A Tonnetz model for pentachords

KEYWORDS. neo-Riemann network, pentachord, contextual group, Tessellation, Poincaré disk, David Lewin, Charles Koechlin, Igor Stravinsky.

ABSTRACT. This article deals with the construction of surfaces that are suitable for representing pentachords or 5-pitch segments that are in the same T/I class. It is a generalization of the well known Öttingen-Riemann torus for triads of neo-Riemannian theories. Two pentachords are near if they differ by a particular set of contextual inversions and the whole contextual group of inversions produces a Tiling (Tessellation) by pentagons on the surfaces. A description of the surfaces as coverings of a particular Tiling is given in the twelve-tone enharmonic scale case.

1. Introduction

The interest in generalizing the Öttingen-Riemann Tonnetz was felt after the careful analysis David Lewin made of Stockhausen's Klavierstück III [25, Ch. 2], where he basically shows that the whole work is constructed with transformations upon the single pentachord $\langle C, C\#, D, D\#, F\#\rangle$. A tiled torus with equal tiles like the usual Tonnetz of Major and Minor triads is not possible by using pentagons (you cannot tile a torus or plane by regular convex pentagons). Therefore one is forced to look at other surfaces and fortunately there is an infinite set of closed surfaces where one can gather regular pentagonal Tilings. These surfaces (called hyperbolic) are distinguished by a single topological invariant: the genus or number of holes the surface has (see Figure 8)[1].

The analysis[2] of Schoenberg's, Opus 23, Number 3, made clear the type of transformations[3] to be used. These are the basic contextual transformations (inversions) we are concerned in this article and they are given in (DEF. 1).

The main result of this paper is given in Theorem 1. We propose a surface of a high genus as a topological model for pentachords satisfying certain conditions. We give a construction of this surface in terms of a covering of another surface of smaller genus which carries a

[1] Not all genera admit a regular pentagonal Tiling.

[2] Prepint by David Lewin cited in [13].

[3] In this work again there is a fundamental generating pentachord $\langle B\flat, D, E, B, C\#\rangle$ and the group of transformations contain the (T/I)-class of this pitch segment.

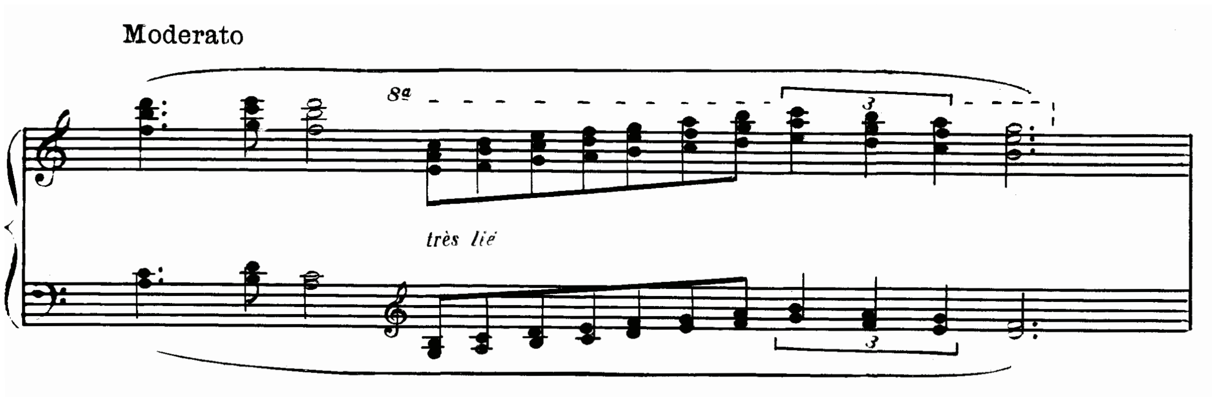

Figure 1: Excerpt of Les Heures Persanes, Op. 65, IV. Matin Frais dans la Haute Vallée by Charles Koechlin. Copyright 1987 by Editions Max Eschig.

pentagonal Tiling on it. The structure of the group of contextual transformations is studied and related to the surface. Many examples of pentachord or 5-pitch segments (and their T/I classes) including the two examples above fit into this kind of surfaces as a regular Tiling.

In a recent paper Joseph Straus [31] gave an interpretation of passages of Schoenberg's Op. 23/3 and Igor Stravinsky's In Memoriam Dylan Thomas. He uses a different set of transformations on pentachords or 5-pitch segments: a combination of inversions and permutations. Straus describes a space for these transformations and it would be interesting to check whether this kind of contextual groups fits into our framework.

Also, the citations in [20, pag. 49] are an useful source of examples for studying 5-pitch segments. For instance, A. Tcherepnin uses the different modes of $\langle C, D, E, F, A \rangle$ and $\langle C, D, E\flat, G, A\flat \rangle$ in his Ops. 51, 52, and 53.

We use the dodecaphonic system. This is best suited for Stockhausen or Schoenberg's works and it translates into numbering pitch classes modulo octave shift by the numbers in $\mathbb{Z}_{12}$. However, as we see in the following passages other systems are admissible. For instance, diatonic with numbers in $\mathbb{Z}_7$.

Example 1. The piece IV of Les Heures Persanes, Op. 65 by Charles Koechlin contains passages like that of Figure 1, where starting with a given pentachord he makes several parallel pitch translates but in a diatonic sense. In this case you have to assign to the notes a number in $\mathbb{Z}_7$ (for instance, $[C, D, E, F, G, A, B] \to [0, 1, 2, 3, 4, 5, 6]$). There are 14 (T/I)-forms for any pentachord.

Example 2. The piano part of Figure 2 from the Ballet Petrushka by Igor Stravinsky also moves pentachords in a parallel diatonic way. The same observations as in Example 1 apply.

In this article we are not going to pursue the case of a pitch system with numbering in a general $\mathbb{Z}_m$. Our approach requires to study each numbering system on a case by case basis. Although, I presume a theory over any $\mathbb{Z}_m$ is possible, but with heavier machinery.

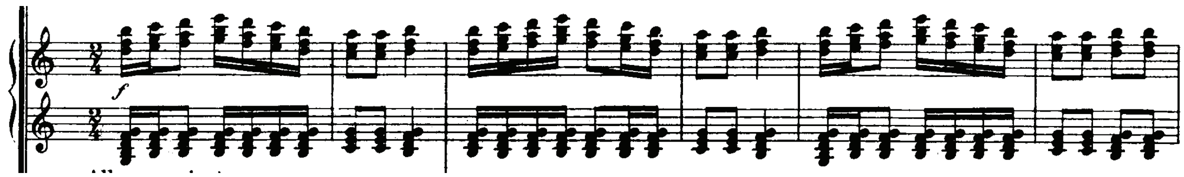

Figure 2: Piano part of Russian Dance, Part I (No. IV) of the Ballet Petrushka by Igor Stravinsky: Petrushka, the Moor and the Ballerina suddenly begin to dance, to great astonishment of the crowd.

In Section 3 we introduce the contextual inversions and translation operators. We give two lemmas relating these operators. Several examples are developed in cases of triads and tetrachords. Tessellations are introduced. In Section 4 surfaces and Triangle groups (groups related to Tilings of a given surface) are explored. Several formulas are given on how to compute the genus in terms of a Tiling of type $\{F, p, q\}$ (F regular convex p-gons where q polygons meet at each vertex). Section 5 is devoted to describe the structure of the transformation group. This group fits into an exact sequence with two terms: a quotient isomorphic to the dihedral $\mathbb{D}_{12}$ (or a subgroup of it) that maps into the triangle group of a surface with minimal Tiling, and an abelian subgroup. Our transformation group is also a semi-direct product of an inversion operator and an abelian group of translation operators. Section 6 contains the main theorem that gives the construction of a tiled surface by pentagons so that all transforms of a given pentachord fit together. This surface is a covering of a genus 13 surface, which is in turn a $2-1$ cover of the genus 4 Bring[4] surface.

2. Tone networks

Tone networks (Tonnetz) where invented by Leonhard Euler as a way of visualizing harmonically related tones by means of a graph where points represent pitches, or by its dual graph, where points represent tones. The musicologist Hugo Riemann extensively used these networks in his theory. More recently, group theorists following David Lewin and Richard Cohn related these ideas with LPR operations [see 3, for a historical perspective]. When we assume enharmonic equivalence and equal-tempered tuning the familiar Öttingen-Riemann Tonnetz associated with Major and Minor triads becomes double periodic and the graph can be wrapped around a torus $\mathbb{T}^2$ in a regular way. That is, the torus surface splits into equilateral (curved) triangle tiles whose sides are the edges of the graph and all of its vertices have 6 incident edges. Such an object is known as a regular tessellation of type $\{3, 6\}$ on the torus.

In a series of papers, Richard Cohn [2, 4] relates the geometry of certain Tonnetze with a kind of voice leadings which he calls "parsimonious". This is moving from one tone to a contiguous tone on the torus Tonnetz. Although this kind of voice leadings seem to have more to do with acoustic or harmonic proximity [33] rather than motion by a few close notes

[4] Surface already introduced in the 19^{th} century that can be regularly tiled into 12 pentagons.

at a time, Tonnetze allow us to decompose Tone space into different fibers: the tori. This brings a sort of better understanding into the geometry of this complex space. The geometry of Tonnetze is spelled out by a set of transformations called contextual transformations (the group generated by LPR operations in case of the Öttingen-Riemann Tonnetz). It was shown by Lewin in his analysis of Schoenberg's Opus 23, No. 3, and in general by Fiore and Satyendra [13] (with a slightly different definition of contextual group) that a quotient of the group of contextual transformations is precisely isomorphic to the dual of the 24 $T-$ and $I-$forms of any pitch segment $\langle x_1, \ldots, x_n \rangle$ [see also 10][5].

3. Pitch classes, Operators and Tilings

Our main objects will be pitch classes of length n. These classes will be vectors whose entries are real numbers representing pitches modulo $12\mathbb{Z}$. A pitch shift of 12 is an octave shift and so pitches live in a circle of length 12 (where $0 \equiv 12$) and pitch classes of length n are elements of an $n-$dimensional torus $\mathbb{T}^n = (\mathbb{R}/12\mathbb{Z})^n$. That is, we regard only the octave shift symmetry "O" from the "OPTIC" set of symmetries[6] considered in [1], [see also 34]. Most of the time we will be concerned with the set of integer pitches $\{0,1,2,3,4,5,6,7,8,9,10,11\} = \mathbb{Z}_{12}$ which are in one to one correspondence with the notes of the chromatic scale $\{C, C\# = D\flat, D, D\# = E\flat, E, F, F\# = G\flat, G, G\# = A\flat, A, A\# = B\flat, B\}$[7]. These twelve number classes allow addition and multiplication operations as usual numbers do, but always computing the positive remainder of the division by 12. For instance $(-7).5 = 5.5 = 25 = 1$; $10 + 7 = 17 = 5$.

3.1. Main Definitions

The usual translation and inversion operators are defined in $\mathbb{Z}_{12}$: $T_n(x) := x+n$, with $n \in \mathbb{Z}_{12}$, $x \in \mathbb{Z}_{12}$ and $I(x) := -x$ with $x \in \mathbb{Z}_{12}$. Also, $I_n(x) := T_n \circ I(x) = -x+n$, $n \in \mathbb{Z}_{12}$ and $x \in \mathbb{Z}_{12}$ is the inversion around n. Remember that when making computations with these operators all of them should be carried out in $\mathbb{Z}_{12}$. The group generated by $\{I, T_1\}$ (the T/I group) is isomorphic to the dihedral group $\mathbb{D}_{12}$. His elements are $\{T_0, T_1, \ldots, T_{11}, I_0 = I, I_1, \ldots, I_{11}\}$ and the following relations hold $I^2 = \mathrm{Id}$, $T_1^{12} = \mathrm{Id}$, $I \circ T_1 \circ I = T_1^{-1}$. Here $\mathrm{Id} = T_0$ is the identity operator and the power $T_1^m = T_m$ means compose T_1 m times.

Each of these operators are defined component-wise on vectors: $T_m\langle x_1, x_2, \ldots, x_n\rangle := \langle T_m(x_1), T_m(x_2), \ldots, T_m(x_n)\rangle$, and $I_m\langle x_1, x_2, \ldots, x_n\rangle := \langle I_m(x_1), I_m(x_2), \ldots, I_m(x_n)\rangle$. On segments of length n we define our contextual operators to be the operators p_{ij} with $i \neq j$, $1 \leqslant i, j \leqslant n$, defined as follows[8]:

$$p_{ij}\langle x_1, x_2, \ldots, x_n\rangle := I_{x_i+x_j}\langle x_1, x_2, \ldots, x_n\rangle. \qquad \text{(DEF. 1)}$$

[5]This paper has a nice account for the dictionary between notes and the integers mod 12.

[6]The vectors will be ordered classes. However, later on we will have to consider the "OP" equivalence, that is octave shift and permutations of the vector entries. This is the symmetrization of the torus $\mathbb{T}^n$: $S(\mathbb{T}^n) = \mathbb{T}^n/S_n$.

[7]For short we will denote $t = 10$, $e = 11$.

[8]These definition already appears in [13] but comes from earlier work by Lewin et al.

All of them have order 2, namely $p_{ij}^2 = \mathrm{Id}$, and they satisfy some relations that we will state later.

We also consider the permutation operators τ_{ij} and σ, $1 \leqslant i < j \leqslant n$. τ_{ij} transposes only the i and j coordinates of the vector while σ ciclicaly permutes each coordinate:

$$\tau_{ij}\langle x_1, \dots, x_i, \dots, x_j, \dots, x_n\rangle := \langle x_1, \dots, x_j, \dots, x_i, \dots, x_n\rangle.$$

$$\sigma\langle x_1, x_2, \dots, x_{n-1}, x_n\rangle := \langle x_2, x_3, \dots, x_n, x_1\rangle.$$

The full symmetric group S_n, which has $n!$ elements, is generated by $\tau_{i,i+1}$ and σ, for any $i \geqslant 1$. Other sets of generators are the following $\{\tau_{12}, \tau_{23}, \dots, \tau_{n-1,n}\}$; $\{\tau_{12}, \tau_{13}, \dots, \tau_{1n}\}$; and any set containing a 2-cycle and an n-cycle.

3.2. Some Lemmas

In order to prove statements about our contextual operators we find useful to represent them by matrices with entries in $\mathbb{Z}_{12}$. Operations with matrices should be carried out in $\mathbb{Z}_{12}$ and statements about matrices will transfer to statements about the elements of the group. So the contextual operators (as well as the elements of S_n) can be thought of as linear operators while the elements of the T/I group are in the affine group. Namely, T_m is translation by the vector $\langle m, m, \dots, m\rangle$ and I_m is inversion plus translation by $\langle m, m, \dots, m\rangle$.

We define a new linear operator. For any set of four indexes i, j, h, k, $1 \leqslant i, j, h, k \leqslant n$ let

$$T_{ij}^{hk}\langle x_1, x_2, \dots, x_n\rangle := T_{x_h + x_k - x_i - x_j}\langle x_1, x_2, \dots, x_n\rangle. \qquad \text{(DEF. 2)}$$

In case one of the upper indices coincides with a lower index we cancel them and write for short $T_j^k = T_{ij}^{ik}$.

Lemma 1. *The contextual operators p_{ij}, $1 \leqslant i < j \leqslant n$, satisfy the following relations.*

$$p_{ij} \circ p_{hk} = T_{ij}^{hk}, \qquad \textit{for any } i, j, k, l, \qquad (1)$$

$$\sigma^{i-1} \circ p_{i,i+1} = p_{12} \circ \sigma^{i-1}, \qquad 2 \leqslant i, \qquad (2)$$

$$\tau_{ij} \circ p_{hj} = p_{hi} \circ \tau_{ij}, \qquad h < i < j, \qquad (3)$$

$$\tau_{2i} \circ p_{1i} = p_{12} \circ \tau_{2i}, \qquad 3 \leqslant i. \qquad (4)$$

Proof. First notice that by (DEF. 1) we have $p_{ij} = p_{ji}$. So we consider those p_{ij} with $i < j$. We have that $p_{ij} \circ p_{hk}\langle x_1, x_2, \dots, x_n\rangle = p_{ij} I_{x_h + x_k}\langle x_1, x_2, \dots, x_n\rangle$. Now $\lambda = I_{x_h+x_k}(x_i) + I_{x_h+x_k}(x_j) = 2(x_h + x_k) - (x_i + x_j)$, and applying I_λ to an element $I_{x_h+x_k}(x_r) = x_h + x_k - x_r$ we get $x_h + x_k - x_i - x_j + x_r$. This shows (1).

Compute $p_{12}\sigma^{i-1}\langle x_1, x_2, \dots, x_n\rangle$. Evaluated on each coordinate x_r gives $\sigma^{i-1}(x_1) + \sigma^{i-1}(x_2) - \sigma^{i-1}(x_r) = x_i + x_{i+1} - \sigma^{i-1}(x_r)$, and this is precisely the left hand operator in (2) when evaluated on the coordinate x_r. This proves (2) in case $i < n$. If $i = n$ we make the convention $i + 1 = 1$ and the result also hods.

Since τ_{ij} exchanges the entries x_i and x_j, the right hand side of (3) on coordinate x_r takes the value $x_h + x_j - x_r$ if r is different from either i or j. On x_i the right hand side is equal to x_h and on x_j is equal to $x_h + x_j - x_i$. An easy checking gives the same values on the left hand side of (3). Finally (4) is a consequence of (3). ▫

Lemma 2. *The set of operators p_{ij}'s and T_{ij}^{hk}'s satisfy the following with respect to a translation T_m and inversion I.*

$$p_{ij} \circ T_m = T_m \circ p_{ij} \qquad \text{for any } i, j, \text{ indices and } m \in \mathbb{Z}_{12}, \tag{5}$$

$$p_{ij} \circ I = I \circ p_{ij} \qquad \text{for any indices } i, j, \tag{6}$$

$$T_{ij}^{hk} \circ T_m = T_m \circ T_{ij}^{hk} \qquad \text{for any } i, j, h, k, \text{ indices and } m \in \mathbb{Z}_{12}, \tag{7}$$

$$T_{ij}^{hk} \circ I = I \circ T_{ij}^{hk} \qquad \text{for any indices } i, j, h, k. \tag{8}$$

Thus, any subgroup generated by a set of T_{ij}^{hk}'s is abelian. Moreover, $T_{ij}^{hk} = T_i^h \circ T_j^k$, $(T_{ij}^{hk})^{-1} = T_{hk}^{ij}$, and the subgroup generated by any p_{uv} and T_{ij}^{hk} is isomorphic to the dihedral group $\mathbb{D}_{12}$.

Proof. To show (5) we check the formula for $m = 1$. $p_{ij}T_1\langle x_1, x_2, \ldots, x_n\rangle = p_{ij}\langle x_1 + 1, x_2 + 1, \ldots, x_n + 1\rangle = \langle x_i + x_j - x_1 + 1, x_i + x_j - x_2 + 1, \ldots, x_i + x_j - x_n + 1\rangle = T_1 p_{ij}\langle x_1, x_2, \ldots, x_n\rangle$. Formula (6) is also straight forward $p_{ij}I\langle x_1, x_2, \ldots, x_n\rangle = p_{ij}\langle -x_1, -x_2, \ldots, -x_n\rangle = \langle -x_i - x_j + x_1, -x_i - x_j + x_2, \ldots, -x_i - x_j + x_n\rangle = Ip_{ij}\langle x_1, x_2, \ldots, x_n\rangle$. By virtue of formula (1) we get (7) and (8) from (5) and (6). Any subgroup generated by a set of T_{ij}^{hk}'s is abelian because of (DEF. 2) and (7). Also, since p_{uv} has order 2 and T_{ij}^{hk} has order 12 in general[9], it will suffice to prove that $p_{uv}T_{ij}^{hk}p_{uv} = (T_{ij}^{hk})^{-1}$. Now $p_{uv}T_{ij}^{hk}p_{uv} = p_{uv}p_{ij}p_{hk}p_{uv} = T_{uv}^{ij}T_{hk}^{uv} = T_{hk}^{ij} = (T_{ij}^{hk})^{-1}$ ▫

3.3. Coxeter groups

As remarked in [3.1] after (DEF. 1), the operators p_{ij}'s are symmetries. However, because of equation (1) they satisfy $(p_{ij}p_{hk})^{12} = \mathrm{Id}$ since $(T_{ij}^{hk})^{12} = T_{x_h+x_k-x_i-x_j}^{12} = T_{12(x_h+x_k-x_i-x_j)} = T_0$. So at least, the group generated by these contextual transformations is a quotient of the Coxeter group abstractly[10] generated by p_{ij}'s and with relations $p_{ij}^2 = (p_{ij}p_{kl})^{12} = \mathrm{Id}$.

Definition 3. A Coxeter group is a group defined by a set of generators $\{R_1, R_2, \ldots, R_s\}$ and relations $\{R_i^2 = (R_iR_j)^{m_{ij}} = \mathrm{Id},\ 1 \leqslant i \leqslant s,\ m_{ij} \geqslant 2$ positive integers for $i < j\}$.

[9] We make the distinction between T_{ij}^{hk} applied to a generic pitch class and to a particular (T/I) set generated by a fixed pitch class $\langle x_1, x_2, \ldots, x_n\rangle$. In the first case, since any values for the entries of the vector can occur (in particular, a vector for which $x_h + x_k - x_i - x_j = 1$), we get that $(T_{ij}^{hk})^{12} = \mathrm{Id}$. As for the second case, the order of T_{ij}^{hk} can be any divisor of 12, namely $2, 3, 4, 6$, or 12.

[10] This means that we are regarding the p_{ij}'s as the generators of a Coxeter group merely with the relations satisfied by Coxeter groups and no extra relation.

Coxeter groups are usually infinite but have nice geometric realizations as reflections about hyperplanes in n-dimensional Euclidean spaces [9, chap. 9]. Also, some of these groups are presented with two generators (one of them a reflection), and in these cases the target space is a surface [9, chap. 8].

The symmetric group S_n can be presented with different sets of generatos and relations [9, chap. 6] and in particular as a quotient of a Coxeter group in the generators $\{\tau_{12}, \tau_{23}, \ldots, \tau_{n-1,n}\}$.

3.4. Regular Tilings

LPR operations on a triadic segment are just the operations induced by the group G_3 generated by $\{p_{12}, p_{13}, p_{23}\}$ [10]. The Öttingen-Riemann torus is constructed by associating with the segment $\langle x_1, x_2, x_3\rangle$ an oriented equilateral triangle with consecutive vertices $\{1, 2, 3\}$ in correspondence with the notes of the segment. Then one reflects this "Tile" along the side ij (meaning reflecting by p_{ij}) and glues the two tiles along the common side. One continues in this way by applying all the elements of G_3 and hopes for the tiles to match together into a surface S.

A Tiling on a surface S by a convex polygon T (for instance a regular polygon) is the action of a group on S such that you can cover the surface S with translations of T by elements of the group in such a way that two tiles are either disjoint, they meet at a common vertex or they meet along a common side (perfectly interlocking with each other). "Tilings" or "Tessellations" with a pattern Tile T on a surface S are not always possible; they depend on the shape of the tile and the topology of the surface. Tilings on the plane $\mathbb{R}^2$ or the sphere $\mathbb{S}^2$ (polyhedra) described in a mathematical fashion were basically initiated by Kepler in his book [23]. Tilings in the plane with different shapes, patterns and group of symmetries are thoroughly studied in [18] and polyhedra in [8]. A more recent account on Tessellations and Symmetries is in the beautiful book by Conway et al. [7].

The problem of finding a Tiling on a surface S by a number F of equal regular (possibly curved) polygons with p sides such that at each vertex the same number q of (incident) polygons meet is called the problem $\{F, p, q\}$ on S. When there is no regarding of the number F of tiles or faces we call it simply the problem $\{p, q\}$ on S. Solving this problem depends on the group of automorphisms that a surface S admits and an attempt to classifying them is done in a series of papers (at least up to genus 101) by Conder et al. [5, 6]. For instance, as well known, the only possible Regular Tessellations on the sphere $\mathbb{S}^2$ are the usual regular simple polyhedra:

1. the regular **tetrahedron** $\{3, 3\}$, with group of symmetries of order 12: A_4 (the even permutations in S_4),

2. the **cube** $\{4, 3\}$ and the **octahedron** $\{3, 4\}$, with group of symmetries of order 24: S_4,

3. the **dodecahedron** $\{5, 3\}$ and the **icosahedron** $\{3, 5\}$, with group of symmetries of order 60: A_5 (the alternating group of S_5).

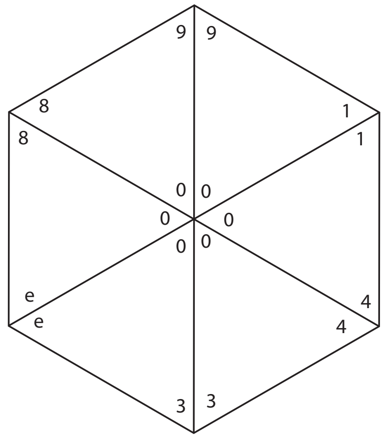

Figure 3: incidence for the segment $\langle 0,1,9\rangle$ at vertex 0

In general, if a finite Tessellation on a surface S is found, the number of faces divides the order of the group of symmetries. On the torus the only possible Regular Tessellations are of type $\{4,4\}$, $\{6,3\}$ and $\{3,6\}$. In this case any number of faces are allowed as long as they close up to a torus.

An exploration of the possibility of drawing Regular Tilings in Computer Graphics is given in the paper by van Wijk [36].

3.5. Trichords

Now we deduce the abstract structure of the group G_3 defined in [3.4] in order to explain the construction of the torus. By formula (2) we have $p_{23} = \sigma^{-1}p_{12}\sigma$ and $p_{13} = \sigma^{-2}p_{12}\sigma^2$. Therefore it is natural to map G_3 into the group generated by $\{p_{12}, \sigma\}$. In order to say something about this new group we try to find relations between p_{12} and σ. Write them as matrix operators: $p_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix}$, $\sigma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. Then $z = \sigma p_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix}$, and one checks by computing successive powers of z that $z^6 = \mathrm{Id}$. So the group generated by $\{p_{12}, \sigma\}$ is contained in the "rotation" group: $\langle A, B \mid A^2 = \mathrm{Id}, B^3 = \mathrm{Id}, (B^{-1}A)^6 = \mathrm{Id}\rangle$. Such groups appear precisely as generator groups of Regular Tilings. The introduction of σ together with the contextual transformations (inversions) p_{ij}'s makes sense geometrically since σ is a rotation of the triangle vertices.

We make the following algebraic convention: glue the triangle associated to the ordered vector $\langle x_1, x_2, x_3\rangle$ with the triangle associated with $\langle x_1, x_4, x_2\rangle$ along the side $\langle x_1, x_2\rangle$, if and only if $x_4 = I_{x_1+x_2}(x_3)$.

Example 3. We show in the following example the successive tiles about the first vertex by applying z: $\langle 0,1,9\rangle \xrightarrow{z} \langle 0,4,1\rangle \xrightarrow{z} \langle 0,3,4\rangle \xrightarrow{z} \langle 0,e,3\rangle \xrightarrow{z} \langle 0,8,e\rangle \xrightarrow{z} \langle 0,9,8\rangle \xrightarrow{z} \langle 0,1,9\rangle$. Pictorically we see this in Figure 3.

By reflecting along the different sides of the triangles we can complete the 24 tiles of the

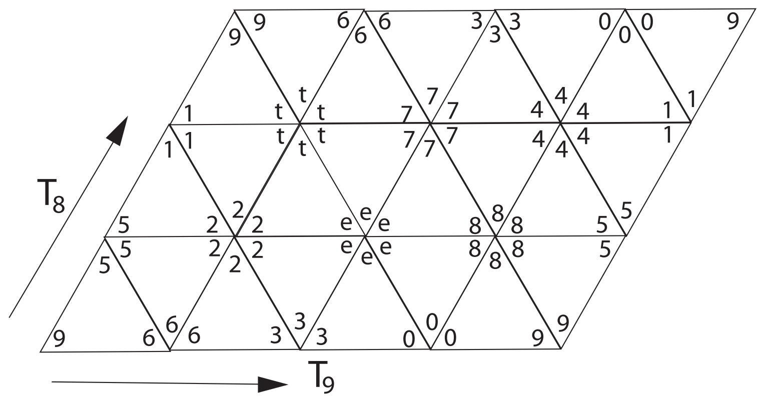

Figure 4: minimal torus for $\langle 0, 1, 9\rangle$

T/I class for $\langle 0, 1, 9\rangle$. In this case $p_{13}p_{23} = T_1^2$ acts as T_1 (translation by 1). So, by acting with G_3 on the given pitch segment, we get the full set of translated and inverted pitch classes associated with $\langle 0, 1, 9\rangle$.

A minimal Tiling containing the 24 elements of the T/I translates of $\langle 0, 1, 9\rangle$ is pictured in Figure 4.

In this example $p_{12}p_{23} = T_1^3 \equiv T_9$, and $p_{12}p_{13} = T_2^3 \equiv T_8$. By lemma 2, T_8 and T_9 commute. They are the usual translations on the torus of Figure 4 moving a unit in each direction. Now, the subgroup generated by T_8 and T_9 is $\mathbb{Z}_3 \oplus \mathbb{Z}_4 \simeq \mathbb{Z}_{12}$, and since G_3 (restricted to the pitch segment) is generated by $\{p_{12}, T_8, T_9\}$, lemma 2 tells us that G_3 is isomorphic to the dihedral group $\mathbb{D}_{12}$. Therefore Figure 4 is a geometrical representation of $G_3 \bullet \langle 0, 1, 9\rangle \cong (T/I) \bullet \langle 0, 1, 9\rangle$[11].

3.6. Tetrachords

A similar analysis for the group G_4 is more involved. This group is generated by $\{p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}\}$. An alternative set of generators is $\{p_{12}, T_2^3, T_2^4, T_1^3, T_1^4, T_{12}^{34}\}$. So, basically we have $\mathbb{Z}_2$ generated by p_{12} and the abelian subgroup generated by $\{T_2^3, T_2^4, T_1^3, T_1^4\}$.

T_{12}^{34} is ruled out since it is the sum of T_1^3 and T_2^4: for these abelian operators which act as translations the composition means sum of translations.

[11] All these considerations should be possible (with little modification in lemmas 1, 2 and definitions 1, 2) for pitch classes generated by translation and τI inversion, where $\tau \in S_n$ is a trasposition: the $(T/\tau I)$ group. For instance if $\tau = R$ is the retrograde then we will have the RI-chains. Geometrical representations with RI-chains involved are in the paper by Joseph Straus [31].

Now, these generators share relations and we would like to find a linearly independent set of generators.

We immediately have the relation $T_2^3 - T_2^4 = T_1^3 - T_1^4 = T_4^3$. Therefore, we are left with three generators $\{T_1^3, T_2^3, T_1^4\}$ which in general are linearly independent. Indeed, let λ_1, λ_2, λ_3 be numbers in $\mathbb{Z}_{12}$ such that $\lambda_1 T_1^3 + \lambda_2 T_2^3 + \lambda_3 T_1^4 = 0$, that is $\lambda_1(x_3 - x_1) + \lambda_2(x_3 - x_2) + \lambda_3(x_4 - x_1) = 0$ for any triple of x_i's in $\mathbb{Z}_{12}$. Clearly the only possible λ_i's are equal to 0. If however we work with the (T/I) set generated by a fixed segment $\langle x_1, x_2, x_3, x_4 \rangle$, then relations will appear and the group will be smaller (a subgroup of $\mathbb{Z}_2 \ltimes (\mathbb{Z}_{12} \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_{12})$).

We cannot map G_4 into a rotation group as in the case of G_3. Indeed, we have by formulas (2) and (4) that $p_{23} = \sigma^{-1} p_{12} \sigma$, $p_{34} = \sigma^{-2} p_{12} \sigma^2$, $p_{14} = \sigma^{-3} p_{12} \sigma^3$, $p_{13} = \tau_{23} p_{12} \tau_{23}$, and $p_{24} = \sigma^{-1} p_{13} \sigma$. Namely, besides the cycle σ and symmetry p_{12} one has to introduce the transposition τ_{23}.

Instead, we consider the subgroup $\widetilde{G}_4$ generated by $\{p_{12}, p_{23}, p_{34}, p_{14}\}$ which maps into the group generated by $\{p_{12}, \sigma\}$. Representing these as matrix operators $p_{12} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix}$, $\sigma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, one gets $z = \sigma p_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, with $z^4 = \mathrm{Id}$.

This corresponds to the rotation group for a Tiling on the plane by squares. Here σ is rotation around the square center and z around each vertex. In the general case $\widetilde{G}_4$ is generated by $\{p_{12}, T_1^3, T_2^4\}$, with T_1^3, T_2^4, linearly independent (namely $\widetilde{G}_4 \cong \mathbb{Z}_2 \ltimes (\mathbb{Z}_{12} \oplus \mathbb{Z}_{12})$).

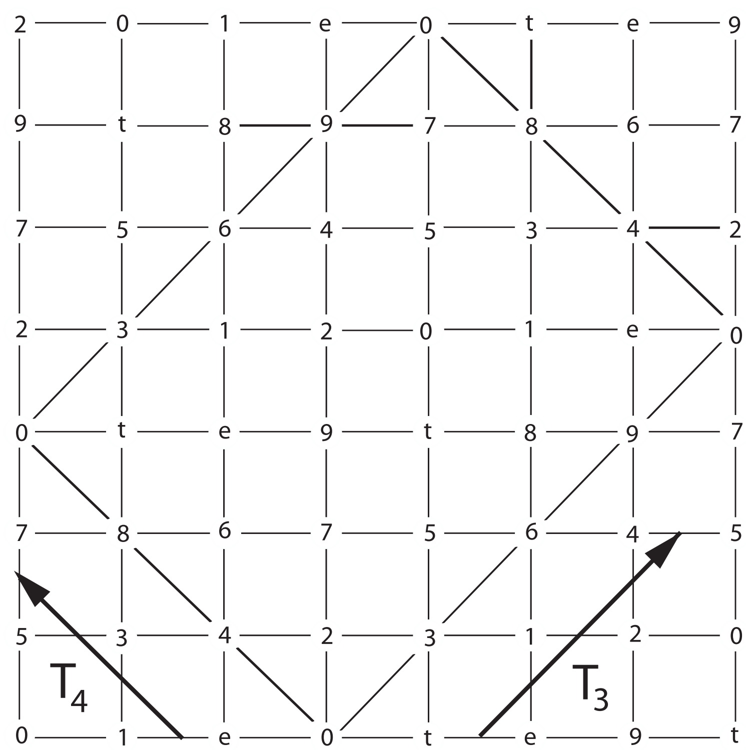

Figure 5: minimal torus for pitch class $\langle 0, 1, 3, 5 \rangle$

Example 2. Restricting to the pitch class $\langle 0, 1, 3, 5 \rangle$ we have the translations generating the abelian subgroup of $\widetilde{G}_4$: $T_1^3 \equiv T_3$ and $T_2^4 \equiv T_4$. This is $\mathbb{Z}_4 \oplus \mathbb{Z}_3 \simeq \mathbb{Z}_{12}$. Figure 5 shows the minimal torus which is generated by the action of $\widetilde{G}_4$. Also in here, the whole set class $sc(0135)$ is obtained. The action along the sides of the square tiles by the generators of $\widetilde{G}_4$ is transversal to the action by $\{T_3, T_4\}$. Notice that these translations have real meaning in terms of numbers and that they define where the Torus sits.

However, there may be degeneracies to this model. For instance, consider the following.

Example 3. The segment $\langle 0,3,6,9\rangle$ whose (T/I) orbit possess several symmetries. Here, the action of $\widetilde{G}_4$ just repeats four squares and its abelian subgroup is generated by (T_6,T_6). We get a torus out of eight square tiles but not the whole orbit $(T/I)\bullet\langle 0,3,6,9\rangle$ (see Figure 6). Even the abelian subgroup of the full G_4 generated by (T_6,T_3,T_9) would not give the whole T/I orbit. In this case we get three connected components: $\widetilde{G}_4\bullet\langle 0,3,6,9\rangle$, $\widetilde{G}_4\bullet\langle 1,4,7,t\rangle$, and $\widetilde{G}_4\bullet\langle 2,5,8,e\rangle$, whose union is the whole set class $sc(0369)$.

4. Triangle groups and Surfaces

It was said in [3.3] that the group generated by the contextual transformations p_{ij}'s is a quotient of a Coxeter group. We tie this with the Triangle groups that we consider as a special kind of Coxeter groups which have geometrical realizations as Tilings on surfaces.

Definition 4. given three positive integers $h,k,l\geqslant 2$, A triangle group $\Delta_g(h,k,l)$ is a group generated by three reflections R_1,R_2,R_3 satisfying the relations $R_1^2=R_2^2=R_3^2=(R_1R_3)^h=(R_3R_2)^k=(R_2R_1)^l=\mathrm{Id}$, and possibly other relations associated with a surface S_g.

These are groups of reflections of regular triangulations on surfaces, that is triangulations associated with Regular Tessellations. Any surface can be subdivided by triangles but not every surface contains a Regular Tessellation by polygons with a certain number of faces (problem $\{F,p,q\}$ cited in [3.4]). For instance, if a surface S_g contains a Regular Tiling by squares[12], then each square is subdivided into triangles with vertices at the center and middle edges as shown in Figure 7. There is a fundamental triangle T (yellow) and its mirror reflections (red, blue and green) about the edges. The composition of two of these reflections gives a rotation around one of the black hinges (the vertices of the triangle). This system of rotations is called a rotation group (as mentioned in [3.5]).

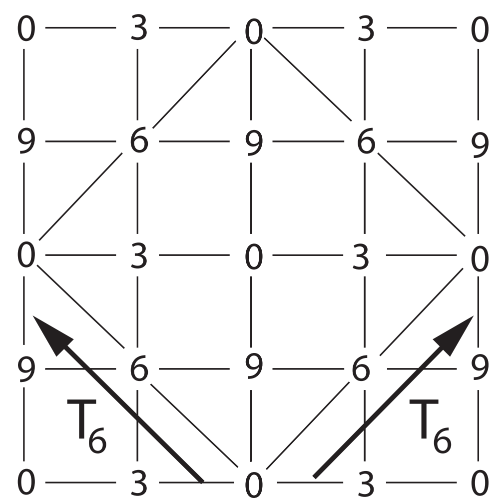

Figure 6: minimal torus for $\langle 0,3,6,9\rangle$

The surface S_g is covered by applying all possible sequences of reflections R_i's to the triangle T or by applying all possible sequences of rotations $\mathcal{R}=R_1R_3$, $\mathcal{S}=R_3R_2$ and $\mathcal{T}=R_2R_1$ (around the triangles vertices) to the fundamental polygon[13].

The group generated by $\{\mathcal{R},\mathcal{S},\mathcal{T}\}$ is an important subgroup of $\Delta_g(h,k,l)$ which has index 2. It is called a von Dyck group and can be characterized as follows $\Delta_g^+(h,k,l)=\{\mathcal{R},\mathcal{S},\mathcal{T},$ such that $\mathcal{R}^h=\mathcal{S}^k=\mathcal{T}^l=\mathcal{RST}=\mathrm{Id}$, plus other relations depending on the surface $S_g\}$.

[12] This only happens if the surface is a Torus or the plane.

[13] This assumes that the Tiling is already given and therefore all the vertices of the triangulation.

There is more structure to this: the fundamental triangle T associated with the triangle group $\Delta_g(h,k,l)$ has angles $\pi/h, \pi/k, \pi/l$.

Spherical triangles satisfy $1/h + 1/k + 1/l > 1$ and only a finite number of configurations[14] are possible. Triangles in the plane or Torus satisfy $1/h + 1/k + 1/l = 1$ and also a finite number of cases are possible. However, the most interesting cases correspond to hyperbolic triangles: those for which the inequality $1/h + 1/k + 1/l < 1$ holds.

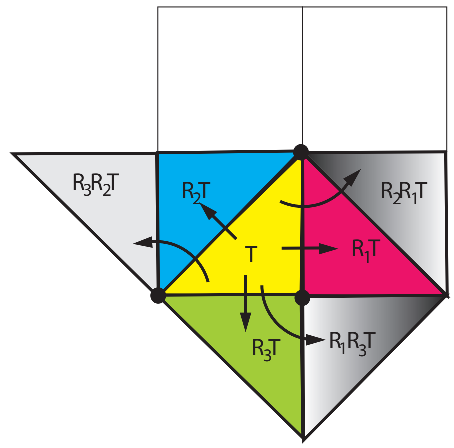

Figure 7: triangle motions for a square

The triangle groups related to the problem $\{F, p, q\}$ are those of the form $\Delta_g(2,q,p)$[15]. Their corresponding von Dyck groups (we write them as $\Delta_g^+(2,q,p)$) can be presented as generated by $\mathcal{R}$ and $\mathcal{S}$, one of them of order 2. Indeed, writing $\mathcal{T} = \mathcal{S}^{-1}\mathcal{R}$, we can present the von Dyck group as follows: $\Delta_g^+(2,q,p) = \{\mathcal{R}, \mathcal{S},$ such that $\mathcal{R}^2 = \mathcal{S}^p = (\mathcal{S}^{-1}\mathcal{R})^q = \mathrm{Id}$, plus relations depending on the surface $S_g\}$.

We will be mainly concerned with orientable[16] closed surfaces (that is without boundary). These surfaces are classified by their genus g (an integer $g \geqslant 0$). This is the number of holes they have or the number of handles attached to a sphere. For instance the genus of a sphere is 0 and that of a torus is 1. All the surfaces of genus $g > 1$ fall into the Hyperbolic realm. Figure 8 pictures a surface of genus 4. If a surface S_g of genus g has a Tiling by convex[17] polygons and V =number of vertices, E =number of edges, F =number of faces for this Tiling, then the Euler-Poincaré Characteristic: $\chi(S_g) := V - E + F$ is an integer number which does not depend on the Tiling considered on S_g. We have the following.

$$\chi(S_g) = V - E + F = 2 - 2g \tag{9}$$

If a surface S_g of genus $g > 1$ has a Tiling by regular $p-$gons with incidence q at each polygon vertex, then there is an induced barycentric Tessellation by hyperbolic right triangles with angles $\pi/p, \pi/q$. These triangles are in correspondence with the elements of the group

[14]These configurations are related to the regular polyhedra [3.4].

[15]This is because when doing barycentric subdividion of a regular polygon we get right triangles and so $h = 2$.

[16]Not those like the Möbius band or the projective plane.

[17]We exclude polygons like Star polygons because they may be misleading when computing the Euler-Poincaré characteristic. The safest way to compute the Euler-Poincaré characteristic is to use a triangulation on S_g.

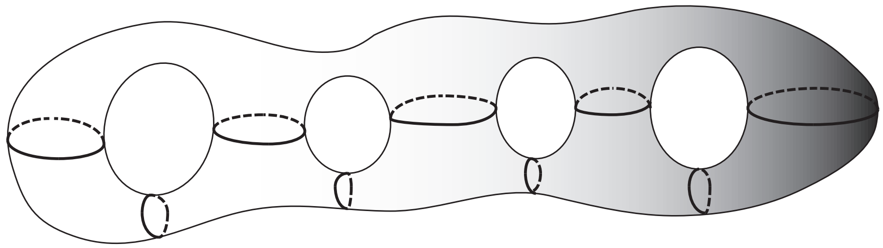

Figure 8: a genus $g = 4$ surface

$\Delta_g(2, p, q)$ and its order can be computed by the formula [see 21]

$$|\Delta_g(2,p,q)| = \frac{(\text{Hyperbolic Area of } S_g)}{(\text{Hyperbolic Area of the } (2,p,q)-\text{triangle})} = \tag{10}$$

$$= \frac{-2\pi\chi(S_g)}{(\frac{\pi}{2} - \frac{\pi}{p} - \frac{\pi}{q})} = \frac{8pq(g-1)}{(p-2)(q-2)-4} \tag{11}$$

Remark 1. Formula (11) holds also for the sphere (just put $g = 0$).

Remark 2. If on a surface S_g of genus g we can solve the problem $\{F, p, q\}$, then by taking apart the polygons and counting edges we get $qV = 2E = pF$. Thus, $\chi(S_g) = 2E(\frac{1}{p} + \frac{1}{q} - \frac{1}{2})$. So, another way of writing Formula (11) is the following $|\Delta_g(2, p, q)| = 4E = 2pF = 2qV$.

Remark 3. As long as we know the number of faces (edges or vertices) of a regular Tessellation of type $\{p, q\}$ on a surface S_g we can compute its genus:

$$g = 1 - \frac{\chi(S_g)}{2} = 1 + \frac{1}{2}(\frac{1}{2} - \frac{1}{p} - \frac{1}{q})pF. \tag{12}$$

Remark 4. There is still another formula which allows us to compute the Euler-Poincaré characteristic in case we know the number of faces F and vertices V of a regular $\{p, q\}$ Tiling on S_g:

$$(p-2)(q-2) = 4\left(1 - \frac{\chi(S_g)}{V}\right)\left(1 - \frac{\chi(S_g)}{F}\right) \tag{13}$$

Remark 5. The full triangle group $\Delta(2, p, q)$[18] takes a fundamental right triangle with angles $\pi/p, \pi/q$ in the Poincaré Disk and moves it to cover the whole Disk giving a pattern that also contains p-gons where q polygons meet at each vertex[19]. One obtains such patterns as pictured in Escher Limit Circle drawings[20]. This group is infinite and for each surface S_g containing a regular $\{p, q\}$ Tiling, the group $\Delta_g(2, p, q)$ is the quotient of $\Delta(2, p, q)$ by a subgroup containing at least the dihedral group preserving the p-gon.

[18] This is Definition 4 without the relations for the surface S_g.
[19] As shown in the picture `(2,3,7)-triangle Tiling`.
[20] See for instance `http://www.josleys.com/show_gallery.php?galid=325`.

5. Group structure and local incidence

Consider now a pitch segment $\langle x_1, x_2, \ldots, x_n\rangle$, $x_i \in \mathbb{Z}_{12}$ for $0 \leqslant i \leqslant n$ and assume that we allow the contextual transformations generated by $\{p_{12}, p_{23}, \ldots, p_{i,i+1}, \ldots, p_{n1}\}$. The group generated by these inversions is denoted $\widetilde{G}_n$. This group is contained in a bigger group G_n generated by all the inversions p_{ij}'s. As in [3.6] $\widetilde{G}_n$ is mapped into the group generated by $\{p_{12}, \sigma\}$ and later on we will show that this is a von Dyck group.

We describe the structure and basis of G_n and $\widetilde{G}_n$.

Proposition 1. *The group of contextual inversions G_n is isomorphic to the semidirect product of $\mathbb{Z}_2$ and the abelian group $\mathbb{Z}_{12}^{n-1}$. Bases for this last abelian group are given by*

$$\{T_1^2, T_1^3, \ldots, T_1^n\} \text{ or } \{T_1^2, T_2^3, \ldots, T_2^n\}.$$

The group of contextual inversions $\widetilde{G}_n$ is isomorphic to $\mathbb{Z}_2 \ltimes \mathbb{Z}_{12}^{n-2}$ if n is even and to $\mathbb{Z}_2 \ltimes \mathbb{Z}_{12}^{n-1}$ in case n is odd. Bases for the abelian parts in these two cases are:

$$\{T_1^3, T_2^4, T_1^5, \ldots, T_2^n\} \quad \text{if } n \text{ is even},$$

$$\{T_1^2, T_1^3, T_2^4, \ldots, T_2^{n-1}, T_1^n\} \quad \text{if } n \text{ is odd}.$$

Proof. G_n is generated by $\{p_{12}, p_{13}, \ldots, p_{1n}, p_{23}, p_{24}, \ldots, p_{2n}, p_{34}, \ldots, p_{3n}, \ldots, p_{n-1,n}\}$ and also by p_{12} and the abelian generators $\{T_2^3, \ldots, T_2^n, T_1^3, \ldots, T_1^n, T_{12}^{34}, \ldots, T_{12}^{3n}, \ldots, T_{12}^{n-1,n}\}$. Since we have in additive notation $T_{ij}^{hk} = T_i^h + T_j^k$ (see Lemma 2), the set $\{T_2^3, \ldots, T_1^n, T_1^3, \ldots, T_1^n\}$ already generates the abelian piece. Now, since $T_1^3 - T_2^3 = \cdots = T_1^n - T_2^n = T_1^2$, we are left with the generators $\{T_1^2, T_1^3, \ldots, T_1^n\}$ which are readily seen to be generically independent. The remaining conclusions about G_n follow from Lemma 2.

$\widetilde{G}_n$ is generated by $\{p_{12}, p_{23}, p_{34}, \ldots, p_{n-1,n}, p_{1n}\}$. So, p_{12} and the abelian piece generated by $\{T_1^3, T_{12}^{34}, \ldots, T_{12}^{n-1,n}, T_2^n\}$ define the whole group. Suppose that $n = 2k$ and write these generators as $\{T_1^3, T_1^3 + T_2^4, T_2^4 + T_1^5, \ldots, T_1^{2k-1} + T_2^{2k}, T_2^{2k}\}$. Then, an equivalent set of generators is $\{T_1^3, T_2^4, T_1^5, T_2^6, \ldots, T_1^{2k-1}, T_2^{2k}\}$, and these are $n-2$ independent generators. Therefore, $\widetilde{G}_n \cong \mathbb{Z}_2 \ltimes \mathbb{Z}_{12}^{n-2}$.

If $n = 2k+1$ the generators of the abelian piece are $\{T_1^3, T_1^3 + T_2^4, T_2^4 + T_1^5, \ldots, T_1^{2k-1} + T_2^{2k}, T_2^{2k} + T_1^{2k+1}, T_2^{2k+1}\}$. We can replace this set with $\{T_1^3, T_2^4, \ldots, T_2^{2k}, T_1^{2k+1}, T_2^{2k+1}\}$ or the equivalent set $\{T_1^2, T_1^3, T_2^4, \ldots, T_2^{2k}, T_1^{2k+1}\}$. These generators are equivalent to any of the given bases for the abelian part of G_{2k+1}. So, in this case $\widetilde{G}_{2k+1} = G_{2k+1}$. □

What is the image of a single pitch segment class $\langle x_1, x_2, \ldots, x_n\rangle$ under $\widetilde{G}_n$? Do we recover the whole set class $(T/I)\bullet\langle x_1, x_2, \ldots, x_n\rangle$? Obviously $\widetilde{G}_n\bullet\langle x_1, x_2, \ldots, x_n\rangle$ is contained in $(T/I)\bullet\langle x_1, x_2, \ldots, x_n\rangle$ because p_{ij}'s are made up of translations T_m's and inversion I.

If n is odd a condition for $\widetilde{G}_n\bullet\langle x_1, x_2, \ldots, x_n\rangle$ to equal $(T/I)\bullet\langle x_1, x_2, \ldots, x_n\rangle$ is the following:

Condition 1. There exist integer numbers $\lambda_2, \lambda_3, \ldots, \lambda_n$, such that

$$\lambda_2(x_2 - x_1) + \lambda_3(x_3 - x_1) + \cdots + \lambda_n(x_n - x_1) \equiv 1 \bmod 12.$$

This makes sure that by applying $T_1 \in \widetilde{G}_n$. We use the equality deduced in Proposition 1 $\widetilde{G}_n = G_n$ and the first basis of G_n. The operator $Op = (T_1^2)^{\lambda_2} \circ (T_1^3)^{\lambda_3} \circ \cdots \circ (T_1^n)^{\lambda_n}$ is translation by 1 when applied to the segment $\langle x_1, x_2, \ldots, x_n\rangle$. Now, the operator $p_{12} \circ (Op)^{(-x_1-x_2)}$ applies like I on the segment.

In case n is even we use

Condition 2. There are integer numbers $\lambda_3, \lambda_4, \ldots, \lambda_n$, such that

$$\lambda_3(x_3 - x_1) + \lambda_4(x_4 - x_2) + \lambda_5(x_5 - x_1) + \cdots + \lambda_{n-1}(x_{n-1} - x_1) + \lambda_n(x_n - x_2) \equiv 1 \bmod 12.$$

The proof that this suffices is similar to that of Condition 1. We observe that since there are many bases for $\widetilde{G}_n$, other conditions are possible to obtain $\widetilde{G}_n \bullet \langle x_1, x_2, \ldots, x_n\rangle = (T/I)\bullet \langle x_1, x_2, \ldots, x_n\rangle$.

As a counterexample, the pitch segment $\langle 0, 2, 4, 6, 8\rangle$ does not satisfy Condition 1. Either we treat it as a pathological case, or we deal with it as a particular case, aside from the considerations we are explaining in this article.

5.1. The gluing procedure

We map $\widetilde{G}_n$ into the group generated by $\{p_{12}, \sigma\}$ as follows (using Formula (2)):

$p_{23} = \sigma^{-1}p_{12}\sigma$, $p_{34} = \sigma^{-2}p_{12}\sigma^2, \ldots, p_{1n} = \sigma^{-(n-1)}p_{12}\sigma^{n-1}$. As we did in [3.5] for triangles, we consider the regular polygon of n sides P_n associated to the segment $\langle x_1, x_2, \ldots, x_n\rangle$, and glue it to the polygon associated with $\sigma p_{12}\langle x_1, x_2, \ldots, x_n\rangle = \langle x_1, I_{x_1+x_2}(x_3), \ldots, I_{x_1+x_2}(x_n), x_2\rangle$ along the side $\langle x_1, x_2\rangle$ (Figure 9). Namely, the element $z = \sigma p_{12}$ is a rotation around the vertex x_1 that takes one polygon into the next and σ^{-1} is rotation of the polygon vertices.

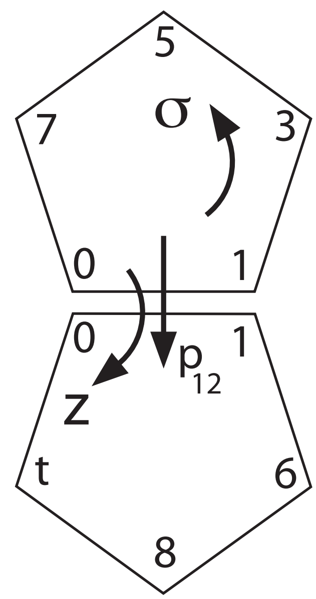

Figure 9: matching a pentagon

The order of z is the incidence at each vertex. For instance, by successively rotating with z the polygon of Figure 9 we get back the pitch $\langle 0, 1, 3, 5, 7\rangle$ after 10 times: $\langle 0, 1, 3, 5, 7\rangle \xrightarrow{z} \langle 0, t, 8, 6, 1\rangle \xrightarrow{z} \langle 0, 2, 4, 9, t\rangle \xrightarrow{z} \langle 0, t, 5, 4, 2\rangle \xrightarrow{z} \langle 0, 5, 6, 8, t\rangle \xrightarrow{z} \langle 0, e, 9, 7, 5\rangle \xrightarrow{z} \langle 0, 2, 4, 6, e\rangle \xrightarrow{z} \langle 0, t, 8, 3, 2\rangle \xrightarrow{z} \langle 0, 2, 7, 8, t\rangle \xrightarrow{z} \langle 0, 7, 6, 4, 2\rangle \xrightarrow{z} \langle 0, 1, 3, 5, 7\rangle$.

We can picture this in the Poincaré Disk with 10 meeting pentagons (see Figure 10).

Transporting these prescriptions along all sides and vertices we get a surface which is regularly tessellated by equal polygons of n sides. If the Tiling is finite the surface is closed since each edge belongs to two and only two faces (there are no free edges belonging to only one polygon) and each vertex has the same number of incident edges (this is the order of z). Thus the group generated by $\{p_{12}, \sigma^{-1}\}$ is a von Dyck group of a regular Tessellation on a surface of genus g since it can be described by generators and relations as $\{p_{12}, \sigma^{-1},$ such that $p_{12}^2 = \sigma^n = (\sigma p_{12})^q = \text{Id}$ plus relations coming from $S_g\}$.

Proposition 2. *Assume the pitch segment $\langle x_1, x_2, \dots, x_n\rangle$ induces a Regular Tessellation by regular polygons of n sides on a closed surface of genus g. Then, the incidence of the polygons at each vertex is n for even n and $2n$ if n is odd. Therefore, the group generated by p_{12} and σ is isomorphic to $\Delta_g^+(2,n,n)$ if n is even and $\Delta_g^+(2,n,2n)$ if n is odd.*

Proof. It is enough to show that $q = n$ if n is even and $q = 2n$ if n is odd. We have that $p_{12}p_{23}\dots p_{1n}{=}p_{12}\sigma^{-1}p_{12}\sigma^{-1}\dots p_{12}\sigma^{-1} = (\sigma p_{12})^{-n} = z^{-n}$. On the other hand, $p_{12}p_{23}p_{34}p_{45}\dots p_{n-1,n}p_{1n}{=}T_1^3p_{12}T_{12}^{34}p_{12}T_{12}^{45}\dots p_{12}T_{12}^{n-1,n}p_{12}T_2^n$. If $n = 2k$ we write $z^{-n} = T_1^3p_{12}T_{12}^{34}p_{12}T_{12}^{45}\dots p_{12}T_{12}^{2k-1,2k}p_{12}T_2^{2k} = T_1^3T_{34}^{12}T_{12}^{45}\dots T_{2k-1,2k}^{12}T_2^{2k} = T_1^3T_3^1T_4^2T_2^4T_1^5\dots T_{2k-1}^1T_{2k}^2T_2^{2k} = \mathrm{Id}$. This shows $z^n = \mathrm{Id}$ for n even. If $n = 2k+1$, $z^{-n} = T_1^3T_{34}^{12}T_{12}^{45}\dots T_{2k-1,2k}^{12}T_{12}^{2k,2k+1}p_{1,2k+1} = T_1^{2k+1}p_{1,2k+1}$. However, by Lemma 2 $z^{-2n} = T_1^{2k+1}p_{1,2k+1}T_1^{2k+1}p_{1,2k+1} = T_1^{2k+1}T_{2k+1}^1 = \mathrm{Id}$. This shows the proposition. □

5.2. Mapping $\widetilde{G}_n$

Let us call by $\mathcal{N}$ the image of $\widetilde{G}_n$ into the von Dyck group. Then it is readily seen that this group is normal. Indeed, $p_{12}\mathcal{N}p_{12} \subseteq \mathcal{N}$ and $\sigma\mathcal{N}\sigma^{-1} \subseteq \mathcal{N}$. We just check this on the generating elements of $\mathcal{N}$. Therefore we can make sense of the quotient group $\Delta_g^+(2,n,n)/\mathcal{N}$ (n even) and the quotient group $\Delta_g^+(2,n,2n)/\mathcal{N}$ (n odd). These quotients are isomorphic to $\mathbb{Z}_m = \langle\widehat{\sigma}$ plus relations induced by $S_g\rangle$ where m divides n. Namely, in both cases $\mathcal{N}$ is a normal subgroup of index m in a von Dyck group. On the other hand we have the exact sequence of groups $1 \to Ab \to \widetilde{G}_n \to \mathcal{N} \to 1$, where Ab is an abelian subgroup of $\widetilde{G}_n$. That is Ab is a subgroup of $\mathbb{Z}_{12}^{n-2}$ for n even and a subgroup of $\mathbb{Z}_{12}^{n-1}$ for n odd.

We give an example to see how these groups look like.

Example 4. Consider the pitch segment $\langle 0,4,7,t,2\rangle$ and use the basis of Proposition 1.

We see that the abelian part of $\widetilde{G}_5$ is generated by the translations in four different directions $T_1^2 \equiv T_4$, $T_1^3 \equiv T_7$, $T_1^4 \equiv T_{10}$, and $T_1^5 \equiv T_2$. In this case, any element of $\widetilde{G}_5$ is written in the form $p_{12}T_4^{\lambda_1}T_7^{\lambda_2}T_{10}^{\lambda_3}T_2^{\lambda_4}$, where the λ_i's are integer exponents. Condition 1 is readily checked and therefore the image $\mathcal{N}$ of $\widetilde{G}_5$ is isomorphic to the dihedral group $\mathbb{D}_{12}$. Indeed, the operator T_i^j is written as $T_i^j = \mathrm{Id} + C_j - C_i$ where C_i is a 5×5 matrix with zero entries everywhere except in he i-th column where all entries are 1. One checks that $(T_i^j)^\lambda = \mathrm{Id} + \lambda(C_j - C_i)$ and these matrices go down to $\mathrm{Id} + \lambda T_{x_j - x_i}$ as they are restricted to the segment (T/I)-orbit. $\mathbb{D}_{12}$ is the group generated by two letters s, t with relations $s^{12} = t^2 = 1$, $tst = s^{-1}$, and the isomorphism with $\mathcal{N}$ is produced by sending $p_{12} \to t$, $T_1 \to s$. Since $\mathcal{N} \simeq \mathbb{D}_{12}$ and has index 5 into $\Delta_g^+(2,5,10)$, then by Formula (11) one gets $24 \times 5 \times 2 = |\Delta_g(2,5,10)| = 20(g-1)$. Thus, a surface containing the group generated by p_{12} and σ must have genus 13. However this is not the surface containing an action of the whole group $\widetilde{G}_5$ acting on the orbit. For this we have to compute the kernel of the morphism $\varphi : \mathbb{Z}^4 \to \mathbb{Z}_{12}$ defined as $\varphi(\lambda_1,\lambda_2,\lambda_3,\lambda_4) = 4\lambda_1 + 7\lambda_2 + 10\lambda_3 + 2\lambda_4$ so that we get the isomorphism $Ab \simeq \mathbb{Z}^4/Ker\varphi$. To determine $Ker\varphi$ we notice that it is generated by the vectors $\{e_1 = (0,10,0,1), e_2 = (0,2,1,0), e_3 = (1,8,0,0), e_4 = (0,12,0,0), e_5 =$

$(3,0,0,0), e_6 = (0,0,6,0), e_7 = (0,0,0,6)\}$. Also $\{e_1, e_2, e_3, e_4\}$ generates $Ker\varphi$ and is an independent basis for it. Any element in $\mathbb{Z}^4$ can be written in terms of this basis of $Ker\varphi$ as follows $(\mu_1, \mu_2, \mu_3, \mu_4) = \mu_1 e_3 + \mu_3 e_2 + \mu_4 e_1 + \frac{1}{12}(\mu_2 - 8\mu_1 - 2\mu_3 - 10\mu_4)e_5$. Clearly, we obtain $Ab \simeq \mathbb{Z}^4/Ker\varphi \simeq \mathbb{Z}_{12}$.

Consider a genus g closed surface S_g on which we have a Tessellation of type $\{n,n\}$ for n even or $\{n,2n\}$ if n is odd. Moreover, let us assume the set of faces contain at least the 24 faces associated with the $(T/I)-$forms of a given pitch class segment $\langle x_1, x_2, \dots x_n\rangle$. That is, $F = 24N$. What would the Euler-Poincaré characteristic and the minimal possible genus of S_g be? This problem depends on the possibility of embedding a Graph on a surface of genus g. A topological connected Graph (i.e. points joined by arc segments) can always be embedded into a surface of some genus g. Even into a non-orientable surface. However determining the minimum embedding genus g is a very difficult problem that belongs to Topological Graph Theory [see 19, chap. 7], [also 17].

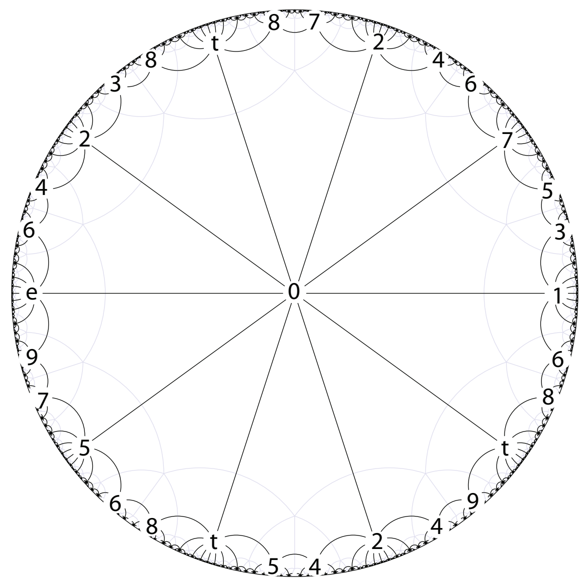

Figure 10: ten pentagons incidence

Putting $p = q = n$ and $F = 24N$ in Formula (12) we get $g = 1 + 6(n-4)N$ for n even. In the n odd case $p = n, q = 2n$ and we get $g = 1 + 6(n-3)N$.

In Table 1 we shows for small n the Tessellation type on S_g for the corresponding pitch class segment and what could the minimal genus be if a Tessellation of that type is found on S_g. Warning: the genera shown in the table only reflect the formulas obtained. This does not mean that you would get a Tiling representing the pitch set class $sc(x_1 x_2 \dots x_n)$ with the minimal genus displayed in the table. The only certain values for this hold in genus 1.

Remark 6. A generic pitch class segment holds a very high genus. Indeed, suppose we can view the group of inversions $\widetilde{G}_n$ as a group of automorphisms on a surface of genus g, then by a famous theorem of Hurwitz, the order of $\widetilde{G}_n$ is bounded above by $84(g-1)$. Thus, in the case of n odd Proposition 1 would tell us that $g \geqslant 1 + \frac{|\mathbb{Z}_2 \ltimes \mathbb{Z}_{12}^{n-1}|}{84} = 1 + \frac{2\times 12^{n-1}}{84}$. This is $g \geqslant 495$ if $n = 5$. However, as seen in Example 4, $\widetilde{G}_5$ cuts down to a group of order 2×12^2 for the pitch segment $\langle 0,4,7,t,2\rangle$, i.e. $g \geqslant 1 + \frac{2\times 12^2}{84} \geqslant 5$. Latter we will see that this example holds a much higher genus.

Table 1: Incidence and possible genera for small pitch class segments

n	pitch segment	Tiling type	genus formula	possible values
3	$\langle x_1, x_2, x_3 \rangle$	$\{3, 6\}$	$g = 1$	1
4	$\langle x_1, x_2, x_3, x_4 \rangle$	$\{4, 4\}$	$g = 1$	1
5	$\langle x_1, x_2, x_3, x_4, x_5 \rangle$	$\{5, 10\}$	$g = 1 + 12N$	13, 25, 37, 49, 61
6	$\langle x_1, x_2, x_3, x_4, x_5, x_6 \rangle$	$\{6, 6\}$	$g = 1 + 12N$	13, 25, 37, 49, 61
7	$\langle x_1, x_2, x_3, x_4, x_5, x_6, x_7 \rangle$	$\{7, 14\}$	$g = 1 + 24N$	25,49, 73, ...
8	$\langle x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \rangle$	$\{8, 8\}$	$g = 1 + 24N$	25,49, 73, ...
9	$\langle x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9 \rangle$	$\{9, 18\}$	$g = 1 + 36N$	37, 73, ...
10	$\langle x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} \rangle$	$\{10, 10\}$	$g = 1 + 36N$	37, 73, ...
11	$\langle x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11} \rangle$	$\{11, 22\}$	$g = 1 + 48N$	49, ...
12	$\langle x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12} \rangle$	$\{12, 12\}$	$g = 1 + 48N$	49, ...

6. Minimal model surface and pentachords

6.1. A Tiling containing 24 pentagons

In a similar way as did in Example 4 we will work with a fixed pitch segment $\langle x_1, x_2, x_3, x_4, x_5 \rangle$ and its (T/I) orbit. We assume that this pitch segment satisfies Condition 1.

Then the image $\mathcal{N}$ of $\widetilde{G}_5$ into the group generated by $\{p_{12}, \sigma\}$ is the dihedral group $\mathbb{D}_{12}$. If a Tiling by pentagons with vertex incidence 10 exists on a genus g surface S_g, then the von Dyck group $\Delta_g^+(2, 5, 10)$ contains $\mathcal{N}$ with index 5. The same computation we did in Example 4 holds and shows that the genus is $g = 13$.

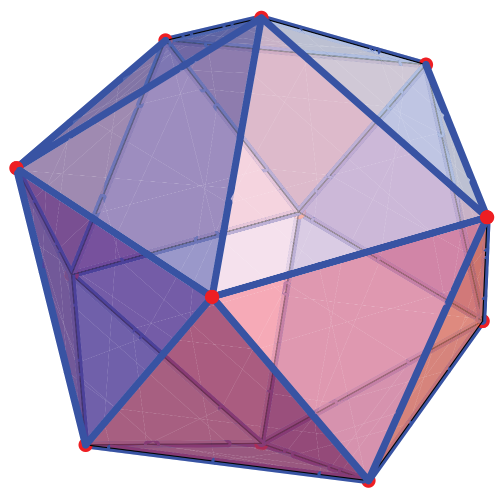

Figure 11: icosahedron

The problem here is the existence of such a Tiling on a surface of genus 13. Fortunately, such a Tiling exists and it is related to a surface already studied by Felix Klein [22] in connection with the solutions of the quintic equation: the Bring surface[21]. The Bring surface is a surface of genus 4 and can be viewed as a triple branched cover of the icosahedron [see 35, for such matters]. The Bring surface possess a Regular Tessellation of type $\{5, 5\}$ consisting of twelve hyperbolic pentagons whose centers map to the vertices of the the sphere

[21]In the literature is called Bring curve. Here we use the dictionary "algebraic complete curves over the complex numbers" ≡ "closed smooth surfaces with complex structure" (or Riemann surfaces, in honor of the mathematician Bernhard Riemann).

with icosahedral Tessellation (an inflated icosahedral balloon framed at its vertices). The 12 vertices of the pentagons are branched (with branch index 2) over the same vertices of the Tessellated sphere (each one fifth $(2\pi/5, 2\pi/10, 2\pi/10)$-hyperbolic triangle of a hyperbolic pentagon maps onto a $(2\pi/5, 2\pi/5, 2\pi/5)$-spherical triangle of the icosahedral Tessellation of the sphere). This Tessellation was explained in W. Threlfall book [32, 35] and can be viewed in Figure 12[22] as its image into the Poincaré Disk.

The pentagonal tiles P_i's are numbered 1 to 12 with repeated tiles in the Figure since we are in the Poincaré Disk. The vertices are the red points labeled by {a, b, c, d, e, f, g, h, i, j, k, m}. The fundamental domain of the surface is the union of the 10 quadrilaterals incident at the center of P_1 and forming a regular icosagon (violet dashed boundary) with identification of sides.

Cutting with scissors this icosagon and gluing equivalent sides of its border we get a genus 4 surface tessellated by 12 pentagons (some pentagons are recovered by rearranging their triangular pieces).

Indeed, this $\{5,5\}$ Tessellation has a set of 12 vertices and 30 edges, so the Euler-Poincaré characteristic is $\chi = 12 - 30 + 12 = -6$, corresponding to genus 4. The relations satisfied by the sides of the icosagon (labeled $\mathcal{A}_i$ and $\mathcal{B}_i$, $i = 1, \ldots, 5$) are given in Threlfall's book [32, see p. 22]. They are $\mathcal{A}_1\mathcal{B}_1\mathcal{A}_2\mathcal{B}_2\mathcal{A}_3\mathcal{B}_3\mathcal{A}_4\ \mathcal{B}_4\mathcal{A}_5\mathcal{B}_5 = \mathrm{Id}$, $\mathcal{A}_1\mathcal{A}_4\mathcal{A}_2\mathcal{A}_5\mathcal{A}_3 = \mathrm{Id}$, and $\mathcal{B}_1\mathcal{B}_3\mathcal{B}_5\mathcal{B}_2\mathcal{B}_4 = \mathrm{Id}$.

Now we proceed to take two copies of this tessellated genus 4 surface, make cuts along 6 disjoint edges in each sheet and glue the sheets along each pair of edges (lips) produced in the cuts. We get in this way a two sheeted cover Γ of the Bring surface B which is branched over the 12 points {a, b, c, d, e, f, g, h, i, j, k, m}. One way of choosing the cuts is the following: edge(a,k)=$P_{10} \cap P_{11}$, edge(b,c)=$P_1 \cap P_3$, edge(d,e)=$P_1 \cap P_6$, edge(f,j)=$P_8 \cap P_{12}$, edge(h,m)=$P_2 \cap P_5$, edge(i,g)=$P_9 \cap P_{12}$. The new surface Γ has an induced Tessellation consisting of 24 hyperbolic pentagons, 12 vertices and 60 edges, but at each vertex 10 pentagons meet. Indeed, as we turn around a vertex we cover 5 pentagons; by crossing the cut we go into the other sheet and with another turn we cover the remaining 5 pentagons that meet around the chosen vertex. Therefore, Γ carries a Tiling of type $\{5,10\}$ and has invariants $\chi(\Gamma) = 12 - 60 + 24 = -24$, and $g = 13$.

On the other hand, Hurwitz formula gives the same result: $\chi(\Gamma) = 2.\chi(B) -$ ramification index $= 2.(-6) - 12.(2-1) = -24$.

From what it was said in this paragraph we have the following:

Proposition 3. *There is a surface Γ of genus 13 which carries a Tiling by 24 hyperbolic pentagons of type $\{5,10\}$. This surface is a 2-cover of the Bring surface B branched over the 12 vertices of the $\{5,5\}$ Tessellation of B. The von Dyck group $\Delta^+_{13}(2,5,10)$ exists and has order 120.*

[22] This is essentially a picture due to Threlfall with some modifications by Weber.

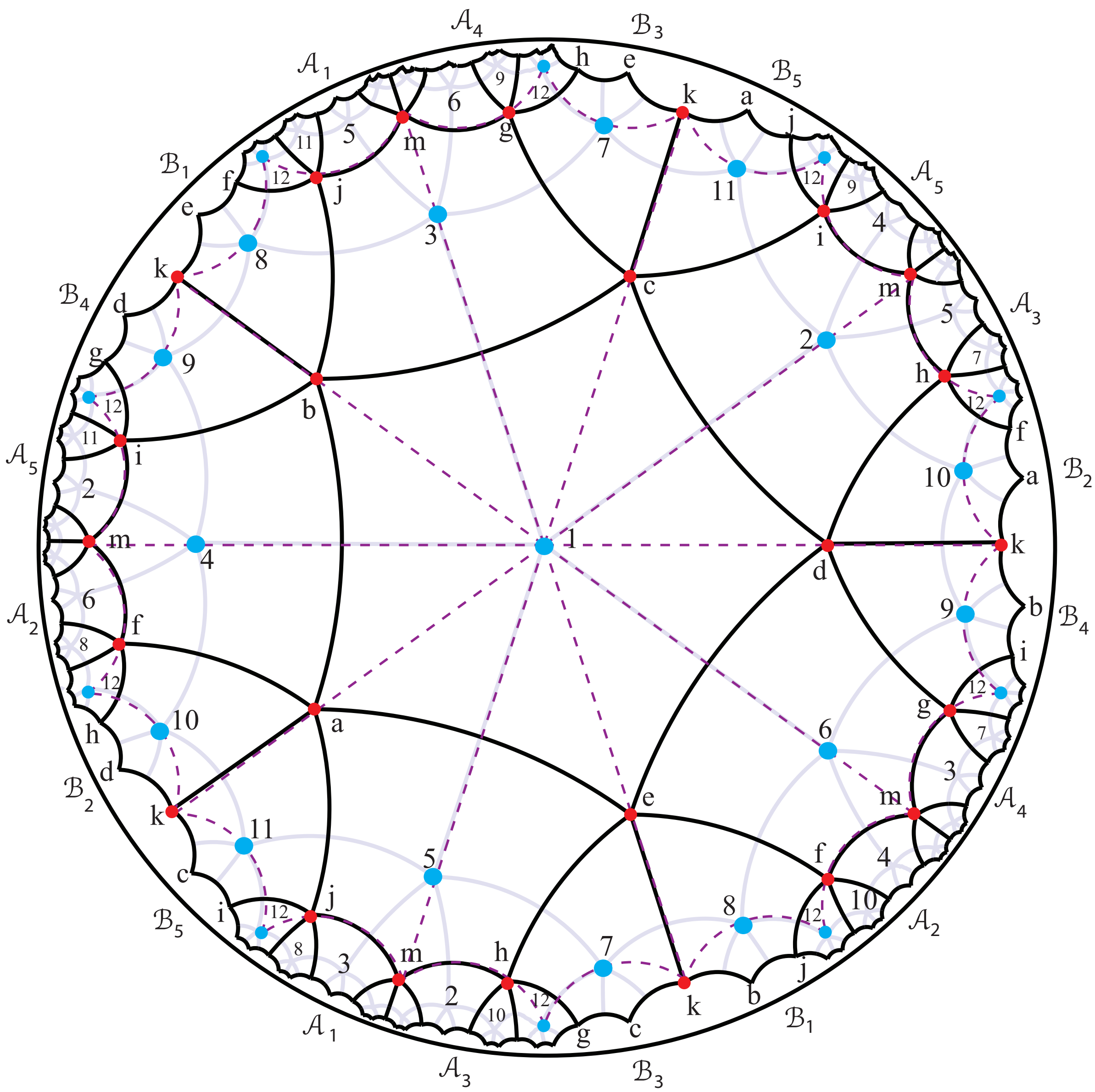

Figure 12: Tessellation of Bring surface by 12 pentagons

6.2. Construction of the surface model

We cannot claim that the genus 13 surface Γ is the model we are looking for. Indeed, by starting with any pentachord pitch class segment from a given tile in Γ and walking trough all tiles by the inversions in $\widetilde{G}_5$ we find inconsistencies. That is clear because as said in Example 4 the surface Γ is not acted by the whole group $\widetilde{G}_5$.

Start with a fundamental tile T: this is a pentagon (hyperbolic) with labeled vertices, and the labels are numbers within the (chromatic) pitch set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, t, e\}$[23]. The labels of T correspond to a given ordered pitch segment $\langle x_1, x_2, x_3, x_4, x_5\rangle$. The group $\widetilde{G}_5$ has the inversions $\mathscr{G} = \{p_{12}, p_{23}, p_{34}, p_{45}, p_{15}\}$ as generator set and relations contained at least in $\mathscr{R} = \{p_{12}^2 = p_{23}^2 = p_{34}^2 = p_{45}^2 = p_{15}^2 = (p_{12}p_{23}p_{34}p_{45}p_{15})^2 = \mathrm{Id},$ commuting relations among the products $p_{ij}p_{kl}\}$.

A walk in $\mathscr{G}$ is a finite sequence $w = (p_1, p_2, \ldots, p_n)$ where $p_i \in \mathscr{G}$ for $i = 1, \ldots, n$. The set $\mathscr{W}$ of all walks[24] has a group structure by concatenation of walks: $(q_1, q_2, \ldots, q_m) \circ (p_1, p_2, \ldots, p_n) = (p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_m)$; the identity of $\mathscr{W}$ is the empty walk $(\ )$[25].

Two walks that differ by having two consecutive equal entries $p \in \mathscr{G}$ more (or less) are said to be equivalent. The set of equivalent classes $\widetilde{\mathscr{W}}$ has a group structure induced by that of $\mathscr{W}$ and each class has a unique member (called reduced walk) without two consecutive $p \in \mathscr{G}$ [27, Ch. 2].

Since we identify pentagonal tiles with ordered sequences $\langle x_1, x_2, x_3, x_4, x_5\rangle$, we act on them with the group $\widetilde{\mathscr{W}}$ as follows: $\overline{w}\bullet\langle x_1, x_2, x_3, x_4, x_5\rangle = (p_1, p_2, \ldots, p_n)\bullet\langle x_1, x_2, x_3, x_4, x_5\rangle = p_n \ldots p_2 p_1 \langle x_1, x_2, x_3, x_4, x_5\rangle$, where $(p_1, p_2, \ldots, p_n)$ is the reduced walk in the class $\overline{w}$ and the last action is the $\widetilde{G}_5$ action.

In this way the defined actions of $\widetilde{\mathscr{W}}$ and $\widetilde{G}_5$ on tiles are both left actions. Moreover, there is a surjective homomorphism $\varphi : \widetilde{\mathscr{W}} \to \widetilde{G}_5$ defined by $\varphi(p_1, p_2, \ldots, p_n) = p_n \ldots p_2 p_1$.

Now starting from a given tile T, consider the union of all translated tiles $\overline{w} \bullet T$ by reduced walks representing the elements $\overline{w} \in \widetilde{\mathscr{W}}$, i.e. $\mathscr{U} = \bigcup_{\overline{w}\in\widetilde{\mathscr{W}}} \overline{w} \bullet T$. Then $\widetilde{\mathscr{W}}$ acts on $\mathscr{U}$ by shuffling the tiles of $\mathscr{U}$[26]. Recalling the gluing procedure we explained in [5.1], we can endow $\mathscr{U}$ with a structure of topological space: it is a simply connected space, a tree of pentagons in which two pentagons $\overline{w_1} \bullet T$ and $\overline{w_2} \bullet T$ are glued along a side if and only if $\overline{w_2} \bullet T = p(\overline{w_1} \bullet T)$ or $\overline{w_1} \bullet T = p(\overline{w_2} \bullet T)$ for some $p \in \mathscr{G}$. Each pentagon bounds on the sides with other 5 pentagons, and in order to visualize and fit an infinite set of pentagons together in that way we need to embed this into an infinite dimensional space.

Let $\mathscr{K}$ be the kernel of φ. Then we construct the quotient space $\widetilde{\mathscr{U}} = \mathscr{U}/\mathscr{K}$ as the set of orbits $\mathscr{K} \bullet u$, with $u \in \mathscr{U}$. This space is a Hausdorff space because the action of $\widetilde{\mathscr{W}}$ (hence that of $\mathscr{K}$) on the locally compact space $\mathscr{U}$ is discrete and proper [see 15, Th. I.6.7].

[23] Actually, any pitch set with abelian group operations on it could be used for this purpose, e.g. a diatonic group $(\mathbb{Z}_7, +)$.

[24] In other contexts these are called words.

[25] This means do not move.

[26] In $\mathscr{U}$ all tiles $\overline{w} \bullet T$ are different and numbered by $\widetilde{\mathscr{W}}$.

Moreover, it has an induced Tiling by the elements of $\widetilde{G}_5 = \mathbb{Z}_2 \ltimes \mathbb{Z}_{12}^4$: $\widetilde{\mathscr{U}} = \bigcup_{g \in \widetilde{G}_5} gT$ is a closed surface.

The restriction of the basis $\{T_1^2, T_1^3, T_1^4, T_1^5\}$ for $\mathbb{Z}_{12}^4$ to T gives a smaller group $\widetilde{G}_T \simeq \mathbb{Z}_2 \ltimes (\mathbb{Z}_{12} \oplus Ab_T)$. Namely, we are assuming that there is an exact sequence $1 \to Ab_T \to \widetilde{G}_T \to \mathbb{D}_{12} \to 1$ and a surjective map $\widetilde{G}_5 \xrightarrow{\pi} \widetilde{G}_T \to 1$. Now, the kernel of π can be pulled back via φ and give the discrete group $\mathscr{H}_T \subset \widetilde{\mathscr{W}}$. We have a new tiled closed surface $\widetilde{\mathscr{V}}_T = \mathscr{U}/\mathscr{H}_T = \bigcup_{g \in \widetilde{G}_T} gT$.

The group $\widetilde{G}_T$ acts on $\widetilde{\mathscr{V}}$ discretely and at most with a finite set of fixed points. Indeed, $\mathbb{Z}_2$ is generated (for instance) by the inversion p_{12} which moves each tile to a different one but has the middle point of the glued tiles gT and $p_{12}gT$ fixed. The translations in the normal subgroup $\mathbb{Z}_{12} \oplus Ab_T$ act without fixed points[27].

We are in the following situation:

1. A tiled closed surface $\widetilde{\mathscr{V}}_T$ with a discrete action of G_T on it.

2. An exact sequence $1 \to Ab_T \to \widetilde{G}_T \to \mathbb{D}_{12} \to 1$ and a discrete action of $\mathbb{D}_{12}$ on a tiled genus 13 surface Γ (see [6.1]).

We want to relate these two pieces of data[28].

The group Ab_T of translations act on $\widetilde{\mathscr{V}}_T$ properly discontinuously[29], thus the map $\widetilde{\mathscr{V}}_T \to \widetilde{\mathscr{V}}_T/Ab_T = \widetilde{\Gamma}$ is a covering map and the closed surface $\widetilde{\Gamma}$ is built with 24 pentagonal tiles and the group $\mathbb{D}_{12}$ acts discretely on $\widetilde{\Gamma}$ by shuffling tiles.

Now if we identify fundamental pentagonal tiles on Γ and $\widetilde{\Gamma}$ and give an isomorphism of the acting groups $\mathbb{D}_{12}$, we can extend this identification to a homeomorphism between Γ and $\widetilde{\Gamma}$ that preserves pentagonal faces, sides and vertices.

Summing-up, we can state the following:

Theorem 1. *Let $\langle x_1, x_2, x_3, x_4, x_5\rangle = T$ be a pitch class satisfying Condition 1 and such that $\widetilde{G}_T$ (the restriction of $\widetilde{G}_5$ to the pitch class) fits into an exact sequence $1 \to Ab_T \to \widetilde{G}_T \to \mathbb{D}_{12} \to 1$, where Ab_T is an abelian group of order n. Then, there is a tiled surface $\widetilde{\mathscr{V}}$ of type $\{5, 10\}$ containing all tile occurrences of T under $\widetilde{G}_T$ assembled so that two pentagons have a common side if and only if one is a transform of the other by an inversion in $\mathscr{G}$. The surface $\widetilde{\mathscr{V}}$ is an n-covering (non-branched) of a surface Γ of genus 13 having a Tiling by 24 hyperbolic pentagons. The Tiling of $\widetilde{\mathscr{V}}$ has $24n$ hyperbolic pentagons and genus$(\widetilde{\mathscr{V}}) = 12n + 1$.*

[27] This is because the tiles associated to $T_i^j\langle a, b, c, d, e\rangle$ and $\langle a, b, c, d, e\rangle$ either are equal (in whose case the translation is trivial) or do not intersect. If they are different and have a common point they must be glued along a side by an inversion p, so that pT_i^j fixes the tile of $\langle a, b, c, d, e\rangle$, and therefore all tiles, but this is impossible for it would mean $T_i^j = p$.

[28] Other foundations for this situation and the construction in [6.2] are found in the paper [30] and bibliography therein and the book [11].

[29] Namely, each point $v \in \widetilde{\mathscr{V}}_T$ has a neighborhood such that all its translates by Ab_T are pairwise disjoint.

Proof. Condition 1 already implies the exactness of the group short sequence. The arguments above and [6.1], [6.2] show that there is a surface $\widetilde{\mathscr{V}}$ with a Tiling by pentagons and an n-covering $\widetilde{\mathscr{V}} \to \Gamma$ where Γ is the genus 13 surface which is a $2-1$ ramified cover of the Bring surface. As Γ has a Tiling of type $\{5, 10\}$ by 24 pentagons, then $\widetilde{\mathscr{V}}$ has $24n$ pentagonal faces and since we have a covering map, the incidence of the pentagons at each pentagon vertex is preserved. Hurwitz formula yield $\chi(\widetilde{\mathscr{V}}) = n\chi(\Gamma) = -24n$ (because there is no ramification) and the genus Formula (12) give $g(\widetilde{\mathscr{V}}) = 12n + 1$, which shows the statement. ▫

Example 5. In the pitch segment $\langle C, E, G, B\flat, D\rangle$ of example 4 we found that $Ab_T \simeq \mathbb{Z}_{12}$. Theorem 1 show that the surface on which the pentagonal tiles fit together has genus $g = 12.12 + 1 = 145$. It is easy to see that the Stockhausen Klavierstück III pentachord $\langle C, C\#, D, D\#, F\#\rangle$, the Schoenberg Op. 23/3 pentachord $\langle B\flat, D, E, B, C\#\rangle$, and the pentachords $\langle C, D, E, F, A\rangle$ and $\langle C, D, E\flat, G, A\flat\rangle$ used by A. Tcherepnin in Ops. 51, 52, and 53, also have the same pattern. Indeed, in each case one proves that $Ab_T \simeq \mathbb{Z}_{12}$ and they satisfy Condition 1. Thus, these pentachords under the contextual inversions of $\widetilde{G}_5 = G_5$ produce a Tiling on a genus 145 surface.

7. Conclusion and Remarks

The genus 145 surface constructed can be viewed as a 24 sheeted covering of the Tiling appearing in Figure 12 (with identifications). Keeping track of the tile and sheet we are in may be kind of difficult in terms of computer graphics, however, a space representation obtained by the methods in [36] or [29] seems to be harder due to the high genus.

This Tonnetz representation may be useful and applicable in describing tone paths through similar pentachords using real time software that involves sound, images and media like Max/MSP/Jitter by Cycling '74[30] .

A surface similar to Γ but in the situation where one has other scales like in Koechlin or Stravinsky's Examples 1, 2 (diatonic $\mathbb{Z}_7$) need to be addressed particularly. Our genus 13 surface would not work since the dihedral group $\mathbb{D}_7$ is not contained in the von Dyck group $\Delta^+_{13}(2, 5, 10)$. So another highly symmetric surface is needed possessing a Tiling $\{5, 10\}$ by pentagons.

Also in order to model hexachords, heptachords, etc. other surfaces like Γ with appropriate Tessellations ($\{6, 6\}$, $\{7, 14\}$, etc.) have to be found. Then our procedures and methods can be applied in a similar fashion.

We notice that what is basically important is not precisely the Tiling of a surface but a graph containing the pitch information (or other musical information) embedded into some space (eventually a surface as we did it here). This reverts the problem of representing some kind of musical data to that of "Graph Drawing"[31] and most papers in Music Theory

[30] http://cycling74.com/.

[31] There is an active community of mathematicians and yearly proceedings on this subject http://www.graphdrawing.org/.

take this approach [24, 4, 12, 16, 28, 14]. We pursued here the embedding of a pentachord network graph into a closed surface, which is in principle a harder problem than a singular Graph Drawing. By representing the dual Tessellations any pentachord is a vertex in the dual graph, edges relate contiguous pentachords and tiles (now decagons) close up the graph to a surface. Here a walk is a real path in this graph from vertex to vertex. This dual graph is very interesting and a well known object in Graph Theory: the Cayley Graph[32] of our group $\widetilde{G}_5$ with generating set $\mathscr{G}$ (or that of the group $\widetilde{G}_T$ with the restricted generators).

For the purpose of algebraic calculations it is easier to consider a multidimensional torus[33] with translations in $\mathbb{Z}_{12}$ (i.e. a higher dimensional analog of the Öttingen-Riemann Tonnetz). This paradigm is understandable to the light of curves and their Jacobians. Any surface (Riemann surface or complete algebraic complex curve) Γ[34] of genus g can be embedded into its Jacobian $Jac^g(\Gamma)$ [26], which is a $2g$-dimensional real torus[35] with algebraic operations. The torus where the translations T_i^j occur is isogenous to the Jacobian of Γ (i.e. either a covering or a quotient of it). So we say that in the Jacobian of Γ one can interpret or merge both, the Tiling of the surface Γ and the translations of the group $\widetilde{G}_5$, as we did in the examples of Figures 4 and 5 for the genus 1 case. Of course, beyond genus 1 we do not have nice graphic representations for these tori.

Assume we have a set of generators $\mathscr{G} = \{p_1, p_2, p_3, p_4, p_5\}$ for the group $\widetilde{G}_5$ so that we can write $p_{i+1} = p_1 t_i$, $i = 1, 2, 3, 4$, where the t_i's are translations in $\mathbb{Z}_{12}$. The group $\widetilde{G}_5$ consists of the reflection $p_1 = p$ and the translations $\{t_1, t_2, t_3, t_4\}$. How should we encode the data for a single path in these generators? For instance $p_5p_3p_2p_4 = pt_4.pt_2.pt_1.pt_3 = t_4^{-1}t_2t_1^{-1}t_3$, where we have used $ptp = t^{-1}$ for any translation. Another example gives $p_5p_3p_2p_4p_5 = pt_4.pt_2.pt_1.pt_3.pt_5 = t_4^{-1}t_2t_1^{-1}t_3t_5^{-1}p$. We can give the following rule in case the reduced path contains only the generators in $\mathscr{G}\backslash\{p\}$.

1. if length of path is even change each occurrence of p_{i+1} in an odd position by t_i^{-1} and each occurrence of p_{i+1} in an even position by t_i.

2. if length of path is odd do the same as in 1. but insert p at the end.

Some other examples when the path starts with p and then follows by elements in $\mathscr{G}\backslash\{p\}$: $pp_5p_4p_3p_2 = p.pt_4.pt_3.pt_2.pt_1 = t_4t_3^{-1}t_2t_1^{-1}p$, or $pp_5p_4p_3 = p.pt_4.pt_3.pt_2 = t_4t_3^{-1}t_2$. In these cases we go by the rule.

1. if length of path is even change each occurrence of p_{i+1} in an odd position by t_i, each occurrence of p_{i+1} in an even position by t_i^{-1} and delete the initial p.

[32]Information on this in [19].

[33]However, we eventually lose the possibility of having a space representation.

[34]Here I am not specifying which Γ to take. It could be the genus 4 Bring surface of our setting, or the genus 13 curve in Proposition 3, or the genus 145 curve of Example 5. Making this precise requires further developement.

[35]$Jac^g(\Gamma) \equiv (\mathbb{R}/\mathbb{Z})^{2g}$

2. if length of path is odd do the same as in 1. but insert p at the end.

In case the path has several occurrences of p, we separate it into segments with a starting p and apply to each successive segment the rules above. An example will clarify this where we have put for short number i instead of p_i: $3241541451323 = (324)(154)(145)(1323) = t_2^{-1}t_1t_3^{-1}p(154)(145)(1323) = t_2^{-1}t_1t_3^{-1}t_4^{-1}t_3t_3t_4^{-1}p(1323) = t_2^{-1}t_1t_3^{-1}t_4^{-1}t_3t_3t_4^{-1}t_2^{-1}t_1t_2^{-1}p = t_1^2t_2^{-3}$ $t_3t_4^{-2}p$.

References

[1] Callender, Clifton, Ian Quinn, and Dmitri Tymoczko. 2008. “Generalized Voice Leading Spaces.” *Science* 320: 346–348.

[2] Cohn, Richard. 1997. “neo-Riemannian Operations, Parsimonious Trichords, and their ‘Tonnetz’ Representations.” *Journal of Music Theory* 41(1): 1–66.

[3] Cohn, Richard. 1998. “Introduction to neo-Riemannian Theory: A Survey and a Historical Perspective.” *Journal of Music Teory* 42(2): 167–180.

[4] Cohn, Richard. 2003. “A Tetrahedral Model of Tetrachordal Voice-Leading Space.” *Music Theory Online* 9(4).

[5] Conder, M., Dobcsányi,, P. 2001. “Determination of all regular maps of small genus.” *Journal of Combinatorial Theory*, Series B 81, 224–242.

[6] Conder, M., 2006. “Orientable regular maps of genus 2 to 101.” http://www.math.auckland.ac.nz/~conder.

[7] Conway, J. H., Burgiel, H., Goodman-Strauss, C. 2008. “The Symmetries of Things.” A.K. Peters.

[8] Coxeter, H. S. M. 1974, “Regular Complex Polytopes.” Cambridge University Press. London and New York.

[9] Coxeter, H. S. M., Moser, W. O. J. 1980. “Generators and Relations for Discrete Groups.” Springer-Verlag, Berlin, 1957. Fourth edition 1980.

[10] Crans, Alissa S. Fiore, Thomas M. and Satyendra, Ramon. 2009. “Musical actions of dihedral groups.” *Amer. Math. Monthly* 116, no. 6, 479–495.

[11] Davis, Michael W. 2008. “’The Geometry and Topology of Coxeter Groups”. Princeton University Press. Princeton and Oxford.

[12] Douthett, Jack and Peter Steinbach. 1998. “Parsimonious Graphs: A Study in Parsimony, Contextual Transformations, and Modes of Limited Transposition”. *Journal of Music Theory* 42/2: 241–63.

[13] Fiore, Thomas M. Satyendra, Ramon. 2005. “Generalized Contextual Groups.” *Music Theory Online* 11(3).

[14] Fitzpatrick, Michael. 2011. "Models and Music Theory: Reconsidering David Lewin's Graph of the 3-2 Cohn Cycle". *Music Theory Online*, 17/1.

[15] Godbillon, Claude. 1971. "Éléments de Topologie Algébrique". Hermann, Paris.

[16] Gollin, Edward. 1998. "Some Aspects of Three-Dimensional 'Tonnetze". *Journal of Music Theory* 42/2: 195–206.

[17] Gross, Jonathan L., Tucker, Thomas W. 2001. "Topological Graph Theory." Dover Publications, New York. First published in 1987 by John Wiley & Sons, New York.

[18] Grünbaum, B., Shepard, G. C. 1986. "Tilings and patterns." W. H. Freeman and Company, New York.

[19] Gross, J., Yellen, J. Editors. 2004. "Handbook of Graph Theory." *Discrete Mathematics and its Applications Series.* CRC Press, London, New York.

[20] Jedrzejewski, Franck. 2006. "Mathematical Theory of Music". *Collections Musique/Sciences.* Editions DELATOUR FRANCE/Ircam-Centre Pompidou.

[21] Karcher, Hermann; Weber, Matthias. "The geometry of Klein's Riemann surface." The eightfold way, 9–49, *Math. Sci. Res. Inst. Publ.*, 35, Cambridge Univ. Press, Cambridge, 1999. Preprint available online: doi=10.1.1.47.1879.

[22] Klein, Felix. 1884. "Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade", Teubner, Leipzig. Reprinted Birkhäuser, Basel, 1993 (edited by P. Slodowy); translated as Lectures on the icosahedron and the solution of equations of the fifth degree, Kegan Paul, London, 1913 (2nd edition); reprinted by Dover, 1953.

[23] Kepler, J. 1619. "Harmonices mundi." Linz. Translated as The harmony of the world by E. J. Aiton et al., Philadelphia, Amer. Philos. Society, 1997.

[24] Lewin, David. 2007. "Generalized Musical Intervals and Transformations." *Oxford University Press.* Oxford and New York. Originally published in 1987 by Yale University Press.

[25] Lewin, David. 2007. "Musical Forms and Transformations. Four Analytic Essays." *Oxford University Press.* Oxford and New York. Originally published in 1993 by Yale University Press.

[26] Mumford, David. 1977. "Curves and their Jacobians". The University of Michigan Press.

[27] Robinson, Derek John Scott. 1995. "A Course in the Theory of Groups". *Graduate Texts in Mathematics.* Springer Verlag, New York, Berlin, Heidelberg. Second edition.

[28] Rockwell, Joti. 2009. "Birdcage Flights: A Perspective on Inter-Cardinality Voice Leading". *Music Theory Online* 15.5.

[29] Séquin, C. H. 2007. "Symmetric embedding of locally regular hyperbolic tilings". *In Proc. BRIDGES 2007 Conference*, San Sebastian, 379–388.

[30] Singerman, David. 1988. "Universal tessellations". *Revista Matematica de la Universidad Complutense de Madrid*. Vol. 1, No. 1, 2, 3, 1988, 111–123.

[31] Straus, Joseph N. 2011. "Contextual-Inversion Spaces." *Journal of Music Theory* 55:1, Spring 2011, 43–88.

[32] Threlfall, W. 1932. "Gruppenbilder", *Abh. Sächs. Akad. Wiss. Leipzig Math.-Natur. Kl.* 41:6 (1932), 1–59.

[33] Tymoczko, Dmitri. 2009. "Three Conceptions of Musical Distance." In *Mathematics and Computation in Music,* eds. Elaine Chew, Adrian Childs, and Ching–Hua Chuan, 258–273. Heidelberg: Springer.

[34] Tymoczko, Dmitri. 2011. "A Geometry of Music : Harmony and Counterpoint in the Extended Common Practice." *Oxford Studies in Music Theory.* Oxford University Press, New York.

[35] Weber, Matthias. 2005. "Kepler's Small Stellated Dodecahedron as a Riemann Surface." *Pacific Journal of Mathematics*, Vol. 220, No. 1, 167–182.

[36] van Wijk, Jarke J. 2009. "Symmetric Tiling of Closed Surfaces: Visualization of Regular Maps." *ACM Transactions on Graphics*, Vol. 28, No. 3, Article 49.