *respectively.*

Proof. One can easily see that

$$\mathcal{L}_{\lambda_0,\mu_0}\mathbf{u}_1 = 0 \quad \text{in } \mathbb{R}^2\backslash\overline{D}, \qquad\qquad \mathcal{L}_{\lambda_1,\mu_1}\mathbf{u}_1 = 0 \quad \text{in } D.$$

It follows from (3.3), (4.39), (5.7), and (5.12) that

$$
\begin{aligned}
\mathbf{u}_1^i - \mathbf{u}_1^e =& \widetilde{\boldsymbol{\mathcal{S}}}_D[\boldsymbol{\psi}^{(1)}] - \boldsymbol{\mathcal{S}}_D[\boldsymbol{\varphi}^{(1)}] + \boldsymbol{\mathcal{S}}_D[\kappa h\boldsymbol{\varphi}] - \widetilde{\boldsymbol{\mathcal{S}}}_D[\kappa h\boldsymbol{\psi}] + \widetilde{\boldsymbol{\mathcal{D}}}^\sharp_D[h\boldsymbol{\psi}]\big|_- - \boldsymbol{\mathcal{D}}^\sharp_D[h\boldsymbol{\varphi}]\big|_+ \\
=& h\frac{\partial \mathbf{H}}{\partial \mathbf{n}} + \boldsymbol{\mathcal{S}}_D^{(1)}[\boldsymbol{\varphi}] - \widetilde{\boldsymbol{\mathcal{S}}}_D^{(1)}[\boldsymbol{\psi}] + \boldsymbol{\mathcal{S}}_D[\kappa h\boldsymbol{\varphi}] - \widetilde{\boldsymbol{\mathcal{S}}}_D[\kappa h\boldsymbol{\psi}] + \widetilde{\boldsymbol{\mathcal{D}}}^\sharp_D[h\boldsymbol{\psi}]\big|_- - \boldsymbol{\mathcal{D}}^\sharp_D[h\boldsymbol{\varphi}]\big|_+ \\
=& h\Big(\frac{\partial \mathbf{H}}{\partial \mathbf{n}} + \frac{\partial \boldsymbol{\mathcal{S}}_D[\boldsymbol{\varphi}]}{\partial \mathbf{n}}\Big|_+ - \frac{\partial \widetilde{\boldsymbol{\mathcal{S}}}_D[\boldsymbol{\psi}]}{\partial \mathbf{n}}\Big|_-\Big) \\
=& h\big(\nabla \mathbf{u}^e \mathbf{n} - \nabla \mathbf{u}^i \mathbf{n}\big) \\
=& h\big(\mathbb{K}_{0,1}\widehat{\nabla}\mathbf{u}^i\big)\mathbf{n} \quad \text{on } \partial D.
\end{aligned}
$$

Using (5.12), we get

$$
\begin{aligned}
\frac{\partial \mathbf{u}_1}{\partial \widetilde{\nu}}\Big|_- - \frac{\partial \mathbf{u}_1}{\partial \nu}\Big|_+ =& \frac{\partial \widetilde{\boldsymbol{\mathcal{S}}}_D[\boldsymbol{\psi}^{(1)}]}{\partial \widetilde{\nu}}\Big|_- - \frac{\partial \boldsymbol{\mathcal{S}}_D[\boldsymbol{\varphi}^{(1)}]}{\partial \nu}\Big|_+ + \frac{\partial \boldsymbol{\mathcal{S}}_D[\kappa h\boldsymbol{\varphi}]}{\partial \nu}\Big|_+ - \frac{\partial \widetilde{\boldsymbol{\mathcal{S}}}_D[\kappa h\boldsymbol{\psi}]}{\partial \widetilde{\nu}}\Big|_- \\
&+ \frac{\partial \widetilde{\boldsymbol{\mathcal{D}}}^\sharp_D[h\boldsymbol{\psi}]}{\partial \widetilde{\nu}}\Big|_- - \frac{\partial \boldsymbol{\mathcal{D}}^\sharp_D[h\boldsymbol{\varphi}]}{\partial \nu}\Big|_+ \\
=& \kappa h\frac{\partial \mathbf{H}}{\partial \nu} - \frac{\partial}{\partial \boldsymbol{\tau}}\Big(h\big(\mathbb{C}_0\widehat{\nabla}\mathbf{H}\big)\boldsymbol{\tau}\Big) - \widetilde{\boldsymbol{\mathcal{K}}}_D^{(1)}[\boldsymbol{\psi}] + \boldsymbol{\mathcal{K}}_D^{(1)}[\boldsymbol{\varphi}] \\
&+ \frac{\partial \boldsymbol{\mathcal{S}}_D[\tau h\boldsymbol{\varphi}]}{\partial \nu}\Big|_+ - \frac{\partial \widetilde{\boldsymbol{\mathcal{S}}}_D[\kappa h\boldsymbol{\psi}]}{\partial \widetilde{\nu}}\Big|_- + \frac{\partial \widetilde{\boldsymbol{\mathcal{D}}}^\sharp_D[h\boldsymbol{\psi}]}{\partial \widetilde{\nu}}\Big|_- - \frac{\partial \boldsymbol{\mathcal{D}}^\sharp_D[h\boldsymbol{\varphi}]}{\partial \nu}\Big|_+ .
\end{aligned}
$$

According to (3.1), (4.36), (5.7), and (5.11) we obtain

$$
\frac{\partial \mathbf{u}_1}{\partial \widetilde{\nu}}\Big|_- - \frac{\partial \mathbf{u}_1}{\partial \nu}\Big|_+ = \frac{\partial}{\partial \boldsymbol{\tau}}\big(h(\mathbb{C}_1\widehat{\nabla}\mathbf{u}^i)\boldsymbol{\tau}\big) - \frac{\partial}{\partial \boldsymbol{\tau}}\big(h(\mathbb{C}_0\widehat{\nabla}\mathbf{u}^e)\boldsymbol{\tau}\big) = \frac{\partial}{\partial \boldsymbol{\tau}}\big(h[\mathbb{C}_1 - \mathbb{M}_{0,1}]\widehat{\nabla}\mathbf{u}^i)\boldsymbol{\tau}\big).
$$

Now, let us check the condition

$$
\boldsymbol{\mathcal{S}}_D[\boldsymbol{\varphi}^{(1)} - \kappa h\boldsymbol{\varphi}](x) \to 0 \quad \text{as } |x| \to \infty. \tag{5.18}
$$

To do this, we rewrite the system of equations (5.12)

$$
\left\{
\begin{aligned}
&\widetilde{\boldsymbol{\mathcal{S}}}_D\big[\boldsymbol{\psi}^{(1)} - \kappa h\boldsymbol{\psi}\big]\Big|_- - \boldsymbol{\mathcal{S}}_D\big[\boldsymbol{\varphi}^{(1)} - \kappa h\boldsymbol{\varphi}\big]\Big|_+ = h\big(\mathbb{K}_{0,1}\widehat{\nabla}\mathbf{u}^i\big)\mathbf{n} - \widetilde{\boldsymbol{\mathcal{D}}}^\sharp_D[h\boldsymbol{\psi}]\big|_- + \boldsymbol{\mathcal{D}}^\sharp_D[h\boldsymbol{\varphi}]\big|_+ \\
&\frac{\partial \widetilde{\boldsymbol{\mathcal{S}}}_D}{\partial \widetilde{\nu}}\big[\boldsymbol{\psi}^{(1)} - \kappa h\boldsymbol{\psi}\big]\Big|_- - \frac{\partial \boldsymbol{\mathcal{S}}_D}{\partial \nu}\big[\boldsymbol{\varphi}^{(1)} - \kappa h\boldsymbol{\varphi}\big]\Big|_+ = \frac{\partial \boldsymbol{\mathcal{D}}^\sharp_D[h\boldsymbol{\varphi}]}{\partial \nu}\Big|_+ - \frac{\partial \widetilde{\boldsymbol{\mathcal{D}}}^\sharp_D[h\boldsymbol{\psi}]}{\partial \tilde{\nu}}\Big|_- \\
&\qquad\qquad + \frac{\partial}{\partial \boldsymbol{\tau}}\Big(h\big(\mathbb{C}_1\widehat{\nabla}\mathbf{u}^i\big)\boldsymbol{\tau}\Big) - \frac{\partial}{\partial \boldsymbol{\tau}}\Big(h\big(\mathbb{C}_0\widehat{\nabla}\mathbf{u}^e\big)\boldsymbol{\tau}\Big).
\end{aligned}
\right. \tag{5.19}
$$

It is clear that

$$
\int_{\partial D}\Big[\frac{\partial}{\partial \boldsymbol{\tau}}\Big(h\big(\mathbb{C}_1\widehat{\nabla}\mathbf{u}^i\big)\boldsymbol{\tau}\Big) - \frac{\partial}{\partial \boldsymbol{\tau}}\Big(h\big(\mathbb{C}_0\widehat{\nabla}\mathbf{u}^e\big)\boldsymbol{\tau}\Big)\Big] \cdot \theta_m \, d\sigma = 0 \quad \text{for } m = 1, 2.
$$

We have

$$\int_{\partial D} \frac{d}{dt}\Big(h\big(\mathbb{C}_0\widehat{\nabla}\mathbf{u}^e(x)\big)\boldsymbol{\tau}(x)\Big)\cdot\theta_3(x)d\sigma = -\int_{\partial D} h(x)\big(\mathbb{C}_0\widehat{\nabla}\mathbf{u}^e(x)\big)\boldsymbol{\tau}(x)\cdot\mathbf{n}(x)d\sigma$$
$$= -\mu_0\int_{\partial D} h(x)\Big(\nabla\mathbf{u}^e(x)+(\nabla\mathbf{u}^e)^T(x)\Big)\boldsymbol{\tau}(x)\cdot\mathbf{n}(x)d\sigma$$
$$= -\mu_0\int_{\partial D} h(x)\Big(\nabla\mathbf{u}^e(x)+(\nabla\mathbf{u}^e)^T(x)\Big)\mathbf{n}(x)\cdot\boldsymbol{\tau}(x)d\sigma$$
$$= -\int_{\partial D} h(x)\frac{\partial\mathbf{u}^e}{\partial\nu}(x)\cdot\boldsymbol{\tau}(x)d\sigma.$$

Similarly, we get

$$\int_{\partial D} \frac{d}{dt}\Big(h(x)\big(\mathbb{C}_1\widehat{\nabla}\mathbf{u}^i(x)\big)\boldsymbol{\tau}(x)\Big)\cdot\theta_3(x)d\sigma = -\int_{\partial D} h(x)\frac{\partial\mathbf{u}^i}{\partial\tilde{\nu}}(x)\cdot\boldsymbol{\tau}(x)d\sigma.$$

Thus

$$\int_{\partial D}\Big[\frac{\partial}{\partial\boldsymbol{\tau}}\Big(h\big(\mathbb{C}_1\widehat{\nabla}\mathbf{u}^i\big)\boldsymbol{\tau}\Big)-\frac{\partial}{\partial\boldsymbol{\tau}}\Big(h\big(\mathbb{C}_0\widehat{\nabla}\mathbf{u}^e\big)\boldsymbol{\tau}\Big)\Big]\cdot\theta_3 d\sigma = \int_{\partial D} h\Big(\frac{\partial\mathbf{u}^e}{\partial\nu}-\frac{\partial\mathbf{u}^i}{\partial\tilde{\nu}}\Big)\cdot\boldsymbol{\tau}d\sigma = 0.$$

Consequently,

$$\frac{\partial}{\partial\boldsymbol{\tau}}\Big(h\big(\mathbb{C}_1\widehat{\nabla}\mathbf{u}^i\big)\boldsymbol{\tau}\Big)-\frac{\partial}{\partial\boldsymbol{\tau}}\Big(h\big(\mathbb{C}_0\widehat{\nabla}\mathbf{u}^e\big)\boldsymbol{\tau}\Big)\in L^2_\Psi(\partial D).$$

By (4.1), $\partial\widetilde{\boldsymbol{\mathcal{S}}}_D[\boldsymbol{\psi}^{(1)}-\kappa h\boldsymbol{\psi}]/\partial\tilde{\nu}\big|_-$ and $\partial\widetilde{\boldsymbol{\mathcal{D}}}^\sharp_D[h\boldsymbol{\psi}]/\partial\tilde{\nu}\big|_-\in L^2_\Psi(\partial D)$. It then follows from (5.19) that $\partial\boldsymbol{\mathcal{S}}_D[\boldsymbol{\varphi}^{(1)}-\kappa h\boldsymbol{\varphi}]/\partial\nu\big|_+ + \partial\boldsymbol{\mathcal{D}}^\sharp_D[h\boldsymbol{\varphi}]/\partial\nu\big|_+\in L^2_\Psi(\partial D)$. Since

$$\boldsymbol{\varphi}^{(1)}-\kappa h\boldsymbol{\varphi}+\frac{\partial}{\partial\boldsymbol{\tau}}\Big(h\langle\boldsymbol{\varphi},\boldsymbol{\tau}\rangle\mathbf{n}+\frac{\lambda_0}{2\mu_0+\lambda_0}h\langle\boldsymbol{\varphi},\mathbf{n}\rangle\boldsymbol{\tau}\Big) = \frac{\partial\boldsymbol{\mathcal{S}}_D}{\partial\nu}\big[\boldsymbol{\varphi}^{(1)}-\kappa h\boldsymbol{\varphi}\big]\Big|_+ + \frac{\partial\boldsymbol{\mathcal{D}}^\sharp_D[h\boldsymbol{\varphi}]}{\partial\nu}\Big|_+$$
$$-\frac{\partial\boldsymbol{\mathcal{S}}_D}{\partial\nu}\big[\boldsymbol{\varphi}^{(1)}-\kappa h\boldsymbol{\varphi}\big]\Big|_- - \frac{\partial\boldsymbol{\mathcal{D}}^\sharp_D[h\boldsymbol{\varphi}]}{\partial\nu}\Big|_-,$$

with $\partial\boldsymbol{\mathcal{S}}_D[\boldsymbol{\varphi}^{(1)}-\kappa h\boldsymbol{\varphi}]/\partial\nu\big|_-, \partial\boldsymbol{\mathcal{D}}^\sharp_D[h\boldsymbol{\varphi}]/\partial\nu\big|_-\in L^2_\Psi(\partial D)$, see (4.1). Then

$$\boldsymbol{\varphi}^{(1)}-\kappa h\boldsymbol{\varphi}+\frac{\partial}{\partial\boldsymbol{\tau}}\Big(h\langle\boldsymbol{\varphi},\boldsymbol{\tau}\rangle\mathbf{n}+\frac{\lambda_0}{2\mu_0+\lambda_0}h\langle\boldsymbol{\varphi},\mathbf{n}\rangle\boldsymbol{\tau}\Big)\in L^2_\Psi(\partial D).$$

Therefore, we have

$$\boldsymbol{\mathcal{S}}_D[\boldsymbol{\varphi}^{(1)}-\kappa h\boldsymbol{\varphi}](x) = \boldsymbol{\Gamma}(x)\int_{\partial D}(\boldsymbol{\varphi}^{(1)}-\kappa h\boldsymbol{\varphi})d\sigma + O(|x|^{-1})$$
$$= \boldsymbol{\Gamma}(x)\int_{\partial D}\Big(\boldsymbol{\varphi}^{(1)}-\kappa h\boldsymbol{\varphi}+\frac{\partial}{\partial\boldsymbol{\tau}}\Big(h\langle\boldsymbol{\varphi},\boldsymbol{\tau}\rangle\mathbf{n}+\frac{\lambda_0}{2\mu_0+\lambda_0}h\langle\boldsymbol{\varphi},\mathbf{n}\rangle\boldsymbol{\tau}\Big)\Big)d\sigma$$
$$-\boldsymbol{\Gamma}(x)\int_{\partial D}\frac{\partial}{\partial\boldsymbol{\tau}}\Big(h\langle\boldsymbol{\varphi},\boldsymbol{\tau}\rangle\mathbf{n}+\frac{\lambda_0}{2\mu_0+\lambda_0}h\langle\boldsymbol{\varphi},\mathbf{n}\rangle\boldsymbol{\tau}\Big)d\sigma + O(|x|^{-1})$$
$$= O(|x|^{-1}) \quad \text{as } |x|\to\infty.$$

Thus $\mathbf{u}_1$ defined by (5.17) satisfies $\mathbf{u}_1(x) = O(|x|^{-1})$ as $|x| \to \infty$. This completes the proof of the theorem 5.5.

The theorem 1.1 immediately follows from the integral representation of $\mathbf{u}_1$ in (5.17) and the theorem 5.4.

5.3 Proof of the theorem 1.2

The following corollary can be proved in exactly the same manner as Theorem 1.1.

Corollary 5.6 *Let* $\mathbf{u}$ *and* $\mathbf{u}_\epsilon$ *be the solutions to* (1.5) *and* (1.7)*, respectively. Let* Ω *be a bounded region outside the inclusion* D*, and away from* ∂D*. For* $x \in \Omega$*, the following pointwise asymptotic expansion holds:*

$$\frac{\partial \mathbf{u}_\epsilon}{\partial \nu}(x) = \frac{\partial \mathbf{u}}{\partial \nu}(x) + \epsilon \frac{\partial \mathbf{u}_1}{\partial \nu}(x) + O(\epsilon^2), \tag{5.20}$$

where the remainder $O(\epsilon^2)$ *depends only on* $\lambda_0, \lambda_1, \mu_0, \mu_1$*, the* $\mathcal{C}^2$*-norm of* X*, the* $\mathcal{C}^1$*-norm of* h*,* $dist(\Omega, \partial D)$*, and* $\mathbf{u}_1$ *is the unique solution of* (1.9)*.*

Let S be a Lipschitz closed curve enclosing D away from ∂D. Let $\mathbf{v}$ be the solution to (1.12). It follows from (1.8), (4.22), and (5.20) that

$$\int_S (\mathbf{u}_\epsilon - \mathbf{u}) \cdot \frac{\partial \mathbf{F}}{\partial \nu} d\sigma - \int_S \big(\frac{\partial \mathbf{u}_\epsilon}{\partial \nu} - \frac{\partial \mathbf{u}}{\partial \nu}\big) \cdot \mathbf{F} d\sigma = \epsilon \int_S \Big(\mathbf{u}_1 \cdot \frac{\partial \mathbf{v}}{\partial \nu} - \frac{\partial \mathbf{u}_1}{\partial \nu} \cdot \mathbf{v}\Big) d\sigma + O(\epsilon^2).$$

By using Lemma 4.2 to the integral on the right-hand side, we get

$$\int_S \Big(\frac{\partial \mathbf{v}}{\partial \nu} \cdot \mathbf{u}_1 - \mathbf{v} \cdot \frac{\partial \mathbf{u}_1}{\partial \nu}\Big) d\sigma = \int_{\partial D} \Big(\frac{\partial \mathbf{v}^e}{\partial \nu} \cdot \mathbf{u}_1^e - \mathbf{v}^e \cdot \frac{\partial \mathbf{u}_1^e}{\partial \nu}\Big) d\sigma.$$

According to the jump conditions for $\mathbf{u}_1$ in (1.9), we deduce that

$$\begin{aligned}\int_S \Big(\frac{\partial \mathbf{v}}{\partial \nu} \cdot \mathbf{u}_1 - \mathbf{v} \cdot \frac{\partial \mathbf{u}_1}{\partial \nu}\Big) d\sigma &= \int_{\partial D} \Big(\frac{\partial \mathbf{v}^i}{\partial \tilde{\nu}} \cdot \mathbf{u}_1^i - \mathbf{v}^i \cdot \frac{\partial \mathbf{u}_1^i}{\partial \tilde{\nu}}\Big) d\sigma \\ &\quad - \int_{\partial D} h\big(\mathbb{K}_{0,1} \widehat{\nabla} \mathbf{u}^i\big)\mathbf{n} \cdot \big(\mathbb{C}_1 \widehat{\nabla} \mathbf{v}^i\big)\mathbf{n} d\sigma \\ &\quad + \int_{\partial D} \frac{\partial}{\partial \boldsymbol{\tau}} \Big(h\big([\mathbb{C}_1 - \mathbb{M}_{0,1}] \widehat{\nabla} \mathbf{u}^i\big)\boldsymbol{\tau}\Big) \cdot \mathbf{v}^i d\sigma. \end{aligned} \tag{5.21}$$

It follows from (4.21) that

$$\int_{\partial D} \Big(\frac{\partial \mathbf{v}^i}{\partial \tilde{\nu}} \cdot \mathbf{u}_1^i - \mathbf{v}^i \cdot \frac{\partial \mathbf{u}_1^i}{\partial \tilde{\nu}}\Big) d\sigma = 0. \tag{5.22}$$

We have

$$\int_{\partial D} \frac{\partial}{\partial \boldsymbol{\tau}} \Big(h\big([\mathbb{C}_1 - \mathbb{M}_{0,1}] \widehat{\nabla} \mathbf{u}^i\big)\boldsymbol{\tau}\Big) \cdot \mathbf{v}^i d\sigma = -\int_{\partial D} h\big([\mathbb{C}_1 - \mathbb{M}_{0,1}] \widehat{\nabla} \mathbf{u}^i\big)\boldsymbol{\tau} \cdot \nabla \mathbf{v}^i \boldsymbol{\tau} d\sigma. \tag{5.23}$$

One can easily check that

$$\big([\mathbb{C}_1 - \mathbb{M}_{0,1}] \widehat{\nabla} \mathbf{u}^i\big)\boldsymbol{\tau} \cdot \nabla \mathbf{v}^i \boldsymbol{\tau} = \big([\mathbb{C}_1 - \mathbb{M}_{0,1}] \widehat{\nabla} \mathbf{u}^i\big)\boldsymbol{\tau} \cdot \widehat{\nabla} \mathbf{v}^i \boldsymbol{\tau}. \tag{5.24}$$

We finally obtain from (5.21)-(5.24) the relationship between traction-displacement measurements and the shape deformation h (1.13).

6 Asymptotic expansion of EMTs

We introduce the notion of EMTs associated with D and Lamé parameters (λ_0, μ_0) for the background and (λ_1, μ_1) for D as follows (see [4, 5]): For multi-index $\alpha \in \mathbb{N}^2$ and $j = 1, 2$, let the pair $(\mathbf{f}_\alpha^j, \mathbf{g}_\alpha^j)$ in $L^2(\partial D) \times L^2(\partial D)$ be the unique solution to

$$\begin{cases} \widetilde{\boldsymbol{\mathcal{S}}}_D[\mathbf{f}_\alpha^j]\big|_- - \boldsymbol{\mathcal{S}}_D[\mathbf{g}_\alpha^j]\big|_+ = x^\alpha \mathbf{e}_j & \text{on } \partial D, \\ \dfrac{\partial \widetilde{\boldsymbol{\mathcal{S}}}_D[\mathbf{f}_\alpha^j]}{\partial \widetilde{\nu}}\Big|_- - \dfrac{\partial \boldsymbol{\mathcal{S}}_D[\mathbf{g}_\alpha^j]}{\partial \nu}\Big|_+ = \dfrac{\partial (x^\alpha \mathbf{e}_j)}{\partial \nu} & \text{on } \partial D. \end{cases} \tag{6.1}$$

Now for multi-index $\beta \in \mathbb{N}^2$, the EMTs are defined by

$$M_{\alpha\beta}^j = (m_{\alpha\beta 1}^j, m_{\alpha\beta 2}^j) := \int_{\partial D} y^\beta \mathbf{g}_\alpha^j(y) d\sigma(y). \tag{6.2}$$

Let $\mathbf{H}(x) = \displaystyle\sum_{j=1}^2 \sum_{\alpha \in \mathbb{N}^2} a_j^\alpha x^\alpha e_j$ and $\mathbf{F}(x) = \displaystyle\sum_{k=1}^2 \sum_{\beta \in \mathbb{N}^2} b_k^\beta x^\beta e_k$ be tow polynomials satisfying $\nabla \cdot \big(\mathbb{C}_0 \widehat{\nabla} \cdot\big) = 0$ in $\mathbb{R}^2$. The EMTs $m_{\alpha\beta k}^j(D)$ associated with D satisfy

$$\sum_{\alpha\beta jk} a_j^\alpha b_k^\beta m_{\alpha\beta k}^j(D) = \int_{\partial D} \mathbf{F}(y) \boldsymbol{\varphi}(y) d\sigma(y), \tag{6.3}$$

where $\boldsymbol{\varphi}$ is defined in (5.8).

The perturbed $m_{\alpha\beta k}^j(D_\epsilon)$ satisfy

$$\sum_{\alpha\beta jk} a_j^\alpha b_k^\beta m_{\alpha\beta k}^j(D_\epsilon) = \int_{\partial D_\epsilon} \mathbf{F}(\tilde{y}) \boldsymbol{\varphi}_\epsilon(\tilde{y}) d\sigma_\epsilon(\tilde{y}), \tag{6.4}$$

where $\boldsymbol{\varphi}_\epsilon$ is defined in (5.6).

The purpose of this section is to prove the asymptotic behavior of $\sum_{\alpha\beta jk} a_j^\alpha b_k^\beta m_{\alpha\beta k}^j(D_\epsilon)$ defined in (6.4) as ϵ tends to zero.

By Taylor expansion, we have

$$\mathbf{F}(\tilde{y}) = \mathbf{F}\big(y + \epsilon h(y)\mathbf{n}(y)\big) = \mathbf{F}(y) + \epsilon h(y) \frac{\partial \mathbf{F}}{\partial \mathbf{n}}(y) + O(\epsilon^2), \quad y \in \partial D.$$

It follows from Lemma 5.3 that

$$\boldsymbol{\varphi}_\epsilon(\tilde{y}) = \boldsymbol{\varphi}_\epsilon\big(y + \epsilon h(y)\mathbf{n}(y)\big) = \boldsymbol{\varphi}(y) + \epsilon \boldsymbol{\varphi}^{(1)}(y) + O(\epsilon^2), \quad y \in \partial D,$$

where $\boldsymbol{\varphi}^{(1)}$ is defined in (5.12).

Recall that $d\sigma_\epsilon(\tilde{y}) = \big(1 - \epsilon\kappa h(y)\big) d\sigma(y) + O(\epsilon^2)$ for $y \in \partial D$. After the change of variables

$\tilde{y} = y + \epsilon h(y)\mathbf{n}(y)$, we get from (6.4) that

$$\sum_{\alpha\beta jk} a_j^\alpha b_k^\beta m_{\alpha\beta k}^j(D_\epsilon) = \int_{\partial D} \Big(\mathbf{F} + \epsilon h \frac{\partial \mathbf{F}}{\partial \mathbf{n}}\Big) \cdot \Big(\boldsymbol{\varphi} + \epsilon \boldsymbol{\varphi}^{(1)}\Big)\big(1 - \epsilon\kappa h\big) d\sigma + O(\epsilon^2)$$

$$= \sum_{\alpha\beta jk} a_j^\alpha b_k^\beta m_{\alpha\beta k}^j(D) + \epsilon \int_{\partial D} \mathbf{F} \cdot \Big(\boldsymbol{\varphi}^{(1)} - \kappa h \boldsymbol{\varphi}\Big) d\sigma$$

$$+ \epsilon \int_{\partial D} h \frac{\partial \mathbf{F}}{\partial \mathbf{n}} \cdot \boldsymbol{\varphi} d\sigma + O(\epsilon^2). \tag{6.5}$$

From (4.5), we have

$$\int_{\partial D} \mathbf{F} \cdot \Big(\boldsymbol{\varphi}^{(1)} - \kappa h \boldsymbol{\varphi}\Big) d\sigma = \int_{\partial D} \mathbf{F} \cdot \Big(\frac{\partial \boldsymbol{\mathcal{S}}_D[\boldsymbol{\varphi}^{(1)} - \kappa h \boldsymbol{\varphi}]}{\partial \nu}\Big|_+ - \frac{\partial \boldsymbol{\mathcal{S}}_D[\boldsymbol{\varphi}^{(1)} - \kappa h \boldsymbol{\varphi}]}{\partial \nu}\Big|_- \Big) d\sigma.$$

By using (4.22) and (5.18), we get

$$\int_{\partial D} \mathbf{F} \cdot \frac{\partial \boldsymbol{\mathcal{S}}_D[\boldsymbol{\varphi}^{(1)} - \kappa h \boldsymbol{\varphi}]}{\partial \nu}\Big|_+ d\sigma = \int_{\partial D} (\mathbf{F} - \mathbf{v}^e) \cdot \frac{\partial \boldsymbol{\mathcal{S}}_D[\boldsymbol{\varphi}^{(1)} - \kappa h \boldsymbol{\varphi}]}{\partial \nu}\Big|_+ d\sigma$$

$$+ \int_{\partial D} \mathbf{v}^e \cdot \frac{\partial \boldsymbol{\mathcal{S}}_D[\boldsymbol{\varphi}^{(1)} - \kappa h \boldsymbol{\varphi}]}{\partial \nu}\Big|_+ d\sigma$$

$$= \int_{\partial D} \Big(\frac{\partial \mathbf{F}}{\partial \nu} - \frac{\partial \mathbf{v}^e}{\partial \nu}\Big) \cdot \boldsymbol{\mathcal{S}}_D[\boldsymbol{\varphi}^{(1)} - \kappa h \boldsymbol{\varphi}] d\sigma$$

$$+ \int_{\partial D} \mathbf{v}^e \cdot \frac{\partial \boldsymbol{\mathcal{S}}_D[\boldsymbol{\varphi}^{(1)} - \kappa h \boldsymbol{\varphi}]}{\partial \nu}\Big|_+ d\sigma.$$

Since, by using (4.21), we get

$$\int_{\partial D} \frac{\partial \mathbf{F}}{\partial \nu} \cdot \boldsymbol{\mathcal{S}}_D[\boldsymbol{\varphi}^{(1)} - \kappa h \boldsymbol{\varphi}] d\sigma - \int_{\partial D} \mathbf{F} \cdot \frac{\partial \boldsymbol{\mathcal{S}}_D[\boldsymbol{\varphi}^{(1)} - \kappa h \boldsymbol{\varphi}]}{\partial \nu}\Big|_- d\sigma = 0.$$

It then follows from (1.12) and (5.19) that

$$\int_{\partial D} \mathbf{F} \cdot \Big(\boldsymbol{\varphi}^{(1)} - \kappa h \boldsymbol{\varphi}\Big) d\sigma$$

$$= \int_{\partial D} \mathbf{v}^e \cdot \frac{\partial \boldsymbol{\mathcal{S}}_D[\boldsymbol{\varphi}^{(1)} - \kappa h \boldsymbol{\varphi}]}{\partial \nu}\Big|_+ d\sigma - \int_{\partial D} \frac{\partial \mathbf{v}^e}{\partial \nu} \cdot \boldsymbol{\mathcal{S}}_D[\boldsymbol{\varphi}^{(1)} - \kappa h \boldsymbol{\varphi}] d\sigma$$

$$= \int_{\partial D} \mathbf{v}^i \cdot \frac{\partial \widetilde{\boldsymbol{\mathcal{S}}}_D[\boldsymbol{\psi}^{(1)} - \kappa h \boldsymbol{\psi}]}{\partial \tilde{\nu}}\Big|_- d\sigma - \int_{\partial D} \frac{\partial \mathbf{v}^i}{\partial \tilde{\nu}} \cdot \widetilde{\boldsymbol{\mathcal{S}}}_D[\boldsymbol{\psi}^{(1)} - \kappa h \boldsymbol{\psi}] d\sigma$$

$$+ \int_{\partial D} \mathbf{v}^i \cdot \frac{\partial \widetilde{\boldsymbol{\mathcal{D}}}_D^\sharp[h\boldsymbol{\psi}]}{\partial \tilde{\nu}}\Big|_- d\sigma - \int_{\partial D} \frac{\partial \mathbf{v}^i}{\partial \tilde{\nu}} \cdot \widetilde{\boldsymbol{\mathcal{D}}}_D^\sharp[h\boldsymbol{\psi}]\big|_- d\sigma$$

$$- \int_{\partial D} \mathbf{v}^e \cdot \frac{\partial \boldsymbol{\mathcal{D}}_D^\sharp[h\boldsymbol{\varphi}]}{\partial \nu}\Big|_+ d\sigma + \int_{\partial D} \frac{\partial \mathbf{v}^e}{\partial \nu} \cdot \boldsymbol{\mathcal{D}}_D^\sharp[h\boldsymbol{\varphi}]\big|_+ d\sigma$$

$$- \int_{\partial D} \mathbf{v}^i \cdot \frac{\partial}{\partial \boldsymbol{\tau}}\Big(h\big([\mathbb{C}_1 - \mathbb{M}_{0,1}]\widehat{\nabla}\mathbf{u}^i\big)\boldsymbol{\tau}\Big) d\sigma + \int_{\partial D} \frac{\partial \mathbf{v}^i}{\partial \tilde{\nu}} \cdot \Big(h(\mathbb{K}_{0,1}\widehat{\nabla}\mathbf{u}^i)\mathbf{n}\Big) d\sigma. \tag{6.6}$$

One can easily see that

$$
\begin{aligned}
&-\int_{\partial D} \mathbf{v}^i \cdot \frac{\partial}{\partial \boldsymbol{\tau}}\Big(h\big([\mathbb{C}_1-\mathbb{M}_{0,1}]\widehat{\nabla}\mathbf{u}^i\big)\boldsymbol{\tau}\Big)d\sigma + \int_{\partial D} \frac{\partial \mathbf{v}^i}{\partial \tilde{\nu}} \cdot \Big(h(\mathbb{K}_{0,1}\widehat{\nabla}\mathbf{u}^i)\mathbf{n}\Big)d\sigma \\
&= \int_{\partial D} h\bigg(\big([\mathbb{C}_1-\mathbb{M}_{0,1}]\widehat{\nabla}\mathbf{u}^i\big)\boldsymbol{\tau}\cdot\nabla\mathbf{v}^i\boldsymbol{\tau} + (\mathbb{K}_{0,1}\widehat{\nabla}\mathbf{u}^i)\mathbf{n}\cdot\big(\mathbb{C}_1\widehat{\nabla}\mathbf{v}^i\big)\mathbf{n}\bigg)d\sigma. \qquad (6.7)
\end{aligned}
$$

We now apply (4.21) to obtain that

$$
\int_{\partial D} \mathbf{v}^i \cdot \frac{\partial \widetilde{\boldsymbol{\mathcal{S}}}_D[\boldsymbol{\psi}^{(1)}-\kappa h\boldsymbol{\psi}]}{\partial \tilde{\nu}}\Big|_- d\sigma - \int_{\partial D} \frac{\partial \mathbf{v}^i}{\partial \tilde{\nu}} \cdot \widetilde{\boldsymbol{\mathcal{S}}}_D[\boldsymbol{\psi}^{(1)}-\kappa h\boldsymbol{\psi}]d\sigma = 0, \qquad (6.8)
$$

$$
\int_{\partial D} \mathbf{v}^i \cdot \frac{\partial \widetilde{\boldsymbol{\mathcal{D}}}^\sharp_D[h\boldsymbol{\psi}]}{\partial \tilde{\nu}}\Big|_- d\sigma - \int_{\partial D} \frac{\partial \mathbf{v}^i}{\partial \tilde{\nu}} \cdot \widetilde{\boldsymbol{\mathcal{D}}}^\sharp_D[h\boldsymbol{\psi}]\big|_- d\sigma = 0. \qquad (6.9)
$$

It follows from (4.21), (4.22), (4.33), and the proposition 4.1 that

$$
\begin{aligned}
&\int_{\partial D} \frac{\partial \mathbf{v}^e}{\partial \nu} \cdot \boldsymbol{\mathcal{D}}^\sharp_D[h\boldsymbol{\varphi}]\big|_+ d\sigma - \int_{\partial D} \mathbf{v}^e \cdot \frac{\partial \boldsymbol{\mathcal{D}}^\sharp_D[h\boldsymbol{\varphi}]}{\partial \nu}\Big|_+ d\sigma \\
&= \int_{\partial D} \frac{\partial \mathbf{v}^e}{\partial \nu} \cdot \boldsymbol{\mathcal{D}}^\sharp_D[h\boldsymbol{\varphi}]\big|_+ d\sigma - \int_{\partial D} (\mathbf{v}^e - \mathbf{F}) \cdot \frac{\partial \boldsymbol{\mathcal{D}}^\sharp_D[h\boldsymbol{\varphi}]}{\partial \nu}\Big|_+ d\sigma - \int_{\partial D} \mathbf{F} \cdot \frac{\partial \boldsymbol{\mathcal{D}}^\sharp_D[h\boldsymbol{\varphi}]}{\partial \nu}\Big|_+ d\sigma \\
&= \int_{\partial D} \frac{\partial \mathbf{F}}{\partial \nu} \cdot \boldsymbol{\mathcal{D}}^\sharp_D[h\boldsymbol{\varphi}]\big|_+ d\sigma - \int_{\partial D} \mathbf{F} \cdot \frac{\partial \boldsymbol{\mathcal{D}}^\sharp_D[h\boldsymbol{\varphi}]}{\partial \nu}\Big|_+ d\sigma \\
&= -\int_{\partial D} \mathbf{F} \cdot \frac{\partial}{\partial \boldsymbol{\tau}}\Big(\langle h\boldsymbol{\varphi}, \boldsymbol{\tau}\rangle\mathbf{n} + \frac{\lambda_0}{2\mu_0+\lambda_0}\langle h\boldsymbol{\varphi}, \mathbf{n}\rangle\boldsymbol{\tau}\Big)d\sigma \\
&\quad + \int_{\partial D} \frac{\partial \mathbf{F}}{\partial \nu} \cdot \boldsymbol{\mathcal{D}}^\sharp_D[h\boldsymbol{\varphi}]\big|_+ d\sigma - \int_{\partial D} \mathbf{F} \cdot \frac{\partial \boldsymbol{\mathcal{D}}^\sharp_D[h\boldsymbol{\varphi}]}{\partial \nu}\Big|_- d\sigma \\
&= \int_{\partial D} \Big(\langle h\boldsymbol{\varphi}, \boldsymbol{\tau}\rangle\langle\nabla\mathbf{F}\boldsymbol{\tau}, \mathbf{n}\rangle + \frac{\lambda_0}{2\mu_0+\lambda_0}\langle h\boldsymbol{\varphi}, \mathbf{n}\rangle\langle\nabla\mathbf{F}\boldsymbol{\tau}, \boldsymbol{\tau}\rangle\Big)d\sigma \\
&\quad + \int_{\partial D} \frac{\partial \mathbf{F}}{\partial \nu} \cdot \boldsymbol{\mathcal{D}}^\sharp_D[h\boldsymbol{\varphi}]\big|_+ d\sigma - \int_{\partial D} \frac{\partial \mathbf{F}}{\partial \nu} \cdot \boldsymbol{\mathcal{D}}^\sharp_D[h\boldsymbol{\varphi}]\big|_- d\sigma \\
&= \int_{\partial D} h\Big(\langle\nabla\mathbf{F}\boldsymbol{\tau}, \mathbf{n}\rangle\boldsymbol{\tau} + \frac{\lambda_0}{2\mu_0+\lambda_0}\langle\nabla\mathbf{F}\boldsymbol{\tau}, \boldsymbol{\tau}\rangle\mathbf{n}\Big) \cdot \boldsymbol{\varphi}\, d\sigma \\
&\quad + \int_{\partial D} h\bigg[\frac{\partial \boldsymbol{\mathcal{S}}_D[\frac{\partial \mathbf{F}}{\partial \nu}]}{\partial \mathbf{n}}\Big|_- - \frac{\partial \boldsymbol{\mathcal{S}}_D[\frac{\partial \mathbf{F}}{\partial \nu}]}{\partial \mathbf{n}}\Big|_+\bigg] \cdot \boldsymbol{\varphi}\, d\sigma.
\end{aligned}
$$

By using (1.4), (4.17), and the identity $\nabla \cdot \mathbf{F} = \langle\nabla\mathbf{F}\mathbf{n}, \mathbf{n}\rangle + \langle\nabla\mathbf{F}\boldsymbol{\tau}, \boldsymbol{\tau}\rangle$, we get

$$
\frac{\partial \boldsymbol{\mathcal{S}}_D[\frac{\partial \mathbf{F}}{\partial \nu}]}{\partial \mathbf{n}}\Big|_- - \frac{\partial \boldsymbol{\mathcal{S}}_D[\frac{\partial \mathbf{F}}{\partial \nu}]}{\partial \mathbf{n}}\Big|_+ = -\nabla\mathbf{F}\mathbf{n} - \langle\nabla\mathbf{F}\boldsymbol{\tau}, \mathbf{n}\rangle\boldsymbol{\tau} - \frac{\lambda_0}{2\mu_0+\lambda_0}\langle\nabla\mathbf{F}\boldsymbol{\tau}, \boldsymbol{\tau}\rangle\mathbf{n},
$$

and hence

$$\int_{\partial D} \frac{\partial \mathbf{v}^e}{\partial \nu} \cdot \boldsymbol{\mathcal{D}}^{\sharp}_D[h\boldsymbol{\varphi}]\big|_+ d\sigma - \int_{\partial D} \mathbf{v}^e \cdot \frac{\partial \boldsymbol{\mathcal{D}}^{\sharp}_D[h\boldsymbol{\varphi}]}{\partial \nu}\Big|_+ d\sigma = -\int_{\partial D} h\frac{\partial \mathbf{F}}{\partial \mathbf{n}} \cdot \boldsymbol{\varphi}\, d\sigma. \tag{6.10}$$

In conclusion, we obtain from (5.24) and (6.5)-(6.10) the theorem 1.3.

In the remaining part of this section we show that the asymptotic expansion in (1.15) coincides with that one obtained in [22, Theorem 3.1]. We can easily see from Proposition 3.1 that (1.15) is equivalent to

$$\begin{aligned}\sum_{\alpha\beta jk} a_j^\alpha b_k^\beta m^j_{\alpha\beta k}(D_\epsilon) = \sum_{\alpha\beta jk} a_j^\alpha b_k^\beta m^j_{\alpha\beta k}(D) \\ + \epsilon \int_{\partial D} h\Big(\big([\mathbb{M}_{1,0} - \mathbb{C}_0]\widehat{\nabla}\mathbf{u}^e\big)\boldsymbol{\tau} \cdot \widehat{\nabla}\mathbf{v}^e\boldsymbol{\tau} - \big(\mathbb{K}_{1,0}\widehat{\nabla}\mathbf{u}^e\big)\mathbf{n} \cdot \big(\mathbb{C}_0\widehat{\nabla}\mathbf{v}^e\big)\mathbf{n}\Big) d\sigma \\ + O(\epsilon^2),\end{aligned}$$

with

$$\begin{aligned}\mathbb{M}_{1,0} - \mathbb{C}_0 &= \frac{2(\lambda_1\mu_0 - \lambda_0\mu_1)}{(\lambda_1 + 2\mu_1)}\mathbf{I}\otimes\mathbf{I} + \frac{4(\mu_1 - \mu_0)(\lambda_1 + \mu_1)}{\lambda_1 + 2\mu_1}\mathbf{I}\otimes(\boldsymbol{\tau}\otimes\boldsymbol{\tau}) \\ &:= \eta\mathbf{I}\otimes\mathbf{I} + \delta\mathbf{I}\otimes(\boldsymbol{\tau}\otimes\boldsymbol{\tau}),\end{aligned} \tag{6.11}$$

and

$$\begin{aligned}-\mathbb{K}_{1,0} &= \frac{(\lambda_1 - \lambda_0)\mu_1 - 2(\mu_1 - \mu_0)(\lambda_1 + \mu_1)}{\mu_1(\lambda_1 + 2\mu_1)}\mathbf{I}\otimes\mathbf{I} + 2\big(1 - \frac{\mu_0}{\mu_1}\big)\mathbb{I} \\ &\quad + 2\frac{(\mu_1 - \mu_0)(\lambda_1 + \mu_1)}{\mu_1(\lambda_1 + 2\mu_1)}\mathbf{I}\otimes(\boldsymbol{\tau}\otimes\boldsymbol{\tau}) \\ &:= \rho\mathbf{I}\otimes\mathbf{I} + \tau\mathbb{I} + \varrho\mathbf{I}\otimes(\boldsymbol{\tau}\otimes\boldsymbol{\tau}).\end{aligned} \tag{6.12}$$

Simple computations, yield

$$\begin{aligned}\big([\mathbb{M}_{1,0} - \mathbb{C}_0]\widehat{\nabla}\mathbf{u}^e\big)\boldsymbol{\tau} \cdot \widehat{\nabla}\mathbf{v}^e\boldsymbol{\tau} &= \eta(\nabla\cdot\mathbf{u}^e)\langle\widehat{\nabla}\mathbf{v}^e\boldsymbol{\tau}, \boldsymbol{\tau}\rangle + \delta\langle\widehat{\nabla}\mathbf{u}^e\boldsymbol{\tau}, \boldsymbol{\tau}\rangle\langle\widehat{\nabla}\mathbf{v}^e\boldsymbol{\tau}, \boldsymbol{\tau}\rangle, \\ -\big(\mathbb{K}_{1,0}\widehat{\nabla}\mathbf{u}^e\big)\mathbf{n} \cdot \big(\mathbb{C}_0\widehat{\nabla}\mathbf{v}^e\big)\mathbf{n} &= \lambda_0\rho(\nabla\cdot\mathbf{u}^e)(\nabla\cdot\mathbf{v}^e) + 2\mu_0\rho\nabla\cdot\mathbf{u}^e\langle\widehat{\nabla}\mathbf{v}^e\mathbf{n}, \mathbf{n}\rangle \\ &\quad + \lambda_0\tau(\nabla\cdot\mathbf{v}^e)\langle\widehat{\nabla}\mathbf{u}^e\mathbf{n}, \mathbf{n}\rangle + 2\mu_0\tau\langle\widehat{\nabla}\mathbf{u}^e\mathbf{n}, \widehat{\nabla}\mathbf{v}^e\mathbf{n}\rangle \\ &\quad + \lambda_0\varrho(\nabla\cdot\mathbf{v}^e)\langle\widehat{\nabla}\mathbf{u}^e\boldsymbol{\tau}, \boldsymbol{\tau}\rangle + 2\mu_0\varrho\langle\widehat{\nabla}\mathbf{u}^e\boldsymbol{\tau}, \boldsymbol{\tau}\rangle\langle\widehat{\nabla}\mathbf{v}^e\mathbf{n}, \mathbf{n}\rangle.\end{aligned}$$

Note that

$$2\mu_0\tau\langle\widehat{\nabla}\mathbf{u}^e\mathbf{n}, \widehat{\nabla}\mathbf{v}^e\mathbf{n}\rangle = \mu_0\tau\widehat{\nabla}\mathbf{u}^e : \widehat{\nabla}\mathbf{v}^e + \mu_0\tau(\nabla\cdot\mathbf{u}^e)\langle\widehat{\nabla}\mathbf{v}^e\mathbf{n}, \mathbf{n}\rangle - \mu_0\tau(\nabla\cdot\mathbf{v}^e)\langle\widehat{\nabla}\mathbf{u}^e\boldsymbol{\tau}, \boldsymbol{\tau}\rangle.$$

Hence

$$\big([\mathbb{M}_{1,0} - \mathbb{C}_0]\widehat{\nabla}\mathbf{u}^e\big)\boldsymbol{\tau} \cdot \widehat{\nabla}\mathbf{v}^e\boldsymbol{\tau} - \big(\mathbb{K}_{1,0}\widehat{\nabla}\mathbf{u}^e\big)\mathbf{n} \cdot \big(\mathbb{C}_0\widehat{\nabla}\mathbf{v}^e\big)\mathbf{n} = \mathbb{S}\widehat{\nabla}\mathbf{u}^e : \widehat{\nabla}\mathbf{v}^e,$$

where

$$\mathbb{S}\widehat{\nabla}\mathbf{u}^e = \lambda_0(\rho+\tau)(\nabla\cdot\mathbf{u}^e)\mathbf{I} + (\lambda_0\varrho - \lambda_0\tau + 2\mu_0\varrho - \mu_0\tau)\langle\widehat{\nabla}\mathbf{u}^e\boldsymbol{\tau},\boldsymbol{\tau}\rangle\mathbf{I} + \eta(\nabla\cdot\mathbf{u}^e)\boldsymbol{\tau}\otimes\boldsymbol{\tau}$$
$$+ (\delta - 2\mu_0\varrho)\langle\widehat{\nabla}\mathbf{u}^e\boldsymbol{\tau},\boldsymbol{\tau}\rangle\boldsymbol{\tau}\otimes\boldsymbol{\tau} + (2\mu_0\rho + \mu_0\tau)(\nabla\cdot\mathbf{u}^e)\mathbf{n}\otimes\mathbf{n} + \mu_0\tau\widehat{\nabla}\mathbf{u}^e. \tag{6.13}$$

It is proved in [22] that

$$\sum_{\alpha\beta jk} a_j^\alpha b_k^\beta m^j_{\alpha\beta k}(D_\epsilon) = \sum_{\alpha\beta jk} a_j^\alpha b_k^\beta m^j_{\alpha\beta k}(D) + \epsilon\int_{\partial D} h\big(\mathbb{M}\widehat{\nabla}\mathbf{u}^e\big) : \widehat{\nabla}\mathbf{v}^e + O(\epsilon^{1+\gamma}),$$

for some positive γ and

$$\mathbb{M}\widehat{\nabla}\mathbf{u}^e := \big(\mathbb{C}_1 - \mathbb{C}_0\big)\mathbb{C}_1^{-1}\Big(\big(\mathbb{K}\widehat{\nabla}\mathbf{u}^e\boldsymbol{\tau}\big)\otimes\boldsymbol{\tau} + \big(\mathbb{C}_0\widehat{\nabla}\mathbf{u}^e\mathbf{n}\big)\otimes\mathbf{n}\Big),$$

where the 4−tensor $\mathbb{K}$ is defined by

$$\mathbb{K} := p\mathbf{I}\otimes\mathbf{I} + 2\mu_0\mathbb{I} + q\mathbf{I}\otimes(\boldsymbol{\tau}\otimes\boldsymbol{\tau}),$$

where

$$p := \frac{\lambda_1(\lambda_0+2\mu_0)}{\lambda_1+2\mu_1} \qquad \text{and} \qquad q := \frac{4(\mu_1-\mu_0)(\lambda_1+\mu_1)}{\lambda_1+2\mu_1}.$$

Denote by

$$\lambda := \frac{\lambda_1-\lambda_0+\mu_1-\mu_0}{2(\lambda_1+\mu_1)} - \frac{\mu_1-\mu_0}{2\mu_1}, \qquad \mu = \frac{\mu_1-\mu_0}{2\mu_1}.$$

It is proved in [1] that

$$\mathbb{M}\widehat{\nabla}\mathbf{u}^e = \big[\lambda(p+\lambda_0+2\mu_0) + 2\mu p - \eta\big](\nabla\cdot\mathbf{u}^e)\mathbf{I} + \lambda q\langle\widehat{\nabla}\mathbf{u}^e\boldsymbol{\tau},\boldsymbol{\tau}\rangle\mathbf{I} + \eta(\nabla\cdot\mathbf{u}^e)\boldsymbol{\tau}\otimes\boldsymbol{\tau}$$
$$+ 2\mu q\langle\widehat{\nabla}\mathbf{u}^e\boldsymbol{\tau},\boldsymbol{\tau}\rangle\boldsymbol{\tau}\otimes\boldsymbol{\tau} + \big[2\mu\lambda_0 + \eta - 2\mu p\big](\nabla\cdot\mathbf{u}^e)\mathbf{n}\otimes\mathbf{n} + 4\mu\mu_0\widehat{\nabla}\mathbf{u}^e. \tag{6.14}$$

Looking at the coefficients in (6.13) and (6.14), we confirm that

$$\mathbb{M} = \mathbb{S} = \frac{\lambda_0(\lambda_1-\lambda_0) + 2\lambda_0(\mu_1-\mu_0)}{\lambda_1+2\mu_1}\mathbf{I}\otimes\mathbf{I} + 2\big(1-\frac{\mu_0}{\mu_1}\big)\frac{\mu_0\lambda_1-\mu_1\lambda_0}{\lambda_1+2\mu_1}\mathbf{I}\otimes(\boldsymbol{\tau}\otimes\boldsymbol{\tau})$$
$$+ \frac{2(\lambda_1\mu_0-\lambda_0\mu_1)}{(\lambda_1+2\mu_1)}(\boldsymbol{\tau}\otimes\boldsymbol{\tau})\otimes\mathbf{I} + 4\big(1-\frac{\mu_0}{\mu_1}\big)\frac{(\mu_1-\mu_0)(\lambda_1+\mu_1)}{\lambda_1+2\mu_1}(\boldsymbol{\tau}\otimes\boldsymbol{\tau})\otimes(\boldsymbol{\tau}\otimes\boldsymbol{\tau})$$
$$+ 2\big(\frac{\mu_0}{\mu_1}\big)\frac{(\lambda_1-\lambda_0)\mu_1 - (\mu_1-\mu_0)\lambda_1}{\lambda_1+2\mu_1}(\mathbf{n}\otimes\mathbf{n})\otimes\mathbf{I} + \frac{2\mu_0(\mu_1-\mu_0)}{\mu_1}\mathbb{I}.$$

Appendix

1) Derivation of the $\dfrac{\partial \boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}]}{\partial \nu}(x)$

We have

$$\begin{aligned}\boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}](x) &= \int_{\partial D} \boldsymbol{\Gamma}(x-y)\boldsymbol{\phi}(y)d\sigma(y)\\ &= \int_{\partial D}\Big(\frac{A}{2\pi}\log|x-y|\boldsymbol{\phi}(y) - \frac{B}{2\pi}\frac{\langle x-y, \boldsymbol{\phi}(y)\rangle}{|x-y|^2}(x-y)\Big)d\sigma(y).\end{aligned}$$

Let $\mathbf{x} := x-y$. By using (2.9), we get

$$\begin{aligned}&\nabla_x\left(\frac{A}{2\pi}\log|\mathbf{x}|\boldsymbol{\phi}(y) - \frac{B}{2\pi}\frac{\langle \mathbf{x}, \boldsymbol{\phi}(y)\rangle}{|\mathbf{x}|^2}\mathbf{x}\right)\\ &\qquad = \frac{A}{2\pi}\frac{\boldsymbol{\phi}(y)\otimes\mathbf{x}}{|\mathbf{x}|^2} - \frac{B}{2\pi}\frac{\langle \mathbf{x}, \boldsymbol{\phi}(y)\rangle}{|\mathbf{x}|^2}\mathbf{I} + \frac{B}{\pi}\frac{\langle \mathbf{x}, \boldsymbol{\phi}(y)\rangle}{|\mathbf{x}|^4}\big(\mathbf{x}\otimes\mathbf{x}\big) - \frac{B}{2\pi}\frac{\mathbf{x}\otimes\boldsymbol{\phi}(y)}{|\mathbf{x}|^2}.\end{aligned}$$

Therefore

$$\begin{aligned}&\nabla_x\Big(\frac{A}{2\pi}\log|\mathbf{x}|\boldsymbol{\phi}(y) - \frac{B}{2\pi}\frac{\langle \mathbf{x}, \boldsymbol{\phi}(y)\rangle}{|\mathbf{x}|^2}\mathbf{x}\Big) + \Big[\nabla_x\Big(\frac{A}{2\pi}\log|\mathbf{x}|\boldsymbol{\phi}(y) - \frac{B}{2\pi}\frac{\langle \mathbf{x}, \boldsymbol{\phi}(y)\rangle}{|\mathbf{x}|^2}\mathbf{x}\Big)\Big]^T\\ &= \frac{(A-B)}{2\pi}\frac{\boldsymbol{\phi}(y)\otimes\mathbf{x}}{|\mathbf{x}|^2} + \frac{(A-B)}{2\pi}\frac{\mathbf{x}\otimes\boldsymbol{\phi}(y)}{|\mathbf{x}|^2} - \frac{B}{\pi}\frac{\langle \mathbf{x}, \boldsymbol{\phi}(y)\rangle}{|\mathbf{x}|^2}\mathbf{I} + \frac{2B}{\pi}\frac{\langle \mathbf{x}, \boldsymbol{\phi}(y)\rangle}{|\mathbf{x}|^4}\big(\mathbf{x}\otimes\mathbf{x}\big),\end{aligned} \tag{6.15}$$

which gives

$$\begin{aligned}&\left(\nabla_x\Big(\frac{A}{2\pi}\log|\mathbf{x}|\boldsymbol{\phi}(y) - \frac{B}{2\pi}\frac{\langle \mathbf{x}, \boldsymbol{\phi}(y)\rangle}{|\mathbf{x}|^2}\mathbf{x}\Big) + \Big[\nabla_x\Big(\frac{A}{2\pi}\log|\mathbf{x}|\boldsymbol{\phi}(y) - \frac{B}{2\pi}\frac{\langle \mathbf{x}, \boldsymbol{\phi}(y)\rangle}{|\mathbf{x}|^2}\mathbf{x}\Big)\Big]^T\right)\mathbf{n}(x)\\ &= \frac{(A-B)}{2\pi}\frac{\langle \mathbf{x}, \mathbf{n}(x)\rangle}{|\mathbf{x}|^2}\boldsymbol{\phi}(y) + \frac{(A-B)}{2\pi}\frac{\langle \mathbf{n}(x), \boldsymbol{\phi}(y)\rangle}{|\mathbf{x}|^2}\mathbf{x} - \frac{B}{\pi}\frac{\langle \mathbf{x}, \boldsymbol{\phi}(y)\rangle}{|\mathbf{x}|^2}\mathbf{n}(x)\\ &\quad + \frac{2B}{\pi}\frac{\langle \mathbf{x}, \boldsymbol{\phi}(y)\rangle\langle \mathbf{x}, \mathbf{n}(x)\rangle}{|\mathbf{x}|^4}\mathbf{x}\\ &= \frac{(A-B)}{2\pi}\frac{\langle \mathbf{x}, \mathbf{n}(x)\rangle}{|\mathbf{x}|^2}\boldsymbol{\phi}(y) + \frac{(A-B)}{2\pi}\frac{\mathbf{x}\otimes\mathbf{n}(x)}{|\mathbf{x}|^2}\boldsymbol{\phi}(y) - \frac{B}{\pi}\frac{\mathbf{n}(x)\otimes\mathbf{x}}{|\mathbf{x}|^2}\boldsymbol{\phi}(y)\\ &\quad + \frac{2B}{\pi}\frac{\langle \mathbf{x}, \mathbf{n}(x)\rangle}{|\mathbf{x}|^4}(\mathbf{x}\otimes\mathbf{x})\boldsymbol{\phi}(y)\\ &= \Big[\frac{(A-B)}{2\pi}\frac{\langle \mathbf{x}, \mathbf{n}(x)\rangle}{|\mathbf{x}|^2}\mathbf{I} + \frac{(A-B)}{2\pi}\frac{\mathbf{x}\otimes\mathbf{n}(x)}{|\mathbf{x}|^2} - \frac{B}{\pi}\frac{\mathbf{n}(x)\otimes\mathbf{x}}{|\mathbf{x}|^2} + \frac{2B}{\pi}\frac{\langle \mathbf{x}, \mathbf{n}(x)\rangle}{|\mathbf{x}|^4}(\mathbf{x}\otimes\mathbf{x})\Big]\boldsymbol{\phi}(y)\\ &:= \mathbf{Q}(\mathbf{x})\boldsymbol{\phi}(y).\end{aligned}$$

Hence, we obtain

$$\Big(\nabla\boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}](x) + \big[\nabla\boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}](x)\big]^T\Big)\mathbf{n}(x) = \int_{\partial D}\mathbf{Q}(x-y)\boldsymbol{\phi}(y)d\sigma(y), \quad x\in\partial D. \tag{6.16}$$

It follows from (2.14) that

$$\nabla_x\cdot\Big(\boldsymbol{\Gamma}(\mathbf{x})\boldsymbol{\phi}(y)\Big) = \langle\nabla_x\cdot\boldsymbol{\Gamma}(\mathbf{x}),\boldsymbol{\phi}(y)\rangle. \tag{6.17}$$

Thus

$$\nabla_x\cdot\Big(\boldsymbol{\Gamma}(\mathbf{x})\boldsymbol{\phi}(y)\Big)\mathbf{n}(x) = \langle\nabla_x\cdot\boldsymbol{\Gamma}(\mathbf{x}),\boldsymbol{\phi}(y)\rangle\mathbf{n}(x) = \Big(\mathbf{n}(x)\otimes\nabla_x\cdot\boldsymbol{\Gamma}(\mathbf{x})\Big)\boldsymbol{\phi}(y).$$

Since

$$\begin{aligned}\nabla_x\cdot\boldsymbol{\Gamma}(\mathbf{x}) &= \nabla_x\cdot\Big(\frac{A}{2\pi}\log|\mathbf{x}|\mathbf{I} - \frac{B}{2\pi}\frac{\mathbf{x}\otimes\mathbf{x}}{|\mathbf{x}|^2}\Big)\\ &= \frac{A}{2\pi}\frac{\mathbf{x}}{|\mathbf{x}|^2} - \frac{B}{2\pi}\Big(-2\frac{(\mathbf{x}\otimes\mathbf{x})\mathbf{x}}{|\mathbf{x}|^4} + \frac{\nabla\mathbf{x}\;\mathbf{x} + \nabla\cdot\mathbf{x}\;\mathbf{x}}{|\mathbf{x}|^2}\Big)\\ &= \frac{A-B}{2\pi}\frac{\mathbf{x}}{|\mathbf{x}|^2},\end{aligned} \tag{6.18}$$

then

$$\nabla_x\cdot\Big(\boldsymbol{\Gamma}(\mathbf{x})\boldsymbol{\phi}(y)\Big)\mathbf{n}(x) = \frac{A-B}{2\pi}\frac{\mathbf{n}(x)\otimes\mathbf{x}}{|\mathbf{x}|^2}\boldsymbol{\phi}(y) := \mathbf{P}(\mathbf{x})\boldsymbol{\phi}(y),$$

and hence

$$\nabla\cdot\boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}](x)\mathbf{n}(x) = \int_{\partial D}\mathbf{P}(x-y)\boldsymbol{\phi}(y)d\sigma(y). \tag{6.19}$$

It then follows from (1.4), (6.16) and (6.19) that

$$\frac{\partial\boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}]}{\partial\nu}(x) = \int_{\partial D}\Big(\lambda_0\mathbf{P}(x-y) + \mu_0\mathbf{Q}(x-y)\Big)\boldsymbol{\phi}(y)d\sigma(y). \tag{6.20}$$

Note that $\lambda_0\mathbf{P}(x-y) + \mu_0\mathbf{Q}(x-y) = \mathbf{K}^T(x-y)$ for $x,y\in\partial D, x\neq y$, where $\mathbf{K}^T$ is defined by (4.8).

2) Derivation of the $\frac{\partial \mathcal{D}_D^\sharp[\phi]}{\partial \nu}(x)$

According to (4.14), we have

$$
\begin{aligned}
&\nabla_x\Big(\frac{\partial \mathbf{\Gamma}(\mathbf{x})}{\partial \mathbf{n}(y)}\boldsymbol{\phi}(y)\Big)\\
&\quad= -\frac{A}{2\pi}\Big[\frac{\boldsymbol{\phi}(y)\otimes\mathbf{n}(y)}{|\mathbf{x}|^2} - 2\frac{\langle \mathbf{x},\mathbf{n}(y)\rangle}{|\mathbf{x}|^4}\boldsymbol{\phi}(y)\otimes\mathbf{x}\Big]\\
&\qquad -\frac{B}{\pi}\Big[\frac{\langle \mathbf{x},\boldsymbol{\phi}(y)\rangle}{|\mathbf{x}|^4}\mathbf{x}\otimes\mathbf{n}(y) + \frac{\langle \mathbf{x},\mathbf{n}(y)\rangle}{|\mathbf{x}|^4}\mathbf{x}\otimes\boldsymbol{\phi}(y) - 4\frac{\langle \mathbf{x},\mathbf{n}(y)\rangle\langle \mathbf{x},\boldsymbol{\phi}(y)\rangle}{|\mathbf{x}|^6}\mathbf{x}\otimes\mathbf{x}\\
&\qquad\qquad + \frac{\langle \mathbf{x},\mathbf{n}(y)\rangle\langle \mathbf{x},\boldsymbol{\phi}(y)\rangle}{|\mathbf{x}|^4}\mathbf{I}\Big]\\
&\qquad +\frac{B}{2\pi}\Big[\frac{\langle \mathbf{n}(y),\boldsymbol{\phi}(y)\rangle}{|\mathbf{x}|^2}\mathbf{I} - 2\frac{\langle \mathbf{n}(y),\boldsymbol{\phi}(y)\rangle}{|\mathbf{x}|^4}\mathbf{x}\otimes\mathbf{x} + \frac{\mathbf{n}(y)\otimes\boldsymbol{\phi}(y)}{|\mathbf{x}|^2} - 2\frac{\langle \mathbf{x},\boldsymbol{\phi}(y)\rangle}{|\mathbf{x}|^4}\mathbf{n}(y)\otimes\mathbf{x}\Big].
\end{aligned}
$$

Then

$$
\begin{aligned}
&\nabla_x\Big(\frac{\partial \mathbf{\Gamma}(\mathbf{x})}{\partial \mathbf{n}(y)}\boldsymbol{\phi}(y)\Big)\mathbf{n}(x)\\
&\quad= -\frac{A}{2\pi}\Big[\frac{\langle \mathbf{n}(x),\mathbf{n}(y)\rangle}{|\mathbf{x}|^2}\mathbf{I} - 2\frac{\langle \mathbf{x},\mathbf{n}(y)\rangle\langle \mathbf{x},\mathbf{n}(x)\rangle}{|\mathbf{x}|^4}\mathbf{I}\Big]\boldsymbol{\phi}(y)\\
&\qquad -\frac{B}{\pi}\Big[\frac{\langle \mathbf{n}(x),\mathbf{n}(y)\rangle}{|\mathbf{x}|^4}\mathbf{x}\otimes\mathbf{x} + \frac{\langle \mathbf{x},\mathbf{n}(y)\rangle}{|\mathbf{x}|^4}\mathbf{x}\otimes\mathbf{n}(x) - 4\frac{\langle \mathbf{x},\mathbf{n}(y)\rangle\langle \mathbf{x},\mathbf{n}(x)\rangle}{|\mathbf{x}|^6}\mathbf{x}\otimes\mathbf{x}\\
&\qquad\qquad + \frac{\langle \mathbf{x},\mathbf{n}(y)\rangle}{|\mathbf{x}|^4}\mathbf{n}(x)\otimes\mathbf{x}\Big]\boldsymbol{\phi}(y)\\
&\qquad +\frac{B}{2\pi}\Big[\frac{\mathbf{n}(x)\otimes\mathbf{n}(y)}{|\mathbf{x}|^2} - 2\frac{\langle \mathbf{x},\mathbf{n}(x)\rangle}{|\mathbf{x}|^4}\mathbf{x}\otimes\mathbf{n}(y) + \frac{\mathbf{n}(y)\otimes\mathbf{n}(x)}{|\mathbf{x}|^2} - 2\frac{\langle \mathbf{x},\mathbf{n}(x)\rangle}{|\mathbf{x}|^4}\mathbf{n}(y)\otimes\mathbf{x}\Big]\boldsymbol{\phi}(y).
\end{aligned}
\tag{6.21}
$$

Likewise, we get

$$
\begin{aligned}
&\Big[\nabla_x\Big(\frac{\partial \mathbf{\Gamma}(\mathbf{x})}{\partial \mathbf{n}(y)}\boldsymbol{\phi}(y)\Big)\Big]^T\mathbf{n}(x)\\
&\quad= -\frac{A}{2\pi}\Big[\frac{\mathbf{n}(y)\otimes\mathbf{n}(x)}{|\mathbf{x}|^2} - 2\frac{\langle \mathbf{x},\mathbf{n}(y)\rangle}{|\mathbf{x}|^4}\mathbf{x}\otimes\mathbf{n}(x)\Big]\boldsymbol{\phi}(y)\\
&\qquad -\frac{B}{\pi}\Big[\frac{\langle \mathbf{x},\mathbf{n}(y)\rangle}{|\mathbf{x}|^4}\mathbf{n}(x)\otimes\mathbf{x} + \frac{\langle \mathbf{x},\mathbf{n}(x)\rangle}{|\mathbf{x}|^4}\mathbf{n}(y)\otimes\mathbf{x} - 4\frac{\langle \mathbf{x},\mathbf{n}(y)\rangle\langle \mathbf{x},\mathbf{n}(x)\rangle}{|\mathbf{x}|^6}\mathbf{x}\otimes\mathbf{x}\\
&\qquad\qquad + \frac{\langle \mathbf{x},\mathbf{n}(y)\rangle\langle \mathbf{x},\mathbf{n}(x)\rangle}{|\mathbf{x}|^4}\mathbf{I}\Big]\boldsymbol{\phi}(y)\\
&\qquad +\frac{B}{2\pi}\Big[\frac{\mathbf{n}(x)\otimes\mathbf{n}(y)}{|\mathbf{x}|^2} - 2\frac{\langle \mathbf{x},\mathbf{n}(x)\rangle}{|\mathbf{x}|^4}\mathbf{x}\otimes\mathbf{n}(y) + \frac{\langle \mathbf{n}(x),\mathbf{n}(y)\rangle}{|\mathbf{x}|^2}\mathbf{I} - 2\frac{\langle \mathbf{n}(x),\mathbf{n}(y)\rangle}{|\mathbf{x}|^4}\mathbf{x}\otimes\mathbf{x}\Big]\boldsymbol{\phi}(y).
\end{aligned}
\tag{6.22}
$$

Using (2.12), (2.13), (2.14), and (4.14), we readily get

$$\nabla_x \cdot \Big(\frac{\partial \mathbf{\Gamma}(\mathbf{x})}{\partial \mathbf{n}(y)}\boldsymbol{\phi}(y)\Big)\mathbf{n}(x) = \Big\langle \nabla_x \cdot \frac{\partial \mathbf{\Gamma}(\mathbf{x})}{\partial \mathbf{n}(y)}, \boldsymbol{\phi}(y)\Big\rangle \mathbf{n}(x)$$

$$= \frac{A-B}{2\pi}\Big[2\frac{\langle \mathbf{x}, \mathbf{n}(y)\rangle}{|x|^4}\mathbf{n}(x)\otimes \mathbf{x} - \frac{\mathbf{n}(x)\otimes\mathbf{n}(y)}{|x|^2}\Big]\boldsymbol{\phi}(y). \tag{6.23}$$

It then follows from (1.4), (4.13), (6.21), (6.22), and (6.23) that

$$\begin{aligned}
&\frac{\partial \boldsymbol{\mathcal{D}}_D^\sharp[\boldsymbol{\phi}]}{\partial \nu}(x) \\
&= \frac{1}{2\pi}\frac{A-B}{A+B}\int_{\partial D}\Big[2\frac{\langle \mathbf{x}, \mathbf{n}(y)\rangle\langle \mathbf{x}, \mathbf{n}(x)\rangle}{|\mathbf{x}|^4} - \frac{\langle \mathbf{n}(x), \mathbf{n}(y)\rangle}{|\mathbf{x}|^2}\Big]\boldsymbol{\phi}(y)d\sigma(y) \\
&\quad + \frac{1}{2\pi}\frac{A-B}{A+B}\int_{\partial D}\Big[2\frac{\langle \mathbf{x}, \mathbf{n}(y)\rangle}{|\mathbf{x}|^4}\Big(\mathbf{x}\otimes\mathbf{n}(x) - \mathbf{n}(x)\otimes\mathbf{x}\Big) \\
&\qquad\qquad - \frac{\mathbf{n}(y)\otimes\mathbf{n}(x) - \mathbf{n}(x)\otimes\mathbf{n}(y)}{|\mathbf{x}|^2}\Big]\boldsymbol{\phi}(y)d\sigma(y) \\
&\quad + \frac{1}{\pi}\frac{2B}{A+B}\int_{\partial D}\Big[4\frac{\langle \mathbf{x}, \mathbf{n}(y)\rangle\langle \mathbf{x}, \mathbf{n}(x)\rangle}{|\mathbf{x}|^6}\mathbf{x}\otimes\mathbf{x} - \frac{\langle \mathbf{n}(x), \mathbf{n}(y)\rangle}{|\mathbf{x}|^4}\mathbf{x}\otimes\mathbf{x} \\
&\qquad\qquad - \frac{\langle \mathbf{x}, \mathbf{n}(x)\rangle}{|\mathbf{x}|^4}\Big(x\otimes\mathbf{n}(y) + \mathbf{n}(y)\otimes\mathbf{x}\Big)\Big]\boldsymbol{\phi}(y)d\sigma(y).
\end{aligned} \tag{6.24}$$

3) Derivation of the $\nabla\nabla\cdot\boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}](x)\cdot\mathbf{n}(x)\mathbf{n}(x)$

It follows from (6.17) and (6.18) that

$$\nabla\cdot\boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}](x) = \frac{(A-B)}{2\pi}\int_{\partial D}\frac{\langle \mathbf{x}, \boldsymbol{\phi}(y)\rangle}{|\mathbf{x}|^2}d\sigma(y).$$

Thus

$$\nabla\nabla\cdot\boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}](x) = \frac{(A-B)}{2\pi}\int_{\partial D}\Big(\frac{\boldsymbol{\phi}(y)}{|\mathbf{x}|^2} - 2\frac{\langle \mathbf{x}, \boldsymbol{\phi}(y)\rangle}{|\mathbf{x}|^4}\mathbf{x}\Big)d\sigma(y),$$

which gives

$$\nabla\nabla\cdot\boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}](x)\cdot\mathbf{n}(x) = \frac{(A-B)}{2\pi}\int_{\partial D}\Big(\frac{\langle\boldsymbol{\phi}(y), \mathbf{n}(x)\rangle}{|\mathbf{x}|^2} - 2\frac{\langle \mathbf{x}, \boldsymbol{\phi}(y)\rangle\langle \mathbf{x}, \mathbf{n}(x)\rangle}{|\mathbf{x}|^4}\Big)d\sigma(y).$$

Thanks to the identity in (2.15), we obtain

$$\nabla\nabla\cdot\boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}](x)\cdot\mathbf{n}(x)\mathbf{n}(x) = \frac{(A-B)}{2\pi}\int_{\partial D}\Big(\frac{\mathbf{n}(x)\otimes\mathbf{n}(x)}{|\mathbf{x}|^2} - 2\frac{\langle \mathbf{x}, \mathbf{n}(x)\rangle}{|\mathbf{x}|^4}\mathbf{n}(x)\otimes\mathbf{x}\Big)\boldsymbol{\phi}(y)d\sigma(y). \tag{6.25}$$

4) Derivation of the $\nabla\Big(\nabla\boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}](x)+\big[\nabla\boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}](x)\big]^T\Big)\mathbf{n}(x)\mathbf{n}(x)$

It follows from (2.15) and (6.15) that

$$\nabla\Bigg(\nabla\Big(\frac{A}{2\pi}\log|\mathbf{x}|\boldsymbol{\phi}(y)-\frac{B}{2\pi}\frac{\langle\mathbf{x},\boldsymbol{\phi}(y)\rangle}{|\mathbf{x}|^2}\mathbf{x}\Big)+\Big[\nabla\Big(\frac{A}{2\pi}\log|\mathbf{x}|\boldsymbol{\phi}(y)-\frac{B}{2\pi}\frac{\langle\mathbf{x},\boldsymbol{\phi}(y)\rangle}{|\mathbf{x}|^2}\mathbf{x}\Big)\Big]^T\Bigg)\mathbf{n}(x)\mathbf{n}(x)$$

$$\begin{aligned}
&=\Bigg[\frac{(A-B)}{2\pi}\Bigg(\frac{\boldsymbol{\phi}(y)\otimes\mathbf{I}+\big(\mathbf{I}\otimes\boldsymbol{\phi}(y)\big)^T}{|\mathbf{x}|^2}-2\frac{\boldsymbol{\phi}(y)\otimes\mathbf{x}\otimes\mathbf{x}+\mathbf{x}\otimes\boldsymbol{\phi}(y)\otimes\mathbf{x}}{|\mathbf{x}|^4}\Bigg)\\
&\quad-\frac{B}{\pi}\Bigg(\frac{\mathbf{I}\otimes\boldsymbol{\phi}(y)}{|\mathbf{x}|^2}-2\frac{\langle\mathbf{x},\boldsymbol{\phi}(y)\rangle}{|\mathbf{x}|^4}\big(\mathbf{I}\otimes\mathbf{x}\big)\Bigg)\\
&\quad+\frac{2B}{\pi}\Bigg(\frac{\mathbf{x}\otimes\mathbf{x}\otimes\boldsymbol{\phi}(y)}{|\mathbf{x}|^4}-4\frac{\langle\mathbf{x},\boldsymbol{\phi}(y)\rangle}{|\mathbf{x}|^6}\big(\mathbf{x}\otimes\mathbf{x}\otimes\mathbf{x}\big)+\frac{\langle\mathbf{x},\boldsymbol{\phi}(y)\rangle}{|\mathbf{x}|^4}\Big((\mathbf{I}\otimes\mathbf{x})^T+\mathbf{x}\otimes\mathbf{I}\Big)\Bigg)\Bigg]\mathbf{n}(x)\mathbf{n}(x)\\
&=\Bigg[\frac{(A-B)}{2\pi}\Bigg(\frac{\boldsymbol{\phi}(y)\otimes\mathbf{n}(x)+\mathbf{n}(x)\otimes\boldsymbol{\phi}(y)}{|\mathbf{x}|^2}-2\frac{\langle\mathbf{x},\mathbf{n}(x)\rangle}{|\mathbf{x}|^4}\Big(\boldsymbol{\phi}(y)\otimes\mathbf{x}+\mathbf{x}\otimes\boldsymbol{\phi}(y)\Big)\Bigg)\\
&\quad-\frac{B}{\pi}\Bigg(\frac{\langle\boldsymbol{\phi}(y),\mathbf{n}(x)\rangle}{|\mathbf{x}|^2}-2\frac{\langle\mathbf{x},\boldsymbol{\phi}(y)\rangle}{|\mathbf{x}^2|}\frac{\langle\mathbf{x},\mathbf{n}(x)\rangle}{|\mathbf{x}|^2}\Bigg)\mathbf{I}\\
&\quad+\frac{2B}{\pi}\Bigg(\frac{\langle\mathbf{n}(x),\boldsymbol{\phi}(y)\rangle}{|\mathbf{x}|^4}\big(\mathbf{x}\otimes\mathbf{x}\big)-4\frac{\langle\mathbf{x},\boldsymbol{\phi}(y)\rangle\langle\mathbf{x},\mathbf{n}(x)\rangle}{|\mathbf{x}|^6}\big(\mathbf{x}\otimes\mathbf{x}\big)\\
&\qquad\qquad+\frac{\langle\mathbf{x},\boldsymbol{\phi}(y)\rangle}{|\mathbf{x}|^4}\Big(\mathbf{n}(x)\otimes\mathbf{x}+\mathbf{x}\otimes\mathbf{n}(x)\Big)\Bigg)\Bigg]\mathbf{n}(x)\\
&=\Bigg[\frac{(A-B)}{2\pi}\Bigg(\frac{\mathbf{I}+\mathbf{n}(x)\otimes\mathbf{n}(x)}{|\mathbf{x}|^2}-2\frac{(\langle\mathbf{x},\mathbf{n}(x)\rangle)^2}{|\mathbf{x}|^4}\mathbf{I}-2\frac{\langle\mathbf{x},\mathbf{n}(x)\rangle}{|\mathbf{x}|^4}\Big(\mathbf{x}\otimes\mathbf{n}(x)\Big)\Bigg)\\
&\quad-\frac{B}{\pi}\Bigg(\frac{\mathbf{n}(x)\otimes\mathbf{n}(x)}{|\mathbf{x}|^2}-2\frac{\langle\mathbf{x},\mathbf{n}(x)\rangle}{|\mathbf{x}|^4}\Big(\mathbf{n}(x)\otimes\mathbf{x}\Big)\Bigg)\\
&\quad+\frac{2B}{\pi}\Bigg(\frac{\langle\mathbf{x},\mathbf{n}(x)\rangle}{|\mathbf{x}|^4}\Big(\mathbf{x}\otimes\mathbf{n}(x)+\mathbf{n}(x)\otimes\mathbf{x}\Big)-4\frac{(\langle\mathbf{x},\mathbf{n}(x)\rangle)^2}{|\mathbf{x}|^6}\big(\mathbf{x}\otimes\mathbf{x}\big)+\frac{\mathbf{x}\otimes\mathbf{x}}{|\mathbf{x}|^4}\Bigg)\Bigg]\boldsymbol{\phi}(y)\\
&:=\mathbf{L}(\mathbf{x})\boldsymbol{\phi}(y).
\end{aligned}$$

Then, we have

$$\nabla\Big(\nabla\boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}](x)+\big[\nabla\boldsymbol{\mathcal{S}}_D[\boldsymbol{\phi}](x)\big]^T\Big)\mathbf{n}(x)\mathbf{n}(x)=\int_{\partial D}\mathbf{L}(x-y)\boldsymbol{\phi}(y)d\sigma(y).$$

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