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Implementation and performance evaluation of an extended precision floating-point arithmetic library for high-accuracy semidefinite programming

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Abstract

Semidefinite programming (SDP) is widely used in optimization problems with many applications, however, certain SDP instances are ill-posed and need more precision than the standard double-precision available. Moreover, these problems are large-scale and could benefit from parallelization on specialized architectures such as GPUs. In this article, we implement and evaluate the performance of a floating-point expansion-based arithmetic library (CAMPARY) in the context of such numerically highly accurate SDP solvers. We plugged-in CAMPARY with the state-of-the-art SDPA solver for both CPU and GPU-tuned implementations. We compare and contrast both the numerical accuracy and performance of SDPA-GMP, -QD and -DD, which employ other multiple-precision arithmetic libraries against SDPA-CAMPARY. We show that CAMPARY is a very good trade-off for accuracy and speed when solving ill-conditioned SDP problems.

Index Terms

floating-point arithmetic, multiple precision library, ill-posed semidefinite programming, GPGPU computing, error-free transform, floating-point expansions

1. Introduction

Nowadays most floating-point (FP) computations use *single*-precision (*binary32*) or *double*-precision (*binary64*) arithmetic. Arithmetic operations on these two formats are very efficiently implemented in currently available processors, either CPUs or GPUs, and are compliant with the IEEE 754-2008 standard for FP arithmetic [1]. Among other requirements, this standard defines five rounding modes (round downwards, upwards, towards zero, to the nearest "ties to even", and to the nearest "ties to away"). An arithmetic operation should return the result as if computed with infinite precision and then the rounding function is applied. Such an operation is said to be correctly rounded. The IEEE 754-2008 standard enforces correct rounding of all basic arithmetic operations (addition, multiplication, division and square root). This improves portability and reliability of numerical software.

These two ubiquitous precision formats are sometimes not sufficient e.g., in some recently studied optimization problems, which come to solving in a very accurate way, numerically sensitive and sometimes large-scale semidefinite optimization problems (SDP). Examples include high-accuracy computation of *kissing numbers* [2]; bounds from binary codes; control theory and structural design optimization (e.g., the wing of Airbus A380) [3]; quantum information and physics [4].

This resulted in a recent increased interest in providing both higher-precision (also called *multiple precision*) and high-performance SDP libraries. To accomplish this, firstly, one has to establish the core mathematical algorithm for numerically solving (say, in "real numbers") the SDP problem: most nowadays solvers employ primal-dual path-following interior-point method (PDIPM) [5]. This algorithm is considered in literature as theoretically mature, is widely accepted and implemented in most state-of-the-art SDP solvers like SDPA [6], CSDP [7], SeDuMi [8], SDPT3 [9]. This algorithm is recalled in Section 2 and we considered the SDPA implementation in this work, without any important modification.

Secondly, the underlying multiple-precision arithmetic operations have to be treated. Several multiple precision arithmetic libraries like GMP and QD were already ported inside SDPA [10]. The resulted implementations are more accurate, yet much more computationally expensive [10]. These libraries and implementations with SDPA are described in Section 3. Finally, since most problems are large-scale, parallelization is also very important. We treat the case of highly parallel architectures of GPUs, for which most multiple precision libraries are not suitable.

The first contribution of this work is to propose

the use of our multiple precision library CAMPARY[1] with SDPA, which results in a better performance vs. accuracy trade-off. Our underlying algorithms based on floating-point expansions are described in Section 4. The second contribution is a multiple precision GPU compatible general matrix multiplication routine that can be used in SDPA. This routine runs at up to 83% of the theoretical GPU peak-performance and allows for an average speedup of one order of magnitude for SDP instances run in multiple precision with SDPA-CAMPARY and GPU support compared to SDPA-CAMPARY on CPU only. Implementation details are provided in Section 5. Benchmarks were performed on well-known ill-conditioned examples from [11] and [3]. The results are discussed in Section 6.

2. Semidefinite programming formulation

Semidefinite programing (SDP) is a convex optimization problem, which can be seen as a natural generalization of linear programming to the cone of symmetric matrices with non-negative eigenvalues, i.e. positive semidefinite matrices. Denote by $\mathbb{R}^{n \times n}$ the space of size $n \times n$ real matrices, $\mathbb{S}^n \subset \mathbb{R}^{n \times n}$ the subspace of real symmetric matrices, equipped with the inner product $\langle A, B \rangle_{\mathbb{S}^n} = \text{tr}(A^T B)$, where $\text{tr}(A)$ is the trace of A. Also, a positive semidefinite matrix A is denoted by $A \succeq O$ (respectively $A \succ O$ for positive definite). A typical SDP program is expressed in its primal-dual form as follows:

$$
\text{(P)} \quad
\begin{aligned}
p^* = {} & \sup_{X \in \mathbb{S}^n} \langle C, X \rangle_{\mathbb{S}^n} \\
\text{s.t. } & \langle A_i, X \rangle_{\mathbb{S}^n} = b_i, i = 1, \ldots, m, \\
& X \succeq O,
\end{aligned}
$$

$$
\text{(D)} \quad
\begin{aligned}
d^* = {} & \inf_{y \in \mathbb{R}^m} b^T y \\
\text{s.t. } & Y := \sum_{i=1}^{m} y_i A_i - C \succeq O,
\end{aligned}
$$

for given $C, A_i \in \mathbb{S}^{n \times n}, i = 1, \ldots, m$ and $b \in \mathbb{R}^m$.

It is however difficult in general to obtain an accurate optimum for a SDP problem. On the one hand, strong duality does not always hold, unlike for linear programs: weak duality is always satisfied, i.e. $p^* \leq d^*$, but sometimes, p^* is strictly less than d^*. Simpler instances are those where strong duality holds and this happens when the feasible set contains a positive definite matrix [12, Thm. 1.3]. In practice, the method of choice for SDP solving is based on interior-point algorithm. This relies on the existence of interior feasible solutions for problems (P) and (D). In such

cases, these two problems are simultaneously solved in polynomial time in the size of parameters of the input problem using the well-established primal-dual path-following interior-point method (PDIPM) [5]. This method relies on the fact that when an interior feasible solution exits, one has the necessary and sufficient condition for X^*, Y^* and y^* to be an optimal solution:

$$X^* Y^* = 0, \; X^* \succ O, \; Y^* \succ O, \tag{1}$$

$$Y^* = \sum_{i=1}^{m} y_i^* A_i - C, \tag{2}$$

$$\langle A_i, X \rangle_{\mathbb{S}^n} = b_i, i = 1, \ldots, m, \tag{3}$$

The complementary slackness condition (1) is replaced by a perturbed one: $X_\mu Y_\mu = \mu I$. This perturbed system has a unique solution and that the central path $\mathcal{C} = \{(X_\mu, Y_\mu, y_\mu) : \mu > 0\}$ forms a smooth curve converging to (X^*, Y^*, y^*) as $\mu \to 0$. So, the main idea is to numerically trace the central path $\mathcal{C}$. This algorithm, implemented among others by SDPA is sketched in Algorithm 1, adapted from [10]. Step 1 computes the search direction based on Mehrotra type predictor-corrector [6]. The stopping criteria (Step 4) depends on several quantities: primal and dual feasibility error are defined as the maximum absolute error appearing in (3) and (2), respectively. The duality gap depends on the absolute or relative difference between the objectives of (P) and (D). Numerical results involving these parameters are given in Section 6.

Algorithm 1 PDIPM Algorithm, adapted from SDPA implementation [10].

Step 0: Choose an initial point $X^0 \succ O, y^0$ and $Y^0 \succ O$. Set $h = 0$ and choose the parameter $\gamma \in (0, 1)$.

Step 1: Compute a search direction (dy, dX, dY).

Step 2: Compute max step length α to keep the positive semidefiniteness: $\alpha = \max\{\alpha \in [0, 1] : Y^h + \alpha dY \succ O, X^h + \alpha dX \succ O\}$.

Step 3: Update the current point: $(y^{h+1}, X^{h+1}, Y^{h+1}) = (y^h, X^h, Y^h) + \gamma \alpha (dy, dX, dY)$.

Step 4: If $(y^{h+1}, X^{h+1}, Y^{h+1})$ satisfies the stopping criteria, output it as a solution. Otherwise, set $h = h + 1$ and return to Step 1.

However, problems which do not have an interior feasible point induce numerical instability and may result in inaccurate calculations or non-convergences. Recently, SPECTRA package [13] proposes to solve such problems with exact rational arithmetic, but the instances treated are small and this package does not aim to be a concurrent of general numerical solvers.

On the other hand, even for problems which have interior feasible solutions, numerical inaccuracies may appear when solving with finite precision due to large condition numbers (higher than 10^{16}, for example) which appear when solving linear equations. This happens, as explained in [10], when approaching optimal solutions: suppose there exist $X^* \succ O$, $Y^* \succ O$ and y^* which satisfy all the constraints in (P) and (D); or better said, when $\mu \to 0$ on the central path. Then, $X^*Y^* = 0$. From this it follows that $\text{rank}(X^*) + \text{rank}(Y^*) \leq n$, which implies that these matrices are usually singular in practice.

In this second case, having an efficient underlying multiple-precision arithmetic is crucial to detect (at least numerically) whether the convergence issue came simply from numerical errors due to lack of precision.

3. Multiple precision arithmetic libraries

Most SDP solvers are based on *single* or *double* floating-point arithmetic, which is very efficient as it is implemented in hardware. An increase in precision, for example to *quad*-precision (*binary128*) or more, comes with a significant performance drawback since this is currently done by software emulation. Arbitrary precision, i.e., the user's ability to choose the precision for each calculation, is now available in most computer algebra systems such as Maple. SPECTRA [13] is implemented with Maple's rational arithmetic. SDPA extended precision versions are implemented using either GMP [14] or QD [15] libraries. GMP offers arbitrary precision by representing floating-point numbers in a so-called *multiple-digit* format. This is sufficiently general for manipulating numbers with tens of thousands of bits, but it is a quite heavy alternative for precisions of few hundreds bits. QD choses to better optimize precisions of 2 or 4 doubles using a different *multiple-component* format. It is known that most operations implemented in this library do not come with proven error bounds and correct or directed rounding is not supported. It is thus usually impossible to assess the final accuracy of these operations.

However, the performance results of QD are very good on tested problems (e.g. on SDP instances [6]). Moreover, only QD is ported on parallel GPU architectures, since floating-point algorithms in the *multiple-digit* format are very complex, and they employ non-trivial memory management. QD code with SDPA shows in practice that it is possible to compute very accurate values even when rounding occurs at the intermediate operation's level. We modified and generalized these algorithms for proving their correctness and keeping good performances.

4. CAMPARY library

For extending the precision, we represent numbers as unevaluated sums of FP numbers. When the sum consists of two terms, one has a *double-double* (DD) number, *triple-double* (TD) for three, *quad-double* for four, and *floating-point expansion* when made up with an arbitrary number of terms. Arithmetic operations with such numbers are based on so-called *error-free transforms* algorithms, which consist in a simple sequence of standard precision optimized hardware FP operations. Knuth's *2Sum* algorithm [16] computes the error-free sum of 2 FP numbers, with 6 FP operations. When the operands are ordered, Dekker's *Fast2Sum* algorithm [17] takes only 3 FP operations. Similarly, when a fused multiply-add [2] (FMA) operator is available, *2ProdFMA* [16] returns the correct rounded product, and the rounding error, in 2 FP operations.

Based on chaining together such error-free transforms (resulting in the so called *distillation* algorithms [18]), efficient basic operations with FP expansions were recently improved and proved to be correct in [19]. These are implemented in CAMPARY, a multiple-precision arithmetic library which targets both CPU applications and applications deployed on NVIDIA GPU platforms (compute capability 2.0 or greater). Both a CPU version (in C++ language) and a GPU version (written in CUDA C programming language [20]) are freely available. This library supports both the *binary64* and the *binary32* formats as basic bricks for the multiple component representation. This allows for extended precisions on the order of a few hundreds of bits, with constraints given by the exponent range of the underlying FP format used: the maximum expansion size is 39 for *double*-precision, and 12 for *single*-precision. Currently, all basic multiple-precision arithmetic operations $(+, -, *, /, \sqrt{})$ are supported. The implementation is very flexible: the precision (FP expansion size) is given as a template parameter and we provide overloaded operators.

For double-double, authors of this article rediscussed, improved and proved in [21], some algorithms initially proposed in QD library. For example, addition of two DD numbers takes 20 FP operations and multiplication 9, when an FMA is available (see Algorithms 6 and 11 in [21]). For higher precisions, we use algorithms from [19], with the technicality that the renormalization of FP expansions is composed of the first two levels of chained 2Sum of Algorithm 6 in [19]. This generality comes with a higher operation count. For addition, Algorithm 4 [19], takes $3n^2 + 10n - 4$

---

2. FMA operator evaluates an expression of the form $xy + t$ with only one final rounding.

double FP additions, while multiplication, Algorithm 5 [19], takes $2n^3 + 2n^2 + 6n - 4$ double FP operations (see Table 1).

n	Addition	Multiplication
2D	20	9
3D	53	86
4D	84	180
5D	121	326
6D	164	536
8D	268	1196

Table 1: Number of operations for addition and multiplication (FMA compatible) with n-doubles formats.

5. Implementation of SDPA-CAMPARY

SDPA-CAMPARY package is built from SDPA-QD/DD package, where QD/DD library is replaced by CAMPARY at the compilation step of SDPA. This is efficiently done since both SDPA and CAM-PARY are coded in C/C++. Starting with version 6.0, SDPA incorporated LAPACK [22] for dense matrix computations, but also exploits sparsity of data matrices and solves large scale SDPs [23]. More recently, MPACK [24] was developed and integrated with SDPA. This is a multiple precision linear algebra package, based on BLAS and LAPACK [22]. For this package, the major change is the underlying arithmetic format, since it supports multiple precision libraries, like GMP, MPFR, and QD. We also integrated CAM-PARY with MPACK.

Moreover, MPACK also provided a GPU tuned implementation in double-double of the Rgemm routine: this is the multiple precision *Real* version of Dgemm, the general double matrix multiplication [25]. This routine is central for other linear algebra operations such as solving linear equations, singular value decomposition or eigenvalue problems. For this, MPACK authors re-implemented parts of QD library for CUDA-compliant code. In our implementation, we used instead our GPU version of CAMPARY.

This routine's implementation in [25] has best practical performance and was intensively tuned for GPU-based parallelism: classical blocking algorithm is employed, and for each element of a block a thread is created; a specific number of threads is allocated per block also. For example, for the NVIDIA(R) Tesla(TM) C2050 GPU, best performance of 16.4GFlops (Giga FP operations per second) is obtained in [25] for $A \times B$, the product of two matrices A and B with block size of: 16×16 for A and 16×64 for B; 256 threads are allocated per block. Shared memory is used for each block. Also, reading is done from texture memory.

In our implementation, we use a similar algorithm, except that reading is done from global memory instead of texture memory, since a texture memory

element (texel) size is limited to *int4* i.e., 128 bits and our implementation is generic for n-double. Matrix block sizes and thread block sizes are similar, since our bench GPU is similar.

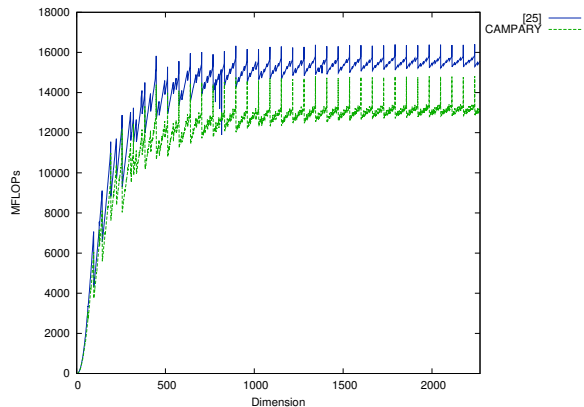Figure 1: Performance of RGEMM with CAMPARY vs [25] in double-double on GPU. Max performance of 14.8GFlops for CAMPARY and 16.4GFlops for [25].
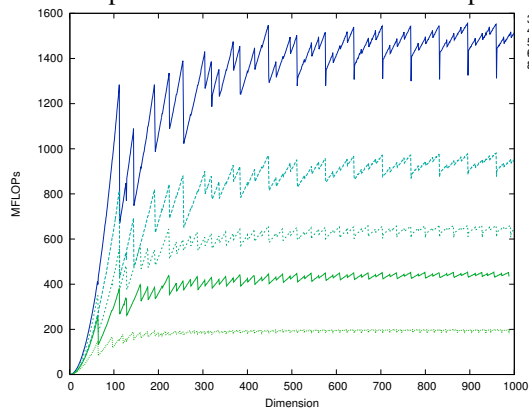

Figure 2: Performance of RGEMM with CAMPARY for n-double on GPU. Maximum performance was 1.6GFlops for TD, 976MFlops for QD, 660MFlops for 5D, 453MFlops for 6D, 200MFlops for 8D.
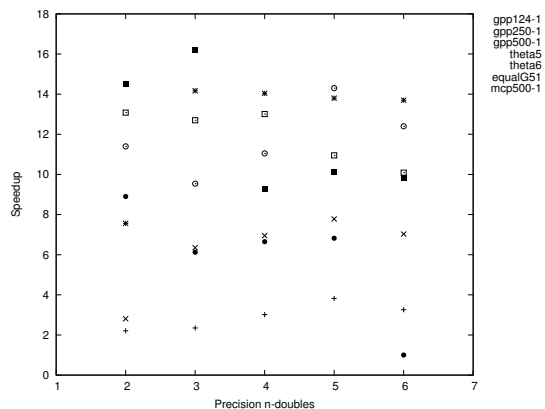

Figure 3: Speedup of SDPA-CAMPARY for n-double with GPU vs CPU. Maximum speedup was 16.2.

Specifically, our GPU NVIDIA(R) Tesla(TM) C2075 is part of the same Fermi architecture, with 448 cores, 1.15 GHz, 32KB of register, 64KB shared memory/L1 cache set by default to 48KB for shared memory and 16KB for L1 cache. The difference is that our GPU has 6GB of global memory, compared to 3GB of C2050. However, this has little importance for the performance results on kernel execution once the global memory has been loaded. With CAMPARY in double-double, the peak kernel performance obtained is 14.8GFlops, see Figure 1. The theoretical peak performance can be obtained as follows: first, consider that in Rgemm operations consist of mainly multiply-add type, so the theoretical peak for multiply-add is of $1.15[\text{GHz}]\times14[\text{SM}]\times32[\text{CUDA cores}]\times(2[\text{Flop}]/2[\text{cycle}]) = 515[\text{Gflops}]$. Now, since addition and multiplication in CAMPARY take $(20+9)$ Flops for double-double, one obtains theoretically: $515.2/29 = 17.8[\text{GFlops}]$. For double-double, our implementation is slower by $\sim 10\%$ than [25]. This can be explained by the generality of our code. Testing our implementation with texture memory, we observed no speedup. On the other hand, in our case, higher precision RGEMM is straightforward. Performance results for n-double RGEMM are shown in Figure 2: the increase in number of additions and multiplications reported in Table 1 fits the decrease of performance when precision is increased. Note that cuBLAS, the NVIDIA GPU linear algebra package does not support precisions higher than double and it is not open source, so we consider it difficult to extend in the present context. In [26], for Fermi GPUs like C2050 or C2075, the peak performance of DGEMM is 302GFlops with cuBlas and 362GFlops with further optimizations, which is 58% and 70% of the theoretical peak performance, so the RGEMM implementation we have is quite efficient with: 83% of theoretical peak performance for DD, respectively 43% for TD, 50% for 4D, 57% for 5D, 61% for 6D, 57% for 8D.

6. Numerical results

In order to assess the performance of the SDPA-CAMPARY solver we look at the results obtained for some standard SDP problems both on CPU and GPU. In particular, on GPU we use the Rgemm routine, with our CAMPARY implementation explained above.

All CPU tests were performed on an Intel(R) Xeon(R) CPU E3-1270 v3 @ 3.50GHz processor, with Haswell micro-architecture which supports hardware implemented FMA instructions. For the GPU tests, we used a NVIDIA(R) Tesla(TM) C2075.

Table 2 shows the results and performance obtained for five well-known problems from the SDPLIB package [11]. We compare the QD library implementation, both double- and quad-double precisions, with the double-, triple-, and quad-double offered by CAMPARY, on CPU. Our double-double implementation outperforms the one of QD's by far, while our quad-double precision performs the same. Triple-double proves to be a good intermediary alternative.

Unlike QD, GMP offers arbitrary precision. Since CAMPARY also has several extended n-double formats, we compare it against GMP in Table 3. We consider three problems from SDPLIB with corresponding precisions of $106, 159, 212, 265, 318$ and 424 bits. For small precisions, up to quad-double, our library performs better or the same on CPU. Performance decreases gradually on CPU when precision increases due to the overhead caused by error propagation and handling inside the FP expansions. However, since CAMPARY is generic and ported on GPU, this performance decrease can be overcome when launching SDPA-CAMPARY with GPU support as seen in Table 3. In Figure 3 we show the speedup obtain for CAMPARY with GPU support, when varying precision, for several problems from the SDPLIB.

For testing the accuracy of our library, we considered several examples from Sotirov's collection [3], which are badly conditioned numerically, and cannot be tackled with double precision only. For instance, a classical problem in coding theory is finding the largest set of binary words with n letters, such that the Hamming distance between two words is at least d. This is reformulated as a maximum stable set problem, which is solved with SDP, according to the seminal work of Schrijver [27], followed by Laurent [28].

In Table 4 we show the performance obtained on the CPU for the Schrijver and Laurent instances from [3]. The comparison is done between SDPA-DD, SDPA-GMP (run with 106 bits of precision) and SDPA-CAMPARY-DD. The results obtained were similar and performance-wise our library performs better. Some instances do not converge when DD precision is used. We also include the results obtained with the SDPA-CAMPARY with triple-double (TD) precision, which has a better performance comparing to SDPA-GMP with 106 bits of precision.

7. Conclusion and future works

We implemented and evaluated the performance of our floating-point expansions-based arithmetic library, CAMPARY, in the context of highly accurate computations needed by SDP solvers for ill-conditioned

Problem	SDPA-DD	SDPA-QD	SDPA-CAMPARY		
			(DD)	(TD)	(QD)
`gpp124-1`		optimal: -7.3430762652465377			
relative gap	$7.72e-04$	$6.75e-13$	$7.72e-04$	$8.33e-12$	$5.42e-18$
p.feas.error	$5.42e-20$	$4.37e-45$	$5.42e-20$	$1.57e-30$	$1.14e-41$
d.feas.error	$4.40e-14$	$1.54e-30$	$3.99e-14$	$2.58e-16$	$3.43e-22$
iteration	24	40	24	38	49
time (s)	3.27	66.37	2.63	35.56	83.92
`gpp250-1`		optimal: $-1.5444916882934067e+01$			
relative gap	$5.29e-04$	$4.32e-13$	$5.19e-04$	$1.03e-12$	$1.89e-17$
p.feas.error	$3.89e-20$	$2.27e-45$	$1.35e-20$	$1.38e-30$	$5.73e-42$
d.feas.error	$9.78e-14$	$1.97e-30$	$1.12e-13$	$3.04e-17$	$3.25e-22$
iteration	25	41	25	44	52
time (s)	26.68	538.39	21.57	321.35	698.34
`gpp500-1`		optimal: $-2.5320543879075787e+01$			
relative gap	$1.008e-03$	$4.72e-13$	$3.72e-04$	$8.89e-12$	$9.30e-18$
p.feas.error	$1.01e-20$	$1.92e-45$	$2.71e-20$	$8.81e-31$	$1.43e-42$
d.feas.error	$5.29e-14$	$2.95e-30$	$3.51e-13$	$1.49e-16$	$5.13e-22$
iteration	25	42	26	41	51
time (s)	207.63	4340.95	172.95	2396.43	5369.79
`qap10`		optimal: $-1.0926074684462389e+03$			
relative gap	$3.84e-05$	$3.18e-14$	$6.94e-05$	$1.08e-09$	$7.22e-14$
p.feas.error	$2.54e-21$	$7.32e-47$	$6.56e-21$	$6.62e-35$	$2.18e-47$
d.feas.error	$4.91e-14$	$8.78e-30$	$1.83e-13$	$2.98e-21$	$3.45e-29$
iteration	20	36	20	27	35
time (s)	27.75	647.96	21.01	262.03	629.81
`hinf3`		optimal: $5.69407780090669388e+01$			
relative gap	$1.35e-08$	$7.93e-3$	$3.25e-05$	$3.98e-26$	$1.98e-31$
p.feas.error	$2.75e-24$	$1.18e-54$	$1.69e-21$	$3.32e-39$	$3.18e-55$
d.feas.error	$3.82e-14$	$6.82e-35$	$3.41e-13$	$2.83e-29$	$4.08e-42$
iteration	30	90	24	48	48
time (s)	0.00	0.18	0.00	0.05	0.09

Table 2: The optimal value, relative gaps, primal/dual feasible errors, iterations and time for solving some problems from SDPLIB by SDPA-QD, -DD, -CAMPARY.

and performance demanding problems. The underlying arithmetic operations of state-of-the-art multiple-precision solver SDPA were replaced with our algorithms. This was possible since both SDPA and CAMPARY are written in open-source and freely available C/C++. We compare and contrast both the numerical accuracy and performance of SDPA-GMP, -QD and -DD, which employ other multiple-precision arithmetic libraries, against SDPA-CAMPARY. We consider both CPU and GPU implementations. We show that CAMPARY is a very good trade-off for accuracy and speed when solving ill-conditioned SDP problems.

One current limitation is that compared to the tuned GPU implementation of double-double RGEMM of [25], our implementation is 10% slower. However, the improvement is that our RGEMM is generic for n-double precision. This allows us to solve very ill-conditioned problems with SDPA and GPU support in a more efficient way: before, these problems could be tackled only with high precision GMP support on CPU only. Our new results show an order of magnitude ($\simeq 10$ times) average speedup when CAMPARY is used with GPU support. A future work is to solve more complicated and large scale very ill-conditioned SDP problems appearing in experimental mathematics, like for instance, the "kissing numbers" problem [2].

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Problem	SDPA-CAMPARY		SDPA-GMP
	CPU	GPU	CPU
`gpp124-1`	optimal: -7.3430762652465377		
precision	DD	DD	$2 * 53$ bits
iteration	24	24	38
time (s)	2.63	1.19	56.31
precision	TD	TD	$3 * 53$ bits
iteration	38	39	48
time (s)	35.56	15.1	78.30
precision	QD	QD	$4 * 53$ bits
iteration	49	57	59
time (s)	83.92	27.9	103.32
precision	5 D	5 D	$5 * 53$ bits
iteration	62	62	77
time (s)	178.23	46.5	149.76
precision	6 D	6 D	$6 * 53$ bits
iteration	77	77	77
time (s)	330.08	101	149.76
precision	8 D	8 D	$8 * 53$ bits
iteration	77	77	77
time (s)	666.60	217	186.32
`gpp250-1`	optimal: $-1.5444916882934067e + 01$		
precision	DD	DD	$2 * 53$ bits
iteration	25	25	39
time (s)	21.57	7.65	455.36
precision	TD	TD	$3 * 53$ bits
iteration	44	39	46
time (s)	321.35	50.6	591.12
precision	QD	QD	$4 * 53$ bits
iteration	52	52	64
time (s)	698.34	100.3	880.90
precision	5 D	5 D	$5 * 53$ bits
iteration	62	62	73
time (s)	1404.48	180.4	1117.60
precision	6 D	6 D	$6 * 53$ bits
iteration	73	73	73
time (s)	2465.30	350.6	1116.72
precision	8 D	8 D	$8 * 53$ bits
iteration	73	73	73
time (s)	4987.770	1047.7	1392.71
`hinf3`	optimal: $5.6940778009669388e + 01$		
precision	DD	DD	$2 * 53$ bits
iteration	24	24	39
time (s)	0.00	0.51	0.06
precision	TD	TD	$3 * 53$ bits
iteration	48	48	47
time (s)	0.05	0.82	0.07
precision	QD	QD	$4 * 53$ bits
iteration	48	48	47
time (s)	0.09	2.01	0.08
precision	5 D	5 D	$5 * 53$ bits
iteration	48	48	47
time (s)	0.16	3.52	0.09
precision	6 D	6 D	$6 * 53$ bits
iteration	48	48	47
time (s)	0.25	5.13	0.09
precision	8 D	8 D	$8 * 53$ bits
iteration	48	48	47
time (s)	0.53	8.11	0.10

Table 3: The optimal value, iterations and time for solving some problems from SDPLIB by SDPA-GMP, -CAMPARY

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Problem	SDPA-DD	SDPA-CAMPARY-DD	SDPA-CAMPARY-TD	SDPA-GMP
Laurent_A(19,6)		optimal: $-2.4414745686616550e-03$		
iteration	92	94	71	73
time (s)	4.3	3.1	18.65	29.16
Laurent_A(26,10)		optimal: $-1.3215201241629400e-05$		
iteration	80	80	123	125
time (s)	12.8	8.68	109.54	173.42
Laurent_A(28,8)		optimal: $-1.1977477306795422e-04$		
iteration	93	100	76	113
time (s)	47.8	36.85	219.46	541.19
Laurent_A(48,15)		optimal: $-2.229e-09$		
iteration	134	134	165	145
time (s)	2204.61	1569.48	14691.92	21695.08
Laurent_A(50,15)		optimal: $-1.9712e-09$		
iteration	142	142	191	154
time (s)	3463.2	2421.86	25773.96	35173.79
Laurent_A(50,23)		optimal: $-2.5985e-13$		
iteration	124	124	155	140
time (s)	342.73	221.32	2333.74	3426.17
Schriver_A(19,6)		optimal: $-1.2790362700180910e+03$		
iteration	40	40	66	95
time (s)	1.59	1.14	14.65	32.21
Schriver_A(26,10)		optimal: $-8.8585714285713880e+02$		
iteration	54	54	127	108
time (s)	7.75	5.2	100.73	134.48
Schriver_A(28,8)		optimal: $-3.2150795825792913e+04$		
iteration	45	45	69	97
time (s)	21.05	15.06	182.25	422.78
Schriver_A(37,15)		optimal: $-1.4006999999999886e+03$		
iteration	58	58	132	116
time (s)	54.86	36.35	683.07	988.21
Schriver_A(40,15)*		optimal: $-1.9e+04$		
iteration	23	23	23	23
time (s)	53.99	35.99	285.3	471.870
Schriver_A(48,15)*		optimal: $-2.56e+06$		
iteration	27	27	27	27
time (s)	432.13	307.88	2260.24	3862.29
Schriver_A(50,15)**		optimal: $-7.6e+06$		
iteration	29	29	29	29
time (s)	694.07	471.57	3695.95	677.830
Schriver_A(50,23)**		optimal: $-5.2e+03$		
iteration	29	29	29	29
time (s)	76.55	47.84	413.31	6370.97

Table 4: The optimal value, iterations and time for solving some ill-posed problems for binary codes by SDPA-DD, -CAMPARY-DD, -CAMPARY-TD, and -GMP-DD. ∗ problems that converge to more than two digits only when using quad-double precision. ∗∗ problems that converge to more than two digits with precision higher that quad-double. The digits written with blue were obtained only when triple-double precision was employed.

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