CLASSIFYING HOPF ALGEBRAS OF A GIVEN DIMENSION

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ABSTRACT. Classifying all Hopf algebras of a given finite dimension over $\mathbb{C}$ is a challenging problem which remains open even for many small dimensions, not least because few general approaches to the problem are known. Some useful techniques include counting the dimensions of spaces related to the coradical filtration [Fu2], [AN1], [BD], studying sub- and quotient Hopf algebras [G], [GV], especially those sub-Hopf algebras generated by a simple subcoalgebra [N5], working with the antipode [Ng1, Ng2, Ng3, Ng4], and studying Hopf algebras in Yetter-Drinfeld categories to help to classify Radford biproducts [ChNg]. In this paper, we add to the classification tools in [BG] and apply our results to Hopf algebras of dimension rpq and $8p$ where p, q, r are distinct primes. At the end of this paper we summarize in a table the status of the classification for dimensions up to 100 to date.

1. Introduction

Let $\Bbbk$ be an algebraically closed field of characteristic 0. The question of classifying all Hopf algebras of a given dimension over $\Bbbk$ goes back to Kaplansky in 1975. To date, there are very few general results. The Kac-Zhu Theorem [Z], states that a Hopf algebra of prime dimension is isomorphic to a group algebra. S.-H. Ng [Ng1] proved that in dimension p^2, the only Hopf algebras are the group algebras and the Taft algebras, using previous results in [AS1], [Mas3]. It is a common belief that a Hopf algebra of dimension pq, where p and q are distinct prime numbers, is semisimple. Hence, it should be isomorphic to a group algebra or a dual group algebra by [EGel1], [GelW], [Mas5], [N2], [So]. This conjecture has been verified for some particular values of p and q, see [AN1, BD, EGel3, Ng2, Ng3, Ng4]. Hilgemann and S.-H. Ng gave the classification of Hopf algebras of dimension $2p^2$ in [HNg] and more recently Cheng and Ng [ChNg] studied the case $4p$, solving the problem for dimension $20, 28$ and 44.

In fact, all Hopf algebras of dimension ≤ 23 are classified: for dimension ≤ 11 the problem was solved by [W]; an alternative proof appears in [Ş]. The classification for dimension 12 was done by [F] in the semisimple case and then completed by [N5] in the general case and for dimension 16 it was solved by [K], [CDR], [B2], [CDMM] and [GV]. For dimension 18 the problem was solved by D. Fukuda [Fu1] and recently Cheng and Ng finished the classification for dimension 20. For the state of the classification of low dimensional Hopf algebras as of 2009, see [B3].

The classification appears more difficult for even dimensions as studied in this article. One reason may be that for H a nonsemisimple Hopf algebra of odd dimension, either H or H^* has a nontrivial grouplike element. The smallest dimension that is still unclassified is 24 and, since the classification for dimension 27 was recently completed in [BG], the next unclassified dimension after 24 is 32.

In this paper we study Hopf algebras over $\Bbbk$ whose dimension is either smaller than 100 or can be decomposed into the product of a small number of prime numbers. In particular, we give some partial results on Hopf algebras of dimension $8p$, with applications to the case of dimension 24, and dimension rpq, where r, p, q are distinct prime numbers. Since there are many results on the classification problem for dimension $4p$ [ChNg] but the complete classification is incomplete, we cannot hope to complete the classification for dimension $8p$. However we can narrow the possibilities.

We will say that a Hopf algebra H is of *type* (r, s) if $|G(H)| = r$ and $|G(H^*)| = s$.

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Theorem A. *Let H be a nonsemisimple Hopf algebra of dimension $8p$ with p an odd prime. If H is not of type (r,s) with r,s powers of 2, $(2p,2)$ or $(2p,4)$, then H is pointed or basic.*

Using counting arguments we can improve the theorem above in case $p=3$.

Theorem B. *Let H be a Hopf algebra of dimension 24 such that the coradical is not a sub-Hopf algebra of H. Then H is of type $(2,2)$, $(2,4)$ or $(6,4)$.*

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This version of the paper contains a correction on the published version. The statement and proof of Proposition 2.17 are changed and the proof of the results that follow from it are corrected accordingly. We thank H.-S. Ng for kindly communicating the gap to us and for the careful reading of our paper.

2. Preliminaries

In this section we introduce notation, recall some previous results which help with the classification of finite dimensional Hopf algebras, see [AN1], [N5], [Ş], [BD], [B2], [GV], [Fu1], and introduce a few new ones. For the general theory of Hopf algebras see [M], [S].

2.1. **Conventions.** Throughout this paper p,q will denote odd prime numbers, C_k the cyclic group of order k and $\mathbb{D}_k$ the dihedral group of order $2k$. Unless otherwise specified, all Hopf algebras in this article are finite dimensional over a field $\Bbbk$ algebraically closed of characteristic zero.

Remark 2.1. By [LR], with the assumptions above, a Hopf algebra is semisimple if and only if it is cosemisimple if and only if S^2, the square of the antipode, is the identity. Thus if L,K are semisimple sub-Hopf algebras of a Hopf algebra H, then $\langle L,K\rangle$, the sub-Hopf algebra of H generated by L and K is semisimple since S_H^2 is the identity on L and on K.

For H a Hopf algebra over $\Bbbk$ then Δ, ε, S denote respectively the comultiplication, the counit and the antipode; $G(H)$ denotes the group of grouplike elements of H; H_0 denotes the coradical; $(H_n)_{n\in\mathbb{N}}$ denotes the coradical filtration of H and L_h (resp. R_h) is the left (resp. right) multiplication in H by h. We say that H is *pointed* if $H_0=\Bbbk G(H)$.

The set of (h,g)*-primitives* (with $h,g\in G(H)$) and set of *skew-primitives* of H are:

$$\begin{aligned}\mathcal{P}_{h,g}(H) &:= \{x\in H \mid \Delta(x)=x\otimes h+g\otimes x\},\\ \mathcal{P}(H) &:= \textstyle\sum_{h,g\in G(H)}\mathcal{P}_{h,g}(H).\end{aligned}$$

We say that $x\in\Bbbk(h-g)$ is a *trivial* skew-primitive; a skew-primitive not contained in $\Bbbk G(H)$ is *nontrivial.*

Let $\mathcal{M}^*(n,\Bbbk)$ denote the simple coalgebra of dimension n^2, dual to the matrix algebra $\mathcal{M}(n,\Bbbk)$. We say that a coalgebra C is a $d\times d$ *matrix-like coalgebra* if C is spanned by elements $(e_{ij})_{1\leq i,j\leq n}$ such that $\Delta(e_{ij})=\sum_{1\leq l\leq n}e_{il}\otimes e_{lj}$ and $\varepsilon(e_{ij})=\delta_{ij}$. If the set $(e_{ij})_{1\leq i,j\leq d}$ of a coalgebra C of dimension d^2 is linearly independent, following Ştefan we call $\mathbf{e}=\{e_{ij}:\ 1\leq i,j\leq d\}$ a *multiplicative matrix* and then $C\simeq\mathcal{M}^*(d,\Bbbk)$ as coalgebras.

Since the only semisimple and pointed Hopf algebras are the group algebras, we shall adopt the convention that 'pointed' means 'pointed nonsemisimple'. Similarly, we say that a finite dimensional Hopf algebra H is *basic* if H is basic as an algebra or copointed as a coalgebra, i.e., all simple H-modules are one-dimensional or the dual is pointed, but H is not the dual of a group algebra.

Recall that a tensor category $\mathcal{C}$ over $\Bbbk$ has the Chevalley property if the tensor product of any two simple objects is semisimple. We shall say that a Hopf algebra H has the *Chevalley property* if the category Comod$\,(H)$ of H-comodules does.

Remarks 2.2. (i) The notion of the Chevalley property in the setting of Hopf algebras was introduced by [AEGel]: it is said in *loc. cit.* that a Hopf algebra has the Chevalley property if the category $\operatorname{Rep}(H)$ of H-modules does.

(ii) Unlike [AEGel], in [CDMM, Section 1], the authors refer to the Chevalley property in the category of H-comodules; this definition is the one we adopt. Note that it is equivalent to say that the coradical H_0 of H is a sub-Hopf algebra.

(iii) If H is semisimple or pointed then it has the Chevalley property.

Let N be a positive integer and let q be a primitive N^{th} root of unity. We denote by T_q the Taft algebra which is generated as an algebra by the elements g and x satisfying the relations $x^N = 0 = 1 - g^N$, $gx = qxg$. Taft Hopf algebras are self-dual and pointed of dimension N^2 with g grouplike and x a $(1,g)$-primitive, i.e., $\Delta(g) = g \otimes g$ and $\Delta(x) = x \otimes 1 + g \otimes x$. If $N = 2$ so that $q = -1$, then T_{-1} is called the Sweedler Hopf algebra and will be denoted H_4 thoughout this article.

2.2. **Spaces of coinvariants.** Let K be a coalgebra with a distinguished grouplike 1. If M is a right K-comodule via δ, then the space of *right coinvariants* is

$$M^{\operatorname{co}\delta} = \{x \in M \mid \delta(x) = x \otimes 1\}.$$

Left coinvariants are defined analogously. If $\pi : H \to K$ is a morphism of Hopf algebras, then H is a right K-comodule via $(1 \otimes \pi)\Delta$. In this case $H^{\operatorname{co}\pi} := H^{\operatorname{co}(1\otimes\pi)\Delta}$ and $H^{co\pi}$ is a subalgebra of H.

We make the following observation.

Lemma 2.3. *Let $\pi : H \to K$ be a Hopf algebra map and let $R := H^{co\pi}$. Then $\pi|_R = \varepsilon|_R$.*

Proof. Let $z \in H^{co\pi}$. Since π is a morphism of Hopf algebras,

$$\Delta_K\pi(z) = (\pi \otimes \pi)\Delta_H(z) = \pi(z_1) \otimes \pi(z_2) = \pi(z) \otimes 1,$$

so that, applying $m_K \circ (\varepsilon_K \otimes \mathrm{id}_K)$ to the equation above, we obtain $\pi(z) = \varepsilon_K(\pi(z)) \in \Bbbk$. Again, since π is a Hopf algebra map, $\varepsilon_K(\pi(z)) = \varepsilon_H(z)$ and so $\pi(z) = \varepsilon_H(z)$. □

2.3. **Extensions of Hopf algebras.** Recall [AD] that an exact sequences of Hopf algebras is a sequence of Hopf algebra morphisms $A \overset{\imath}{\hookrightarrow} H \overset{\pi}{\twoheadrightarrow} B$ where A, H, B are any Hopf algebras, $\imath$ is injective, π is surjective, $\pi\imath = \varepsilon_A$, $\ker\pi = A^+H$ and $A = H^{co\pi}$. An exact sequence is called *central* if A is contained in the centre of H.

The next result will be useful throughout. For a proof see [GV, Lemma 2.3].

Lemma 2.4. *If $\pi : H \to B$ is an epimorphism of Hopf algebras then* $\dim H = \dim H^{\operatorname{co}\pi} \dim B$. *Moreover, if $A = H^{\operatorname{co}\pi}$ is a sub-Hopf algebra of H then the sequence $A \overset{\imath}{\hookrightarrow} H \overset{\pi}{\twoheadrightarrow} B$ is exact.* □

The following proposition tells us how to compute, in a particular case, the dimension of the coradical of H^* using exact sequences.

Proposition 2.5. *Let Γ be a finite group and $A \hookrightarrow H \twoheadrightarrow \Bbbk\Gamma$ an exact sequence of Hopf algebras. Then* $\dim(H^*)_0 = \dim(H/\operatorname{rad} H) = |\Gamma| \dim(A^*)_0$.

Proof. The statement follows from the proof of [GV, Lemma 5.9]. The idea is the following: since the sequence is exact, H is the Γ-crossed product $A * \Gamma$. Let $g \in \Gamma$, then the weak action of g on A defines an algebra map and consequently $\operatorname{rad} A$ is stable by Γ. Then $\operatorname{rad} A * \Gamma$ is a nilpotent ideal of $A * \Gamma$ and $\operatorname{rad} A * \Gamma \subseteq \operatorname{rad} H$. Since $H/(\operatorname{rad} A * \Gamma)$ is semisimple, it follows that $\operatorname{rad} H \subseteq \operatorname{rad} A * \Gamma$ and hence $\dim(H^*)_0 = \dim(H/\operatorname{rad} H) = \dim(A * \Gamma/\operatorname{rad} A * \Gamma) = |\Gamma| \dim(A/\operatorname{rad} A) = |\Gamma| \dim(A^*)_0$. □

2.4. **Yetter-Drinfel'd modules.** For H any Hopf algebra, a left Yetter-Drinfeld module M over H is a left H-module $(M, \cdot)$ and a left H-comodule (M, δ) such that for all $h \in H, m \in M$,

$$\delta(h \cdot m) = h_1 m_{(-1)} \mathcal{S}(h_3) \otimes h_2 \cdot m_{(0)},$$

where $\delta(m) = m_{(-1)} \otimes m_{(0)}$. We will denote this category by ${}^H_H\mathcal{YD}$.

2.5. **On the coradical filtration.** We begin by recalling a description of the coradical filtration due to Nichols. More detail can be found in [AN1, Section 1].

Let D be a coalgebra over $\Bbbk$. Then there exists a coalgebra projection $\pi : D \to D_0$ from D to the coradical D_0 with kernel I, see [M, 5.4.2]. Define the maps

$$\rho_L := (\pi \otimes \mathrm{id})\Delta : D \to D_0 \otimes D \qquad \text{and} \qquad \rho_R := (\mathrm{id} \otimes \pi)\Delta : D \to D \otimes D_0,$$

and let P_n be the sequence of subspaces defined recursively by

$$\begin{aligned} &P_0 = 0,\\ &P_1 = \{x \in D:\ \Delta(x) = \rho_L(x) + \rho_R(x)\} = \Delta^{-1}(D_0 \otimes I + I \otimes D_0),\\ &P_n = \{x \in D:\ \Delta(x) - \rho_L(x) - \rho_R(x) \in \sum_{1 \le i \le n-1} P_i \otimes P_{n-i}\}, \quad n \ge 2. \end{aligned}$$

Then by a result of Nichols, $P_n = D_n \cap I$ for $n \ge 0$, see [AN1, Lemma 1.1]. Suppose that $D_0 = \bigoplus_{\tau \in \mathcal{I}} D_\tau$, where the D_τ are simple coalgebras of dimension d_τ^2. Any D_0-bicomodule is a direct sum of simple D_0-sub-bicomodules and every simple D_0-bicomodule has coefficient coalgebras D_τ, D_γ and has dimension $d_\tau d_\gamma = \sqrt{\dim D_\tau \dim D_\gamma}$ for some $\tau, \gamma \in \mathcal{I}$, where d_τ, d_γ are the dimensions of the associated comodules of D_τ and D_γ, respectively.

Now suppose H is a Hopf algebra. Then H_n, P_n are H_0-sub-bicomodules of H via ρ_R and ρ_L. As in [AN1], [Fu1], for all $n \ge 1$ we denote by $P_n^{\tau,\gamma}$ the isotypic component of the H_0-bicomodule of P_n of type the simple bicomodule with coalgebra of coefficients $D_\tau \otimes D_\gamma$. If $D_\tau = \Bbbk g$ for g a grouplike, we use the superscript g instead of τ. If the simple subcoalgebras are $S(D_\tau)$, $S(D_\gamma)$, (respectively gD_τ, $D_\tau g$ for g grouplike) we write $P_n^{S\tau,S\gamma}$,(respectively $P_n^{g\tau,g\gamma}$,$P_n^{\tau g,\gamma g}$.) For D_τ, D_γ simple coalgebras we denote $P^{\tau,\gamma} = \sum_{n \ge 0} P_n^{\tau,\gamma}$.

Similarly, for Γ a set of grouplikes of H, let $P^{\Gamma,\Gamma}$ denote $\sum_{g,h \in \Gamma} P^{g,h}$ and let $H^{\Gamma,\Gamma} := P^{\Gamma,\Gamma} \oplus \Bbbk\Gamma$. If $\mathcal{D}, \mathcal{E}$ are sets of simple subcoalgebras, let $P^{\mathcal{D},\mathcal{E}}$ denote $\sum_{D \in \mathcal{D}, E \in \mathcal{E}} P^{D,E}$. Since $H_n = H_0 \oplus P_n$, we have that $H = H_0 \oplus \sum_{\tau,\gamma} P^{\tau,\gamma}$.

Following D. Fukuda, we say that the subspace $P_n^{\tau,\gamma}$ is *nondegenerate* if $P_n^{\tau,\gamma} \nsubseteq P_{n-1}$. The following results are due to D. Fukuda; note that (ii) is a generalization of [AN1, Cor. 1.3] for $n > 1$.

Lemma 2.6. (i) [Fu2, Lemma 3.2] *If the subspace $P_n^{\tau,\gamma}$ is nondegenerate for some $n > 1$, then there exists a set of simple coalgebras $\{D_1, \cdots, D_{n-1}\}$ with P_i^{τ,D_i}, $P_{n-i}^{D_i,\gamma}$ nondegenerate for all $1 \le i \le n$.*

(ii) [Fu2, Lemma 3.5] *For S the antipode in the Hopf algebra H and $g \in G(H)$,*

$$\dim P_n^{\tau,\gamma} = \dim P_n^{S\gamma,S\tau} = \dim P_n^{g\tau,g\gamma} = \dim P_n^{\tau g,\gamma g}.$$

(iii) [Fu2, Lemma 3.8] *Let C, D be simple subcoalgebras such that $P_m^{C,D}$ is nondegenerate. If* $\dim C \neq \dim D$ *or* $\dim P_m^{C,D} - P_{m-1}^{C,D} \neq \dim C$ *then there exists a simple subcoalgebra E such that $P_\ell^{C,E}$ is nondegenerate for some $\ell \ge m+1$.* □

The following facts about dimensions from [AN1] will be useful later.

Lemma 2.7. [AN1] *Let H be a Hopf algebra with $G := G(H)$. Then for $n \ge 0$, $d \ge 1$, $|G|$ divides* $\dim H_n$ *and* $\dim H_{0,d}$*, where $H_{0,d}$ denotes the direct sum of the simple subcoalgebras of H of dimension d^2. Also $H_n = H_0 \oplus P_n$ so that $|G|$ divides* $\dim P_n$. □

It is well-known (see for example [AN1]) that if a Hopf algebra H has a nontrivial skew primitive element, then $\dim H$ must be divisible by a square. More precisely we have the following lemma.

Lemma 2.8. *Let H be a Hopf algebra with $|G(H)| = m$ and* $\dim H = mn$ *where m, n are relatively prime. Then H has no nontrivial skew-primitive element.*

Proof. Suppose that x is a nontrivial skew-primitive element in H and let L be the sub-Hopf algebra of H generated by x and $G(H)$. By [M, 5.5.1], L is pointed. By [AN1, Proposition 1.8], $\dim L$ is divisible by rm where $r \neq 1$ is a positive integer dividing m. Then $\dim H = mn$ is divisible by rm, contradicting the fact that $(m, n) = 1$. □

The next proposition generalizes results of Beattie and Dăscălescu [BD] and gives a lower bound for the dimension of a finite dimensional Hopf algebra without nontrivial skew-primitive elements.

Proposition 2.9. [BG, Proposition 3.2] *Let H be a non-cosemisimple Hopf algebra with no nontrivial skew-primitives.*

(i) *For any $g \in G(H)$ there exists a simple subcoalgebra C of H of dimension > 1 such that $P_1^{C,g} \neq 0$, $P_k^{C,D}$ is nondegenerate for some $k > 1$ and D a simple subcoalgebra of the same dimension as C, and $P_m^{g,h}$ is nondegenerate for some $m > 1$ and h grouplike.*

(ii) *Suppose $H_0 \simeq \Bbbk G \oplus \sum_{i=1}^{t} \mathcal{M}^*(n_i, \Bbbk)$ with $t \geq 1$, $2 \leq n_1 \leq \ldots \leq n_t$. Then*

$$\dim H \geq \dim(H_0) + (2n_1 + 1)|G| + n_1^2.$$

□

2.6. **Matrix-like coalgebras.** The next theorem due to Ştefan has been a key component for several classification results.

Theorem 2.10. [Ş, Thm. 1.4] *Let D be the simple coalgebra $\mathcal{M}^*(2, \Bbbk)$.*

(i) *For f an antiautomorphism of D such that $\operatorname{ord}(f^2) = n < \infty$ and $n > 1$, there exist a multiplicative matrix $\mathbf{e}$ in D and a root of unity ω of order n such that*

$$f(e_{12}) = \omega^{-1} e_{12}, \quad f(e_{21}) = \omega e_{21}, \quad f(e_{11}) = e_{22}, \quad f(e_{22}) = e_{11}.$$

(ii) *For f be an automorphism of D of finite order n, there exist a multiplicative matrix $\mathbf{e}$ on D and a root of unity ω of order n such that $f(e_{ij}) = \omega^{i-j} e_{ij}$.* □

Now we recall some useful results on matrix-like coalgebras. In [BD] all 2×2 matrix-like coalgebras of dimension less than 4 were described; we summarize in the following theorem.

Theorem 2.11. [BD, Thm. 2.1] *Let D be a 2×2 matrix-like coalgebra of dimension less than 4. If $\dim D = 1, 2$ then D has a basis of grouplike elements. If $\dim D = 3$, then D has a basis $\{g, h, x\}$ where g, h are grouplike and x is (g, h)-primitive.* □

We end this section with the following lemma.

Lemma 2.12. *Let $\pi : H \to H_4$ be a Hopf algebra epimorphism. If H is generated by a simple subcoalgebra D of dimension 4, then $\dim {}^{\mathrm{co}\,\pi}D \geq 2$ or $\dim D^{\mathrm{co}\,\pi} \geq 2$.*

Proof. Since H_4 is pointed, $\dim \pi(D) < 4$. Moreover, since D generates H as an algebra, $\pi(D)$ generates H_4. Then by Theorem 2.11, $\dim \pi(D) = 3$. Let $\{e_{ij}\}_{1 \leq i,j \leq 2}$ be a multiplicative matrix of D, then $\{\pi(e_{ij})\}_{1 \leq i,j \leq 2}$ is a linearly dependent set.

As in the proof of Theorem 2.11, see [BD, Thm. 2.1], we divide the proof into two cases.

Case 1: The set $\{\pi(e_{11}), \pi(e_{12}), \pi(e_{21})\}$ is linearly independent.

Then $\pi(e_{22}) = \pi(e_{11}) + a\pi(e_{12}) + b\pi(e_{21})$ for scalars a, b. By comparing $\Delta\pi(e_{22})$ and $\pi \otimes \pi \circ \Delta(e_{22})$ one sees that $ab = -1$ so that there exists $0 \neq b \in \Bbbk$ such that $\pi(e_{22}) = \pi(e_{11}) + b\pi(e_{12}) - \frac{1}{b}\pi(e_{21})$. Then it is straightforward to verify that the linearly independent elements $h_1 = \pi(e_{11}) - \frac{1}{b}\pi(e_{21})$ and $h_2 = \pi(e_{11}) + b\pi(e_{12})$ are grouplike.

Suppose that $h_1 = 1$. Then it is straightforward to show that $t_1 = e_{11} - \frac{1}{b} e_{21} \in {}^{\mathrm{co}\,\pi}D$.

Let $s_1 = e_{22} - be_{12}$ and note that s_1, t_1 are linearly independent. Then $\pi(s_1) = \pi(e_{22}) - b\pi(e_{12}) = \pi(e_{11}) + b\pi(e_{12}) - \frac{1}{b}\pi(e_{21}) - b\pi(e_{12}) = \pi(e_{11}) - \frac{1}{b}\pi(e_{21}) = \pi(t_1) = 1$. Then a computation similar to that for t_1, shows that $s_1 \in {}^{\operatorname{co}\pi}D$. Thus $\dim {}^{\operatorname{co}\pi}D \geq 2$.

If $h_2 = 1$ then define $t_2 = e_{11} + be_{12}$ and $s_2 = e_{22} + \frac{1}{b}e_{21}$. A computation similar to that above shows that $t_2, s_2 \in D^{co\pi}$ so that $D^{co\pi}$ has dimension at least 2.

Case 2: The set $\{\pi(e_{11}), \pi(e_{12}), \pi(e_{22})\}$ is linearly independent.

Then there exist $a, b \in \Bbbk$ such that $\pi(e_{21}) = a\pi(e_{11}) + b\pi(e_{12}) - a\pi(e_{22})$. If $a \neq 0$, then the case reduces to Case 1. If $a = 0$, then $\pi(e_{21}) = b\pi(e_{12})$ and by comparing $\Delta\pi(e_{21})$ and $\Delta\pi(be_{12})$, we see that $b = 0$ so that $\pi(e_{21}) = 0$. Thus $\Delta(\pi(e_{11})) = \pi(e_{11}) \otimes \pi(e_{11})$, $\Delta(\pi(e_{22})) = \pi(e_{22}) \otimes \pi(e_{22})$ and $\Delta(\pi(e_{12})) = \pi(e_{11}) \otimes \pi(e_{12}) + \pi(e_{12}) \otimes \pi(e_{22})$, which implies that $G(H_4) = \langle \pi(e_{11}), \pi(e_{22}) \rangle$. If $\pi(e_{11}) = 1$, then $e_{11} \in D^{\operatorname{co}\pi}$, for

$$(1 \otimes \pi)\Delta(e_{11}) = (1 \otimes \pi)(e_{11} \otimes e_{11} + e_{12} \otimes e_{21}) = e_{11} \otimes \pi(e_{11}) + e_{12} \otimes \pi(e_{21}) = e_{11} \otimes 1.$$

Moreover, the element $e_{11} + e_{21} \in D^{\operatorname{co}\pi}$, since

$$\begin{aligned}(1 \otimes \pi)\Delta(e_{11} + e_{21}) &= (1 \otimes \pi)(e_{11} \otimes e_{11} + e_{12} \otimes e_{21} + e_{21} \otimes e_{11} + e_{22} \otimes e_{21} \\ &= e_{11} \otimes \pi(e_{11}) + e_{12} \otimes \pi(e_{21}) + e_{21} \otimes \pi(e_{11}) + e_{22} \otimes \pi(e_{21}) = e_{11} \otimes 1 + e_{21} \otimes 1.\end{aligned}$$

Thus, $\dim D^{\operatorname{co}\pi} \geq 2$. The case $\pi(e_{22}) = 1$ is completely analogous and implies that $\dim {}^{\operatorname{co}\pi}D \geq 2$, taking the elements e_{22} and $e_{22} + e_{21}$. □

Note that if D in Lemma 2.12 is stable under S^2 then the proof can be simplified considerably. For then we may choose a multiplicative matrix for D consisting of eigenvectors for S^2 and we must have that $\pi(e_{ii}) \in \Bbbk C_2 \subset H_4$.

2.7. **Hopf algebras generated by a simple subcoalgebra.** In this subsection we summarize some known facts about Hopf algebras generated by simple subcoalgebras of dimension 4. The most important is the next proposition, due to Natale but derived from a result of Ştefan [Ş].

Proposition 2.13. [N5, Prop. 1.3]. *Suppose that H is nonsemisimple Hopf algebra generated by a simple subcoalgebra of dimension 4 which is stable by the antipode. Then H fits into a central exact sequence $\Bbbk^G \overset{\imath}{\hookrightarrow} H \overset{\pi}{\twoheadrightarrow} A$, where G is a finite group and A^* is a pointed nonsemisimple Hopf algebra.* □

We have the following useful results from [GV]. If H is a Hopf algebra, $\mathcal{Z}(H)$ denotes its centre.

Lemma 2.14. [GV, Lemma 4.2] *Let $\pi : H \to K$ be a morphism of Hopf algebras such that $\pi(g) = 1$ for some $g \in G(H)$, $g \neq 1$. Suppose that H is generated by a simple subcoalgebra of dimension 4 stable by L_g. Then $\pi(H) \subseteq \Bbbk G(K)$.*

The same holds with R_g instead of L_g; or with $\operatorname{ad}_\ell(g)$ or $\operatorname{ad}_r(g)$ if $g \notin \mathcal{Z}(H)$. □

Lemma 2.15. [GV, Lemma 4.3] *Let $\pi : H \to K$ be an epimorphism of Hopf algebras and assume that K is nonsemisimple. Suppose that H is generated by a simple subcoalgebra of H of dimension 4 stable by S_H^2. Then $\operatorname{ord} S_H^2 = \operatorname{ord} S_K^2$.* □

Remark 2.16. (*i*) If H is generated as an algebra by $C \oplus D$ with $S(C) = D$, then C also generates H as an algebra, since the sub-bialgebra generated by C is finite dimensional and thus a sub-Hopf algebra.

(*ii*) Suppose that H is generated by a simple subcoalgebra C of dimension 4 stable by S_H^2. If H_4 is a sub-Hopf algebra of H^*, then $S_H^4 = \operatorname{id}$. For the inclusion $H_4 \hookrightarrow H^*$ induces a Hopf algebra surjection $H \twoheadrightarrow H_4$ and by Lemma 2.15 the claim follows.

We end this section with the following proposition.

Proposition 2.17. *Let $\pi : H \twoheadrightarrow A$ be a Hopf algebra epimorphism and assume* $\dim H = 2\dim A$*. Then $H^{\mathrm{co}\,\pi} = \Bbbk\{1, x\}$ with x a $(1, g)$-primitive element with $\pi(x) = 0$, $g \in G(H)$ and* $\operatorname{ord} g = 2n$ *with $n \geq 1$. If x is trivial, i.e., $x \in \Bbbk G(H)$, then H fits into an exact sequence of Hopf algebras $\Bbbk C_2 \hookrightarrow H \twoheadrightarrow A$. Otherwise $x^2 = 0$, $xg = -gx$ and x, g generate a pointed sub-Hopf algebra of dimension $4n$; in particular, $4|\dim H$.*

Proof. The statement follows from the proof of [HNg, Prop. 1.3]. We reproduce part of the proof. Let $R = H^{\mathrm{co}\,\pi}$; it is known that R is a left coideal subalgebra and $\dim R = 2$. Let $x \in R \smallsetminus \{0\}$ such that $\varepsilon(x) = 0$. Then $R = \Bbbk\{1, x\}$ and $\Delta(x) = a \otimes 1 + b \otimes x$ for some $a, b \in H$. Since $x \in R$, it follows that $x = a$ and $\pi(x) = 0$. The coassociativity of Δ implies that b is grouplike. Denote $g = b$, then $x \in P_{1,g}$ and x is a skew-primitive element. If $x = \alpha(1-g)$ for some $\alpha \in \Bbbk^*$, then $g^2 = 1$ and $R \simeq \Bbbk C_2$ implying that H fits into an exact sequence of Hopf algebras $\Bbbk C_2 \hookrightarrow H \twoheadrightarrow A$.

Assume x is non-trivial and let $m = \operatorname{ord} g$. As R is a subalgebra stable by the adjoint action of g, it follows that $gxg^{-1} = \beta x$ with $\beta^m = 1$ and $\Delta(x^2) = x^2 \otimes 1 + (1+\beta)xg \otimes x + g^2 \otimes x^2$. Let $x^2 = \alpha 1 + \gamma x$. Then applying ε, we see that $\alpha = 0$ and $x^2 = \gamma x$ so that $\Delta(x^2) = \gamma x \otimes 1 + \gamma g \otimes x$. Thus $\beta = -1$ and $\gamma g \otimes x = g^2 \otimes \gamma x$ so that $\gamma = 0$. Since $\beta = -1$, $2|m$ and the subalgebra generated by g and x is a pointed sub-Hopf algebra of H of dimension $2m = 4n$ for some $n \geq 1$. □

2.8. **Hopf algebras of dimension $4p$.** This section contains a brief overview of what is known for dimension $4p$. Knowledge of the classification in this dimension is of course necessary to understand dimension $8p$ which we study in the last section of this note. Recall that for dimension 12 the classification is due to [AN1], [F], [N5].

Up to isomorphism, the semisimple Hopf algebras of dimension $4p$ consist of group algebras and their duals, and also of two self-dual Hopf algebras, constructed by Gelaki in [Gel1], which we will denote by $\mathcal{G}_1$ and $\mathcal{G}_2$. Both have group of grouplikes of order 4 with $G(\mathcal{G}_1) \cong C_4$ and $G(\mathcal{G}_2) \cong C_2 \times C_2$.

2.8.1. *Nonsemisimple pointed Hopf algebras of dimension* $4p$. All pointed Hopf algebras of dimension $4p$ have group of grouplikes isomorphic to C_{2p} and are described in [AN1, A.1].

In particular, let $\mathcal{A}$ be a pointed Hopf algebra of dimension $4p$. Then, with g denoting a generator of C_{2p}, and ξ a primitive pth root of unity, $\mathcal{A}$ is isomorphic to exactly one of the following.

$$\begin{aligned} \mathcal{A}(-1,0) := \quad & \Bbbk\langle g, x \mid g^{2p} - 1 = x^2 = gx + xg = 0\rangle, \\ & \Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes 1 + g \otimes x. \end{aligned}$$

$$\begin{aligned} \mathcal{A}(-1,0)^* := \quad & \Bbbk\langle g, x \mid g^{2p} - 1 = x^2 = gx + \xi xg = 0\rangle, \\ & \Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes 1 + g^p \otimes x. \end{aligned}$$

$$\begin{aligned} \mathcal{A}(-1,1) := \quad & \Bbbk\langle g, x \mid g^{2p} - 1 = x^2 - g^2 + 1 = gx + xg = 0\rangle, \\ & \Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes 1 + g \otimes x. \end{aligned}$$

$$\begin{aligned} H_4 \otimes \Bbbk C_p := \quad & \Bbbk\langle g, x \mid g^{2p} - 1 = x^2 = gx + xg = 0\rangle, \\ & \Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes 1 + g^p \otimes x. \end{aligned}$$

Note that $H_4 \otimes \Bbbk C_p$ is self-dual. The Hopf algebra $\mathcal{A}(-1,1)$ is a nontrivial lifting of $\mathcal{A}(-1,0)$ and has nonpointed dual. The nonpointed Hopf algebra $\mathcal{A}(-1,1)^*$ contains a copy of the Sweedler Hopf algebra and as a coalgebra $\mathcal{A}(-1,1)^* \cong H_4 \otimes \mathcal{M}^*(2,\Bbbk)^{p-1}$. The Hopf algebras $\mathcal{A}(-1,0)$ and $\mathcal{A}(-1,1)$ do not have sub-Hopf algebras isomorphic to H_4 but $\mathcal{A}(-1,0)^*$ and $H_4 \otimes \Bbbk C_p$ do. In all four cases, $S^4 = \mathrm{id}$. In Section 4 we will use this notation for these pointed Hopf algebras.

2.8.2. *Nonsemisimple nonpointed Hopf algebras of dimension* $4p$. These Hopf algebras were studied in [ChNg] with the classification completed for $p = 3, 5, 7, 11$. The main theorems of [ChNg] are:

Theorem 2.18. [ChNg, Theorem I] *For H a nonsemisimple Hopf algebra of dimension $4p$, then H is pointed if and only if $|G(H)| > 2$.*

Theorem 2.19. [ChNg, Theorem II] *For H a nonsemisimple Hopf algebra of dimension $4p$ where $p \leq 11$ is an odd prime, then H or H^* is pointed.*

2.8.3. *Applications of counting arguments.* We end this section by applying the preceding preliminary material to give some simple alternate arguments for known facts about nonsemisimple nonpointed Hopf algebras of dimension $4p$.

Proposition 2.20. *Let H be a nonsemisimple nonpointed Hopf algebra of dimension $4p$. Then $|G(H)| \notin \{p, 2p\}$.*

Proof. Suppose that $|G(H)| = p$ and then $H_0 = \Bbbk G(H) \oplus (\oplus_{i=1}^{t} D_i^{n_i})$ where the D_i are simple subcoalgebras of dimension d_i^2 with $1 < d_1 < d_2 < \ldots < d_t$. If p divides some d_i, then $\dim H_0 \geq p + p^2 = p(1+p) \geq 4p$ since $p \geq 3$, a contradiction. Thus $(p, d_i) = 1$ for all i and p must divide n_i by Lemma 2.7. Then $\dim H_0 \geq p + 4p > 4p$, a contradiction. An analogous proof shows that $|G(H)| \neq 2p$. □

Proposition 2.21. *Suppose that H is generated as an algebra by a simple coalgebra D of dimension 4 which is stable by the antipode. Then H^* is pointed.*

Proof. By Proposition 2.13, H fits into a central exact sequence of Hopf algebras

$$\Bbbk^G \hookrightarrow H \twoheadrightarrow A,$$

with A nonsemisimple and A^* pointed. Since $\dim A$ divides $4p$, then $\dim A \in \{1, 2, 4, p, 2p, 4p\}$. We will show that $\dim A = 4p$ so that $H \cong A$ and thus H^* is pointed.

Since H is nonsemisimple, $\dim A > 1$. If $\dim A = 2, p$ or $2p$, then A is semisimple by the classification of Hopf algebras of these dimensions, see [Ş], [W], [Z], [Ng3]. Since $\Bbbk^G$ is semisimple, this would imply that H is also semisimple, a contradiction.

If $\dim A = 4$, then $\dim \Bbbk^G = p$ and this implies that $\Bbbk^G \simeq \Bbbk C_p$. Thus $C_p \subseteq G(H)$ so that $|G(H)| = p$ or $2p$, a contradiction by Proposition 2.20. Thus $A = H$ and H^* is pointed. □

3. Hopf algebras of dimension rpq

In this section, H will be a Hopf algebra of dimension rpq, with $r < p < q$ primes. Recently, Etingof, Nikshych and Ostrik [ENO], finished the classification of the semisimple Hopf algebras of dimension rpq and rp^2. Specifically, they prove that all semisimple Hopf algebras of these dimensions can be obtained as abelian extensions (Kac Algebras). Then, the classification follows by a result of Natale [N1]. **Thus, we will assume that H is nonsemisimple.** One purpose of this section is to apply counting arguments in the style of D. Fukuda as we did in [BG].

Remark 3.1. Recall that by Lemma 2.8 a nonsemisimple Hopf algebra H of dimension rpq is nonpointed, has no pointed sub-Hopf algebras and has no pointed quotient Hopf algebras. In particular, H cannot be generated by a simple 4-dimensional subcoalgebra C stable under the antipode. For then by Proposition 2.13, $H \cong \Bbbk^G$, which is semisimple, a contradiction. □

Also H cannot have the Chevalley property. The proof is based on the proof of [AN1, Lemma A.2].

Proposition 3.2. *No nonsemisimple Hopf algebra H of dimension rpq has the Chevalley property.*

Proof. Suppose that H has the Chevalley property. Then $\dim H_0|\dim H$ and since H is not pointed or cosemisimple, $1 < \dim H_0 < \dim H$. Then $\dim H_0 = st$, where $s,t \in \{r,p,q\}$ and $s < t$. But by [EGel1], [So], [N1] or if $s = 2$ by [Ng3], all semisimple Hopf algebras of dimension st are trivial, *i.e.* isomorphic to a group algebra or the dual of a group algebra. Hence $H_0 \simeq \Bbbk^F$ with F a nonabelian group of order st; in particular, $s|(t-1)$. Consider now the coradical filtration on H and the associated graded Hopf algebra $\operatorname{gr} H$. Then write $\operatorname{gr} H \simeq R\#\Bbbk^F$ with R the *diagram* of H. Then $(\operatorname{gr} H)^* \simeq R^*\#\Bbbk F$, which implies that $(\operatorname{gr} H)^*$ is pointed. This cannot occur, since $\dim(\operatorname{gr} H)^* = \dim \operatorname{gr} H = \dim H = rpq$. Hence, H_0 cannot be a sub-Hopf algebra. □

We note that all Hopf algebras of dimension $30 = 2 \cdot 3 \cdot 5$ are group algebras or duals of group algebras by [Fu3] but the classification of the other Hopf algebras of dimension rpq with $rpq < 100$, namely dimensions $42, 66, 70$ and 78 is still open. We make a few observations about these cases.

Remark 3.3. (i) A nonsemisimple Hopf algebra of dimension $2pq$ cannot have a semisimple sub-Hopf algebra A of dimension pq. For if this were the case, there would be a Hopf algebra epimorphism $\pi : H^* \to A^*$ and we apply Proposition 2.17. Since H^* has no nontrivial skew-primitive elements, $(H^*)^{co\pi} = \Bbbk C_2$ and we have an exact sequence of Hopf algebras $\Bbbk C_2 \hookrightarrow H^* \twoheadrightarrow A^*$. Thus if A, and thus A^*, were semisimple, H^* and H would be also.

(ii) Suppose that H is nonsemisimple of dimension $2pq$ where all Hopf algebras of dimension pq are semisimple. Then by (i) above, H has no sub-Hopf algebras of dimension pq. Suppose that $|G(H)| > 2$ and let C be a simple subcoalgebra of dimension greater than 1. We will show that C generates H.

Indeed, let $\mathcal{C} := \langle C \rangle$ be the sub-Hopf algebra generated by C. Then $\dim \mathcal{C} \in \{2p, 2q, 2pq\}$. If $\mathcal{C} \neq H$, then $\mathcal{C} \cong \Bbbk^{\mathbb{D}_m}$ with $m \in \{p,q\}$. Then H is generated by $\mathcal{C}$ and $\Bbbk G(H)$, so by Remark 2.1, H is semisimple, a contradiction.

Lemma 3.4. *Suppose that H has dimension $2pq$ with $2 < p < q$. Then*

(i) $|G(H)| \neq pq$.
(ii) *If $p \leq 7$ then $|G(H)| \neq 2q$ and if $q \leq 7$ then $|G(H)| \neq 2p$.*
(iii) *If $p \leq 5$ then $|G(H)| \neq q$.*

Proof. (i) The statement was proved in Remark 3.3(i).

(ii) If $|G(H)| = 2p$, since for all d, by Lemma 2.7, $2p$ divides $\dim H_{0,d} = nd^2$ for some $n \geq 1$ then $\dim H_0 \geq 2p + 4p = 6p$. Then by Proposition 2.9(ii) and Lemma 2.7, $\dim H \geq 6p + 2p(5) + 4p = 20p$, a contradiction if $q \leq 7$.

If $|G| = 2q$ and $p \leq 7$ the argument is the same.

(iii) Assume $|G(H)| = q$. For $G, \mathcal{D}$ as above, $\dim H_0 \geq q + 4q = 5q$, $2 \dim P_1^{G,\mathcal{D}} \geq 4q$, $\dim P^{G,G}$ and $\dim P^{\mathcal{D},\mathcal{D}}$ must be divisible by q and so $\dim H \geq 11q > 2(5)q$, a contradiction. □

Corollary 3.5. (i) *If $\dim H = 42$, then $|G(H)| \notin \{21, 14, 7, 6\}$.*
(ii) *If $\dim H = 70$, then $|G(H)| \notin \{35, 14, 10, 7\}$.*
(iii) *If $\dim H = 66$, then $|G(H)| \notin \{33, 22, 11\}$.*
(iv) *If $\dim H = 78$, then $|G(H)| \notin \{39, 26, 13\}$.* □

Next we show that for Hopf algebras of dimension 66, $G(H)$ does not have order 6.

Lemma 3.6. *If $\dim H = 6p$ with $p < 13$, then $|G(H)| \neq 6$.*

Proof. First we suppose that H has a simple subcoalgebra of dimension 4 and consider various cases. Let $G := G(H)$, the group of grouplikes of H of order 6, and let $\mathcal{D}$ denote the set of simple subcoalgebras of dimension 4.

(i) Suppose that $H_0 = \Bbbk G \oplus \mathcal{M}^*(2, \Bbbk)^3$ so that $\dim H_0 = 18$. Since, by Remarks 3.1 and 3.3, no $D \in \mathcal{D}$ is stable by the antipode, then no $D \in \mathcal{D}$ can be fixed by S^2 either. Thus if $P_1^{1,D}$ is nondegenerate, so are $P_1^{1,S^2(D)}$, $P_1^{S(D),1}$ and $P_1^{S^3(D)=D,1}$ and $2\dim P_1^{G,\mathcal{D}} \geq 2(6)4 = 48$. Since $P^{G,G}$ has nonzero dimension divisible by 6 and $P^{\mathcal{D},\mathcal{D}}$ has nonzero dimension divisible by 12, then the dimension of H is at least $18 + 48 + 6 + 12 = 84$, a contradiction.

(ii) Suppose that $H_0 = \Bbbk G \oplus \mathcal{M}^*(2, \Bbbk)^{3t}$ with $t \geq 2$ so that $\dim H_0 = 6 + 4(3t) \geq 30$. Since for some integers $\ell, m, n \geq 1$, $2\dim P^{G,\mathcal{D}} = 24\ell$, $\dim P^{G,G} = 6m$, $\dim P^{\mathcal{D},\mathcal{D}} = 12n$, then $\dim H \geq 72$, so that we obtain a contradiction if $p < 13$.

(iii) Suppose that $H_0 = \Bbbk G \oplus \mathcal{M}^*(2, \Bbbk)^{3t} \oplus E_1 \ldots \oplus E_N$ where $t, N \geq 1$ and the E_i are simple subcoalgebras of dimension greater than 4. Let $\mathcal{E}$ denote the set of E_i and $\mathcal{D}$ the set of simple subcoalgebras of dimension 4. Then $\dim H_0 \geq 6 + 12 + 18 = 36$. If $P^{G,\mathcal{E}} \neq 0$, then $2\dim P^{G,\mathcal{E}} \geq 2(6)(3) = 36$, $\dim P^{\mathcal{E},\mathcal{E}} \geq 9$ and so $\dim H \geq 81$, contradiction. Thus $P^{G,\mathcal{D}} \neq 0$. If $t = 1$, then as in (i) above $2\dim P_1^{G,\mathcal{D}} \geq 2(6)4 = 48$, so that $\dim H \geq 36 + 48 = 84$, a contradiction. If $t \geq 2$, then $\dim H_0 \geq 48$ and $2\dim P_1^{G,\mathcal{D}} \geq 24$ so that $\dim H_1 \geq 72$. But $P_2^{\mathcal{D},\mathcal{D}}$, $P_2^{G,G}$ are nondegenerate, so that $\dim H_2 \geq 80$, a contradiction.

Now suppose that H has no simple subcoalgebras of dimension 4 and $H_0 = \Bbbk G \oplus E_1 \ldots \oplus E_t$ where the E_i are simple subcoalgebras of dimension at least 9 so that $\dim H_0 \geq 6 + 18 = 24$. Let $\mathcal{E}$ denote the set of simple coalgebras E_i. Then $2\dim P_1^{G,\mathcal{E}} \geq 2(6)(3) = 36$, $\dim P^{G,G} \geq 6$, $\dim P^{\mathcal{E},\mathcal{E}} \geq 9$ and must be divisible by 6 so that $\dim P^{\mathcal{E},\mathcal{E}} \geq 12$. But also $\dim P^{\mathcal{E},\mathcal{E}}$ must be a sum of squares larger than 4 so that $\dim P^{\mathcal{E},\mathcal{E}} > 12$. Thus $\dim H > 24 + 36 + 6 + 12 = 78$, a contradiction. □

Note that in the proof above, the only place where $p \neq 13$ was used was in case (ii). There if $p = 13$ we must have that $\ell = n = 1$ and $m = 2$.

Next we show that for dimension 70 the group of grouplikes must have order 1 or 2.

Lemma 3.7. *If* $\dim H = 70$ *then* $G(H) \ncong C_5$.

Proof. Again, we suppose first that H has a simple subcoalgebra of dimension 4 and consider various cases. Let $\mathcal{D}$ denote the set of simple subcoalgebras of dimension 4 and let $G := G(H) \cong C_5$.

(i) Suppose that $H_0 = \Bbbk C_5 \oplus D_1 \oplus \ldots \oplus D_5$ where $D_i \cong \mathcal{M}^*(2, \Bbbk)$. Since no D_i is stable under the antipode, we may assume that $S(D_i) = D_{i+1}$, subscripts modulo 5. For if $S^2(D_1) = D_1$, then S would permute D_3, D_4, D_5. But by Corollary 3.5, $|G(H^*)| \in \{1, 2, 5\}$ and so 3 does not divide the order of S. Thus by Proposition 2.9(i), P_1^{1,D_i} is nondegenerate for all i and $2\dim P^{G,\mathcal{D}} \geq 2(5)(10) = 100$, a contradiction.

(ii) Suppose that $H_0 = \Bbbk C_5 \oplus \mathcal{M}^*(\Bbbk, 2)^{5t}$ where $t > 1$. Then $\dim H_0 \geq 5 + 10(4) = 45$, $2\dim P^{G,\mathcal{D}} \geq 2(5)(2) = 20$, $\dim P^{G,G} \geq 5$, and $\dim P^{\mathcal{D},\mathcal{D}} \geq 4$, so that $\dim H \geq 74$.

(iii) Let $H_0 = \Bbbk C_5 \oplus \mathcal{M}^*(2, \Bbbk)^{5t} \oplus E$, where $t \geq 1$ and $0 \neq E$ is a sum of simple subcoalgebras E_i of dimension greater than 4. Let $\mathcal{E}$ denote the set of E_i. If the dimensions of any of the E_i are relatively prime to 5, then $\dim H_0 \geq 5 + 20 + 5(9) = 70$, a contradiction. The only remaining case is $H_0 = \Bbbk C_5 \oplus \mathcal{M}^*(2, \Bbbk)^5 \oplus \mathcal{M}^*(5, \Bbbk)$; here $t = 1$ or else $H = H_0$. Then $\dim H_0 = 50$, $2\dim P_1^{G,\mathcal{D}} \geq 2(5)(2) = 20$ and this is a contradiction since $H \neq H_1$.

Thus H cannot have a simple subcoalgebra of dimension 4. The only other possibilities for H_0 are $H_0 = \Bbbk C_5 \oplus \mathcal{M}^*(3, \Bbbk)^5$ and $H_0 = \Bbbk C_5 \oplus \mathcal{M}^*(5, \Bbbk)^t$ with $t = 1, 2$. In the first case, $\dim H_0 = 50$, and for $\mathcal{E}$ the set of simple subcoalgebras of dimension 9, $2\dim P^{G,\mathcal{E}} \geq 2(5)(3) = 30$, a contradiction. In the second case, first let $t = 1$ and here let $\mathcal{E}$ be the set of simple subcoalgebras of dimension 25. Then $\dim H_0 = 30$ and $2\dim P^{G,\mathcal{E}} \geq 2(5)(5) = 50$, a contradiction. The proof for $t = 2$ is the same. □

Corollary 3.8. *If H is a nonsemisimple Hopf algebra of dimension* 70*, then* $|G(H)| = 1, 2$. □

Remark 3.9. (i) To summarize, we have that for H of dimension $42, 66$, $|G(H)| \in \{1, 2, 3\}$, for H of dimension 70, $|G(H)| \in \{1, 2\}$ and for H of dimension 78, $|G(H)| \in \{1, 2, 3, 6\}$.

(ii) If $\dim H = 42$ and $G(H) \cong C_3$, then dimension arguments such as those above show that H has following form: $H_0 \cong \Bbbk C_3 \oplus C$ with $C \cong \mathcal{M}^*(3, \Bbbk)$, $2 \dim P^{G,C} = 18$, $\dim P^{G,G} = 3$, $\dim P^{C,C} = 9$.

4. Hopf algebras of dimension $8p$

In this section we prove some results for Hopf algebras of dimension $8p$.

4.1. **Hopf algebras of dimension 8.**

The structure of Hopf algebras of dimension 8 or dimension $4p$ naturally plays a role in the classification of Hopf algebras of dimension $8p$. Hopf algebras of dimension $4p$ were discussed in Section 2.8.1, including a complete description of the pointed ones.

For dimension 8 the semisimple Hopf algebras are group algebras, duals of group algebras or the noncommutative noncocommutative semisimple Hopf algebra of dimension 8, denoted by A_8 [Mas2]. This Hopf algebra is self-dual and $G(A_8) \cong C_2 \times C_2$; it is constructed as an extension of $\Bbbk[C_2 \times C_2]$ by $\Bbbk C_2$. All other Hopf algebras of dimension 8 are pointed or basic.

Let ξ be a primitive 4^{th} root of 1. By [Ş], every pointed nonsemisimple Hopf algebra of dimension 8 is isomorphic to exactly one of the Hopf algebras listed below:

$$\begin{aligned} \mathcal{A}_2 := \ & \Bbbk\langle g, x, y \mid g^2 - 1 = x^2 = y^2 = gx + xg = gy + yg = xy + yx = 0\rangle, \\ & \Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes 1 + g \otimes x, \quad \Delta(y) = y \otimes 1 + g \otimes y. \end{aligned}$$

$$\begin{aligned} \mathcal{A}'_4 &:= \Bbbk\langle g, x \mid g^4 - 1 = x^2 = gx + xg = 0\rangle, \\ & \qquad \Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes 1 + g \otimes x; \end{aligned}$$

$$\begin{aligned} \mathcal{A}''_4 &:= \Bbbk\langle g, x \mid g^4 - 1 = x^2 - g^2 + 1 = gx + xg = 0\rangle, \\ & \qquad \Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes 1 + g \otimes x; \end{aligned}$$

$$\begin{aligned} \mathcal{A}'''_{4,\xi} &:= \Bbbk\langle g, x \mid g^4 - 1 = x^2 = gx - \xi xg = 0\rangle, \\ & \qquad \Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes 1 + g^2 \otimes x; \end{aligned}$$

$$\begin{aligned} \mathcal{A}_{2,2} &:= \Bbbk\langle g, h, x \mid g^2 = h^2 = 1,\ x^2 = gx + xg = hx + xh = gh - hg = 0\rangle, \\ & \qquad \Delta(g) = g \otimes g, \quad \Delta(h) = h \otimes h, \quad \Delta(x) = x \otimes 1 + g \otimes x. \end{aligned}$$

Except for $\mathcal{A}''_4$, these pointed Hopf algebras have pointed duals. We have the following isomorphisms: $\mathcal{A}_2 \simeq (\mathcal{A}_2)^*$, $\mathcal{A}'''_{4,\xi} \simeq \mathcal{A}'''_{4,-\xi} \simeq (\mathcal{A}'_4)^*$ and $\mathcal{A}_{2,2} \simeq (\mathcal{A}_{2,2})^*$ [Ş]. Moreover, one can check case by case that $\mathcal{A}_2$, $\mathcal{A}'''_{4,\xi}$ and $\mathcal{A}_{2,2}$ have sub-Hopf algebras isomorphic to H_4 and $\mathcal{A}'_4, \mathcal{A}''_4$ do not.

Let $\mathcal{K} = (\mathcal{A}''_4)^*$. Up to isomorphism $\mathcal{K}$ is the only Hopf algebra of dimension 8 which is neither semisimple nor pointed. The next remark is essentially [GV, Lemma 3.3].

Remark 4.1. (i) $\mathcal{K}$ is generated as an algebra by the elements a, b, c, d satisfying the relations

$$\begin{array}{lllll} ab = \xi ba & ac = \xi ca & 0 = cb = bc & cd = \xi dc & bd = \xi db \\ ad = da & ad = 1 & 0 = b^2 = c^2 & a^2 c = b & a^4 = 1 \end{array}$$

(ii) The elements $a = e_{11}, b = e_{12}, c = e_{21}, d = e_{22}$ form a matrix-like basis and

$$\Delta(a^2) = a^2 \otimes a^2 \text{ and } \Delta(ac) = ac \otimes a^2 + 1 \otimes ac.$$

(iii) $\mathcal{K} \simeq H_4 \oplus \mathcal{M}^*(2, \Bbbk)$ as coalgebras.

Using Remark 4.1, one sees that $\mathcal{K}$ is a finite dimensional quotient of the quantum group $\mathcal{O}_\xi(SL_2)$; this is consistent with Proposition 2.13.

4.2. **Nonsemisimple Hopf algebras of dimension $8p$.** Throughout this section, unless otherwise stated, we will assume that H is a nonsemisimple nonpointed nonbasic Hopf algebra of dimension $8p$. Also recall that p denotes an odd prime. Our strategy will be to study the possible orders for the grouplikes in H where $\dim H = 8p$. In this section we prove Theorem A.

4.2.1. *Group of grouplikes divisible by p.* In this subsection we concentrate on general results for Hopf algebras of dimension $8p$ with $|G(H)|$ divisible by p.

Proposition 4.2. *$|G(H)| \neq 8p, 4p$ or p.*

Proof. For H non-cosemisimple, $|G(H)| \neq 8p$. If $|G(H)| = 4p$, since H is not pointed, $H_0 = \Bbbk G(H) \oplus E$ with E the sum of simple coalgebras of dimension bigger than 1. Since $4p$ must divide $\dim(E)$ by Lemma 2.7, then we must have $H = H_0$, impossible because H is not cosemisimple.

If $|G(H)| = p$, then H has no nontrivial skew-primitives by Lemma 2.8. Now we use counting arguments as in the previous sections. Suppose that $H_0 = \Bbbk G(H) \oplus D_1^{s_1} \oplus \dots D_t^{s_t}$ for D_i simple of dimension n_i^2 and $2 \leq n_1 < \dots < n_t$. Let $\mathcal{D}$ denote the set of simple coalgebras D_i. Then by Proposition 2.9(i) and Lemma 2.6(ii), $2 \dim P^{C_p, \mathcal{D}} \geq 2pn_1$. If p divides n_1, then $\dim H \geq \dim H_0 + 2 \dim P^{C_p, \mathcal{D}} \geq p + p^2 + 2p^2 = p(1 + 3p) > 8p$ since $p \geq 3$. If $(p, n_1) = 1$, then p divides s_1 and $\dim H \geq p + 4p + 4p > 8p$. Thus in each case, we arrive at a contradiction. □

Thus, if $|G(H)| = 2p$, then H cannot have the Chevalley property. For, suppose that H_0 is a sub-Hopf algebra of H. Since H is not pointed or cosemisimple, $\dim H_0 = 4p$. Since the semisimple Hopf algebras $\mathcal{G}_i$ have grouplikes of order 4, then $H_0 \cong \Bbbk^{\Gamma}$ where Γ is a nonabelian group of order $4p$. But then $H_0 = \Bbbk^{\Gamma} \cong \Bbbk G(H_0) \oplus D$ where $|G(H_0)| = 2p$ and D is a sum of simple coalgebras. By Lemma 2.7, D is a sum of matrix coalgebras of dimension $d > 1$ and $2p = nd^2$ which is impossible.

Remark 4.3. As in the proof above, we use Lemma 2.8 together with counting arguments to eliminate the possibility that $|G(H)| = 8$ for some small dimensions. Let $\dim H = 8p$ with $p \in \{3, 5, 7\}$ and suppose $|G(H)| = 8$. By Lemma 2.8, H has no nontrivial skew primitive elements. Since $\dim H_0 = 8 + 8m$ for some integer $m \geq 1$, by Lemma 2.9(ii), we have that $\dim H \geq 16 + 40 + 4 = 60$.

The next proposition shows that type $(8, 2p)$ is impossible.

Proposition 4.4. *If $|G(H)| = 2p$ then H^* has no semisimple sub-Hopf algebra L of dimension 8.*

Proof. Suppose H^* contains a semisimple sub-Hopf algebra L of dimension 8 and let Γ be a subgroup of $G(H)$ of order p. Since L^* is semisimple and has no grouplike elements of order p, $\Bbbk\Gamma \hookrightarrow H \twoheadrightarrow L^*$ is an exact sequence of Hopf algebras. This implies that H is semisimple, a contradiction. □

The next proposition determines the coalgebra structure of H when $|G(H)| = 2p$.

Proposition 4.5. *Suppose $|G(H)| = 2p$.*

(i) *H contains a pointed sub-Hopf algebra $\mathcal{A}$ of dimension $4p$ and as a coalgebra $H \cong \mathcal{A} \oplus \mathcal{M}^*(2, \Bbbk)^p$.*
(ii) *If H^* is generated by a simple subcoalgebra of dimension 4 fixed by $S_{H^*}^4$ then S_H has order 4.*

Proof. (i) Since H is not pointed, $H_0 = \Bbbk G(H) \oplus D_1 \oplus \dots \oplus D_t$ where the D_i are simple coalgebras of dimension greater than 1. Suppose that $H_{0,mp} \neq 0$ where $m \geq 1$. Then $\dim(H_{0,mp}) \geq 2p^2$ and thus $\dim(H_0) \geq 2p + 2p^2 = p(2 + 2p) \geq 8p$ since $p \geq 3$, and this is impossible since H is nonsemisimple. If $H_{o,d} \neq 0$ for $(d, p) = 1$ and $d > 2$ then $\dim(H_0) \geq 2p + pd^2 = p(2 + d^2) > 8p$ which is also impossible. Thus $D_i = \mathcal{M}^*(2, \Bbbk)$ for all i, and $H_0 \cong \Bbbk G(H) \oplus \mathcal{M}^*(2, \Bbbk)^p$ as coalgebras.

By Proposition 2.9(ii), H has a nontrivial skew-primitive x and x together with $G(H)$ generates a pointed sub-Hopf algebra $\mathcal{A}$ of H of dimension $4p$ and (i) is proved.

(ii) By (i) there is a Hopf algebra projection $\pi : H^* \to \mathcal{A}^*$ for $\mathcal{A}$ the pointed Hopf algebra of dimension $4p$ from (i). Then $S_{\mathcal{A}}$ and $S_{\mathcal{A}^*}$ have order 4. Suppose $D \cong \mathcal{M}^*(2,\Bbbk) \subset H^*$ is stable under $S^4_{H^*}$ and generates H^*, and suppose that $S^4_{H^*}$ has order $N > 1$. Let $\mathbf{e}$ be a multiplicative matrix for D as in Theorem 2.10 such that $S^4_{H^*}(e_{ij}) = \omega^{i-j}e_{ij}$ where ω is a primitive N^{th} root of unity. Then if $i \neq j$, $\pi(e_{ij}) = 0$ and thus $\dim \pi(D) < 3$. By Theorem 2.11, $\pi(D) \subseteq G(\mathcal{A}^*)$ so that $\pi(D)$ does not generate $\mathcal{A}^*$, contradicting the fact that D generates H^*. □

Next we show that if $|G(H)| = 2p$, then H^* cannot contain a copy of the Sweedler Hopf algebra.

Proposition 4.6. *Assume $|G(H)| = 2p$. Then H^* has no sub-Hopf algebra isomorphic to H_4.*

Proof. If H^* contains a sub-Hopf algebra isomorphic to H_4, there exists a Hopf algebra epimorphism $\pi : H \to H_4$. Then, by Lemma 2.4, $\dim H^{\mathrm{co}\,\pi} = \dim {}^{\mathrm{co}\,\pi}H = 2p$. Let $G(H) = \langle c \rangle \cong C_{2p}$ and let $\Gamma = \langle c^2 \rangle \cong C_p$. Since p is odd, we have that $\Bbbk\Gamma$ is included both in $H^{\mathrm{co}\,\pi}$ and ${}^{\mathrm{co}\,\pi}H$.

On the other hand, Proposition 4.5 implies that $H \simeq \mathcal{A} \oplus D$ where $D = D_1 \oplus \cdots \oplus D_p$, with $D_j \simeq \mathcal{M}^*(2,\Bbbk)$, for all $1 \leq j \leq p$. We will prove that for every j, $1 \leq j \leq p$, $\dim D_j^{\mathrm{co}\,\pi} \geq 2$ or $\dim {}^{\mathrm{co}\,\pi}D_j \geq 2$. This fact leads to a contradiction. Indeed, suppose that for n of the D_j, $\dim D_j^{\mathrm{co}\,\pi} \geq 2$ and for the remaining $p-n$ coalgebras D_j, $\dim {}^{\mathrm{co}\,\pi}D_j \geq 2$. Since p is odd, either $2n > p$ or $2(p-n) > p$ so that either $\dim^{co\pi} D > p$ or $\dim D^{co\pi} > p$. Since $\Bbbk\langle c^2 \rangle$ lies in both the left and right coinvariants, this implies that either $\dim^{co\pi} H > 2p$ or $\dim H^{co\pi} > 2p$, and this gives the desired contradiction.

Fix a simple subcoalgebra D_j and let $K = \langle D_j \rangle$, the sub-Hopf algebra of H generated by D_j. Clearly, $\dim K = 8, 2p, 4p$ or $8p$. We write π also for $\pi|_K$ when the meaning is clear. If π maps K onto H_4, then the result follows from Lemma 2.12; in particular, $\dim K \neq 8p$. If $\pi(K) = \Bbbk$, then $\pi|_K = \varepsilon_K$. Hence for $\mathbf{d} = (d_{ij})$ a multiplicative matrix for D_j, $\pi(d_{ij}) = \delta_{ij}$ and D_j lies in both the left and right coinvariants. It remains to consider the case when $\pi(K) = \Bbbk G(H_4) = \Bbbk\langle g \rangle$ where g generates $G(H_4) \simeq C_2$.

Assume $\dim K = 8$. Since K is nonpointed, by Subsection 4.1 we have that $K = \mathcal{K} = (\mathcal{A}_4'')^* \cong L \oplus D_j$ as coalgebras where $L \cong H_4$. Since $G(L) \subset G(H)$, we have that $c^p \in K$. Suppose $\pi(c^p) = 1$. Since $\pi(c)$ is a grouplike element and $|G(H_4)| = 2$, we have that $\pi(c) = 1$. If x is a nontrivial skew-primitive in $H_4 \subset K$ such that $\pi(x) = 0$, then c, x lie in both ${}^{co\pi}H$ and $H^{co\pi}$, contradicting the fact that the dimension of the coinvariants is $2p$. Thus $\pi(c^p) = \pi(c) = g$, ${}^{co\pi}L = \{1, gx\}$, $L^{co\pi} = \{1, x\}$. Since $\dim {}^{co\pi}K = \dim K^{co\pi} = 4$, then we must have that $\dim {}^{co\pi}D_j = \dim D_j^{co\pi} = 2$.

Next we will show that if $\dim \pi(K) = 2$ then K cannot have dimension $4p$ or $2p$. Suppose that $\dim K = 4p$. Then $\dim K^{co\pi} = 2p = \dim {}^{co\pi}K$ so that $K^{co\pi} = H^{co\pi}$ and the same for the left coinvariants. Thus $\Bbbk\langle c^2 \rangle \cong \Bbbk C_p \subset K$. If K is nonpointed semisimple, by the classification of semisimple Hopf algebras of dimension $4p$ in Section 2.8, p does not divide the order of $G(K)$ either if K is the dual of a group algebra or if K is one of the semisimple Hopf algebras in [Gel1]. If K is not semisimple then by Theorem 2.18, K is pointed, a contradiction.

Finally, suppose now that $\dim K = 2p$ so that $K \cong \Bbbk^{\mathbb{D}_p}$ and $\dim K^{co\pi} = p$. Let $\tilde{K} = \langle K, \Gamma \rangle$ be the sub-Hopf algebra of H generated by K and $\Bbbk\langle c^2 \rangle$. Since $\tilde{K}$ is semisimple, then $\tilde{K} \neq H$ and so has dimension $4p$. But $\tilde{K}$ is then a nonpointed semisimple sub-Hopf algebra of H of dimension $4p$ with a grouplike of order p, and this is impossible by the proof in the paragraph above. □

The next proposition shows that type $(2p, r)$ can occur only if $r = 2, 4$.

Proposition 4.7. *Suppose $|G(H)| = 2p$. Then*

(i) *H fits into an exact sequence of Hopf algebras $\mathcal{A} \hookrightarrow H \twoheadrightarrow \Bbbk C_2$, where $\mathcal{A}$ is a pointed Hopf algebra of dimension $4p$. In particular, $G(H)$ is cyclic.*

(ii) *If $\mathcal{A}^*$ is nonpointed, i.e., $\mathcal{A} \cong \mathcal{A}(-1,1)$, then $\dim H_0^* = 8p-4$, $G(H^*) \cong C_4$ and H^* has a sub-Hopf algebra isomorphic to $\mathcal{A}_4''$.*

(iii) *If $\mathcal{A}^*$ is pointed then $\dim H_0^* = 4p$ and $|G(H^*)|$ is 2 or 4. If H^* has a nontrivial skew-primitive element, then H^* has a sub-Hopf algebra isomorphic to $\mathcal{A}_4''$.*

Proof. (i) Proposition 4.5 implies that $H \simeq \mathcal{A} \oplus \mathcal{M}^*(2,k)^p$, with $\mathcal{A}$ a pointed Hopf algebra of dimension $4p$. Dualizing this inclusion we get a Hopf algebra epimorphism $\pi : H^* \twoheadrightarrow \mathcal{A}^*$ and $\dim H^* = 2\dim \mathcal{A}^*$. Thus by Proposition 2.17, $R := (H^*)^{co\pi} = \Bbbk\{1, x\}$ with x a (possibly trivial) $(1,g)$-primitive element for some grouplike $g \in G(H^*)$ with $\operatorname{ord} g = 2n$, $n \geq 1$. Since $\dim H^* = 8p$, we have that $\operatorname{ord} g = 2, 4$ or $2p$.

Assume $\operatorname{ord} g = 2$. If x is a nontrivial skew-primitive, then by Proposition 2.17, H^* has a sub-Hopf algebra isomorphic to H_4 and this is impossible by Proposition 4.6. Thus $x \in \Bbbk G(H^*)$ and R is a Hopf algebra isomorphic to the group algebra $\Bbbk C_2$. In particular, H fits into the exact sequence of Hopf algebras $\mathcal{A} \hookrightarrow H \twoheadrightarrow \Bbbk C_2$.

We show now that the other cases are not possible. Assume $\operatorname{ord} g = 4$. Then by Proposition 2.17, x must be nontrivial and H^* contains a pointed sub-Hopf algebra L of dimension 8. By inspection on the pointed Hopf algebras of dimension 8 in Section 4.1, we must have that $L \simeq \mathcal{A}_4'$ and consequently L^* is pointed. This implies that $\Bbbk C_p \hookrightarrow H \twoheadrightarrow L^*$ is an exact sequence of Hopf algebras and by [G, Thm. 2.1] H would be pointed, a contradiction.

Finally assume that $\operatorname{ord} g = 2p$. Then $G(H^*) \cong C_{2p}$ and by Proposition 4.5(i), as coalgebras $H^* \simeq \mathcal{B} \oplus \mathcal{M}^*(2,\Bbbk)^p$ with $\mathcal{B}$ a pointed Hopf algebra of dimension $4p$. Let π be the Hopf algebra map from H onto $\mathcal{B}^*$. By Proposition 2.17, we have that $H^{\mathrm{co}\,\pi} = \Bbbk\{1, x\}$ with x a skew-primitive element. If x is trivial, we have that $\Bbbk C_2 \hookrightarrow H \twoheadrightarrow \mathcal{B}^*$ is an exact sequence of Hopf algebras and consequently $\Bbbk C_2$ is normal in H. This implies that $\Bbbk C_2$ is also normal in $\mathcal{A}$ and $\mathcal{A}$ fits into an exact sequence of Hopf algebras $\Bbbk C_2 \hookrightarrow \mathcal{A} \to K$, where $\dim K = 2p$ so that K is semisimple. Thus $\mathcal{A}$ is also semisimple, a contradiction. Therefore $x \in P_1(H)$ but $x \notin H_0$. Since $\pi(x) = 0$, $\pi(P_1(H)) = 0$ and in this case $\mathcal{B}^*$ would be the image of the coradical and hence cosemisimple, which is also a contradiction.

(ii) Suppose now that $\mathcal{A}^*$ is nonpointed. Recall from Subsection 2.8.1 that then $\mathcal{A} \cong \mathcal{A}(-1,1)$ and $\mathcal{A}^* \simeq H_4 \oplus \mathcal{M}^*(2,\Bbbk)^{p-1}$ as coalgebras. Hence $\dim(\mathcal{A}^*)_0 = 4p-2$ and by Proposition 2.5, $\dim(H^*)_0 = 8p-4$. Thus H^* contains a nontrivial skew-primitive element, since otherwise Proposition 2.9 gives a contradiction. Thus $|G(H^*)| > 1$. Since $\dim H^* - \dim(H^*)_0 = 4$ is divisible by $|G(H^*)|$ we have that $|G(H^*)|$ is 2 or 4. But if $G(H^*) \cong C_2$ or if $G(H^*) \cong C_2 \times C_2$, then H^* would contain a sub-Hopf algebra isomorphic to H_4, and this is impossible by Proposition 4.6.

Thus H^* has a pointed sub-Hopf algebra L with $G(L) \cong C_4$ and so L has dimension 8. Then there is a Hopf algebra epimorphism $\rho : H \to L^*$ and $H^{co\rho} \cong \Bbbk C_p$ so that we have an exact sequence of Hopf algebras $\Bbbk C_p \hookrightarrow H \twoheadrightarrow L^*$, and dualizing we obtain the exact sequence $L \hookrightarrow H^* \twoheadrightarrow \Bbbk C_p$. By Proposition 2.5, $6p = \dim H_0 = p \dim L_0^*$. Thus we must have that $\dim L_0^* = 6$ and L^* cannot be pointed. We must have that $L^* \cong \mathcal{K}$ and $L \cong \mathcal{A}_4''$.

(iii) Now suppose that $\mathcal{A}^*$ is pointed so that $G(\mathcal{A}^*) \cong C_{2p}$ by Subsection 2.8.1. Then again using Proposition 2.5 we have that $\dim H_0^* = 4p$. If $|G(H^*)| \neq 2, 4$, since H^* has a grouplike of order 2, by Proposition 4.2 and Proposition 4.4, $|G(H^*)|$ must be $2p$. Then $H_0^* = \Bbbk C_{2p} \oplus E$ where $\dim E = 2p$

and E is a sum of simple subcoalgebras of dimension greater than 1. No simple subcoalgebra can have dimension divisible by p since $p^2 > 2p$. But if a simple subcoalgebra has dimension d^2 with $1 < d$ and $(d,p) = 1$ then H_0^* must contain at least p such simple coalgebras and $d^2 p > 2p$, a contradiction.

If H^* has a nontrivial skew-primitive element then the same argument as in (ii) above shows that H^* has a sub-Hopf algebra isomorphic to $\mathcal{A}''_4$. □

Corollary 4.8. *If $|G(H)| = 2p$ with $p = 3$ or 5, then H^* has a sub-Hopf algebra isomorphic to $\mathcal{A}''_4$.*

Proof. It suffices to show that H^* has a nontrivial skew-primitive element and then the statement follows from Proposition 4.7. We may assume that we are in case (iii) of Proposition 4.7, so that $\dim H_0^* = 4p$.

Let $p = 3$. By Proposition 2.9, if H^* has no skew-primitive, then for $|G(H^*)| \geq 2$, $24 = \dim H^* \geq 12 + 2(5) + 4 = 26$, a contradiction.

If $p = 5$, and $|G(H^*)| = 2$, then $\dim H_0^* = 20$ forces $H_0^* \cong kC_2 \oplus \mathcal{M}^*(3, \Bbbk)^2$. Then if H^* has no nontrivial skew-primitive, Proposition 2.9 implies that $40 \geq 20 + 2(7) + 9 = 43$, a contradiction. If $p = 5$ and $|G(H^*)| = 4$, then Proposition 2.9 implies that $40 \geq 20 + 4(5) + 4 = 44$, again a contradiction. □

Now we can give the proof of Theorem A.

Proof of Theorem A. Let H be a nonsemisimple Hopf algebra of dimension $8p$. By Proposition 4.2 we have that $|G(H)| \in \{1, 2, 4, 8, 2p\}$. If $|G(H)| = 2p$, then by Proposition 4.7 we have that $2 \leq |G(H^*)| \leq 4$ and the theorem is proved. □

4.2.2. *Further results for some specific primes.* In this section, we improve the results of Theorem A for some specific primes p.

Proposition 4.9. *Suppose $|G(H)| = 2p$, $G(H^*) \cong C_4 = \langle g \rangle$ and H^* contains a simple subcoalgebra D of dimension 4. Also assume that $(H^*)_0$ is not a sub-Hopf algebra of H^*, i.e., H^* does not have the Chevalley property. Then*

(i) *D generates H^* as a Hopf algebra;*

(ii) *D is not fixed by L_{g^2}, R_{g^2}, i.e., by left or right multiplication by g^2. If $g^2 \notin \mathcal{Z}(H^*)$ then D is also not fixed by the adjoint action of g^2.*

Proof. (i) Let $L = \langle D \rangle$ be the sub-Hopf algebra of H^* generated by D, then L is a nonpointed Hopf algebra of dimension $8, 2p, 4p$ or $8p$. We will show that each dimension except $8p$ is impossible.

Suppose the dimension of L is 8. Then, by Section 4.1, either $G(L) \cong C_2 \times C_2$, impossible since $G(H^*) \cong C_4$ or else L contains a copy of H_4, impossible by Proposition 4.6.

Suppose the dimension of L is $2p$ so that $L \cong \Bbbk^{\mathbb{D}_p}$. Let $\mathcal{L} = \langle L, \Bbbk\langle g \rangle \rangle$ be the semisimple sub-Hopf algebra of H^* generated by L and by g, a generator of $G(H^*)$. Then the dimension of $\mathcal{L}$ is divisible by $2p$ and by 4 so it must be $4p$. Suppose that $\mathcal{L}$ is the self-dual semisimple Hopf algebra of dimension $4p$ from [Gel1] with grouplikes cyclic of order 4. Then, if $G(H) = \langle h \rangle \cong C_{2p}$ so that $\langle h^2 \rangle \cong C_p$ and π is the Hopf algebra projection from H onto $\mathcal{L}^* \cong \mathcal{L}$, $\pi(h^{2n}) = 1$ for $0 \leq n \leq p-1$. This contradicts Lemma 2.4 which states that the dimension of $H^{co\pi}$ is 2.

Finally suppose that $\mathcal{L} \cong \Bbbk^\Gamma$ for Γ a nonabelian group of order $4p$. Then there is a Hopf algebra projection π from H onto $\Bbbk\Gamma$ and by Proposition 2.17, $H^{co\pi} = \Bbbk\{1, x\}$ where $0 \neq x$ is skew-primitive. If x is trivial, then $H^{co\pi} \cong \Bbbk C_2$. But then by Lemma 2.4, the sequence $\Bbbk C_2 \overset{\imath}{\hookrightarrow} H \overset{\pi}{\twoheadrightarrow} \Bbbk\Gamma$ is exact so that H is semisimple, a contradiction. Thus, by the proof of Proposition 2.17, x is a nontrivial $(1,b)$-primitive with $b = h$ or $b = h^p$ and $x^2 = 0$. Let $b = h$. Thus, since $\{h^i x | 0 \leq i \leq 2p-1\}$ is a set of $2p$ linearly independent nontrivial skew-primitive elements of H, all with square 0, H cannot have a pointed sub-Hopf algebra isomorphic to $\mathcal{A}(-1,1)$. Let $b = h^p$, a grouplike of order 2. Then H contains a sub-Hopf algebra isomorphic to H_4 and again cannot have a pointed sub-Hopf algebra isomorphic to $\mathcal{A}(-1,1)$. Thus $\mathcal{A}$ has pointed dual and so we are in Case (iii) of Proposition 4.7. Then $\dim H_0^* = 4p$ and so $\mathcal{L} = H_0^*$. Since we assumed that H^* does not have the Chevalley property, this is a contradiction.

Suppose the dimension of L is $4p$. By its construction L is not pointed. Also L cannot be semisimple by the arguments in the case above where $\dim \mathcal{L} = 4p$. If L is basic then $L \cong \mathcal{A}(-1,1)^* \cong H_4 \oplus \mathcal{M}^*(2,\Bbbk)^{p-1}$, which is impossible since H^* does not contain a copy of H_4. Thus both L and L^* are nonsemisimple, nonpointed so that by Theorem 2.18, $|G(L)|, |G(L^*)| \leq 2$. But since L is a sub-Hopf algebra of H^*, we have a Hopf algebra epimorphism $\pi : H \twoheadrightarrow L^*$ with $\dim H^{\operatorname{co}\pi} = 2$. This implies that $\pi(h^2) \neq 1$ and consequently $p \leq |G(L^*)| \leq 2$, a contradiction. Thus, this case is also impossible and we have proved (i), namely that $L = H$.

(ii) Now suppose that $g^2 L = L$; if L is stable under R_{g^2} or $\operatorname{ad}_\ell(g^2)$ with $g^2 \notin \mathcal{Z}(H^*)$, the argument is the same. Let $\mathcal{A} \subset H$ be the $4p$-dimensional pointed sub-Hopf algebra of H from Proposition 4.5; there is a Hopf algebra epimorphism $\pi : H^* \to \mathcal{A}^*$. If $\mathcal{A}^*$ is pointed, then $G(\mathcal{A}^*) \cong C_{2p}$, otherwise $G(\mathcal{A}^*) \cong C_2$. In either case, $\pi(g^2) = 1$. Then Lemma 2.14 implies that $\pi(H^*) \subseteq \Bbbk G(\mathcal{A}^*)$, and this contradiction finishes the proof. □

Corollary 4.10. *Assume $|G(H)| = 2p$ with $p = 3, 7, 11$ and suppose that the $4p$ dimensional pointed sub-Hopf algebra $\mathcal{A}$ of H from Proposition 4.5 has pointed dual. If H^* does not have the Chevalley property, then $G(H^*) \ncong C_4$.*

Proof. Suppose that $G(H^*) = \langle g \rangle \cong C_4$. With the notation of Proposition 4.7, the assumption that $\mathcal{A}^*$ is pointed means that we are in Case (iii) so that $\dim H_0^* = 4p$. With notation as in Proposition 4.9, it remains to show that for $p = 3, 7, 11$, then H^* has a simple subcoalgebra of dimension 4 stable under L_{g^2} and that will give a contradiction.

If $p = 3$, $H_0^* \cong \Bbbk C_4 \oplus \mathcal{M}^*(2,\Bbbk)^2$ so since the order of g is 4, the statement is clear.

If $p = 7$ then either $H_0^* \cong \Bbbk C_4 \oplus \mathcal{M}^*(4,\Bbbk) \oplus \mathcal{M}^*(2,\Bbbk)^2$ or else $H_0^* \cong \Bbbk C_4 \oplus \mathcal{M}^*(2,\Bbbk)^6$ and in either case, the statement follows.

If $p = 11$ then $H_0^* \cong \Bbbk C_4 \oplus D$ where D is one of the following: $\mathcal{M}^*(2,\Bbbk)^{10}$ or $\mathcal{M}^*(3,\Bbbk)^4 \oplus \mathcal{M}^*(2,\Bbbk)$ or $\mathcal{M}^*(4,\Bbbk) \oplus E$ where E has dimension 24. Then $E \cong \mathcal{M}^*(2,\Bbbk)^6$ or $E \cong \mathcal{M}^*(2,\Bbbk)^2 \oplus \mathcal{M}^*(4,\Bbbk)$. In any case, H^* has a simple 4-dimensional subcoalgebra stable under L_{g^2}. □

Corollary 4.11. *Let $\dim H = 24$ and suppose H is of type $(6,4)$ and H^* does not have the Chevalley property. Then H fits into an exact sequence $\mathcal{A}(-1,1) \hookrightarrow H \twoheadrightarrow \Bbbk C_2$, in other words, we are in Case (ii) of Proposition 4.7. Then we have that either $H_0^* \cong \Bbbk C_4 \oplus \mathcal{M}^*(2,\Bbbk)^4$ or else $H_0^* \cong \Bbbk C_4 \oplus \mathcal{M}^*(4,\Bbbk)$.*

Proof. The statement follows from Corollary 4.8 and Corollary 4.10. □

4.3. **Generalizations of results of Cheng and Ng.** In this section we generalize some results of [ChNg] to study Hopf algebras of dimension $8p$ with group of grouplikes of order 2^i. We assume throughout this section that H is nonsemisimple, nonpointed, nonbasic and has dimension $8p$.

The following propositions are similar to [ChNg, Prop. 3.2].

Proposition 4.12. (i) *If H contains a pointed sub-Hopf algebra K of dimension 8, then $G(H) = G(K)$.*

(ii) *Assume $H \simeq R\#K$ where K is pointed and basic of dimension 8, and R is a braided Hopf algebra of dimension p in ${}^K_K\mathcal{YD}$. Then $G(H) \cong G(H^*)$ so that H is of type $(4,4)$ or type $(2,2)$.*

(iii) *Suppose that $|G(H)| = 2^t$ for $t \in \{1,2,3\}$ and suppose that H^* contains a sub-Hopf algebra L of dimension 8 so that there is a Hopf algebra epimorphism $\pi : H \to L^*$. Then π is an injective Hopf algebra map from $\Bbbk G(H)$ to $\Bbbk G(L^*)$.*

(iv) *Suppose that H contains a pointed sub-Hopf algebra K of dimension 8 with $|G(K)| = 4$ and H^* contains a pointed sub-Hopf algebra L of dimension 8. Then $K \cong L^*$ and $H \cong R\#K$ where R is a braided Hopf algebra in ${}^K_K\mathcal{YD}$ of dimension p.*

Proof. (i) Suppose H has a grouplike element g such that $g \notin G(K)$. Then $\langle g, K\rangle$, the sub-Hopf algebra of H generated by g and K, is pointed and has dimension greater than 8 and divisible by 8, so must be all of H. This is a contradiction since H is not pointed.

(ii) Assume $H \simeq R\#K$ with K and K^* pointed. By (i), $G(H) = G(K)$. Since $H^* \simeq R^*\#K^*$, and K^* is pointed by assumption, then again by (i), $G(H^*) = G(K^*)$. By Section 4.1, since K is pointed and basic, then $G(K) \cong G(K^*)$ and thus $G(H) \cong G(H^*)$.

(iii) Dualizing the inclusion $L \subset H^*$, we get a Hopf algebra epimorphism $\pi : H \longrightarrow L^*$. Since $\dim L^* = 8$, $\dim R = p$ where $R = H^{co\pi}$ is the algebra of coinvariants. Suppose that $\pi(g) = 1$ for some $g \in G(H)$ and let $\Gamma = \langle g\rangle$. Then $\Bbbk\Gamma \subset R$ and R is a left $(H, \Bbbk\Gamma)$-Hopf module where the left action of $\Bbbk\Gamma$ on R is left multiplication. Then by the Nichols-Zoeller theorem, R is a free $\Bbbk\Gamma$-module which is impossible unless $\Gamma = \{1\}$. Thus π is an injective Hopf algebra map on $\Bbbk G(H)$ as claimed.

(iv) Let $\pi : H \longrightarrow L^*$ and $R = H^{co\pi}$ as in the proof of (iii). Let x be a nontrivial $(g,1)$-primitive in K. We wish to show that $\pi(x)$ is a nontrivial skew-primitive in L^* and then π will be an isomorphism from K to L^*, proving the statement.

By (i), $G(H) = G(K)$ and $G(H^*) = G(L)$. By (iii), since $|G(H)| = 4$, $|G(L^*)| \geq 4$, and since L is pointed, by the description of the duals of pointed Hopf algebras of dimension 8 in Section 4.1, L^* must also be pointed. Again, by Section 4.1, $G(L) \cong G(L^*)$. Let G denote $G(H) \cong G(K) \cong G(L) \cong G(L^*) \cong G(H^*)$.

By (iii), $\pi(x)$ is $(g,1)$-primitive. Suppose that $\pi(K) \subseteq \Bbbk G \subset L^*$. Then $\pi(x) = \lambda(g-1)$ with $\lambda \in \Bbbk$. But this implies that $\pi(x^2) = \lambda^2(g^2 - 2g + 1)$, which is only possible if $\lambda = 0$ since $x^2 = 0$ or $x^2 = g^2 - 1$. Thus $\Bbbk\{1, x\} \subset R = H^{co\pi}$. On the other hand, if V denotes the vector space with basis $\{hx^i | h \in G(H), h \neq 1, i = 0, 1\}$, then $V \cap R = \{0\}$. Since $\dim R = p$, there is some $0 \neq z \in R$ such that $z \notin K$. Then $\langle K, z\rangle$, the sub-Hopf algebra generated by K and z, has dimension greater than 8 and divisible by 8 so is all of H. By Lemma 2.3, $\pi(z) \in \Bbbk$. Thus $\pi(H) \subseteq \Bbbk G$, a contradiction, and so $\pi(z)$ is a nontrivial skew-primitive in L^*. □

Corollary 4.13. *Suppose that H is of type $(4,4)$ and H, H^* each have a nontrivial skew-primitive element. Then $H \cong R\#K$ where K, K^* are pointed Hopf algebras of dimension 8 and R is a braided Hopf algebra in ${}^K_K\mathcal{YD}$ of dimension p.*

Proof. By Proposition 4.12(iv), it remains only to show that H, H^* have pointed sub-Hopf algebras of dimension 8. Let $K = \langle G(H), x\rangle$, the sub-Hopf algebra of H generated by $G(H)$ and a nontrivial skew-primitive element. Then $\dim K < 8p$ and is divisible by 4 so is either 8 or $4p$. Since all pointed Hopf algebras of dimension $4p$ have group of grouplikes of order $2p$ (see Section 2.8.1), $\dim K = 8$. Similarly H^* has a pointed sub-Hopf algebra of dimension 8. □

The following proposition follows the proof of [ChNg, Thm. 3.1].

Proposition 4.14. *Let K be a Hopf algebra and R be a braided Hopf algebra in ${}^K_K\mathcal{YD}$ of odd dimension. If the order of the antipode in the bosonization $R\#K$ is a power of 2, then R and R^* are semisimple.*

Proof. Let $H = R\#K$ be the Radford biproduct or bosonization of R with K. As R is stable by $\mathcal{S}_H^2$, by [AS1, Thm. 7.3] it suffices to prove that $\mathrm{Tr}(\mathcal{S}_H^2|_R) \neq 0$. Clearly, the order of $\mathcal{S}_H^2|_R$ divides the order of $\mathcal{S}_H^2$ and hence is a power of 2. If $\mathrm{Tr}(\mathcal{S}_H^2|_R) = 0$, then by [Ng1, Lemma 1.4] $\dim R$ is even, a contradiction. Thus R is semisimple. The same proof holds for R^* since $H^* \simeq R^*\#K^*$. □

Recall that H nonsemisimple of dimension 24 with $|G(H)| = 4$ has a nontrivial skew-primitive element by Proposition 2.9. Then Corollary 4.13 and Proposition 4.14 imply the next statement.

Corollary 4.15. *Suppose* $\dim H = 24$ *and H is of type* $(4, 4)$. *Then $H \simeq R\#K$ with K and K^* pointed Hopf algebras of dimension* 8 *and R a semisimple Hopf algebra in ${}^K_K\mathcal{YD}$ of dimension* 3. □

The following lemmata generalize results of Cheng and Ng used to study H_4-module algebras, in particular [ChNg, 3.4,3.5].

Lemma 4.16. *Let K be a pointed Hopf algebra generated by grouplikes and skew-primitives, and let A be a finite dimensional left K-module algebra. If A is a semisimple algebra and e is a central idempotent of A such that the two-sided ideal $I = Ae$ is stable by the action of $G(K)$, then I is a K-submodule of A with $g \cdot e = e$ for all $g \in G(K)$ and $x \cdot e = 0$ for any skew-primitive element x.*

Proof. Write $e = e_1 + \cdots + e_t$ as a sum of orthogonal primitive central idempotents. Since I is stable under the action of $G(K)$, then $G(K)$ permutes the primitive idempotents $e_1, \ldots, e_t$ and hence $g\cdot e = e$ for all $g \in G(K)$.

Let x be a $(1, g)$-primitive. Then $x \cdot e = x \cdot e^2 = (x \cdot e)e + (g \cdot e)(x \cdot e) = 2(x \cdot e)e$. Thus $x \cdot e \in I$ so that $x \cdot e = (x \cdot e)e$ and then $x \cdot e = 2(x \cdot e)$ implying that $x \cdot e = 0$. Moreover, since $x \cdot (ae) = (x \cdot a)e$ for all $a \in A$, it follows that I is stable under the action of x and since K is generated by grouplikes and skew-primitives, I is a K-submodule of A. □

Lemma 4.17. *Let K be a pointed Hopf algebra with abelian group of grouplikes. Let A be a semisimple braided Hopf algebra in ${}^K_K\mathcal{YD}$. If I is a one-dimensional ideal of A, then $x\cdot I = 0$ for all skew-primitive elements x of K.*

Proof. Let x be a $(1, g)$-primitive element of K and denote by $\overline{K}$ the pointed sub-Hopf algebra of K generated by x and g. Note that since $G(K)$ is abelian, then $gxg^{-1} = \chi(g)x = \omega x$ for some character χ of G and $1 \neq \omega$ an N-th root of unity with $N = \operatorname{ord} g$.

Since A is semisimple, $I = Ae_1 = \Bbbk e_1$ for some central primitive idempotent. Thus we need to prove that $x \cdot e_1 = 0$. If $g \cdot e_1 = e_1$, then the result follows from Lemma 4.16 using $\overline{K}$ instead of K. Assume $g \cdot e_1 \neq e_1$ and let $e_1, \ldots, e_t$ be representatives of the set $\{g^i \cdot e_1\}_{0\leq i<N}$ of primitive central idempotents of A; in particular t divides N and $g \cdot e_t = e_1$. Let $e = e_1 + \cdots + e_t$, then $\overline{I} = Ae$ is a two-sided ideal of A which is stable under the action of $\Gamma = \langle g\rangle$. Hence, by Lemma 4.16 we have that $x \cdot e = 0$. Since $\Delta(x) = x \otimes 1 + g \otimes x$, we have that $x \cdot e_i = x \cdot e_i^2 = \alpha_{i,i}e_i + \alpha_{i,i+1}e_{i+1}$ and $x \cdot e_t = \alpha_{t,t}e_t + \alpha_{t,1}e_1$ for some $\alpha_{ij} \in \Bbbk$. Using that $x \cdot e = 0$ we get that $\alpha_{i-1,i} + \alpha_{i,i} = 0$ and $\alpha_{t,1} + \alpha_{1,1} = 0$ for all $2 \leq i \leq t$. On the other hand, using that $gxg^{-1} = \omega x$ we obtain that $\omega\alpha_{i,i} = \alpha_{i-1,i-1}$ for all $2 \leq i \leq t$, $\omega\alpha_{i,i+1} = \alpha_{i-1,i}$ for all $2 \leq i \leq t - 1$ and $\omega\alpha_{1,1} = \alpha_{t,t}$, $\omega\alpha_{1,2} = \alpha_{t,1}$ and $\omega\alpha_{t,1} = \alpha_{t-1,t}$. Hence $\alpha_{1,2} = \omega^{-1}\alpha_{t,1} = -\omega^{-1}\alpha_{1,1}$ and $x \cdot e_1 = \alpha_{1,1}(e_1 - \omega^{-1}e_2)$.

Denote by λ_A the right integral of A. Then by [FMS, Thm. 5.8, Rmk. 5.9], see also [ChNg, Eq. (3.4)], for $k \in K$, $a \in A$, we have $\lambda_A(k\cdot a) = \varepsilon_K(k)\lambda_A(a)$. Then $g\cdot e_1 = e_2$ implies that $\lambda_A(e_1) = \lambda_A(e_2)$

and consequently

$$0 = \varepsilon_K(x)\lambda_A(e_1) = \lambda_A(x \cdot e_1) = \alpha_{1,1}(\lambda_A(e_1) - \omega^{-1}\lambda_A(e_2)) = \alpha_{1,1}(1 - \omega^{-1})\lambda_A(e_1).$$

This implies that $\alpha_{1,1} = 0$, since $\omega^{-1} \neq 1$ and $\lambda_A(e_1) \neq 0$ because the kernel of a right integral does not contain any nontrivial ideal. Hence $x \cdot e_1 = 0$ and the lemma is proved. □

Proposition 4.18. *Suppose that $H \cong R\#K$ where K is a pointed Hopf algebra of dimension 8, and R is a Hopf algebra of dimension p in ${}^K_K\mathcal{YD}$ such that $x \cdot R = 0$ for some $(1, g)$ primitive $x \in K$, $x \cdot R$ being the adjoint action of x on R.*

- (i) *If $|G(H)| = 4$, suppose furthermore that K is basic and the condition above holds for R^* and K^*, i.e., for y some nontrivial $(1, h)$-primitive in K^*, then $y \cdot R^* = 0$. Then, H and H^* have the Chevalley property and, in the notation of Section 4.1, we have*
  - (a) *$G(H) \cong C_4$, and $K \cong \mathcal{A}'_4$ or $K \cong \mathcal{A}'''_{4,\xi}$; or*
  - (b) *$G(H) \cong C_2 \times C_2$ and $K \cong \mathcal{A}_{2,2}$.*
- (ii) *If $|G(H)| = 2$ then there is a Hopf algebra epimorphism $\pi : H \to A$ where A is a Hopf algebra of dimension $4p$ which is nonsemisimple, nonpointed and nonbasic. Thus if $p \leq 11$, this situation cannot occur.*

Proof. We note that by Proposition 4.12(i), $G(K) = G(H)$.

(i) Let J be the Hopf ideal of H generated by x. Since $x \cdot R = 0$, then $xR = -gRx \subseteq Hx$ and so $J = Hx$. As a $\Bbbk$-space, $J = Span\{r_i hx : r_1, \dots, r_p$ a basis for $R, h \in G(H)\}$ and so the dimension of J is at most $4p$. Then $\dim H/J \geq 4p$ and divides $8p$ since $(H/J)^*$ is isomorphic to a sub-Hopf algebra of H^*. Thus $\dim J = \dim H/J = 4p$. Then there is a Hopf algebra epimorphism $\pi : H \to A$ where $A := H/J$ is a Hopf algebra of dimension $4p$ and $H^{co\pi} = \Bbbk\{1, x\}$. By Proposition 2.17, $x^2 = 0$ so that $K \cong \mathcal{A}'_4$ or $\mathcal{A}'''_{4,\xi}$ if $G(H) \cong C_4$ and $K \cong \mathcal{A}_{2,2}$ if $G(H) \cong C_2 \times C_2$. Since $(\mathcal{A}'_4)^* \simeq \mathcal{A}'''_{4,\xi}$, in the former case, $H^* \simeq R^*\#\mathcal{A}'''_{4,\xi}$ or $R^*\#\mathcal{A}'_4$, and since $\mathcal{A}_{2,2}$ is self-dual, we have $H^* \cong R^*\#\mathcal{A}_{2,2}$ in the later.

Since $\pi : H \to A := H/J$ is injective on $\Bbbk G(H)$, 4 divides $|G(A)|$. Thus, by Theorem 2.18, if A is not semisimple, A is pointed. But every pointed Hopf algebra of dimension $4p$ has group of grouplikes of order $2p$ which is not divisible by 4, so A must be semisimple.

Since $\mathcal{A}_{2,2}$ is self-dual, we have $H^* \cong R^*\#\mathcal{A}_{2,2}$. The same argument as for H then gives us a Hopf algebra epimorphism from H^* to a semisimple Hopf algebra B of dimension $4p$ with coinvariants $\{1, y\}$ where y is $(1, h)$-primitive.

Then B^* is isomorphic to a sub-Hopf algebra of H, call it L. Since L is cosemisimple, $L \subseteq H_0$ and we wish to show equality. Since L has dimension $4p$, the sub-Hopf algebra $\langle L, x\rangle$ of H generated by L and x is all of H. Since $\pi(x) = 0$, this means that by dimensions π is injective on L and so $\pi : L \cong A$ is a Hopf algebra isomorphism. This implies that $H \cong S\#A$ where $S = \Bbbk\{1, x\}$ is a braided Hopf algebra in ${}^A_A\mathcal{YD}$ and thus H has the Chevalley property. Reversing the roles of H^* and H in the above argument we get that H^* also has the Chevalley property.

(ii) Now suppose that $|G(K)| = 2$ and $K \cong \mathcal{A}_2$. Then $G(K) = \langle g\rangle$ and K is generated by g and two $(1, g)$-primitives, x and x'. Let J be the Hopf ideal of H generated by x and as in (i), $J = Hx$. Thus as a $\Bbbk$-space, $J = Span\{r_i g^j z | r_1, \dots, r_p$ a basis for $R, j = 0, 1, z \in \{x, x'x\}\}$. Thus $\dim J \leq 4p$ so that $\dim H/J \geq 4p$ and is a divisor of $8p$ so $\dim H/J = 4p$ and as above there is a Hopf algebra epimorphism $\pi : H \to A$ where $A := H/J$ is a Hopf algebra of dimension $4p$ and $H^{co\pi} = \{1, x\}$. Since $\pi(x'), \pi(g)$ generate a sub-Hopf algebra of A isomorphic to H_4, then A is not semisimple. If A^* is pointed, then H^* has grouplikes of order $2p$. This is a contradiction since $H^* \cong R^*\#K^*$ with $K^* \cong \mathcal{A}_2$, so that by Proposition 4.12(i), $G(H^*) = G(\mathcal{A}_2) \cong C_2$. Suppose that A is pointed. Since A^* is not pointed, then $A \cong \mathcal{A}(-1,1)$ in the notation of Section 2.8.1. But this is impossible since $\mathcal{A}(-1,1)$ has no sub-Hopf algebra isomorphic to H_4. By Theorem 2.19, for $p \leq 11$, A is either semisimple, pointed or basic. □

Corollary 4.19. *Suppose $H \cong R\#K$ where K is a pointed Hopf algebra of dimension 8, and R is commutative and semisimple.*

(i) If $|G(H)| = 2$, then there is a Hopf algebra map π from H onto a Hopf algebra A of dimension $4p$ which is nonsemisimple, nonpointed and nonbasic.

(ii) If $|G(H)| = 4$ and furthermore K is basic and R^* is commutative and semisimple, then H and H^* have the Chevalley property.

Proof. It remains only to show that under the given conditions there is a $(1,g)$-primitive x such that $x \cdot R = 0$. Since R is semisimple commutative, R can be written as the sum of one-dimensional simple ideals Re_i with e_i a central primitive idempotent. Now apply Lemma 4.17 and Proposition 4.18. □

Corollary 4.20. *If $\dim H = 24$ and H has type $(4,4)$, then H and H^* have the Chevalley property.*

Proof. By Corollary 4.15, $H \cong R\#K$ where K, K^* are pointed Hopf algebras of dimension 8, R is a semisimple braided Hopf algebra in ${}^K_K\mathcal{YD}$ of dimension 3, and R^* is a semisimple braided Hopf algebra in ${}^{K^*}_{K^*}\mathcal{YD}$ of dimension 3. Since all simple representations of R and R^* must be one-dimensional, R, R^* are commutative and the result follows from Corollary 4.19. □

Remark 4.21. Suppose that H is of type $(2^i, 2^j)$, has dimension 24 and $H \cong R\#K$ where K is pointed of dimension 8. Then $|G(H)| \neq 2$. For by Proposition 4.14, R and R^* are semisimple and thus, since both have dimension 3, they are commutative also. Then the conditions of Proposition 4.18 hold.

The next remark summarizes the results proved for various particular dimensions $8p$, with H nonsemsimple, nonpointed, nonbasic as assumed throughout this section.

Remark 4.22.
(i) From Remark 4.3, if $\dim H = 24, 40, 56$, then $|G(H)| \neq 8$.
(ii) From Corollary 4.8, if $\dim H = 24, 40$, then type $(2p, 2)$ is impossible and for type $(2p, 4)$, $G(H^*) \cong C_4$.
(iii) From Corollary 4.10, if $p = 3, 7, 11$, $|G(H)| = 2p$, H^* does not have the Chevalley property, and H does not contain a copy of $\mathcal{A}(-1,1)$, i.e., we are in Case (iii) of Proposition 4.7, then $G(H^*) \ncong C_4$.
(iv) From Corollary 4.11 if $\dim H = 24$ and H has type $(6,4)$ then if H^* does not have the Chevalley property, then H has a sub-Hopf algebra isomorphic to $\mathcal{A}(-1,1)$, $\dim H_0^* = 20$ and as coalgebras, either $H^* \cong \mathcal{A}_4'' \oplus \mathcal{M}^*(2,\Bbbk)^4$ or $H^* \cong \mathcal{A}_4'' \oplus \mathcal{M}^*(4,\Bbbk)$.
(v) By Corollary 4.20, if $\dim H = 24$ and H does not have the Chevalley property, then H is not of type $(4,4)$.

4.4. **Hopf algebras of dimension** 24. In this subsection we specialize to the case of $p = 3$, $\dim H = 24$. Unless otherwise stated, throughout this section H will denote a Hopf algebra without the Chevalley property.

Our first result is a general statement for all Hopf algebras of dimension $8p$ and will need the following remark about nonabelian groups of order $4p$.

Remark 4.23. Suppose that L is a nonabelian group of order $4p$, p an odd prime. Then unless $p = 3$ and $L = \mathbb{A}_4$, L has a normal subgroup N of order p. (This follows from the Sylow Theorems; see, for example, [L, p. 34].) Then there is a Hopf algebra map from $\Bbbk L$ to $\Bbbk(L/N)$ where L/N is a group of order 4. Dualizing we see that $\Bbbk^L$ contains a sub-Hopf algebra isomorphic to a group algebra of dimension 4 and thus $G(\Bbbk^L)$ is a group of order 4.

Proposition 4.24. *Let H be a nonsemisimple, nonpointed nonbasic Hopf algebra with $\dim H = 8p$ and suppose H has a simple subcoalgebra D of dimension 4 stable under the antipode. Then H has a nontrivial grouplike element of order 2.*

Proof. Let $\mathcal{H}$ denote the sub-Hopf algebra of H generated by D. Then $\dim \mathcal{H} \neq 2, 4, p$ and so $\dim \mathcal{H} = 8, 2p, 4p$ or $8p$.

If $\dim \mathcal{H} = 8$, then by the classification of Hopf algebas of dimension 8, [W], [Ş], $\mathcal{H} \cong \Bbbk[C_2 \times C_2] \oplus \mathcal{M}^*(2, \Bbbk)$ as coalgebras if $\mathcal{H}$ is semisimple and $\mathcal{H} \cong H_4 \oplus \mathcal{M}^*(2, \Bbbk)$ if $\mathcal{H}$ is basic. In either case, $\mathcal{H}$, and thus H, contains a grouplike element of order 2.

If $\dim \mathcal{H} = 2p$, then by [Ng3], $\mathcal{H}$ is semisimple, so that $\mathcal{H} = \Bbbk^{\mathbb{D}_p}$ and has a grouplike of order 2.

Now suppose that $\dim \mathcal{H} = 4p$. By Proposition 2.13, $\mathcal{H}$ fits into a central exact sequence:

$$\Bbbk^G \overset{i}{\hookrightarrow} \mathcal{H} \overset{\pi}{\twoheadrightarrow} A$$

for a group G and A a nonsemisimple basic Hopf algebra. Then $|G| \in \{1, 2, 4, p, 2p, 4p\}$. If $|G| = 1$, then $\mathcal{H}$ is nonpointed nonsemisimple but has pointed dual, so by Subsection 2.8.1, $\mathcal{H} \cong \mathcal{A}(-1, 1)^* \cong H_4 \oplus \mathcal{M}^*(2, \Bbbk)^{p-1}$ as coalgebras and consequently has a grouplike element of order 2. If $|G| \in \{2, 4, 2p\}$, then $\Bbbk^G$ has also a grouplike element of order 2. If $|G| = p$, then p divides $|G(\mathcal{H})|$ and $|G(H)|$ so that by Proposition 4.2, $G(H) \cong C_{2p}$ and H has a grouplike of order 2. If $|G| = 4p$, then $\mathcal{H} = \Bbbk^G$ for G a nonabelian group of order $4p$. By Remark 4.23, $\Bbbk^G$ has a group of grouplikes of order 4 unless $p = 3$, $G = \mathbb{A}_4$ and the dimension of $\mathcal{H}$ is 12. But if $\mathcal{H} = \Bbbk^{\mathbb{A}_4}$ does not have a grouplike of order 2, then as a coalgebra $\Bbbk^{\mathbb{A}_4} \cong \Bbbk C_3 \oplus \mathcal{M}^*(3, \Bbbk)$. But $\mathcal{H}$ has a simple subcoalgebra of dimension 4, so this case is impossible.

Finally, assume that D generates H so that as above, we have an exact sequence $\Bbbk^G \overset{i}{\hookrightarrow} H \overset{\pi}{\twoheadrightarrow} A$ for a group G and A a nonsemisimple basic Hopf algebra. Since H is assumed to be nonbasic, then $|G| \neq 1$, and since H is nonsemsimple, $|G| \neq 8p$. The argument above shows that if $|G| \in \{2, 4, p, 2p\}$, then H has a grouplike element of order 2. If $|G|$ is 8 or $4p$, then A has dimension p or 2 respectively and so must be semisimple. This would imply that H is semisimple, a contradiction. □

Lemma 4.25. *If $\dim H = 24$ then H has a grouplike element of order 2.*

Proof. By Proposition 4.2, $G(H) \ncong C_p = C_3$ so it suffices to show that H has a nontrivial grouplike element, i.e., that H_0 is not of the form $\Bbbk \cdot 1 \oplus E$ where E is a sum of simple subcoalgebras of dimension greater than 1. Suppose that

$$H_0 = \Bbbk \cdot 1 \oplus \oplus_{i=1}^{t} D_i \text{ where } D_i \cong \mathcal{M}^*(n_i, \Bbbk) \text{ and } n_j \leq n_{j+1}.$$

By Proposition 2.9, $\dim H_0 \leq 15$ so that the possibilities for H_0 are $H_0 = \Bbbk \cdot 1 \oplus \mathcal{M}^*(2, \Bbbk)^s$ with $s = 1, 2, 3$, $H_0 = \Bbbk \cdot 1 \oplus \mathcal{M}^*(3, \Bbbk)$ or $H_0 = \Bbbk \cdot 1 \oplus \mathcal{M}^*(2, \Bbbk) \oplus \mathcal{M}^*(3, \Bbbk)$. If H has a simple subcoalgebra of dimension 4 stable under the antipode then by Proposition 4.24, H has a grouplike element of order 2. If $H_0 = \Bbbk \cdot 1 \oplus \mathcal{M}^*(3, \Bbbk)$ then Proposition 2.9 implies that $\dim H \geq 26$, a contradiction. Thus only the cases $H_0 = \Bbbk \cdot 1 \oplus \mathcal{M}^*(2, \Bbbk)^s$ with $s = 2, 3$ and $S(D_i) \neq D_i$ remain.

Suppose that $H_0 = \Bbbk \cdot 1 \oplus \sum_i D_i$ with $D_i \cong \mathcal{M}^*(2, \Bbbk)$ and $S(D_i) = D_j$ for some $j \neq i$. Note that $\dim H_0 > 8$. Let $\mathcal{D}$ denote the set of D_i. Then since 4 divides $\dim D_i$, $2 \dim P^{1,\mathcal{D}}$, and $\dim P^{\mathcal{D},\mathcal{D}}$, then

4 divides $1+\dim P^{1,1}$ and $\dim P^{1,1}\geq 3$. Thus by Lemma 2.6, $P_{\ell}^{1,1}$ is nondegenerate for some $\ell>2$. Then $P_m^{1,D_i}, P_1^{D_i,1}, P_1^{1,S(D_i)}$ are nondegenerate for $m=\ell-1\geq 2$, some i. Then $2\dim P^{1,\mathcal{D}}\geq 8$. Since $P_1^{D_i,1}$ and $P_m^{S(D_i),1}$ are nondegenerate then $P^{D_i,\mathcal{D}}$ and $P^{S(D_i),\mathcal{D}}$ are nondegenerate and $\dim P^{\mathcal{D},\mathcal{D}}\geq 8$. But this is impossible if $\dim H=24$. □

Remark 4.26. Similar arguments to the proof of Lemma 4.25 apply if $\dim H=4n$ and $H_0=\Bbbk\cdot 1\oplus\sum_{i=1}^{t} D_i$ with $D_i=\mathcal{M}^*(2,\Bbbk)$ and $D_i\neq S(D_i)$ for all i. Let $\mathcal{D}$ denote the set of D_i. Then $2\dim P^{1,\mathcal{D}}+\dim P^{\mathcal{D},\mathcal{D}}\geq 20$ where $\mathcal{D}$ denotes the set of simple 4-dimensional subcoalgebras.

For, we may suppose that $P^{1,1}=P_{\ell}^{1,1}$ with $\ell\geq 3$. Then $P_1^{1,C}, P_{\ell-1}^{C,1}, P_1^{S(C),1}, P_{\ell-1}^{C,E}, P_{\ell-2}^{C,D}, P_1^{D,1}$ are nondegenerate for some $C,D,E\in\mathcal{D}$ so that $2\dim P^{1,\mathcal{D}}\geq 8$ and $\dim P^{\mathcal{D},\mathcal{D}}\geq 8$. Furthermore, since $\ell-1\geq 2$, then $P_1^{C,X}, P_{\ell-2}^{X,E}$ are nondegenerate for some coalgebra X. If $\dim X=1$, then $P_1^{C,1}, P_1^{S(C),1}, P_{\ell-1}^{C,1}$ are nondegenerate and $2\dim P^{1,\mathcal{D}}\geq 12$. If $\dim X=4$, then $P_1^{C,X}, P_{\ell-1}^{C,E}, P_{\ell-2}^{C,D}$ are nondegenerate and the statement follows.

We finish the section with the proof of Theorem B.

Proof of Theorem B. Let $\dim H=24$ and suppose that H does not have the Chevalley property. Then $|G(H)|\neq 1,3,8,12$ or 24, by Lemma 4.25, Remark 4.3 and Proposition 4.2. Since $|G(H)|$ divides $\dim H$, we have that $|G(H)|\in\{2,4,6\}$ and by Remark 4.22, the proof is complete. □

5. Open cases

The following table enumerates all open cases in the classification of Hopf algebras of dimension less than 100 up to isomorphism. In this table, p is arbitrary, not necessarily odd.

$\dim H$	**Semisimple**	**Pointed**	**Chevalley**	**Other**
p	**Completed:** All trivial [Z]	**None**	**None**	**None:** [Z]
$2p$ p odd	**Completed:** All trivial [Mas5][1]	**None**	**None**	**None:** [Ng3]
p^2	**Completed:** All trivial [Mas3]	**Completed:** $\exists\ p-1$, the Taft Hopf algebras [AS1]	**None**	**None:** [Ng1]
pq	**Completed:** All trivial [Mas2, Ng3, EGel1, GelW, So, N2]	**None**	**None**	**None:** for $p < q \leq 4p+11$ [Ng4] **Open:** 87, 93.
p^3	**Completed:** $p=2$, $\exists$ 1 [KacP] [Mas2] p odd , $\exists\ p+1$ [Mas4]	**Completed:** $p=2$, $\exists$ 5 [Ş] p odd $\exists\ (p-1)(p+9)/2$ [AS2, CD, ŞvO]	**None**	**None** : 8 [W], [Ş] 27 [G], [BG]
$2p^2$ p odd	**Completed:** $\exists$ 2, they are duals [Mas1], [N1]	**Completed:** $\exists\ 4(p-1)$ [AN1, A.1]	**None**	**None:**[2][HNg]
pq^2 p odd	**Completed:**[3] [Gel1, Mas1, N1, N2, N3, ENO]	**Completed:** $\exists\ 4(q-1)$ [AN1, A.1]	**None:** [AN1, Lemma A.2]	**Completed:** 12 [N5] 20, 28, 44 [ChNg] **Open:** 45, 52, 63, 68, 75, 76, 92, 99.
pqr	**Completed:**[4] [N1, N2, ENO]	**None**	**None:** Prop. 3.2	**Completed:** 30 [Fu3] **Open:** 42, 66, 70, 78
p^4	**Completed:** $p=2$, $\exists$ 16 [K, Theorem 1.2] **Open:** 81	**Completed:** 16; $\exists$ 29 [CDR] [5] **Completed:** p odd [AS2]. Infinite nonisomorphic families exist [AS2], [BDG], [Gel2][10]	**Completed:** 16 [CDMM] $\exists$ 2 selfdual, coradical A_8 **Open:** 81	**Completed:** 16 [GV] **Open:** 81
p^3q	**Open**	**Completed:**[6] 24, 40, 54, 56 [Gr1] **Open:** 88 [9]	**Open**	**Open:** 24, 40, 54, 56, 88.
p^2q^2	**Open**	**Completed:** 36 [Gr1] **Open:** 100 [9]	**Open**	**Open**: 36, 100
p^2qr	**Open**	**Completed:** 60 [Gr1] **Open:** 84, 90 [9]	**Open**	**Open:** 60, 84, 90
p^3q^2	**Open**	**Open**[7]	**Open** [8]	**Open:** 72
p^n $n=5,6$	**Open**	**Completed:** 32. [Gr2] Infinite families of nonisomorphic Hopf algebras exist. [Gr2], [B1][10] **Open:** 64	**Open**	**Open:** 32, 64
p^4q	**Open**	**Completed:** 48 [Gr1] **Open:** 80	**Open**	**Open:** 48, 80
p^5q	**Open**	**Open**[6]	**Open**	**Open:** 96

Table 1. Hopf algebras of dimension ≤ 100

[1]Dimension 6 was classified in [Mas2].

[2]The classification for dimension $18 = 2(3^2)$ was completed in [Fu1].

[3]The complete classification of semisimple Hopf algebras of dimension $12 = 3(2^2)$ is given in [F].

[4]The complete classification of semisimple Hopf algebras of dimension 30 and 42 is given in [N4].

[5]The duals to these are explicitly constructed in [B3].

[6]Pointed Hopf algebras H with $\frac{\dim H}{|G(H)|} < 32$ or $\frac{\dim H}{|G(H)|} = p^3$ were classified in [Gr1].

[7]Pointed Hopf algebras with nonabelian grouplikes known to exist by [AHS] dimension p^3p^2 , [FG] dimension p^5q.

[8]Nonpointed Hopf algebras with Chevalley property known to exist [AV1, AV2].

[9]$\dim p^3q, p^2q^2, p^2qr$: For dimensions 88, 100, 84, 90, the classification of the pointed Hopf algebras was completed for those with coradical a group algebra of order a power of 2 in [Ni] and [Gr1].

[10]The families of nonisomorphic pointed Hopf algebras of dimension 81 consist of quasi-isomorphic Hopf algebras [Mas6] but the duals of the families of nonisomorphic pointed Hopf algebras of dimension 32 give an infinite family of non-quasi-isomorphic Hopf algebras [EGel2].

The columns from left to right describe the classification of Hopf algebras which are semisimple, pointed nonsemisimple, nonsemisimple nonpointed with the Chevalley property, etc. We call a Hopf algebra *trivial* if it is a group algebra or the dual of a group algebra. For dimension mn^2, pointed Hopf algebras always exist; just take $\Bbbk C_m \otimes T_q$ where q is a primitive nth root of unity.

Note that by [AN1, Prop. 1.8], a Hopf algebra of square-free dimension cannot be pointed. Also note if for every divisor m of some dimension n the only semisimple Hopf algebras of dimension m are the group algebras, then there are no Hopf algebras of dimension n with the Chevalley property. For example, this is why there are no nonpointed Hopf algebras of dimension p^3 with the Chevalley property.

Examples of nonpointed but basic Hopf algebras do exist. They are given by duals of nontrivial liftings which are not Radford bosonizations. See for example [B2].

In general, this table does not contain references to partial results for a particular dimension even though the literature may contain some. For example the general classification for dimension 24 is listed only as Open. Also when a general result has been proven, the table cites only that result. For example, [HNg] is cited for the result that all Hopf algebras of dimension $2p^2$, p odd, are semisimple or pointed; the specific case of dimension 18 was proved in [Fu1]. We have attempted to include references to some specific cases in the footnotes but make no claim that these are complete.

References

[AD] N. Andruskiewitsch and J. Devoto J, *Extensions of Hopf algebras*, Algebra i Analiz **7**, No. 1 (1995), 22-61; translation in St. Petersburg Math. J. **7** (1996), no. 1, 17–52.

[AEGel] N. Andruskiewitsch, P. Etingof and S. Gelaki, *Triangular Hopf algebras with the Chevalley property*, Michigan Math. J. **49**, no. 2 (2001), 277–298.

[AHS] N. Andruskiewitsch, I. Heckenberger and H.-J. Schneider, *The Nichols algebra of a semisimple Yetter-Drinfeld module*, Amer. J. Math., vol. 132, no. 6, December 2010, 1493–1547.

[AN1] N. Andruskiewitsch and S. Natale, *Counting arguments for Hopf algebras of low dimension*, Tsukuba Math J. **25**, No. 1 (2001), 187–201.

[AN2] ———, *Examples of self-dual Hopf algebras*, J. Math. Sci. Univ. Tokyo **6** (1999), 181-215.

[AS1] N. Andruskiewitsch and H.-J. Schneider, *Hopf algebras of order p^2 and braided Hopf algebras of order p*, J. Algebra **199** (1998), no. 2, 430–454.

[AS2] ———, *Lifting of quantum linear spaces and pointed Hopf algebras of order p^3*, J. Algebra **209** (1998), no. 2, 658–691.

[AV1] N. Andruskiewitsch and C. Vay, *Finite dimensional Hopf algebras over the dual group algebra of the symmetric group in three letters*, Comm. Algebra **39** (2011), 4507–4517.

[AV2] ———, *On a family of Hopf algebras of dimension 72*, Bulletin of the Belgian Mathematical Society, to appear. Preprint: arXiv:1105.0394.

[B1] M. Beattie, *An isomorphism theorem for Ore extension Hopf algebras*, Comm. Algebra, **28** (2) (2000) 569-584.

[B2] ———, *Duals of pointed Hopf algebras*, J. Algebra **262** (2003), 54–76. Corrigendum J. Algebra **320** (2008), 3849.

[B3] ———, *A survey of Hopf algebras of low dimension*, Acta Appl. Math. **108** (2009), no. 1, 19–31.

[BD] M. Beattie and S. Dăscălescu, *Hopf algebras of dimension 14*, J. London Math. Soc. II Ser. **69**, No.1 (2004), 65–78.

[BDG] M. Beattie, S. Dăscălescu and L. Grünenfelder, *On the number of types of finite dimensional Hopf algebras*, Invent. Math. **136** 1-7, 1999.

[BG] M. Beattie and G.A. García, *Techniques for classifying Hopf algebras and applications to dimension p^3*, Comm. Algebra, to appear.

[CD] S. Caenepeel and S. Dascalescu, *Pointed Hopf algebras of dimension p^3*, J. Algebra **209** (1998), no. 2, 622–634.

[CDR] S. Caenepeel, S. Dăscălescu and Ş. Raianu, *Classifying pointed Hopf algebras of dimension 16*, Comm. Algebra **28**, No. 2 (2000), 541–568.

[CDMM] C. Calinescu, S. Dăscălescu, A. Masuoka, C. Menini, *Quantum lines over non-cocommutative cosemisimple Hopf algebras*, J. Algebra **273** (2004), 753–779.

[ChNg] Yi-lin Cheng and Siu-Hung Ng, *On Hopf algebras of dimension* $4p$, J. Algebra **328** (2011), 399–419.

[EGel1] P. Etingof and S. Gelaki, *Semisimple Hopf algebras of dimension pq are trivial*, J. Algebra **210** (1998), 664–669.

[EGel2] ———, *On families of triangular Hopf algebras*, Int. Math. Res. Not. IMRN **1**4(2002), 757-768.

[EGel3] ———, *On Hopf algebras of dimension pq*, J. Algebra **277**, no. 2 (2004), 668–674.

[ENO] P. Etingof, V. Nikshych and V. Ostrik, *Weakly group-theoretical and solvable fusion categories*, Advances in Mathematics, **226** (2011), no. 1, 176–205.

[FG] F. Fantino and G. A. García, *On pointed Hopf algebras over dihedral groups*, Pacific J. Math, Vol. **252** (2011), no. 1, 69–91.

[FMS] D. Fischman, S. Montgomery and H.-J. Schneider, *Frobenius extensions of subalgebras of Hopf algebras*, Trans. Amer. Math. Soc. **349** (1997), no. 12, 4857–4895.

[F] N. Fukuda, *Semisimple Hopf algebras of dimension* 12, Tsukuba J. Math. **21** (1997), 43–54.

[Fu1] D. Fukuda, *Classication of Hopf algebras of dimension* 18, Israel J. Math. **168** (2008), 119–123.

[Fu2] ———, *Structure of coradical filtration and its application to Hopf algebras of dimension pq*. Glasg. Math. J. **50** (2008), no. 2, 183–190.

[Fu3] ———, *Hopf algebras of dimension 30*, J. Algebra Appl. **9** (2010), no. 1, 11–15.

[G] G. A. García, *On Hopf algebras of dimension* p^3, Tsukuba J. Math. 29 (2005), no. 1, 259–284.

[GV] G. A. García and C. Vay, *Hopf algebras of dimension 16*, Algebr. Represent. Theory, **13** (2010), 383–405.

[Gel1] S. Gelaki, *Quantum groups of dimension* pq^2, Israel J. Math. **102** (1997), 227–267.

[Gel2] ———, *Pointed Hopf algebras and Kaplansky's* 10*th conjecture*, J. Algebra **209**, (1998) 635-657 .

[GelW] S. Gelaki and S. Westreich, *On semisimple Hopf algebras of dimension pq*, Proc. Am. Math. Soc. **128** (2000), 39–47. (Corrigendum: Proc. Am. Math. Soc. **128** (2000), 2829–2831.)

[Gr1] M. Graña, *On Nichols algebras of low dimension*, Contemp. Math. **267** (2000), 111–134.

[Gr2] ———, *Pointed Hopf algebras of dimension 32*, Comm. Algebra **28** (6) (2000), 2935–2976.

[HNg] M. Hilgemann and S.-H. Ng, *Hopf algebras of dimension* $2p^2$, J. Lond. Math. Soc. (2) **80** (2009), no. 2, 295–310.

[KacP] G. Kac and V. Paljutkin, *Finite ring groups*, Trudy Moskov. Mat. Obš č. **15** (1966), 224–261.

[K] Y. Kashina, *Classification of semisimple Hopf algebras of dimension* 16, J. Algebra **232** (2000), 617–663.

[KMM] Y. Kashina, G. Mason and S. Montgomery, *Computing the Frobenius-Schur indicator for abelian extensions of Hopf algebras*, J. Algebra **251** (2002), 888–913.

[L] S. Lang, *Algebra*. Revised third edition. Graduate Texts in Mathematics, 211. Springer-Verlag, New York, 2002.

[LR] R. Larson and D. Radford, *Finite dimensional cosemisimple Hopf algebras in characteristic* 0 *are semisimple*, J. Algebra **117** (1988), 267–289.

[Mas1] A. Masuoka, *Some further classification results on semisimple Hopf algebras*, Comm. Algebra **24** (1996), 307–329.

[Mas2] ———, *Semisimple Hopf algebras of dimension* 6*,* 8, Israel J. Math. **92** (1995), 361–373.

[Mas3] ———, *The* p^n *theorem for semisimple Hopf algebras*, Proc. Amer. Math. Soc. **124** no. 3 (1996), 735–737.

[Mas4] ———, *Self dual Hopf algebras of dimension* p^3 *obtained by extension*, J. Algebra **178** (1995), 791–806.

[Mas5] ———, *Semisimple Hopf algebras of dimension* $2p$, Comm. Algebra **23** (1995), 1931–1940.

[Mas6] ———, *Defending the negated Kaplansky conjecture*, Proc. Amer. Math. Soc. **129** (11), (2001), 3185-3192.

[M] S. Montgomery, *Hopf Algebras and their Actions on Rings*, CMBS Reg. Conf. Ser. in Math. **82**, Amer. Math. Soc., 1993.

[N1] S. Natale, *On semisimple Hopf algebras of dimension* pq^2, J. Algebra **221** (1999), 242–278.

[N2] ———, *On semisimple Hopf algebras of dimension* pq^2*, II*, Algebr. Represent. Theory **5** (3), (2001), 277–291.

[N3] ———, *On semisimple Hopf algebras of dimension* pq^r, Algebr. Represent. Theory **7** (2) (2004), 173–188.

[N4] ———, *Semisolvability of semisimple Hopf algebras of low dimension*, Mem. Amer. Math. Soc., **186** (2007), no. 874.

[N5] ———, *Hopf algebras of dimension 12*, Algebr. Represent. Theory **5**, No. 5 (2002), 445–455.

[Ng1] S-H. Ng, *Non-semisimple Hopf algebras of Dimension* p^2, J. Algebra **255**, no. 1 (2002), 182–197.

[Ng2] ______, *Hopf algebras of dimension pq*, J. Algebra **276**, no. 1 (2004), 399–406.

[Ng3] ______, *Hopf algebras of dimension* $2p$, Proc. Amer. Math. Soc. **133**, no. 8 (2005), 2237–2242 (electronic).

[Ng4] ______, *Hopf algebras of dimension pq, II*, J. Algebra **319**, no. 7 (2008), 2772–2788.

[Ni] W. Nichols, *Bialgebras of type one.* Comm. Algebra **6**, no. 15 (1978), 1521–1552.

[So] Y. Sommerhäuser, *Yetter-Drinfel'd Hopf algebras over groups of prime order*, Lecture Notes in Mathematics, 1789. Springer-Verlag, Berlin, 2002. iv+158 pp.

[Ş] D. Ştefan, *Hopf algebras of low dimension*, J. Algebra **211** (1999), 343–361.

[ŞvO] D. Stefan and F. van Oystaeyen, *Hochschild cohomology and coradical filtration of pointed coalgebras: Applications*, J. Algebra **210** (1998), no. 2, 535–556.

[S] M. Sweedler, *Hopf algebras*, Benjamin, New York, (1969).

[W] R. Williams, *Finite dimensional Hopf algebras*, Ph. D. Thesis, Florida State University (1988).

[Z] Y. Zhu, *Hopf algebras of prime dimension*, Int. Math. Res. Not. **1** (1994), 53–59.

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