The magnetic flow on the manifold of oriented geodesics of a three dimensional space form

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Abstract

Let M be the three dimensional complete simply connected manifold of constant sectional curvature $0, 1$ or -1. Let $\mathcal{L}$ be the manifold of all (unparametrized) complete oriented geodesics of M, endowed with its canonical pseudo-Riemannian metric of signature $(2, 2)$ and Kähler structure J. A smooth curve in $\mathcal{L}$ determines a ruled surface in M.

We characterize the ruled surfaces of M associated with the magnetic geodesics of $\mathcal{L}$, that is, those curves σ in $\mathcal{L}$ satisfying $\nabla_{\dot{\sigma}}\dot{\sigma} = J\dot{\sigma}$. More precisely: a time-like (space-like) magnetic geodesic determines the ruled surface in M given by the binormal vector field along a helix with positive (negative) torsion. Null magnetic geodesics describe cones, cylinders or, in the hyperbolic case, also cones with vertices at infinity. This provides a relationship between the geometries of $\mathcal{L}$ and M.

Key words and phrases:[1] manifold of oriented geodesics, Hermitian symmetric space, magnetic flow, ruled surface, horospherical distribution

1 Introduction

For $\kappa = 0, 1, -1$, let M_κ be the three dimensional complete simply connected manifold of constant sectional curvature κ, that is, $\mathbb{R}^3$, $\mathbb{S}^3$ and the hyperbolic space $\mathbb{H}^3$. Let $\mathcal{L}_\kappa$ be the manifold of all (unparametrized) complete oriented geodesics of M_κ. We may think of an element c in $\mathcal{L}_\kappa$ as the equivalence class of unit speed geodesics $\gamma : \mathbb{R} \to M_\kappa$ with image c such that $\{\dot{\gamma}(s)\}$ is a positive basis of $T_{\gamma(s)}c$ for all s.

Let γ be a complete unit speed geodesic of M_κ and let $\mathcal{J}_\gamma$ be the space of all Jacobi fields along γ which are orthogonal to γ. There exists a well-defined canonical isomorphism

$$T_\gamma : \mathcal{J}_\gamma \to T_{[\gamma]}\mathcal{L}_\kappa, \qquad T_\gamma(J) = \left.\frac{d}{dt}\right|_0 [\gamma_t], \tag{1}$$

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where γ_t is any variation of γ by unit speed geodesics associated with J (see [8]).

A pseudo-Riemannian metric of signature $(2,2)$ can be defined on $\mathcal{L}_\kappa$ as follows [9]: For $X \in T_{[\gamma]}\mathcal{L}_\kappa$, the square norm $\|X\| = \langle X, X\rangle$ is well defined by

$$\|X\| = \langle \dot\gamma \times J, J'\rangle, \tag{2}$$

where $X = T_\gamma(J)$, the cross product $\times$ is induced by a fixed orientation of M_κ and J' denotes the covariant derivative of J along γ. Indeed, the right hand side of (2) is a constant function. In the following, for any vector X, we will denote $\|X\| = \langle X, X\rangle$ and $|X| = \sqrt{|\langle X, X\rangle|}$. Recall that X is null, time-like or space-like if $\|X\| = 0$, $\|X\| < 0$ or $\|X\| > 0$, respectively.

Let $[\gamma] \in \mathcal{L}_\kappa$ and let R_γ be the rotation in M_κ fixing γ through an angle of $\pi/2$. This rotation induces an isometry $\widetilde{R}_\gamma$ of $\mathcal{L}_\kappa$ whose differential at $[\gamma]$ is a linear isometry of $T_{[\gamma]}\mathcal{L}_\kappa$ squaring to $-\mathrm{id}$. This yields a complex structure J on $\mathcal{L}_\kappa$. With the metric defined above, $\mathcal{L}_\kappa$ is Kahler.

A *magnetic geodesic* σ of $\mathcal{L}_\kappa$ is a curve satisfying $\nabla_{\dot\sigma}\dot\sigma = J\dot\sigma$. These curves have constant speed, so they will be null, time-like or space-like.

A smooth curve in $\mathcal{L}_\kappa$ determines a ruled surface in M_κ. For $\kappa = 0, -1$, a generic geodesic of $\mathcal{L}_\kappa$ describes a helicoid in M_κ [5, 4, respectively]. Our purpose is to characterize the ruled surfaces in M_κ associated with the magnetic geodesics of $\mathcal{L}_\kappa$. For $v \in TM_\kappa$, γ_v denotes the geodesic of M_κ with initial velocity v.

Theorem 1 *A generic magnetic geodesic σ of $\mathcal{L}_\kappa$ describes the ruled surface in M_κ given by the binormal vector field of a helix. More precisely, σ is a time-like (space-like) magnetic geodesic of $\mathcal{L}_\kappa$ if and only if σ has the form*

$$\sigma(t) = [\gamma_{B(t)}], \tag{3}$$

where B is the binormal vector field of a helix in M_κ with curvature k, speed $1/k$ and positive (negative) torsion, for some $k > 0$.

Now we study null magnetic geodesics in $\mathcal{L}_{-1} = \mathcal{L}(\mathbb{H}^3)$. We recall some concepts related with the hyperbolic space (see for instance [3]).

Two unit speed geodesics γ and α of $\mathbb{H}^3$ are said to be asymptotic if there exists a positive constant C such that $d(\gamma(s), \sigma(s)) \le C$, $\forall s \ge 0$. Two unit vectors $v, w \in T^1\mathbb{H}^3$ are said to be asymptotic if the corresponding geodesics γ_v and γ_w have this property.

A point at infinity for $\mathbb{H}^3$ is an equivalence class of asymptotic geodesics of $\mathbb{H}^3$. The set of all points at infinity for $\mathbb{H}^3$ is denoted by $\mathbb{H}^3(\infty)$ and has a canonical differentiable structure diffeomorphic to the 2-sphere. The equivalence class represented by a geodesic γ is denoted by $\gamma(\infty)$, and the equivalence class represented by the oppositely oriented geodesic $s \mapsto \gamma(-s)$ is denoted by $\gamma(-\infty)$.

Given $v \in T^1\mathbb{H}^3$, the *horosphere* $H(v)$ is the limit of metric spheres $\{S_n\}$ in $\mathbb{H}^3$ that pass through the foot point of v as the centers $\{p_n\}$ of $\{S_n\}$ converge to $\gamma_v(\infty)$. Below we present a more precise definition.

Let $\psi^{\pm} : \mathcal{L}(\mathbb{H}^3) \to \mathbb{H}^3(\infty)$ be the smooth functions given by $\psi^{\pm}([\gamma]) = \gamma(\pm\infty)$ and let $\mathcal{D}^{\pm}$ be the distributions on $\mathcal{L}(\mathbb{H}^3)$ given by $\mathcal{D}^{\pm}_{[\gamma]} = \operatorname{Ker}(d\psi^{\pm}_{[\gamma]})$. These distributions are called *the horospherical distributions* on $\mathcal{L}(\mathbb{H}^3)$.

Cones with vertices at infinity: Let $x \in \mathbb{H}^3(\infty)$ and let $v_o \in T^1\mathbb{H}^3$ such that $\gamma_{v_o}(\pm\infty) \in x$. Let $t \mapsto v(t)$ be a curve in $T^1\mathbb{H}^3$ such that $v(0) = \pm v_o$, $v(t)$ is asymptotic to $\pm v_o$ for all $t \in \mathbb{R}$ and the foot points of $v(t)$ lie on a circle of geodesic curvature $\pm k$ (with $k > 0$) and speed $1/k$ in the horosphere determined by $\pm v_o$. Under these conditions we say that the curve in $\mathcal{L}(\mathbb{H}^3)$ given by $t \mapsto [\gamma_{\pm v(t)}]$ describes a *forward cone with vertex at* x (*for* $+$) or *a backward cone with vertex at* x (*for* $-$). These cones can be better visualized in the upper half space model of $\mathbb{H}^3$ (in particular $\mathbb{H}^3(\infty) = \{z = 0\} \cup \{\infty\}$): Let $\gamma^{\pm}_t(s) = \left(\frac{1}{k}\cos(t), \pm\frac{1}{k}\sin(t), e^{\pm s}\right)$. A curve σ in $\mathcal{L}(\mathbb{H}^3)$ describes a cone with forward (respectively, backward) vertex at ∞ if it is $Sl(2,\mathbb{C})$-congruent to $t \mapsto [\gamma^+_t]$ (respectively, to $t \mapsto [\gamma^-_t]$).

Theorem 2 *A null magnetic geodesic of $\mathcal{L}(\mathbb{H}^3)$ describes in $\mathbb{H}^3$ a cylinder, a cone with vertex at $p \in \mathbb{H}^3$ or a cone with vertex at infinity. More precisely, if σ is a curve in $\mathcal{L}(\mathbb{H}^3)$, then*

a) *σ is a null magnetic geodesic with $\dot{\sigma}(0) \in \mathcal{D}^{\pm}_{\sigma(0)}$ if and only if σ describes a cone with vertex at $\sigma(0)(\pm\infty)$ (forward for $+$ and backward for $-$);*

b) *σ is a null magnetic geodesic with $\dot{\sigma}(0) \notin \mathcal{D}^{\pm}_{\sigma(0)}$ if and only if σ either has the form*

$$\sigma(t) = [\gamma_{B(t)}], \tag{4}$$

where B is the binormal vector field of a helix h in $\mathbb{H}^3$ with curvature k, speed $1/k$ and zero torsion (in particular, h is contained in a totally geodesic surface S and B is normal to S and parallel along h), or σ has the form

$$\sigma(t) = [\gamma_{v(t)}], \tag{5}$$

where v is a curve with geodesic curvature k and speed $1/k$ in $T^1_p\mathbb{H}^3$, for some $p \in \mathbb{H}^3$, for certain $k > 0$.

Theorem 3 *The ruled surfaces associated with null magnetic geodesics of $\mathcal{L}_\kappa$ for $\kappa = 0, 1$ are described in an analogous manner as in the previous theorem, except that case* a) *is empty. Besides, for $\kappa = 1$, a null magnetic geodesic has simultaneously the forms* (4) *and* (5).

2 Preliminaries

For the simultaneous analysis of the three cases $\kappa = 0,1,-1$, we consider the standard presentation of M_κ as a submanifold of $\mathbb{R}^4$. That is, $\mathbb{R}^3 = \{(1, x) \in \mathbb{R}^4 \mid x \in \mathbb{R}^3\}$, $\mathbb{S}^3 = \{x \in \mathbb{R}^4 \mid |x|^2 = 1\}$ and $\mathbb{H}^3 = \{x \in \mathbb{R}^4 \mid -x_0^2 + x_1^2 + x_2^2 + x_3^2 = -1 \text{ and } x_0 > 0\}$.

Let G_κ be the identity component of the isometry group of M_κ, that is, $G_0 = SO_3 \ltimes \mathbb{R}^3$, $G_1 = SO_4$ and $G_{-1} = O_o(1,3)$. We consider the usual presentation of G_0 as a subgroup of $Gl_4(\mathbb{R})$. The group G_κ acts on $\mathcal{L}_\kappa$ as follows: $g \cdot [\gamma] = [g \circ \gamma]$. This action is transitive and smooth.

If we denote by $\mathfrak{g}_\kappa$ the Lie algebra of G_κ we have that

$$\mathfrak{g}_\kappa = \left\{ \begin{pmatrix} 0 & -\kappa x^t \\ x & B \end{pmatrix} \mid x \in \mathbb{R}^3,\ B \in so_3 \right\}.$$

Let γ_o be the geodesic in M_κ with $\gamma_o(0) = e_0$ and initial velocity $e_1 \in T_{e_0}M_\kappa$, where $\{e_0, e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{R}^4$. For A, $B \in \mathbb{R}^{2\times 2}$, let $\operatorname{diag}(A,B) = \begin{pmatrix} A & 0_2 \\ 0_2 & B \end{pmatrix}$, where 0_2 denotes the 2×2 zero matrix. Then the isotropy subgroup of G_κ at $[\gamma_o]$ is

$$H_\kappa = \{\operatorname{diag}(R_\kappa(t), B) \mid t \in \mathbb{R},\ B \in SO_2\},$$

where

$$R_0(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix},\quad R_1(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix},\quad R_{-1}(t) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}. \tag{6}$$

Let $j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The Lie algebra of H_κ is

$$\mathfrak{h}_\kappa = \{\operatorname{diag}(r_\kappa(t), sj) \mid s,t \in \mathbb{R}\},$$

where $r_\kappa(t) = \begin{pmatrix} 0 & -\kappa t \\ t & 0 \end{pmatrix}$. We may identify $\mathcal{L}_\kappa$ with G_κ/H_κ via the diffeomorphism

$$\phi : G_\kappa/H_\kappa \to \mathcal{L}_\kappa, \qquad \phi(gH_\kappa) = g \cdot [\gamma_o]. \tag{7}$$

For $x, y \in \mathbb{R}^2$ we denote $Z(x,y) = \begin{pmatrix} 0_2 & (-\kappa x, -y)^t \\ (x,y) & 0_2 \end{pmatrix}$. Let

$$\mathfrak{p}_\kappa = \left\{ Z(x,y) \in \mathfrak{g}_\kappa \mid x, y \in \mathbb{R}^2 \right\},$$

which is an $\operatorname{Ad}(H_\kappa)$-invariant complement of $\mathfrak{h}_\kappa$.

For $\kappa = 0, 1$, we consider on $\mathfrak{g}_\kappa$ the inner product such that $\mathfrak{h}_\kappa \perp \mathfrak{p}_\kappa$, $\|Z(x,y)\| = \det(x,y)$ and

$$\|\operatorname{diag}(r_\kappa(t), sj)\| = -ts.$$

(for $\kappa = 0$, we have learnt of this inner product from [6, page 499]). On $\mathfrak{g}_{-1}$ we consider the Killing form ($\mathfrak{h}_\kappa \perp \mathfrak{p}_\kappa$ also holds). For $\kappa = 0, 1, -1$, this inner product on $\mathfrak{g}_\kappa$ induces on G_κ a bi-invariant metric. Thus, there exists an unique pseudo-Riemannian metric on $\mathcal{L}_\kappa \simeq G_\kappa/H_\kappa$ such that $\pi : G_\kappa \to G_\kappa/H_\kappa$ is a pseudo-Riemannian submersion. For $\kappa = 0, 1$, this metric on $\mathcal{L}_\kappa$ coincides with the given in (2), see Lemma 5 b). For $\kappa = -1$, the metric on $\mathcal{L}_{-1}$ associated with the Killing form is different from the one in (2).

However, the magnetic geodesics of either metric on $\mathcal{L}_{-1}$ are the same. This follows since the geodesics are the same (see [8]), so the Levi-Civita connections coincide.

Let us call $A = \operatorname{diag}(0_2, j)$, which is in the center of $\mathfrak{h}_\kappa$. We have that ad_A is orthogonal and $\operatorname{ad}_A^2 = -\operatorname{id}$ in $\mathfrak{p}_\kappa$. Hence, ad_A induces a complex structure on G_κ/H_κ. A straightforward computation shows that it coincides, via ϕ in (7), with the complex structure given in the introduction. With the metric above and this complex structure, $\mathcal{L}_\kappa$ is a Hermitian symmetric space.

As a direct application of a result by Adachi, Maeda and Udagawa in [1] (see also [2], Remark 1) we have

Theorem 4 *Let σ be a magnetic geodesic of G_κ/H_κ with initial conditions $\sigma(0) = H_\kappa$ and $\dot\sigma(0) = X \in \mathfrak{p}_\kappa$. Then $\sigma(t) = \pi(\exp t(X + A))$.*

As we saw in (1), $\mathcal{J}_{\gamma_o}$ is isomorphic to $T_{[\gamma_o]}\mathcal{L}_\kappa \cong \mathfrak{p}_\kappa$. In the next Lemma we relate $\mathfrak{p}_\kappa$ and $\mathcal{J}_{\gamma_o}$ explicitly, involving the matrix A.

Lemma 5 *Let $Z = Z(x, y) \in \mathfrak{p}_\kappa$.*
a) *The Jacobi field $J(s) = \frac{d}{dt}\big|_0 \exp t(Z+A)\cdot\gamma_o(s)$ in $\mathcal{J}_{\gamma_o}$ is the unique one that satisfies $J(0) = (0, 0, x)^t$ and $J'(0) = (0, 0, y)^t$.*
b) *$T_{\gamma_o}(J) = d(\phi\circ\pi)Z$ and its norm is $\|d(\phi\circ\pi)Z\| = \det(x, y)$.*

Proof. For each κ, we consider the following parameterization of γ_o:

$$\begin{array}{rcll} \gamma_o(s) & = & (1, s, 0, 0), & \text{if} \quad \kappa = 0; \\ \gamma_o(s) & = & (\cos s, \sin s, 0, 0), & \text{if} \quad \kappa = 1; \\ \gamma_o(s) & = & (\cosh s, \sinh s, 0, 0), & \text{if} \quad \kappa = -1. \end{array}$$

Given $Z = Z(x, y) \in \mathfrak{p}_\kappa$, the Jacobi field along γ_o defined by $J(s) = \frac{d}{dt}\big|_0 \exp t(Z + A)\cdot\gamma_o(s)$ belongs to $\mathcal{J}_{\gamma_o}$, because for all $s \in \mathbb{R}$,

$$\langle J(s), \dot\gamma_o(s)\rangle = \langle (Z + A)(\gamma_o(s)), \dot\gamma_o(s)\rangle = 0\,,$$

since $(Z + A)(\gamma_o(s))$ is orthogonal to e_0 and e_1, while $\dot\gamma_o(s)$ has non zero components only in these two directions.

One verifies easily that $J(0) = (Z + A)(e_0) = (0, 0, x)^t$. On the other hand,

$$\begin{aligned} J'(0) &= \left.\frac{D}{\partial s}\right|_0 \left.\frac{\partial}{\partial t}\right|_0 \exp t(Z + A)\cdot\gamma_o(s) \\ &= \left.\frac{D}{\partial t}\right|_0 \exp t(Z + A)(e_1) = (Z + A)(e_1) = (0, 0, y)^t\,. \end{aligned}$$

Besides,

$$\begin{aligned} T_{\gamma_o}(J) &= \left.\frac{d}{dt}\right|_0 [\exp t(Z + A)\cdot\gamma_o] = \left.\frac{d}{dt}\right|_0 \phi(\exp t(Z + A)H_\kappa) \\ &= \left.\frac{d}{dt}\right|_0 \phi(\pi(\exp t(Z + A))) = d\phi\circ d\pi Z, \end{aligned}$$

where the last equality holds since $A \in \mathfrak{h}_\kappa$. Finally, the norm (2) of $d(\phi \circ \pi)Z$ equals

$$\|d(\phi \circ \pi)Z\| = \langle \dot{\gamma}_o(0) \times J(0), J'(0) \rangle = \det(x, y)$$

and the assertions of b) are verified. □

Let $Z(x,y) \in \mathfrak{p}_\kappa$ and let $h = \operatorname{diag}(R_\kappa(t), B) \in H_\kappa$, where $B \in SO_2$ and

$$R_\kappa(t) = \begin{pmatrix} c_\kappa(t) & -\kappa s_\kappa(t) \\ s_\kappa(t) & c_\kappa(t) \end{pmatrix}$$

is as in (6). Then $\mathrm{Ad}(h)Z(x,y) = Z(Bx_t, By_t)$, where

$$x_t = c_\kappa(t)x - s_\kappa(t)y, \quad y_t = \kappa s_\kappa(t)x + c_\kappa(t)y.$$

We denote by ϵ_1 and ϵ_2 the vectors of the canonical basis of $\mathbb{R}^2$.

Lemma 6 *Let $Z(x,y) \neq 0$ in $\mathfrak{p}_\kappa$.*
a) *If $\{x,y\}$ is a linearly independent set of $\mathbb{R}^2$, then there exists $h \in H_\kappa$ such that $\mathrm{Ad}(h)Z(x,y) = Z(a\epsilon_1, b\epsilon_2)$, with $a > 0$ and $b \neq 0$, for $\kappa = 0, \pm 1$.*
b) *If $\kappa = 0, 1$ and $\{x,y\}$ is a linearly dependent set of $\mathbb{R}^2$, then there exists $h \in H_\kappa$ such that either $\mathrm{Ad}(h)Z(x,y) = Z(0, b\epsilon_2)$, with $b \neq 0$, or $\mathrm{Ad}(h)Z(x,y) = Z(a\epsilon_1, 0)$, with $a > 0$. This is true for $\kappa = -1$ if in addition $|x| \neq |y|$.*
c) *For $\kappa = 1$, there exists $h \in H_\kappa$ such that $\mathrm{Ad}(h)Z(\epsilon_1, 0) = Z(0, \epsilon_2)$.*

Proof. For the proof of a), as $\{x,y\}$ is a linearly independent set, then for $\kappa = 0, \pm 1$ there exists $t \in \mathbb{R}$ such that $\langle x_t, y_t \rangle = 0$. Indeed, for each κ, this is equivalent to fact that the equation

$$\begin{array}{rll} c_3 - c_2 t = 0 & \text{if} & \kappa = 0; \\ \frac{1}{2}(c_1 - c_2)\sin(2t) + c_3\cos(2t) = 0 & \text{if} & \kappa = 1; \\ -\frac{1}{2}(c_1 + c_2)\sinh(2t) + c_3\cosh(2t) = 0 & \text{if} & \kappa = -1 \end{array}$$

has a real solution, where $c_1 = \langle x, x \rangle$, $c_2 = \langle y, y \rangle$ and $c_3 = \langle x, y \rangle$. But the linear independence of x and y determines the existence of the solution in each case. Then, we can take $B \in SO_2$ such that $Bx_t = a\epsilon_1$, with $a > 0$ and $By_t = b\epsilon_2$, with $b \neq 0$. Therefore the isometry $h = \operatorname{diag}(R_\kappa(t), B) \in H_\kappa$ satisfies $\mathrm{Ad}(h)Z(x,y) = Z(a\epsilon_1, b\epsilon_2)$.

For the proof of b), first we suppose that $x = 0$ or $y = 0$ (but not both zero since $Z(x,y) \neq 0$). Let $B \in SO_2$ such that $Bx = a\epsilon_1$ with $a > 0$, if $x \neq 0$, and in the case that $y \neq 0$, let $B \in SO_2$ such that $By = b\epsilon_2$, with $b \neq 0$. Then we can take $h = \operatorname{diag}(I, B) \in H_\kappa$.

Now, let $x \neq 0$ and $y \neq 0$. So $x = \lambda y$ or $y = \lambda x$, with $\lambda \neq 0$. We suppose that $y = \lambda x$ (for $x = \lambda y$ the argument is similar). In the cases $\kappa = 0, 1$ there exists $t \in \mathbb{R}$ such that $x_t = 0$. In fact, from the hypothesis and some computations, $t \in \mathbb{R}$ is obtained by solving

$$1 - \lambda t = 0, \quad \text{if } \kappa = 0 \qquad \text{and} \qquad \cos t - \lambda \sin t = 0, \quad \text{if } \kappa = 1.$$

Thus, taking $B \in SO_2$ such that $By_t = b\epsilon_2$ (with $b \neq 0$ as $y_t \neq 0$), we have that $h = \operatorname{diag}(R_\kappa(t), B) \in H_\kappa$ satisfies $\operatorname{Ad}(h)Z(x,y) = Z(0, b\epsilon_2)$.

For $\kappa = -1$, as in the cases $\kappa = 0, 1$, we find $t \in \mathbb{R}$ such that either $x_t = 0$ or $y_t = 0$ by solving

$$\cosh t - \lambda \sinh t = 0, \qquad \text{and} \qquad -\sinh t + \lambda \cosh t = 0,$$

respectively. But these equations have a solution if and only if $\lambda \neq \pm 1$. That is, if and only if $|x| \neq |y|$. Hence, taking $B \in SO_2$ such that either $By_t = b\epsilon_2$ or $Bx_t = a\epsilon_1$ (with $a > 0$; here again we have that $x_t \neq 0$), as appropriate. Then $h = \operatorname{diag}(R_{-1}(t), B) \in H_{-1}$ is as desired in this case.

For part c), we observe that $h = \operatorname{diag}(R_1(\pi/2), B) \in H_1$, where $B \in SO_2$ takes ϵ_1 to ϵ_2, satisfies $\operatorname{Ad}(h)Z(\epsilon_1, 0) = Z(0, \epsilon_2)$. □

Remark. The previous lemma corresponds, geometrically, with the fact of finding $s \in \mathbb{R}$ at which the Jacobi field associated with $Z(x,y)$ (given by Lemma 5) and its covariant derivative are orthogonal.

Recall that if h is a regular curve in M_κ of constant speed a, then the Frenet frame of h is

$$T(t) = \tfrac{1}{a}\,\dot{h}(t), \qquad N(t) = \dot{h}'(t) / \left|\dot{h}'(t)\right|, \qquad B(t) = T(t) \times N(t) \tag{8}$$

(here the prime denotes the covariant derivative along h), and its curvature and torsion are given by

$$k(t) = \tfrac{1}{a^2}\,\left|\dot{h}'(t)\right|, \qquad \tau(t) = -\tfrac{1}{a}\,\langle B'(t), N(t)\rangle. \tag{9}$$

For each $g \in G_\kappa$ we have that g is an isometry of $\mathcal{L}_\kappa$ and preserves the Hermitian structure. Hence, g takes magnetic geodesics to magnetic geodesics.

3 Time- and space-like magnetic geodesics

Proof of Theorem 1. Let $Z \in \mathfrak{p}_\kappa$ be the initial velocity of σ, with $\|Z\| \neq 0$. First, we consider the case $Z = Z(a\epsilon_1, b\epsilon_2)$, with $a > 0$ and $b \neq 0$.

For each $t \in \mathbb{R}$, let $\alpha(t) = \exp t(Z + A)$. By Theorem 4 and the diffeomorphism ϕ in (7), we know that $\sigma(t) = \alpha(t) \cdot [\gamma_o]$, that is, $\sigma(t) = [\alpha(t) \cdot \gamma_o]$.

Let h be the curve in M_κ given by $h(t) = \alpha(t)(e_0)$. As α is a one-parameter subgroup of isometries of M_κ, we have that h is a curve with constant curvature and torsion, thus h is a helix in M_κ.

Let us see that $\sigma(t) = [\gamma_{B(t)}]$, where $B(t)$ is the binormal field of h. For each $t \in \mathbb{R}$, the initial velocity of the geodesic $\alpha(t) \cdot \gamma_o$ is $d(\alpha(t))(e_1)$, hence $\sigma(t) = [\gamma_{d(\alpha(t))(e_1)}]$. Then, we have to verify that $B(t) = d(\alpha(t))(e_1)$, for all $t \in \mathbb{R}$. Since $\alpha(t)$ is an isometry that preserves the helix and takes the Frenet frame at $t = 0$ to the Frenet frame at t, is suffices to show that $B(0) = e_1$.

By the usual identifications, since $\alpha(t)$ is a linear transformation, we can write $d(\alpha(t))(e_1) = \alpha(t)(e_1)$, so

$$\dot{h}(t) = \alpha(t)((Z+A)e_0) \qquad \text{and} \qquad \dot{h}'(t) = [\alpha(t)((Z+A)^2 e_0)]^{\mathrm{T}},$$

where T denotes the tangent projection. Since

$$\dot{h}(0) = (Z+A)e_0 = ae_2,$$

$$\dot{h}'(0) = \left[(Z+A)^2 e_0\right]^T = [-\kappa a^2 e_0 + ae_3]^T = ae_3$$

and $\alpha(t)$ is an isometry, we have $\left|\dot{h}(t)\right| = a = \left|\dot{h}'(t)\right|$. By the computation before and (8) we obtain

$$B(0) = \frac{1}{a^2}\,\dot{h}(0) \times \dot{h}'(0) = e_1.$$

Since $B'(t) = [\alpha(t)((Z+A)e_1)]^T$ (recall that we have just proven that $B(t) = \alpha(t)(e_1)$), then $B'(0) = be_3$. Besides, using (8) and the previous computations it follows that $N(0) = e_3$. Therefore, by (9) we have that the curvature and torsion of h are equal to

$$k = 1/a, \quad \tau = -b/a. \tag{10}$$

The assertion regarding the sign of the torsion is immediate from Lemma 5 b) and (10). Thus, the theorem is proved in this particular case.

Now, let σ be a magnetic geodesic with $\sigma(0) = [\gamma]$ and initial velocity with non zero norm. Since G_κ acts transitively on $\mathcal{L}_\kappa$, there is an isometry g such that $g \cdot [\gamma] = [\gamma_o]$. So, the magnetic geodesic $g \cdot \sigma$ also has initial velocity with non zero norm and $g \cdot \sigma(0) = [\gamma_o]$. By Lemma 5 b), if $d(\phi \circ \pi)Z(x,y)$ is the initial velocity of $g \cdot \sigma$, we have that the vectors $\{x, y\}$ are linearly independent. Then, by Lemma 6 a), there exists $h \in H_\kappa$ such that $\mathrm{Ad}(h)Z(x,y) = Z(a\epsilon_1, b\epsilon_2)$, with $a > 0$ and $b \neq 0$. Since $((h \circ g) \cdot \sigma)'(0) = d(\phi \circ \pi)(\mathrm{Ad}(h)Z(x,y))$, the curve $(h \circ g) \cdot \sigma$ is a magnetic geodesic of the type studied above. Therefore, σ has the form (3).

Conversely, let h be a helix in M_κ with curvature $k > 0$, non zero torsion τ and speed $1/k$. Let $\{T, B, N\}$ be the Frenet frame of h. As M_κ is a simply connected manifold of constant curvature, we have that there exists an isometry g of M_κ preserving the orientation such that $g(h(0)) = e_0$ and its differential at $h(0)$ takes $B(0)$ to e_1, $T(0)$ to e_2 and $N(0)$ to e_3.

Let $a = 1/k$ and $b = -\tau/k$. Let $Z = Z(a\epsilon_1, b\epsilon_2) \in \mathfrak{p}_\kappa$. We consider, for each $t \in \mathbb{R}$, $\alpha(t) = \exp t(Z+A)$. According to computations from the first part of the proof, both helices have initial position e_0, curvature k, torsion τ, speed $1/k$ and the same Frenet frame at $t = 0$. Hence $(g \circ h)(t) = \alpha(t)e_0$. So, if we call $\bar{B}$ the binormal field of $g \circ h$, we have that $\bar{B}(t) = d(\alpha(t))e_1$, for all t. Finally, since the curve $[\gamma_{\bar{B}(t)}]$ is a magnetic geodesic in $\mathcal{L}_\kappa$ and

$$[\gamma_{B(t)}] = [\gamma_{dg^{-1}\bar{B}(t)}] = g^{-1} \cdot [\gamma_{\bar{B}(t)}],$$

we obtain that $[\gamma_{B(t)}]$ is a magnetic geodesic. $\square$

4 Null magnetic geodesics

We deal first with the hyperbolic case. We use the notation given in the introduction and we recall from [3] certain properties of horospheres and related concepts. To simplify the notation we omit the subindex $\kappa = -1$.

Let γ be a geodesic of $\mathbb{H}^3$. Then, for each $p \in \mathbb{H}^3$ there exists a unique unit speed geodesic α of $\mathbb{H}^3$ such that $\alpha(0) = p$ and α is asymptotic to γ. Let $v \in T^1\mathbb{H}^3$. If p is any point of $\mathbb{H}^3$, then $v(p)$ denotes the unique unit tangent vector at p that is asymptotic to v. The Busemann function $f_v : \mathbb{H}^3 \to \mathbb{R}$ is defined by

$$f_v(p) = \lim_{t \to +\infty} d(p, \gamma_v(t)) - t,$$

and satisfies $\mathrm{grad}_p\,(f_v) = -v(p)$. The *horosphere* determined by v is given by

$$H(v) = \{q \in M : f_v(q) = 0\}.$$

The Jacobi vector fields orthogonal to $\dot{\gamma_o}$ have the form

$$J(s) = e^s U(s) + e^{-s} V(s), \tag{11}$$

where U and V are parallel vector fields along γ_o and orthogonal to $\dot{\gamma_o}$.

A Jacobi vector field Y along a geodesic γ of $\mathbb{H}^3$ is said to be *stable* (*unstable*) if there exists a constant $c > 0$ such that

$$|Y(s)| \ \leq c \qquad \forall s \geq 0 \qquad (\forall s \leq 0).$$

In what follows we shall denote by $\hat{\pi}$ the canonical projection from $T\mathbb{H}^3$ onto $\mathbb{H}^3$. We recall that in the introduction we have defined the smooth maps $\psi^\pm : \mathcal{L}\,(\mathbb{H}^3) \to \mathbb{H}^3(\infty)$ by $\psi^\pm[\gamma] = \gamma(\pm\infty)$ and the distributions $\mathcal{D}^\pm$ in $\mathcal{L}\,(\mathbb{H}^3)$ given by $\mathcal{D}^\pm_{[\gamma]} = \mathrm{Ker}\ (d\psi^\pm_{[\gamma]})$. We need to relate the distributions $\mathcal{D}^\pm$ with distributions $\bar{\mathcal{E}}^\pm$ and $\mathcal{E}^\pm$ on G and $T^1\mathbb{H}^3$, respectively.

Let $\bar{\mathcal{E}}^\pm$ be the left invariant distribution on G defined at $I \in G$ by

$$\bar{\mathcal{E}}^\pm_I = \left\{ Z(u, \mp u) \in \mathfrak{p} \mid u \in \mathbb{R}^2 \right\}.$$

As the canonical action of G on $T^1\mathbb{H}^3$ is transitive, the projection $\bar{p} : G \to T^1\mathbb{H}^3$ given by $\bar{p}(g) = dg_{e_0} e_1$ is a submersion. Since given $v \in T^1\mathbb{H}^3$ there exists $g \in G$ such that $\bar{p}(g) = v$, we define:

$$\mathcal{E}^\pm(v) = (d\bar{p}\ \bar{\mathcal{E}}^\pm)(\bar{p}(g)) = d\bar{p}_g(\bar{\mathcal{E}}^\pm_g).$$

We have that $\mathcal{E}^\pm$ determines a well defined distribution on $T^1\mathbb{H}^3$, which is called the *horospherical distribution* on $T^1\mathbb{H}^3$. This distribution has the following property: if $t \mapsto v(t)$ is a curve in $T^1\mathbb{H}^3$ tangent to the distribution $\mathcal{E}^\pm$, then $\hat{\pi}(v(t))$ is in the horosphere $H\,(\pm v\,(0))$.

Lemma 7 *Let $Z \in \bar{\mathcal{E}}^\pm_I$. For each $t \in \mathbb{R}$, let $\gamma^\pm_t(s) = \exp t\,(Z + A) \cdot \gamma_o(\pm\, s)$. Then the geodesics $\gamma^\pm_t$ are asymptotic to each other for all $t \in \mathbb{R}$.*

Proof. Let J be the Jacobi vector field associated with the variation by geodesics $t \mapsto \gamma_t^{\pm}$. By Lemma 5 a), $J(0) = -J'(0)$. Hence, by (11) we have that $J(s) = e^{-s}U(s)$, where U is a parallel vector field along γ_o orthogonal to $\dot{\gamma_o}$. Thus, J is a stable vector field, that is, there exists $c > 0$ such that $|J(s)| \leq c \; \forall s \geq 0$.

We have to show that given t_0, $t_1 \in \mathbb{R}$ with $t_0 < t_1$, there exists $N > 0$ such that

$$d(\gamma_{t_0}^{\pm}(s), \gamma_{t_1}^{\pm}(s)) \leq N \qquad \forall \; s \geq 0.$$

For fixed s,

$$d(\gamma_{t_0}^{\pm}(s), \gamma_{t_1}^{\pm}(s)) \leq \text{length}\left([t_0, t_1] \ni t \longmapsto \gamma_t^{\pm}(s)\right) = \int_{t_0}^{t_1} \left| \frac{d}{dt} \gamma_t^{\pm}(s) \right| dt.$$

For each $t \in \mathbb{R}$, let $J_t(s) = \frac{d}{dt}\gamma_t^{\pm}(s)$. We observe that $J_{t'+t}(s) = d\exp(t'Z)J_t(s)$ for all t, t'. Since $\exp(t'Z)$ is an isometry, we have $|J_t(s)| = |J(s)|$. Therefore,

$$\int_{t_0}^{t_1} |J_t(s)| dt = \int_{t_0}^{t_1} |J(s)| dt \leq c(t_1 - t_0)$$

for all $s \geq 0$. Then, we may take $N = c(t_1 - t_0) > 0$. □

We consider the projection $p : T^1\mathbb{H}^3 \to \mathcal{L}(\mathbb{H}^3)$, $p(v) = [\gamma_v]$. We call $\bar{\mathcal{D}}^{\pm}$ the distribution on $\mathcal{L}(\mathbb{H}^3)$ p-related with $\mathcal{E}^{\pm}$ (well defined). More specifically, given $[\gamma] \in \mathcal{L}(\mathbb{H}^3)$ and $v \in T^1\mathbb{H}^3$ such that $p(v) = [\gamma]$,

$$\bar{\mathcal{D}}^{\pm}([\gamma]) = dp_v \; \mathcal{E}_v^{\pm}.$$

Proposition 8 *Let $\mathcal{D}^{\pm}$ and $\bar{\mathcal{D}}^{\pm}$ be the distributions on $\mathcal{L}(\mathbb{H}^3)$ defined above. Then $\mathcal{D}^{\pm} = \bar{\mathcal{D}}^{\pm}$.*

Proof. Since $\mathcal{D}^{\pm}$ and $\bar{\mathcal{D}}^{\pm}$ are G-invariant, it is enough to show $\mathcal{D}_{[\gamma_o]}^{\pm} = dp_{(e_0,e_1)}(\mathcal{E}_{(e_0,e_1)}^{\pm})$ (we observe that $\bar{p}(I) = (e_0, e_1)$ and $p(e_0, e_1) = [\gamma_o]$).

Let $Z \in \bar{\mathcal{E}}_I^{\pm}$. We take the curve in $\mathcal{L}(\mathbb{H}^3)$ given by $\alpha(t) = \exp tZ \cdot [\gamma_o]$. As $\alpha(t) = p \circ \bar{p}(\exp tZ)$, we have that $\alpha(0) = [\gamma_o]$ and $\dot{\alpha}(0) = d(p \circ \bar{p})_I Z$. That is, $\dot{\alpha}(0) \in dp_{(e_0,e_1)}(\mathcal{E}_{(e_0,e_1)}^{\pm})$. Besides,

$$\left.\frac{d}{dt}\right|_0 \exp tZ \cdot \gamma_o(s) = \left.\frac{d}{dt}\right|_0 \exp t(Z + A) \cdot \gamma_o(s), \tag{12}$$

since both Jacobi fields have the same initial conditions. Hence, Lemma 7 applies to the geodesics $\gamma_t^{\pm}(s) = \exp tZ \cdot \gamma_o(\pm s)$. Thus, $\psi^{\pm} \circ \alpha$ is constant. Then $(d\psi^{\pm})_{[\gamma_o]}(\dot{\alpha}(0)) = 0$, that is, $\dot{\alpha}(0) \in \mathcal{D}_{[\gamma_o]}^{\pm}$.

On the other hand, let $\varphi : T_{e_0}^1\mathbb{H}^3 \to \mathcal{L}(\mathbb{H}^3)$, $\varphi(v) = [\gamma_v]$, be the submanifold whose image $\mathcal{L}_{e_0}(\mathbb{H}^3)$ consists of all the oriented geodesics passing through e_0. Besides, $H(\infty)$ is a manifold with the differentiable structure (well defined) such that $F_{e_0} : T_{e_0}^1\mathbb{H}^3 \to$

$H(\infty)$ given by $F_{e_0}(v) = \gamma_v(\infty)$ is a diffeomorphism. Then, since $\psi^+|_{\mathcal{L}_{e_0}(\mathbb{H}^3)} \circ \varphi = F_{e_0}$, we have that $(d\psi^+)_{[\gamma_o]}$ is surjective. Now, $(d\psi^-)_{[\gamma_o]}$ is also surjective because ψ^- is the composition of ψ^+ with the diffeomorphism of $\mathcal{L}(\mathbb{H}^3)$ assigning $[\gamma^{-1}]$ to $[\gamma]$. Therefore, $\dim \mathcal{D}^{\pm}_{[\gamma_o]} = \dim \bar{\mathcal{D}}^{\pm}_{[\gamma_o]}$ and equality follows. □

The word *cylinder* in the statement of Theorem 2 refers to a ruled surface determined by a parallel vector field along a curve c of constant geodesic curvature k contained in a totally geodesic surface in M_κ (and normal to it), as explained. For $\kappa = -1$, this ruled surface is diffeomorphic to $S^1 \times \mathbb{R}$ if $|k| > 1$; otherwise it is diffeomorphic to a plane.

Proof of Theorem 2 a). By Lemma 5 b), we have that every element of $\mathcal{D}^{\pm}_{[\gamma]}$ is null. As G acts transitively on $\mathcal{L}(\mathbb{H}^3)$ and by the G-invariance of the horospherical distributions, we may suppose without loss of generality that $\sigma(0) = [\gamma_o]$, hence $\dot{\sigma}(0) \in \mathcal{D}^{\pm}_{[\gamma_o]}$. By Proposition 8, there exists $Z \in \bar{\mathcal{E}}^{\pm}_I$ such that $\dot{\sigma}(0) = (dp)_{(e_0,e_1)}(d\bar{p})_I Z$. Thus, by Theorem 4, $\sigma(t) = [\exp t(Z + A) \cdot \gamma_o]$.

We assume that $Z \in \bar{\mathcal{E}}^{+}_I$. Let us show that σ describes a forward cone with vertex at $\gamma_o(+\infty)$. In a similar way, if $Z \in \bar{\mathcal{E}}^{-}_I$, then σ describes a backward cone with vertex at $\gamma_o(-\infty)$.

We consider the geodesics $\gamma_t(s) = \exp t(Z + A) \cdot \gamma_o(s)$ of $\mathbb{H}^3$. As $Z \in \bar{\mathcal{E}}^{+}_I$, by Lemma 7, we have that the geodesics γ_t are asymptotic to each other for all t. Hence, $z(t) = \dot{\gamma}_t(0)$ is a curve in $T^1\mathbb{H}^3$ of asymptotic vectors to e_1.

Let $c(t) = \hat{\pi}(z(t)) = \exp t(Z + A)(e_0)$. In order to see that $c(t) \in H(e_1)$ for all t, we observe that

$$\frac{d}{dt} f_{e_1}(c(t)) = (df_{e_1})_{c(t)} \dot{c}(t) = \langle \mathrm{grad}_{c(t)}(f_{e_1}), \dot{c}(t) \rangle. \tag{13}$$

Since $\mathrm{grad}_p(f_v) = -v(p)$ we have that

$$\mathrm{grad}_{c(t)}(f_{e_1}) = -z(t) = -d\left(\exp t(Z + A)\right) e_1.$$

On the other hand,

$$\dot{c}(t) = d(\exp t(Z + A))(Z + A)e_0.$$

Since $\exp t(Z+A)$ is an isometry and observing that $(Z+A)e_0$ and e_1 are perpendicular ($Z \in \bar{\mathcal{E}}^{+}_I$), it follows that the expression in (13) is equal to $-\langle e_1, (Z + A)(e_0) \rangle = 0$. Then, $f_{e_1}(c(t)) = f_{e_1}(e_0) = 0$ for all t, that is, $c(t) \in H(e_1)$ for all t.

Now, as c is the orbit through e_0 of a one-parameter subgroup of isometries of G preserving $H(e_1)$, its geodesic curvature and speed are constant. If $Z = Z(u, -u)$ for certain $0 \neq u \in \mathbb{R}^2$, we obtain that the speed of c is $|u|$. For each $v \in T^1H^3$ we consider on $H(v)$ the orientation given by $-\mathrm{grad}\, f_v$. The geodesic curvature of c is then

$$k = \langle -\mathrm{grad}_{e_0}(f_{e_1}), \dot{c}(0) \times \dot{c}'(0) \rangle / |u|^3 = 1/|u|,$$

since $\dot{c}(0) = (Z + A)e_0$ and $\dot{c}'(0) = \left((Z+A)^2 e_0\right)^{\mathrm{T}}$. As for each $v \in T^1\mathbb{H}^3$, $H(v)$, with the induced metric of $\mathbb{H}^3$, is isometric to $\mathbb{R}^2$, we have that $c(t)$ runs along a circle on $H(e_1)$ of geodesic curvature $k = 1/|u| > 0$ and speed $1/k = |u|$.

Besides, $\sigma(t) = [\gamma_{z(t)}]$. Thus we have that all conditions are satisfied in order to assert that σ describes a forward cone with vertex at $\gamma_o(+\infty)$.

Conversely, let σ be a curve in $\mathcal{L}(\mathbb{H}^3)$ that describes a forward cone with vertex at infinity. As G acts transitively on the positively oriented frame bundle, and also each element of G takes horospheres to horospheres, preserving their orientation, we may suppose that $\sigma(t) = [\gamma_{v(t)}]$, where $v(t)$ is a curve in $T^1\mathbb{H}^3$ of asymptotic vectors to $v(0) = e_1$ and $c(t) = \hat{\pi}(v(t))$ is a curve of geodesic curvature k and speed $1/k$ in $H(e_1)$ with $\dot{c}(0) = \frac{1}{k}e_2$, for some $k > 0$. Let $Z = Z(\frac{1}{k}\epsilon_1, -\frac{1}{k}\epsilon_1) \in \bar{\mathcal{E}}_I^+$. We define

$$\bar{c}(t) = \exp t(Z + A)(e_0) \qquad \text{and} \qquad \bar{v}(t) = d\left(\exp t(Z + A)\right)(e_1).$$

We showed above that $\bar{c}(t)$ is a curve of geodesic curvature k and speed $\frac{1}{k}$ in $H(e_1)$. Moreover, $\bar{c}(0) = e_0$ and the initial velocity of $\bar{c}$ is $\frac{1}{k}e_2$. So, we obtain that $\bar{c} = c$. This implies, together with the identities $\hat{\pi} \circ \bar{v} = \bar{c}$ and $\hat{\pi} \circ v = c$, that $\hat{\pi} \circ \bar{v} = \hat{\pi} \circ v$.

According to the first part of the proof, $\bar{v}$ and v are curves of asymptotic vectors to e_1. Hence, $-\bar{v}(t) = \mathrm{grad}_{\bar{c}(t)}(f_{e_1}) = -v(t)$. Therefore, $[\gamma_{\bar{v}(t)}] = [\gamma_{v(t)}]$, which is a null magnetic geodesic with initial velocity in the horospherical distribution since $[\gamma_{v(t)}] = [\exp t(Z + A) \cdot \gamma_o]$. $\square$

Proof of Theorem 2 b). We suppose first that σ is a null magnetic geodesic such that $\sigma(0) = [\gamma_o]$ and $\dot{\sigma}(0) = d(\phi \circ \pi)Z(a\epsilon_1, 0)$, with $a > 0$. The expression (4) and the relation between the speed and curvature of h are obtained as in the prove of Theorem 1. By (10) we know that the torsion of h is $\tau = -b/a = 0$ (since $b = 0$). Thus h is contained in a totally geodesic surface S of $\mathbb{H}^3$ and B is normal to S.

Now, we suppose that $\dot{\sigma}(0) = d(\phi \circ \pi)Z$, where $Z = Z(0, b\epsilon_2)$ with $b \neq 0$. By Theorem 4 we have that $\sigma(t) = [\alpha(t) \cdot \gamma_o]$, where $\alpha(t) = \exp t(Z + A)$. Since $Z + A$ is in the Lie algebra of the isotropy subgroup H of G at $e_0 \in \mathbb{H}^3$, we get that $\alpha(t)$ fixes e_0. Moreover, if v is the curve in $T^1_{e_0}\mathbb{H}^3$ given by $v(t) = d(\alpha(t))\, e_1$, then

$$\sigma(t) = [\alpha(t) \cdot \gamma_o] = [\gamma_{v(t)}],$$

since the initial velocity of the geodesic $\alpha(t) \cdot \gamma_o$ is $v(t)$, for each $t \in \mathbb{R}$.

Furthermore, as v is the orbit through e_1 of a one-parameter subgroup of H (the canonical differential action of G on $T^1_{e_0}\mathbb{H}^3$), then v has constant speed and constant geodesic curvature in $T^1_{e_0}\mathbb{H}^3 \cong \mathbb{S}^2$. Easy computations yield

$$\dot{v}(0) = (0, 0, b)^t \qquad \text{and} \qquad \ddot{v}(0) = \left(-b^2, -b, 0\right)^t.$$

So, the speed of v is $|b|$ and its geodesic curvature is

$$k = \langle v(0), \dot{v}(0) \times \ddot{v}(0) \rangle / |b|^3 = 1/|b|$$

(we consider the orientation of the sphere given by the unit normal field pointing outwards). Thus, v is a curve in $T^1_{e_0}\mathbb{H}^3$ of geodesic curvature $k > 0$ and speed $1/k$. Consequently, σ has the form (5).

Now, let σ be a null magnetic geodesic such that $\sigma(0) = [\gamma]$ and $\dot{\sigma}(0) \notin \mathcal{D}^{\pm}_{[\gamma]}$. As G acts transitively on $\mathcal{L}(\mathbb{H}^3)$ and by the G-invariance of the horospherical distributions, we may suppose that $\sigma(0) = [\gamma_o]$ and $\dot{\sigma}(0) \notin \mathcal{D}^{\pm}_{[\gamma_o]}$. Let $Z = Z(x, y) \in \mathfrak{p}$ such that $\dot{\sigma}(0) = d(\phi \circ \pi)Z$. By Lemma 5 b), as the norm of the initial velocity of σ is zero, we have that x and y are linearly dependent, and since $d(\phi \circ \pi)Z \notin \mathcal{D}^{\pm}_{[\gamma_o]}$, we also have $|x| \neq |y|$. Now, the isometries in Lemma 6 b) take σ to magnetic geodesics of the particular types studied above. Therefore, σ has the form (4) or has the form (5), as desired.

Conversely, given a helix h in $\mathbb{H}^3$ with curvature k, speed $1/k$ and torsion $\tau = 0$, the proof that the expression (4) is a magnetic geodesic is identical to the proof of the converse of Theorem 1. As h has zero torsion, the initial velocity of the magnetic geodesic in (4) is not in the distributions $\mathcal{D}^{\pm}$.

Let v be a curve in $T^1_p\mathbb{H}^3$ with geodesic curvature $k > 0$ and speed $1/k$. Let g be the isometry of $\mathbb{H}^3$ preserving the orientation such that $g(p) = e_0$, $dg(v(0)) = e_1$ and $dg(\dot{v}(0)) = be_3$, for certain $b > 0$. Hence, $g \cdot v$ is a curve in $T^1_{e_0}\mathbb{H}^3$ having the same geodesic curvature and the same speed as v, and also $b = 1/k$. As we showed above, $\bar{v}$ is a curve in $T^1_{e_0}\mathbb{H}^3$ with $\bar{v}(0) = g \cdot v(0)$ and with the same initial velocity and geodesic curvature that $g \cdot v$. By uniqueness, we have that $\bar{v} = g \cdot v$. To complete the proof we observe that $g \cdot [\gamma_{v(t)}] = [\gamma_{g \cdot v(t)}] = [\gamma_{\bar{v}(t)}]$. □

Proof of Theorem 3. Lemma 6 b) implies that the analogue of Theorem 2 a) is empty for the cases $\kappa = 0, 1$. The proof of the fact that every curve σ in $\mathcal{L}_\kappa$ is a null magnetic geodesic if and only if σ has the form (4) or (5) is similar to that of Theorem 2 b).

We check the last statement of the theorem. Without lost of generality, we consider only null magnetic geodesics passing through $[\gamma_o]$ at $t = 0$. We observe that if, in particular, σ is a magnetic geodesic with initial velocity $d(\phi \circ \pi)Z(a\epsilon_1, 0)$, with $a > 0$, (that is, σ has the form (4)), then by Lemma 6 c) there exists $h \in H_1$ such that Ad $(h)Z(a\epsilon_1, 0) = Z(0, a\epsilon_2)$. Hence, $h \cdot \sigma$ is a null magnetic geodesic with initial velocity $d(\phi \circ \pi)Z(0, a\epsilon_2)$, and then it has the form (5). So, σ also has this form. □

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