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THE FINITE MODEL PROPERTY FOR THE VARIETY OF HEYTING ALGEBRAS WITH SUCCESSOR

J.L. CASTIGLIONI AND H.J. SAN MARTÍN

Abstract. The finite model property of the variety of S-algebras was proved by X. Caicedo using Kripke model techniques of the associated calculus. A more algebraic proof, but still strongly based on Kripke model ideas, was given by Muravitskii. In this article we give a purely algebraic proof for the finite model property which is strongly based on the fact that for every element x in a S-algebra the interval $[x, S(x)]$ is a Boolean lattice.

1. Introduction

In [4], Kuznetsov introduced an operation on Heyting algebras as an attempt to build an intuitionistic version of the provability logic of Gödel-Löb, which formalizes the concept of provability in Peano arithmetic. This unary operation, which we shall call *successor* [1], was also studied by Caicedo and Cignoli in [1] and by Esakia in [3]. In particular, Caicedo and Cignoli considered it as an example of an implicit compatible operation on Heyting algebras.

The successor, S, can be defined on the variety of Heyting algebras by the following set of equations:

(S1): $x \leq S(x)$,
(S2): $S(x) \leq y \vee (y \to x)$,
(S3): $S(x) \to x = x$.

There is at most one operation satisfying the previous equations. We shall call *S-algebra* to a Heyting algebra endowed with its successor function, when it exists.

The finite model property of the variety of S-algebras was proved by X. Caicedo in [2], using Kripke model techniques of the associated calculus. A more algebraic proof, but still strongly based on Kripke model ideas, was given by Muravitskii in [5]. In this article we give a purely algebraic proof for the finite model property which is strongly based on the fact that for every element x in a S-algebra the interval $[x, S(x)]$ is a Boolean lattice.

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2. The finite model property

Let T be the type of Heyting algebras with successor built in the usual way from the operation symbols $\wedge$, $\vee$, $\rightarrow$, 0 and S corresponding to meet, join, implication, bottom and successor, respectively. Write $T(X)$ for the term algebra of type T with variables in the set X. It is well known that any function $v : X \rightarrow H$, with H a S-algebra, may be extended to a unique homomorphism $v : T(X) \rightarrow H$.

Write $\mathcal{SH}$ for the variety of S-algebras. Recall that $\mathcal{SH}$ is said to have the *finite model property* (FMP) if for every $\psi \in T(X)$ there is a S-algebra H and a homomorphism $v : T(X) \rightarrow H$ such that if $v(\psi) \neq 1$ then there is a S-finite algebra L and a homomorphism $w : T(X) \rightarrow \mathrm{L}$ such that $w(\psi) \neq 1$. Let us prove algebraically that $\mathcal{SH}$ has the FMP.

If M is a bounded distributive lattice and $N \subseteq M$, we write $\langle N \rangle$ to indicate the bounded sublattice generated by N. In particular the bottom and the top of $\langle N \rangle$ and M are the same. Recall that if M is a finite distributive lattice then M is a Heyting algebra. Moreover, M is a S-algebra. If $\{M_i\}_i$ is a family of S-algebras we write $\rightarrow_i$ for the implication in M_i and S^i for the successor in M_i.

Note that for any sublattice L of a Heyting algebra H, if x, y and $x \rightarrow y \in L$, then $x \rightarrow y$ is the relative pseudocomplement of x with respect to y in L. This holds because for every $z \in L$, $z \wedge x \leq y$ iff $z \leq x \rightarrow y$, and this property completely characterizes the relative pseudocomplement. The following lemma is a particular instance of the previous remark.

Lemma 1. *Let M_1 be a finite distributive lattice and M_2 a S-algebra such that M_1 is a bounded sublattice of M_2. If $x, y, x \rightarrow_2 y \in M_1$ then $x \rightarrow_2 y = x \rightarrow_1 y$.*

Lemma 2. *Let M_1 be a finite bounded lattice and M_2 a S-algebra such that M_1 is a bounded sublattice of M_2. If $x, S^2(x) \in M_1$ then $S^1(x) \leq S^2(x)$.*

Proof. Let $x, S^2(x) \in M_1$. For every $y \in M_1$ we have that $S^1(x) \leq y \vee (y \rightarrow_1 x)$. In particular it holds for $y = S^2(x)$. Hence we have that

$$S^1(x) \leq S^2(x) \vee (S^2(x) \rightarrow_1 x). \tag{1}$$

As $x, S^2(x), S^2(x) \rightarrow_2 x = x \in M_1$, by Lemma 1 we have that $S^2(x) \rightarrow_1 x = S^2(x) \rightarrow_2 x = x$. Thus by equation (1) we conclude that $S^1(x) \leq S^2(x) \vee x = S^2(x)$. □

If H is a Heyting algebra and $a, b \in H$ with $a \leq b$, we write $[a, b]$ for the set $\{x \in H : a \leq x \leq b\}$. We say that $[a, b]$ as sublattice of H is Boolean if for every $x \in [a, b]$ there is a $x^c \in [a, b]$ such that $x \wedge x^c = a$ and $x \vee x^c = b$.

Next lemma is a particular case of the following observation. Since for any interval $[a, b]$ in a Heyting algebra and for any $x, y, z \in [a, b]$ we have $z \wedge x \leq y$ iff $z \leq x \rightarrow y$ iff $z \leq b \wedge (x \rightarrow y)$ and $b \wedge (x \rightarrow y) \in [a, b]$, we have that the lattice $[a, b]$ is a Heyting algebra in its own right, with residuum $x \rightarrow_* y := b \wedge (x \rightarrow y)$.

Lemma 3. *Let H be a Heyting algebra and $a, b \in H$ with $a \leq b$ such that $[a, b]$ as sublattice of H is Boolean. If $x \in [a, b]$ then $x^c = b \wedge (x \rightarrow a)$.*

Lemma 4. *If H is a S-algebra and $a \in H$ then $[a, S(a)]$ as sublattice of H is Boolean. In particular, for every $x \in [a, S(a)]$ the complement of x, for which we write x^a, coincides with $(x \to a) \wedge S(a)$.*

Proof. Let $x \in [a, S(a)]$. A direct computation proves that $x \wedge x^a = x \wedge a \wedge S(a) = a$ and $x \vee x^a = x \vee ((x \to a) \wedge S(a)) = (x \vee (x \to a)) \wedge (x \vee S(a)) = S(a)$. □

Definition 1. *Let $\psi \in T(X)$, H a S-algebra and $v : T(X) \to H$ a homomorphism. Let $\to$ and S be the implication and the successor of H respectively. If $\psi_1, \ldots, \psi_n$ are the subformulas of ψ, for $i = 1, \ldots, n$ we define $\hat{a}_i$ as $v(\psi_i)$ and then we consider the sets $A = \{\hat{a}_1, \ldots, \hat{a}_n\} \subseteq H$, $\mathrm{L}_0 = \langle A \rangle$ and $B = \{a \in A : S(a) \in A\}$. Considering a list $a_1, \ldots, a_k$ for the elements of B (in case that $B \neq \emptyset$), we define recursively the sets*

$$K_i = \{(x \to a_i) \wedge S(a_i) : x \in \mathrm{L}_{\mathrm{i}-1} \cap [a_i, S(a_i)]\},$$
$$\mathrm{L}_{\mathrm{i}} = \langle \mathrm{L}_{\mathrm{i}-1} \cup K_i \rangle\,,$$

for $i = 1, \ldots, k$.

Note that every $a_i, S(a_i) \in \mathrm{L}_0$ and that every L_{i} is a finite distributive lattice, $K_i \subseteq \mathrm{L}_{\mathrm{i}}$ and $\mathrm{L}_{\mathrm{i}-1} \subseteq \mathrm{L}_{\mathrm{i}}$.

Lemma 5. *Let H, A, B and L_i, for $i = 0, \ldots, k$ be as in Definition 1, and assume that $B \neq \emptyset$. Then, for every $i = 1, \ldots, k$, $\mathrm{L}_{\mathrm{i}} \cap [a_i, S(a_i)]$ as a sublattice of L_{i} is Boolean. In particular, for every $x \in [a_i, S(a_i)] \cap \mathrm{L}_{\mathrm{i}}$ we have that the complement of x in $[a_i, S(a_i)] \cap \mathrm{L}_{\mathrm{i}}$ is x^{a_i}. Moreover, $x^{a_i} = (x \to_i a_i) \wedge S(a_i)$.*

Proof. For $i = 1, \ldots, k$ define $B_i = \mathrm{L}_{\mathrm{i}} \cap [a_i, S(a_i)]$, and let $z \in B_i$. Then z can be written as $\bigvee_l \bigwedge_m x_{lm}$, for finitely many $x_{lm} \in \mathrm{L}_{\mathrm{i}-1} \cup K_i$. Note that $z = \bigvee_l \bigwedge_m z_{lm}$, with $z_{lm} = (x_{lm} \vee a_i) \wedge S(a_i)$, so $z_{lm} \in B_i$. Using that $z_{lm} \in [a_i, S(a_i)]$, by the Lemma 4 we have that $(z_{lm})^{a_i}$ is the complement of z_{lm} in the Boolean algebra $[a_i, S(a_i)]$. In the following we will prove that every $(z_{lm})^{a_i} \in B_i$.

If $x_{lm} \in \mathrm{L}_{\mathrm{i}-1}$ then $z_{lm} \in \mathrm{L}_{\mathrm{i}-1}$. Hence $z_{lm} \in \mathrm{L}_{\mathrm{i}-1} \cap [a_i, S(a_i)]$, so $(z_{lm})^{a_i} = (z_{lm} \to a_i) \wedge S(a_i) \in K_i \subseteq \mathrm{L}_{\mathrm{i}}$ and in consequence it belongs to B_i.

If $x_{lm} \in K_i$ then $x_{lm} = (x \to a_i) \wedge S(a_i)$, for some $x \in \mathrm{L}_{\mathrm{i}-1} \cap [a_i, S(a_i)]$. Thus $z_{lm} = (x \to a_i) \wedge S(a_i) = x^{a_i}$, so $(z_{lm})^{a_i} = (x^{a_i})^{a_i} = x \in \mathrm{L}_{\mathrm{i}-1} \cap [a_i, S(a_i)] \subseteq B_i$.

We have proved that $(z_{lm})^{a_i}$ is the complement of z_{lm} in B_i. An easy computation proves that $\bigwedge_l \bigvee_m (z_{lm})^{a_i}$ is the complement of z in B_i, and hence B_i is a Boolean algebra. Besides as B_i is a Boolean sublattice of L_{i},we conclude that $z^{a_i} = (z \to_i a_i) \wedge S(a_i)$ (by Lemma 3). □

Proposition 1. *With the notation and hypothesis of Lemma 5, it holds that, for every $i, j = 1, \ldots, k$ such that $i \leq j$, we have that $\mathrm{L}_{\mathrm{j}} \cap [a_i, S(a_i)]$ as sublattice of L_{j} is Boolean. In particular, for every $x \in \mathrm{L}_{\mathrm{j}} \cap [a_i, S(a_i)]$ we have that the complement of x in $\mathrm{L}_{\mathrm{j}} \cap [a_i, S(a_i)]$ is equal to x^{a_i}. Moreover, $x^{a_i} = (x \to_i a_i) \wedge S(a_i)$.*

Proof. Fix a natural number i, $i \leq k$. We will prove by induction that the property holds for every j such that $i \leq j \leq k$. The case $j = i$ follows from Lemma 5. Suppose that $\mathrm{L}_{\mathrm{h}} \cap [a_i, S(a_i)]$ is a Boolean algebra for some h such that $i \leq h < k$. We will show that $\mathrm{L}_{\mathrm{h}+1} \cap [a_i, S(a_i)]$ is a Boolean algebra.

A direct computation proves that the function $f_h : \mathrm{L_{h+1}} \cap [a_{h+1}, S(a_{h+1})] \to \mathrm{L_{h+1}} \cap [a_i, S(a_i)]$, given by $f_h(x) = (x \vee a_i) \wedge S(a_i)$, is a homomorphism of lattices. Let $z \in \mathrm{L_{h+1}} \cap [a_i, S(a_i)]$, so z can be written as $\bigvee_l \bigwedge_m x_{lm}$, for finitely many $x_{lm} \in \mathrm{L_h} \cup K_{h+1}$. In particular $z = \bigvee_l \bigwedge_m z_{lm}$, with $z_{lm} = (x_{lm} \vee a_i) \wedge S(a_i)$. To prove that $\mathrm{L_{h+1}} \cap [a_i, S(a_i)]$ is a Boolean algebra it is enough to prove that z_{lm} has complement in $\mathrm{L_{h+1}} \cap [a_i, S(a_i)]$.

If $x_{lm} \in \mathrm{L_h}$ then $z_{lm} \in \mathrm{L_h} \cap [a_i, S(a_i)]$. By inductive hypothesis we have that $\mathrm{L_h} \cap [a_i, S(a_i)]$ is a Boolean algebra, so $z_{lm}^{a_i} \in \mathrm{L_h} \cap [a_i, S(a_i)] \subseteq \mathrm{L_{h+1}} \cap [a_i, S(a_i)]$.

We consider the case $x_{lm} \in K_{h+1}$. In particular, $x_{lm} \in \mathrm{L_{h+1}} \cap [a_{h+1}, S(a_{h+1})]$. Hence $z_{lm} = f_h(x_{lm}) \in \mathrm{L_{h+1}} \cap [a_i, S(a_i)]$. We define the elements

$$\alpha = f_h(a_{h+1}),\ \omega = f_h(S(a_{h+1})),\ u = z_{lm} = f_h(x_{lm}),\ \overline{u} = f_h(x_{lm}^{a_{h+1}}),$$

$$v = (\omega^{a_i} \vee \overline{u}) \wedge \alpha^{a_i}.$$

The element v belongs to $\mathrm{L_{h+1}} \cap [a_i, S(a_i)]$. It is clear that $v \in [a_i, S(a_i)]$. Besides as $a_{h+1}, a_i, S(a_{h+1}), S(a_i) \in \mathrm{L_0}$ we have that $\alpha, \omega \in \mathrm{L_i}$, so $\alpha, \omega \in \mathrm{L_i} \cap [a_i, S(a_i)]$. Using Lemma 5 we have that $\alpha^{a_i}, \omega^{a_i} \in \mathrm{L_i} \cap [a_i, S(a_i)] \subseteq \mathrm{L_{h+1}} \cap [a_i, S(a_i)]$. As $\overline{u} \in \mathrm{L_{h+1}} \cap [a_i, S(a_i)]$ we have that $v \in \mathrm{L_{h+1}} \cap [a_i, S(a_i)]$. In the following we will prove that $u \vee v = S(a_i)$ and $u \wedge v = a_i$.

Using that $a_{h+1} \le x_{lm} \le S(a_{h+1})$ we have that

$$\alpha \le u \le \omega.$$

Then using that f_h is a homomorphism of lattices we have that

$$\begin{aligned} v \vee u &= ((\omega^{a_i} \vee \overline{u}) \wedge \alpha^{a_i}) \vee u = (\omega^{a_i} \vee \overline{u} \vee u) \wedge (\alpha^{a_i} \vee u) = (\omega^{a_i} \vee \omega) \wedge (\alpha^{a_i} \vee u) \\ &= S(a_i) \wedge (\alpha^{a_i} \vee u) \ge S(a_i) \wedge (\alpha^{a_i} \vee \alpha) = S(a_i) \wedge S(a_i) = S(a_i). \end{aligned}$$

Thus $u \vee v = S(a_i)$. On the other hand,

$$\begin{aligned} v \wedge u &= ((\omega^{a_i} \vee \overline{u}) \wedge \alpha^{a_i}) \wedge u = \alpha^{a_i} \wedge ((\omega^{a_i} \wedge u) \vee (\overline{u} \wedge u)) = \alpha^{a_i} \wedge ((\omega^{a_i} \wedge u) \vee \alpha) \\ &\le \alpha^{a_i} \wedge ((u^{a_i} \wedge u) \vee \alpha) = \alpha^{a_i} \wedge (a_i \vee \alpha) = \alpha^{a_i} \wedge \alpha = a_i. \end{aligned}$$

Thus $u \wedge v = a_i$. Therefore $\mathrm{L_{h+1}} \cap [a_i, S(a_i)]$ is a Boolean algebra. □

Theorem 6. *$\mathcal{SH}$ has the* FMP*.*

Proof. Let $\psi \in T(X)$, H a S-algebra and $v : T(X) \to H$ a homomorphism such that $v(\psi) \neq 1$. Let $\to$ and S be the implication and the successor of H respectively. We will prove that there is a finite S-algebra L and $w : T(X) \to \mathrm{L}$ a homomorphism such that $w(\psi) \neq 1$.

Let $\psi_1, \ldots, \psi_n$ be all the subformulas of ψ. For $i = 1, .., n$ we define $\hat{a}_i = v(\psi_i)$. In the following we will use the notation given in Definition 1.

If $B = \emptyset$ then we can take $L = L_0$; so let us assume in what follows that B is non-void.

Every $\mathrm{L_i}$ is a finite S-algebra. We will prove that $S^1(a_1) = S(a_1)$. As $S(a_1) \in \mathrm{L_0}$ we have that $S(a_1) \in \mathrm{L_1}$. Thus by Lemma 2 it holds that $S^1(a_1) \le S(a_1)$, so $S^1(a_1) \in \mathrm{L_1} \cap [a_1, S(a_1)]$. By Proposition 1 we have that

$$(S^1(a_1))^{a_1} = (S^1(a_1) \to_1 a_1) \wedge S(a_1) = a_1 \wedge S(a_1) = a_1. \tag{2}$$

Hence $S^1(a_1) = S(a_1)$.

In a similar way we can prove that $S^2(a_2) = S(a_2)$. Note that by Lemma 2 and Proposition 1 we have that $S(a_1) = S^2(a_1)$. Iterating this argument we obtain that $\mathrm{L} = \mathrm{L_k}$ is a finite bounded sublattice of H that satisfies the following two conditions:

(1) If $a, b, a \to b \in \mathrm{L}$ then $a \to b = a \to_k b$ (by Lemma 1).
(2) For every $i = 1, \ldots, k$, $S(a_i) = S^k(a_i)$.

Let V the set of propositional variables that appear in ψ. We define a function $w : X \to \mathrm{L}$ in the following way:

$$w(x) = \begin{cases} v(x) & \text{if } x \in V, \\ 0 & \text{if } x \notin V. \end{cases}$$

This function may be extended to a unique homomorphism $w : T(X) \to \mathrm{L}$. By an easy induction on formulas one can prove that $w(\psi_i) = v(\psi_i)$, for $i = 1, \ldots, n$. Therefore $w(\psi) = v(\psi) \neq 1$. $\square$

Take α and β in $T(X)$. Note that an equation $\alpha \approx \beta$ holds in a S-algebra H if and only if $\alpha \to \beta \approx 1$ holds in H; and the latter is equivalent to requiring that for any homomorphism $v : T(X) \to H$, $v(\alpha \to \beta) = 1$.

Corollary 7. *The variety $\mathcal{SH}$ is generated by its finite members.*

Proof. Let H be an S-algebra and let us assume that the equation $\alpha \approx \beta$ does not hold in H. By the previous remark, this implies the existence of a homomorphism $v : T(X) \to H$, such that $v(\alpha \to \beta) \neq 1$. By Theorem 6, there are a finite S-algebra L and a homomorphism $w : T(X) \to L$, such that $w(\alpha \to \beta) \neq 1$.

Using the previous remark again, this implies that $\alpha \approx \beta$ does not hold in the finite algebra L. $\square$

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J. L. Castiglioni
Departamento de Matemática,
Facultad de Ciencias Exactas, UNLP
Casilla de correos 172,
1900 La Plata, Argentina.
jlc@mate.unlp.edu.ar

H. J. San Martín
Departamento de Matemática,
Facultad de Ciencias Exactas, UNLP
Casilla de correos 172,
1900 La Plata, Argentina.
hsanmartin@mate.unlp.edu.ar

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