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First passage time on pattern formation in a non–local Fisher population dynamics

Research Article

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Abstract: We use stochastic dynamics to develop the patterned attractor of a non–local extended system. This is done analytically using the stochastic path perturbation approach scheme, where a theory of perturbation in the small noise parameter is introduced to analyze the random escape of the stochastic field from the unstable state. Emphasis is placed on the specific mode selection that these types of systems exhibit. Concerning the stochastic propagation of the front we have carried out Monte Carlo simulations which coincide with our theoretical predictions.

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1. Introduction

Competitive interactions are ubiquitous in Nature. One important, highly studied type of model used to understand these interactions are the so-called Lotka–Volterra (or Fisher) equations [1, 2].

This paradigm has been used extensively in ecology, mode competition in optical systems [3], and as competing technologies, a very different system, but one showing similar characteristics [4, 5].

The classical equations of this model assume that individuals follow a random walk in order to diffuse or disperse to nearby locations. This assumption is a very good approximation in many scenarios, but it cannot be used when individuals compete in a finite neighborhood, i.e. to model this kind of system the local competition interaction is replaced by a non–local operator.

Incorporating these *non–local* effects in a reaction–diffusion model, has been recently applied towards the end of improving the growth dynamics of a given population: for example, a petri dish bacteria culture in which the diffusion of nutrients and/or the release of toxic substances can cause non–locality in the interactions [6, 7]. Notice that in all these models the environmental fluctuations are

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not taken into account, and the dynamics are completely deterministic. A more realistic framework can be thought of as an extension of the previous work but with the introduction of an additive Gaussian noise in space and time with intensity $\sqrt{\epsilon}$ as follows

$$\partial_t u(x,t) = D\partial_{xx}u(x,t) + au(x,t) - bu(x,t)\int_{\mathcal{D}} G\left(x-x'\right)u(x',t)dx' + \sqrt{\epsilon}\xi(x,t). \tag{1}$$

This stochastic partial differential equation is completely characterized by giving the statistics of the field $\xi(x,t)$. In the present work we adopt the white-noise cumulants [8, 9]:

$$\langle \xi(x,t)\rangle = 0; \qquad \langle \xi(x,t)\xi(x',t')\rangle = \delta(x-x')\delta(t-t').$$

The kernel $G(x)$ in the non–local interaction is adopted to be *symmetric* and *normalized* in the domain of interest $\mathcal{D}$. We are interested in the stochastic pattern formation description of the (positive) density field $u(x,t)$. The deterministic version of model, Eq. (1), has two homogeneous steady states: $u_0 = \{0, a/b\}$. In the *local* case the states $\{0, a/b\}$ constitute unstable and stable fixed points respectively; note that in the *non–local* case Fisher's model is non–variational.

2. Non–local instability

The instability can be characterized by doing the usual linear Fourier analysis. Consider for example a small perturbation around the homogeneous state $u(x,t) = u_0 + U(x,t)$ with $\|U(x,t)\| \sim \mathcal{O}\left(\sqrt{\epsilon}\right)$. From these considerations and using natural boundary conditions it is simple to see that the instability of the fixed points $\{0, a/b\}$ are characterized by

$$u_0 = 0, \text{ unstable if } k < \sqrt{a/D}, \tag{2}$$

$$u_0 = a/b, \text{ unstable if } k < \sqrt{-aG(k)/D}, \tag{3}$$

where $G(k) \equiv \int_{-\infty}^{\infty} G(x)e^{ikx}dx \in \mathcal{R}_e$ is the *continuous* Fourier transform of the kernel $G(x)$. From (3) we realize that a non-homogeneous pattern may grow from the homogeneous fully populated state a/b, if the Fourier transform of the kernel is negative for some Fourier mode.

It bears mention that an alternating function $G(k)$ (i.e., ranging from positive to negative values) is a typical characteristic of the Fourier transform of any *abrupt* positive symmetric function $G(x)$. By abrupt behavior we mean that the positive kernel goes to zero in a finite domain.

In what follows we introduce a theory of perturbation to characterize the Mean First Passage Time (MFPT) associated with the formation of a non-homogeneous pattern from the homogeneous steady state $u_0 = a/b$. We also carry out numerical simulations and compare these results with our theoretical predictions.

We now adopt periodic boundary condition on the macroscopic interval $\mathcal{D} \equiv [-1, 1]$. Thus any characteristic length should be compared with the typical scaling size adopted for the system, i.e., $L = 1$. Therefore we now introduce the *discrete* Fourier transform in the non–local Fisher equation (1), using

$$\begin{aligned} u(x,t) &= \sum_{n=-\infty}^{\infty} A_n(t)\exp\left(ik_n x\right) \\ \xi(x,t) &= \sum_{n=-\infty}^{\infty} \xi_n(t)\exp\left(ik_n x\right) \\ G(x) &= \sum_{n=-\infty}^{\infty} G_n \exp\left(ik_n x\right), \end{aligned}$$

where $k_n = n\pi,\ n = 0, 1, 2, 3...$, and noting that

$$\begin{aligned} \int_{-1}^{1} e^{i(m+n)\pi x}dx &= 2\delta_{m+n,0} \\ \int_{-1}^{1} G(x)\,dx &= 1, \end{aligned}$$

we obtain the following set of coupled Fourier modes

$$\frac{dA_n}{dt} = \left(-Dk_n^2 + a\right)A_n - 2b\sum_{l=-\infty}^{l=\infty} A_{n-l}A_l G_l + \sqrt{\epsilon}\xi_n(t), \tag{4}$$

where: $G_n = \frac{1}{2}\int_{-1}^{1} e^{-in\pi x}G(x)dx$.

From this equation and linear stability analysis it is simple to see that the homogenous state $u_0 = a/b$ will be unstable under small perturbations of the form: $u(x,t) = a/b + \sqrt{\epsilon}e^{\varphi t + ik_n x}$, if

$$\varphi = -Dk_n^2 - 2aG_n > 0. \tag{5}$$

This dispersion relation is analogous to the one obtained by using the *continuous* Fourier wave number k; see Eq. (3), the factor 2 in (5) comes from the finite character of the domain $\mathcal{D}$. Note that any typical length scale characterizing the abrupt condition of the kernel $G(x)$ will appear in the final expression of G_n.

3. The normal form

Motivated by the analysis of instability from (5), we introduce now the notation k_{me} to represent the most unstable discrete Fourier mode that will grow from any small perturbation, and we specify the evolution for the A_0 mode coupled with the non-homogenous ones. Therefore from (4) we can rewrite the deterministic coupled dynamics in the form

$$\frac{dA_0}{dt} = aA_0 - bA_0^2 - 2b\sum_{l\neq 0} A_{-l}A_l G_l, \tag{6}$$

$$\frac{dA_m}{dt} = a(m)A_m - bA_mA_0\left(1+2G_m\right) - 2b\sum_{l\neq 0,m} A_{m-l}A_lG_l;\;\; m\neq 0, \tag{7}$$

where

$$a(m) \equiv \left(-Dk_m^2 + a\right).$$

Near criticality, when the initial condition is $u(x,0) = a/b$, and adopting the parameters in such a way that there will be only one Fourier unstable mode k_{me}, we get for short time horizons the evolution $A_0(\Delta t) \sim a/b + 2bA_{me}(0)^2\left|G_{me}\right|\Delta t$, with $\left|A_{me}(0)\right| \sim \mathcal{O}\left(\sqrt{\epsilon}\right) \ll a/b$ indicating that $A_0(t)$ growths linearly in time. On the other hand $A_{me}(t)$ grows exponentially in the form $A_{me}(\Delta t) \sim A_{me}(0)\exp\left[\left(-Dk_{me}^2 + 2a\left|G_{me}\right|\right)\Delta t\right]$, as can be seen from (7). Therefore we can introduce fast and slow variables to solve the set of coupled equations (6) and (7). The simplest way to do this is to approximate $A_0(t)$ as a constant during the short time when $A_{me}(t)$ grows exponentially. Therefore we can now introduce what we call the Minimum Coupling Approximation (MCA); this approximation allows us to derive from (6) and (7) the set of coupled equations

$$0 \simeq aA_0 - bA_0^2 - 2bA_{me}^2G_{me}, \tag{8}$$

$$\frac{dA_{me}}{dt} \simeq a(me)A_{me} - bA_{me}A_0\left(1+2G_{me}\right), \tag{9}$$

where we have used the symmetry of the deterministic system to adopt $\mathcal{R}_e[A_{me}] = \mathcal{R}_e[A_{-me}]$ and $\mathcal{I}_m[A_{me}] = -\mathcal{I}_m[A_{-me}]$. From (8) we can solve A_0 as a function of A_{me} and introduce this solution in the equation for dA_{me}/dt; after some algebra we get the *normal form* characterizing the pattern formation from the Fourier mode k_{me}

$$\frac{dX}{dt} = \alpha X - \beta X^3, \tag{10}$$

where

$$\alpha = \left(-Dk_{me}^2 + 2a\left|G_{me}\right|\right) > 0,$$

$$\beta = 2a\left(1-2\left|G_{me}\right|\right)(b/a)^2\left|G_{me}\right| > 0.$$

The saturation term in (10) is valid as long as $\left|G_{me}\right| < 1/2$. If this is not the case we have to write the next term $\sim X^5$ in order to introduce the saturation contribution in the normal form. Interestingly, the stationary state of the MCA set (8) and (9) is

$$A_{me}^{\infty} = \sqrt{\frac{\left(aA_0 - bA_0^2\right)}{2bG_{me}}} \tag{11}$$

$$A_0^{\infty} = \frac{a(m_e)}{b\left(2G_{me}+1\right)}.$$

The stationary state of (10) is $X^+ = \sqrt{\alpha/\beta}$, so comparing X^+ vs A_{me}^{∞} gives us a criterion to establish the accuracy of the normal form (10). In order to fulfill this last criterion we must assure that $a(m_e) \approx a$ and $2G_{me} \ll 1$. In the present work, with the aim of providing an example, we use a square kernel for the non–local interaction model, defined as

$$G(x) = \frac{1}{2w}\left[\Theta(w-x)\Theta(w+x)\right] \to G_n = \frac{1}{2}\frac{\sin n\pi w}{nkw}, \tag{12}$$

where $\Theta(x)$ is the step function. Then for this particular case, we finally get that

$$\frac{1}{w} \ll k_{me} \equiv m_e\pi \ll \sqrt{a/D}.$$

This means that the Fourier wave length of the unstable mode, $\lambda_{me} = 2\pi/k_{me}$, should be much larger than the diffusion length measured in units of the linear relaxation time a^{-1} and much shorter than the non–local range w of the kernel. Any other abrupt kernel may be worked in a similar manner.

4. The stochastic path perturbation approach

The analysis of the stochastic passage times near criticality can be done by introducing the stochastic path perturbation approach (SPPA) [10, 11] in the dynamics of the stochastic version of the normal form (10), i.e.

$$\frac{dX}{dt} = \alpha X - \beta X^3 + \sqrt{\epsilon}\,\xi(t), \tag{13}$$

where $\xi(t)$ is Gaussian white noise. In the small noise approximation, the SPPA consists of obtaining information about the first passage time statistics without solving the Fokker–Planck equation. This is done by analyzing the stochastic realizations of the process under study, when they are written in terms of Wiener paths.

Introducing the non-linear transformation

$$X(t)=\frac{Z(t)}{Y(t)}$$

in (13) gives the following set of coupled equivalent equations (in the Stratonovich calculus [9])

$$\frac{dZ}{dt} = \alpha Z(t)+\sqrt{\epsilon}Y(t)\xi(t);\ Z(0)=0, \qquad (14)$$

$$\frac{dY}{dt} = \beta\frac{Z(t)^2}{Y(t)};\ Y(0)=1. \qquad (15)$$

This set of equations can be solved iteratively for small noise $\epsilon\to 0$. At short time putting $Y(t)\simeq 1$ in (14) we get

$$Z(t)\simeq\sqrt{\epsilon}e^{\alpha t}h(t);\ \text{with } h(t)\equiv\int_0^t e^{-\alpha s}\xi(s)ds, \qquad (16)$$

where in order to assure the positivity of $X(t)$ the new stochastic process $h(t)$ must be bounded to positive support. Then, introducing (16) in (15) and neglecting $Y(t)$ in the denominator we get

$$Y(t)\simeq 1+\beta\epsilon\int_0^t e^{2\alpha s}\ h^2(s)ds. \qquad (17)$$

From equations (16) and (17) it is possible to analyze the escape of the stochastic trajectories from the initial condition of $\mathcal{O}\left(\sqrt{\epsilon}\right)$ to $\mathcal{O}(1)$. Then we get for the transient behavior of the unstable Fourier amplitude $A_{me}(t)\sim X(t)$ the expression

$$X(t)\simeq\frac{\sqrt{\epsilon}\ e^{\alpha t}\ h(t)}{1+\beta\epsilon\int_0^t e^{2\alpha s}h^2(s)ds}. \qquad (18)$$

For $\alpha t\gg 1$ the stochastic process $h(t)$ relaxes very quickly to its stationary state $h(\infty)\equiv\Omega$, from which we can introduce our next approximation in (18) in order to obtain the analytical First Passage Time Distribution (FPTD) for the random escapes from $X(t)\sim\sqrt{\epsilon}$ to $\sim\mathcal{O}(1)\sim\sqrt{\alpha/\beta}$. Approximating the stochastic process $h(t)$ by the random variable Ω in (18) we get

$$X(t)\simeq\frac{\sqrt{\epsilon}\ e^{\alpha t}\ \Omega}{1+\beta\epsilon\Omega^2\left(e^{2\alpha t}-1\right)/2\alpha}. \qquad (19)$$

The numerator of (19) contains all the information to solve the random escape analysis from the linear instability of the normal form, and the denominator introduces the first correction from the nonlinear contribution appearing in (10). Then it is simple to see that the random escape time t_e (the random first passage time) is given here as a random variable transformation from Ω

$$t_e=\frac{1}{\alpha}\ln\left(\frac{K}{\Omega\sqrt{\epsilon}}\right), \qquad (20)$$

where $K=\sqrt{\alpha/\beta}$. In order to improve this approximation we can go one step further and extract the random variable transformation from the denominator of (19); we get the same transformation as before, but with a renormalized value for $K=2\sqrt{\alpha/\beta}$.
Using Wiener's integrals we calculate the stationary distribution of $h(t)$, with a positive support, and then we get for the Gaussian random variable Ω the Probability Distribution Function (PDF)

$$P(\Omega)=2\sqrt{\frac{\alpha}{\pi}}\exp\left(-\alpha\Omega^2\right),\ \Omega\in(0,\infty). \qquad (21)$$

Therefore the FPDT is given from (20) by using the PDF of Ω

$$P(t_e) = \int\delta\left(t_e-t_e(\Omega)\right)P(\Omega)\ d\Omega \qquad (22)$$

$$= \frac{2K}{\mathcal{N}}\frac{\alpha^{3/2}\exp\left(-\alpha t_e\right)}{\sqrt{\pi\epsilon}}\exp\left[-\frac{K^2\exp\left(-2\alpha t_e\right)}{\epsilon/\alpha}\right]$$

with $K=2\sqrt{\alpha/\beta}$. Note that this formula has only one important universal parameter $\tilde{K}=K\sqrt{\alpha/\epsilon}$, so by introducing the change of variable $\tau_e=\alpha t_e$ we obtain a universal dimensionless expression for the FPDT of the most unstable Fourier mode k_{me}

$$P(\tau_e)=\frac{2\tilde{K}}{\mathrm{erf}\left(\tilde{K}\right)\sqrt{\pi}}\exp\left[-\tau_e-\tilde{K}^2\exp\left(-2\tau_e\right)\right],$$

$$\tilde{K}=2\alpha\left(\beta\epsilon\right)^{-1/2},\ \tau_e=\alpha t_e. \qquad (23)$$

In order to perform the numerical simulations we have solved the stochastic evolution equation for the field $u(x,t)$ in real space from (1), starting from the initial condition $u(x,0)=a/b$. The parameters in the non–local Fisher equation are fixed near criticality and in order to have only one unstable Fourier mode k_{me} we chose those shown in Table 1. Using these parameters we get from the (square) Fourier kernel the following critical value $G_2=-0.108118$, i.e., the second mode is unstable, and so $\varphi>0$; see Table 2.
Then in the simulation the values of the noise intensity were chosen to be: $\epsilon=\{10^{-2},10^{-3},10^{-4}\}$.

Table 1. Parameters near criticality used.

Physical parameters	Decription
$a = 1$	Linear growth rate
$b = 1$	Nonlinear coupling parameter
$D = 5 \times 10^{-4}$	Diffusion coeficient
$L = 1$	Macroscopic size system
$w = 0.7$	Cut-off in the non-local interaction range

Figure 1 depicts $P(\tau_e)$ curves for different values of noise. Also the corresponding Monte Carlo simulations for the onset of the stochastic pattern formation, from the unstable fully populated homogeneous state a/b, are compared with our theoretical prediction. Despite the many approximations that we have introduced to calculate an analytical first passage time the density is good enough to predict an accurate MFPT even when the shape of the profile is not entirely in agreement with the simulation of the escape from the homogeneous initial condition $u(x, t = 0) \sim a/b$ to the pattern formation characterized by $u(x, t) \sim A_0(t) + A_{me}(t) \exp(ik_{me}x) + A_{-me}(t) \exp(-ik_{me}x)$.

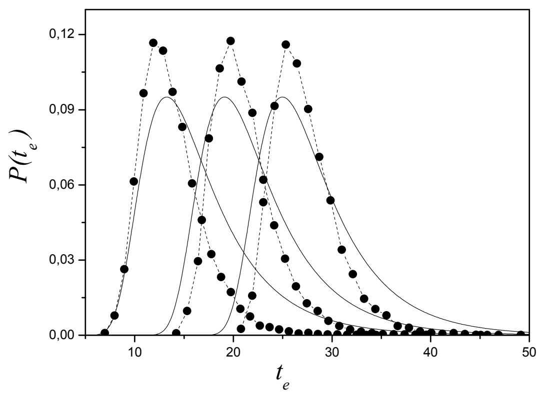

Figure 1. Probability Distribution Function for the scape time $P(t_e)$. Continous line: theoretical distributions from Eq. (22). Dots: simulations of Eq. (1), the dashed lines connecting dots are a guide to the eye only. The parameter values used are shown in Table 1, with $\epsilon = 10^{-2}, 10^{-3}$ and 10^{-4}, from left to right.

4.1. The mean first passage time

From equations (22, 23) we can calculate the mean first passage time. This value is a good statistical characteristic to determine the time scale in which the non-homogeneous pattern will appear from criticality. We get

$$\langle \tau_e \rangle = \int_0^\infty \tau_e P(\tau_e)\, d\tau_e \simeq \ln\left(\tilde{K}\right) + \frac{\mathcal{E} + \ln 4}{2\,\mathrm{erf}\left(\tilde{K}\right)}, \quad \tilde{K} \gg 1,$$

where here $\mathcal{E} = 0.577215\cdots$ is the Euler constant. We compare this prediction, $\langle \tau_e \rangle$, with numerical simulations obtained from the non–local Fisher equation, $\langle \tau_e \rangle_{MC}$; Table 3 shows a high degree of agreement.

It is important to note here that in order to make the simulations we adopt the value a/b as a threshold to count when the stochastic realizations $u(x, t)$ escape from $\mathcal{O}(\sqrt{\epsilon})$ to $\mathcal{O}(1)$.

5. Conclusions

In the present work we have studied the stochastic escape of an homogeneous stationary state toward a patterned final attractor. We emphasize that the specific mode selection that this type of system exhibits allows us to study the dynamics of unstable non–homogeneous modes, beyond the classical homogeneous–like escape.

Using the stochastic path perturbation approach we were able to find the analytic expression for the escape times distribution. We have found a high level of agreement with the Monte Carlo simulation.

Note from Eq. (10) that because the instability is linear, a deterministic path starting from $X(t = 0) = \sqrt{\epsilon}$ will take an infinite time to reach the basis of the attractor $X^+ = \sqrt{\alpha/\beta}$; this is the reason why the very definition of escape time has a margin of indeterminacy. Clearly, the adiabatic-like approximation that we refer to as the *Minimum Coupling Approximation* in the deterministic Eqs. (8-9), which lead to the normal form, Eq.(10), for the unstable mode A_{me}, cannot be used as a valid approximation in the entire interval from $A_{me} = 0$ all the way up to the true stationary state of the deterministic Eq.(1). If the instability were nonlinear as in the case when $\left(-Dk_{me}^2 + 2a\,|G_{me}|\right) = 0$ the perturbation of an escape time must be derived in a different way. This case is beyond the scope of the present approximation and it will be presented in a future contribution.

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Table 2. Critical parameters.

Physical parameters	Decription
$G_2 = \frac{1}{2}\frac{\sin 2\pi w}{2\pi w}$	Fourier mode of the square non-local Kernel
$\varphi = -D(2\pi)^2 - 2aG_2 > 0.$	Phase in the small wave perturbation

Table 3. Mean first passage time: prediction and Monte Carlo (MC) simulations.

Physical parameters	Decription		
	$\epsilon = 10^{-2}$	$\epsilon = 10^{-3}$	$\epsilon = 10^{-4}$
$\langle \tau_e \rangle$	16.5	22.3	28.2
$\langle \tau_e \rangle_{MC}$	13.5 ± 3.2	21.0 ± 3.2	27.2 ± 3.3

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