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# *Stable Sets for Exchange Economies with Interdependent Preferences*

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**Abstract**

We introduce the notion of stable sets with externalities and address the existence problem. The importance of this solution concept is related to the fact that the existence of core allocations for exchange economies is not in general assured in a framework with more than two traders.

**JEL Classification**: C71, D51, D70.

**Keywords**: Stable sets; interdependent preferences; core; types of agents.

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# 1 Introduction

In a companion paper (see, Graziano et al., 2015), we analyzed the notion of stable sets à la von Neumann and Morgenstern in a general equilibrium context represented by exchange economies with asymmetrically informed agents and an exogenous rule that regulates the information sharing among agents. Two different frameworks were taken into account: a model without expectations and a model with expected utilities. For the former, we showed that the set V of all individually rational, Pareto optimal, symmetric allocations is the unique stable set of symmetric allocations. For the latter, an example was presented which showed that the same set V is not externally stable and a weaker result was proved. In the present paper, we analyze a framework where there is a consumption externality; that is, traders do not care only about their own consumption but may have interdependent preferences.

Precisely, we are interested here in economies with a finite number $l$ of commodities and a finite set $N$ formed by $n$ agents, possibly partitioned into types, whose utility functions depend not only on their own consumption bundles but on the consumption profile of all traders in the economy. The utility function $U_i$ of trader $i \in N$ is thus given by:

$$U_i : \mathbb{R}^{l \cdot n}_{+} \longrightarrow \mathbb{R}.$$

In this framework, each agent $i \in N$ chooses, according to his preferences, a distribution of goods $x \in \mathbb{R}^{l \cdot n}_{+}$ among all traders in the market. This way of modeling preferences captures public good models and can also be increasingly encountered in social and behavioral economics when dealing with reciprocity and/or fairness issues (see, among others, Sobel (2005), Dufwenberg et al. (2011), Velez (2016)).

We also suppose that the consumption set of agent $i \in N$ is a subset of $\mathbb{R}^{l}_{+}$ that can vary according to the coalition he joins. The notation $X_i(S)$ will be used to denote the consumption set of agent $i$ when he belongs to coalition $S$. This way of modelling consumption sets, which has been also adopted for different purposes in del Mercato (2006), recovers the traditional case where $X_i(S) = \mathbb{R}^{l}_{+}$, for all $i \in N$ and for all $S \subseteq N$, and is general enough to cover the asymmetric information framework with an exogenous rule that regulates information sharing among agents. In such a context, traders' consumption sets do not coincide with the positive orthant of the commodity space: due to the information constraints, they are proper subsets of the positive orthant, they differ from agent to agent depending on the initial information and, moreover, they can vary according to what coalition is joined and what are the opportunities of communication within a coalition.

The notion of stable set has been first introduced by von Neumann and Morgenstern

(1944); both for games and exchange economies, it can be easily formalized and does not present ambiguity once a coalitional dominance relation has been defined over the set of allocations. In particular, it hinges on two requirements of self-consistency which can be interpreted as acceptable behaviors within a society. For the framework of an exchange economy, a non empty set $V$ of individually rational allocations is said to be stable if it is internally and externally stable. Internal stability means that no allocation in $V$ is dominated by any other allocation in $V$. External stability requires that every allocation that does not belong to $V$ is dominated by some element in $V$.

The argument behind these two conditions can be summarized as follows: internal stability guarantees that, once an outcome within $V$ is selected, there is no interest in deviating toward any other outcome; according to the second condition, on the contrary, every outcome which is not in $V$ is unstable because it is dominated by some element in $V$.

When compared with the more traditional core notion, it is clear that only one of these two properties, that is, the internal stability, holds for the core. With regard to the external stability, a core outcome must be undominated by any other outcome, including those that can, in their turn, be dominated. Stable sets can be considered as arising exactly by this conceptual deficiency of the core[1].

For a context where there is an externality in consumption, the ease in defining a stable set contrasts with some difficulties in defining a coalitional dominance relation between two allocations. Such difficulties are mainly related to the fact that several possibilities can be considered. Generally speaking, a feasible allocation $y$ is dominated by another feasible allocation $x$ if some coalition $S$ prefers the latter distribution, in the sense that all its members gain higher utility. However, given that the utility of each trader depends on the consumption bundles of every other agents in the economy, the issue becomes more subtle: in fact, several reactions by the complementary coalition $N \setminus S$ can be considered which give rise to different scenarios.

In the present paper we adopt a taxonomy based on the viewpoint of the blocking coalition $S$ towards the reaction of the counter–coalition $N \setminus S$: we distinguish between a pessimistic (or conservative) and an optimistic (or non-conservative) attitude of the deviating coalition $S$. This distinction results into three different notions of dominance relations that we name $\alpha$, $\alpha_1$ and $\alpha_2$ dominance.

In the first notion we consider, a blocking coalition $S$ maintains a pessimistic attitude; that is, it considers as possible reactions by the complementary coalition $N \setminus S$ all redis-

[1]Some controversy regarding the very logic under the stable set concept has been arisen by Harsanyi (1974) that formulated the notion of sophisticated stable sets. The flaw that the new concept aims to amend is the following: in the stable set logic a deviation cannot be considered as valid if there is a further deviation toward some stable outcome.

tributions of the quantity $\sum_{i\in N\setminus S} e_i$ among its members. Formally, a coalition S blocks an allocation y if there is a feasible reallocation $x^S$ within the coalition that makes every trader in S better off with $x$ than $y$, regardless of how agents outside the coalition may redistribute their initial endowment among themselves. This notion is customarily referred to as $\alpha$–blocking[2]. The complete "optimistic" counterpart of the previous dominance mechanism is also considered, where a coalition $S$ blocks $y$ assuming that the complementary coalition $N \setminus S$ does not react and can only consume its initial endowment. This notion of dominance is referred to as $\alpha_1$–dominance throughout the paper[3]. Finally, under the $\alpha_2$–dominance, the deviating coalition $S$ considers all feasible redistributions by the complementary coalition $N \setminus S$ as possible reactions; however, maintaining an optimistic attitude, it is willing to deviate towards the new allocation $x$ as soon as at least one of these redistributions makes each of its members better off.
Notice that when agents are selfish, that is, their utility function depends only on their own consumption bundle, all previous dominance relations are equivalent.

A second element is added into the previous taxonomy which concerns the notion of agents' types. Types have been frequently considered when studying stable sets, particularly in order to treat the case of a non–atomic continuum of agents where each commodity is initially owned by only one type of agents (see, for example, Hart (1974) and Einy and Shitovitz (2003)). Here, we introduce types into our analysis because the issues of what bundles have to be assigned to traders with the same characteristics and how these traders influence reciprocally their utilities appear to be particularly important when there is a consumption externality.
When agents are selfish, there is no ambiguity on this point: two agents are said to be of the same type if they are identical as regards their initial endowments and their utility functions. With externalities in consumption, different definitions of what is intended for types can be properly conceived which differ in the way they deal with the utility functions. In the present paper we consider two possible definitions of type.
The first definition is completely in line with the selfish framework: we say that two agents are of the same type if they have exactly the same attributes, that is, the same initial endowment, the same consumption set in the grand coalition and the same (in-

[2]In connection with this terminology, the $\alpha$-core as a cooperative solution for a game with interdependent preferences, has been introduced in Aumann (1961) and studied in Scarf (1971). Yannelis (1991) uses a different definition for the $\alpha$-core of a pure exchange economy corresponding to the one that we follow in the present paper.

[3]This notion of dominance has been introduced by Chander and Tulkens (1995) under the name of $\gamma$–dominance.

terdependent) utility function. According to this point of view, the model is considered as an abstraction of the selfish one and the emphasis is on the utility that agents derive from the allocation as a whole. The second definition of type is more in the spirit of interdependence; in this case, agents of the same type may have utility functions with different functional forms and, as a consequence, they may get different utility levels from the same consumption profile. However, such utility functions have to exhibit some form of symmetry formalized by two requirements. The first one is an intra–type condition: the utility level of two traders of the same type must be the same after switching their consumption bundles and leaving everything else unchanged. The second requirement, which is an across–type condition, states that the utility of each agent does not change after permuting consumption bundles of traders of a different type; the interpretation is that the external effect produced by traders of the same type on that agent is anonymous. For instance, with two types and one commodity, we can consider the following interdependent utility functions for traders of the first type, denoted by $(1, 1)$ and $(1, 2)$:

$$U_{11}(x_{11}, x_{12}, x_{21}, x_{22}) = x_{11} + 2x_{12} + x_{21} + x_{22}$$

$$U_{12}(x_{11}, x_{12}, x_{21}, x_{22}) = 2x_{11} + x_{12} + x_{21} + x_{22}$$

Despite their utility functions are different, agent $(1, 1)$ and agent $(1, 2)$ can be considered of the same type from the preferences viewpoint.
By using the second notion of types, contrary to what happens with the first definition, the selfish model adopted in Einy and Shitovitz (2003) can be recovered.

For all the scenarios resulting from the previous taxonomy, we provide conditions ensuring that the set formed by all the individually rational, Pareto optimal, possibly type–symmetric allocations, denoted by $C_e(E)$ and called extreme core in the paper, is internally and externally stable, and therefore the unique stable set in the market. Results are provided even in the case where deviating coalitions may assume different attitudes with respect to the internal and the external stability, that is, the stability of the set $C_e(E)$ with respect to crossed dominance relations is also analyzed. As a by product of our results, we identify conditions under which the difference in pessimistic and optimistic attitudes by deviating coalitions does not affect the stable set solution.
Two points are worth noting about the results in this paper. First, the importance of the theorems about the stability of $C_e(E)$ is increased by the fact that the more traditional notion of core can frequently be empty in a framework with more than two agents and interdependent preferences (see, Holly (1994), Martins-da-Rocha and Yannelis (2011),

Dufwenberg et al. (2011)) [4]. Second, and more technically, a slightly different set of assumptions is used in each scenario resulting from the taxonomy. In particular, for the $\alpha$–dominance relation and the first notion of type, we completely dispense with the assumptions of monotonicity and concavity which are indispensable in the selfish models. The assumptions of monotonicity and concavity formulated on the whole allocations of resources are in general too stringent as they rule out specific behaviors that an agent may have towards the consumption of other agents. The assumption of strict quasi–concavity, with respect to the consumption bundle of each type, is adopted when working with the $\alpha$–dominance and the second notion of type. Also in this case, the use of monotonicity is limited to the special case of partially interdependent preferences: with partially interdependent preferences, due to monotonicity, the assumption of strict quasi–concavity, can be replaced by quasi–concavity.

The paper proceeds in the following order. In Section 2 we outline the economic model and we define all dominance relations and the two notions of agents' types. In Section 3 the existence and uniqueness of stable sets with respect to each dominance relation are faced; this section is divided into two subsections, each devoted to one kind of agents' type. Finally, Section 4 presents a final remark about the asymmetric information framework and the conclusions.

## 2 The economic model

We consider a pure exchange economy $E$ characterized by a finite population of agents, indexed by $i \in N = \{1, \ldots, n\}$, and a finite number $l$ of commodities.

$\mathbb{R}^l$ is the commodity space and every subset of $N$ is referred to as a coalition.

We suppose that the consumption set of agent $i \in N$ is a subset of $\mathbb{R}^l_+$ that can vary according to the coalition he joins. The notation $X_i(S)$ will be used to denote the consumption set of agent $i$ when he takes part of coalition $S$.

This way of modelling consumption sets recovers the traditional case where $X_i(S) = \mathbb{R}^l_+$, for all $i \in N$ and for all $S \subseteq N$. On the other hand, it is general enough to cover also an asymmetric information framework with an exogenous rule that regulates information sharing among agents. In such a context, in fact, traders' consumption sets do not coincide with the positive orthant of the commodity space: due to the information constraints, they are smaller subsets of it, they differ from agent to agent depending on the initial

[4] On the non-cooperative side, the existence of Nash-Walras equilibria for exchange economies where there is an external restriction for the consumption of goods has been recently proved by Hervès-Beloso et al. (2012).

information and, moreover, they can vary according to what coalition is joined and what are the opportunities of communication within the coalition (see Section 4 for details).
The initial endowment of physical resources for agent $i \in N$ is modelled by a vector in $\mathbb{R}^l_+$, denoted by $e_i$.
Throughout the paper, we assume that $e_i \in X_i(S)$, whichever coalition $S$ is considered with $i \in S$. The interpretation for this position is clear: the initial endowment is always available for trader $i$, irrespective of what are the coalitions he joins. Moreover, we also suppose that $X_i(N)$ is closed, convex and $0 \in X_i(N)$, for every $i \in N$.
In the economic model we consider, there is an externality in consumption. Precisely, agents are not necessarily self-interested but they care about others. Formally, the preferences of each trader $i \in N$ are specified via an interdependent utility function $U_i$ which depends not only on his own consumption bundle, but also on the consumption of the other traders in the economy; that is:

$$U_i : \mathbb{R}^{l \cdot n}_+ \longrightarrow \mathbb{R}.$$

In this framework each agent $i \in N$ chooses, according to his preferences, a distribution of goods $x \in \mathbb{R}^{l \cdot n}_+$ among all traders in the market.
Hereafter, the situation in which the utility of agent $i$ only depends on his contingent consumption bundle $x_i \in \mathbb{R}^l_+$ will be referred to as the selfish case.
Denoting by $\mathcal{P}_i(N)$ the set of all subsets $S$ of $N$ such that $i \in S$ and by $\mathcal{P}(\mathbb{R}^l_+)$ the set of all subsets of $\mathbb{R}^l_+$, $X_i : \mathcal{P}_i(N) \to \mathcal{P}(\mathbb{R}^l_+)$ , $i \in N$ is the correspondence which associates to each coalition $S$ the consumption set $X_i(S) \subseteq \mathbb{R}^l_+$ that trader $i$ has when he joins coalition $S$.
The exchange economy with externalities $E$ is thus formalized by the following collection:

$$E = \{N = \{1, \dots, n\};\ (X_i)_{i \in N};\ (U_i, e_i)_{i \in N}\}.$$

In the sequel, another element will be added to such description, that is, agents' types.
The following notions of assignment, allocation and individually rational allocation will be used throughout the paper.

**Definition** 2.1 (Assignment and allocation) Given a coalition $S \subseteq N$, an **assignment for** $S$ is a vector $y^S = (y_i)_{i \in S}$ such that:

**i)** $y_i \in X_i(S)$, for every $i \in S$ (*consumption set feasibility*);

**ii)** $\sum_{i \in S} y_i = \sum_{i \in S} e_i$ (*physical feasibility*).

An **allocation** for the economy $E$ is an assignment $x = (x_i)_{i \in N} \in \mathbb{R}^{l \cdot n}_+$ for the grand coalition $N$.

Given a coalition $S$, the notation $x = (y^S, z^{N\setminus S})$ will denote the vector of $\mathbb{R}_+^{l\cdot n}$ whose components are equal to $y_i$ if $i \in S$ and $z_i$ if $i \in N \setminus S$.

We adopt the following notion of individual rationality introduced by Yannelis (1991):

**Definition** 2.2 (Individually rational allocation) An **allocation** $x$ is said to be individually rational if, for every $i \in N$:

$$U_i(x) \geq U_i(e)\,,$$

where $e = (e_1, \ldots, e_n)$ is the initial endowment allocation.

We will denote by $I$ the set of all the individually rational allocations for the economy $E$.

**Remark 2.1** One point related to the previous definition is worth noting. For the concept of individual rationality defined above the following property holds true: if $x$ is an individually rational allocation and $y$ is such that $U_i(y) \geq U_i(x), \forall i \in N$, then $y$ is individually rational.
That is, the property of being individually rational is inherited through the dominance via the grand coalition $N$. The same implication, which is "natural" in selfish models, may not be true in an economy with externalities when different notions of individual rationality are considered (see, for example, Le Van et al. (2001)) .

## 2.1 Dominance relations, core and stable set notions

We define now the dominance relations that will be used throughout the paper and represent the basis for both the equilibrium concepts of core and stable sets.
Generally speaking, given two allocations $x$ and $y$, we say that the alternative allocation $x$ dominates the status quo $y$ if there exists a coalition $S \subseteq N$ such that $x$ is feasible for $S$ and ensures a better outcome to each one of its members. It is clear that the externality in consumption much influences the last point and leads to several distinct definitions. In fact, since utility functions depend on the consumption bundles of all traders, a potentially blocking coalition $S$ has to take into account the reaction that the complementary coalition $N \setminus S$ might oppose to a deviation $x$ from the status quo $y$. Among these reactions, the favorable ones for $S$ are those associated to a redistribution $z^{N\setminus S}$ of resources within $N \setminus S$ which guarantees, via the externality effect of the new allocation $(x^S, z^{N\setminus S})$, a higher utility level to each member of $S$.
For the safe of clarity, we stress that the definitions we consider in this paper derive from

a classification based on the following two elements: 1. the blocking coalition $S$ may have different viewpoints about what reactions are possible for the counter–coalition $N \setminus S$; 2. the blocking coalition $S$ may have different viewpoints about what reactions, among the possible ones, suffice to deviate.
As concerns the first element, the blocking coalition $S$ can maintain:

**a.** a pessimistic attitude; that is, $S$ considers all the redistributions $z^{N\setminus S}$ of the initial endowments as possible reactions by $N \setminus S$;

**b.** an optimistic attitude; that is, $S$ considers that $N \setminus S$ does not react to its deviation and just sticks to its initial endowment $e^{N\setminus S}$.

As concerns the second element, the deviating coalition $S$ can consider as enough for a deviation from $y$ to $x$:

**c.** the extreme situation in which all possible reactions $z^{N\setminus S}$ by $N \setminus S$ ensure a favorable redistribution $(x^S, z^{N\setminus S})$ for each member of $S$;

**d.** the fact that at least one of the possible reactions $z^{N\setminus S}$ by $N \setminus S$ determines a favorable redistribution $(x^S, z^{N\setminus S})$ for each member of $S$.

Next scheme summarizes the combinations one gets by pairing the previous elements and shows the greek letter that will be used hereafter to denominate the associated dominance relations:

| Expected reactions by $N \setminus S$ | Favorable reactions required to deviate | Dominance relation |
|---|---|---|
| all feasible redistributions | all feasible redistributions | $\alpha$ |
| no feasible redistribution | initial endowment distribution | $\alpha_1$ |
| all feasible redistributions | at least one feasible redistribution | $\alpha_2$ |

Formally, the dominance relations we will consider are defined as follows:

**Definition** 2.3 ($\alpha$-Dominance) Let $x$ and $y$ be two allocations of the economy $E$. We say that $x$ $\alpha$–dominates $y$, denoted by $x \succ_\alpha y$, if there exists a non empty coalition $S \subseteq N$ such that:

**a.** $x^S$ is an assignment for $S$;

**b.** for all $i \in S$ and for every allocation $z$ with $z^S = x^S$, it holds that $U_i(x^S, z^{N\setminus S}) > U_i(y)$.

**Definition** 2.4 ($\alpha_1$-DOMINANCE) Let $x$ and $y$ be two allocations of the economy $E$. We say that $x$ $\alpha_1$–dominates $y$, denoted by $x \succ_{\alpha_1} y$, if there exists a non empty coalition $S \subseteq N$ such that:

**a.** $x^S$ is an assignment for $S$;

**b.** $U_i(x^S, e^{N \setminus S}) > U_i(y)$, for all $i \in S$.

**Definition** 2.5 ($\alpha_2$-DOMINANCE) Let $x$ and $y$ be two allocations of the economy $E$. We say that $x$ $\alpha_2$–dominates $y$, denoted by $x \succ_{\alpha_2} y$, if there exists a non empty coalition $S \subseteq N$ such that:

**a.** $x^S$ is an assignment for $S$;

**b.** for all $i \in S$ and for an allocation $z$ with $z^S = x^S$, it holds that $U_i(x^S, z^{N \setminus S}) > U_i(y)$.

In the sequel, the expression "the coalition $S$ blocks the allocation $y$ through $x$" will be also used to express the same concepts of dominance defined above. Moreover, coalition $S$ will be referred to as a "blocking coalition" and sometime, for the sake of clarity, it will be explicitly included into the notation by writing $x \succ_\beta^S y$, where $\beta \in \{\alpha,\, \alpha_1,\, \alpha_2\}$.

Some remarks are in order.

In each of the previous dominance relations, no form of cooperation between the deviating coalition and the outsiders is assumed. Therefore, all of them reduce to the standard blocking mechanism in the case of selfish models.
The $\alpha$-dominance relation has been first introduced by Yannelis (1991); its optimistic counterpart, the $\alpha_1$ dominance relation, in which agents outside the deviating coalition are assumed to consume their initial endowments, has been studied in Chander and Tulkens (1995); the $\alpha_2$–dominance has been introduced by Hervés-Beloso and Moreno García (2016) in order to define the optimistic core as contrasted with the $\alpha$–core which is referred to as pessimistic.
Given two allocations $x$ and $y$, the following implications hold true:

$$x \succ_\alpha y \Longrightarrow x \succ_{\alpha_1} y \Longrightarrow x \succ_{\alpha_2} y \,.$$

In all the dominance relations we consider, the distribution of items reached after that blocking takes place, that is, $(x^S, z^{N \setminus S})$ or $(x^S, e^{N \setminus S})$, is physically feasible for the grand coalition $N$. By contrast, a framework where the complementary coalition $N \setminus S$ is supposed to have no freedom and/or possibility to react and, as a consequence, no market

feasibility condition is taken into account, is analyzed in Dufwenberg et al. (2011).
Finally, note that in the $\alpha$–dominance relation the allocation $x$ itself and the allocation that allots $x_i$ to agents in $S$ and the initial endowment $e_i$ to agents in $N \setminus S$ can always be considered among the allocations $z$ which appear in the definition.
Based on these coalition dominance relations, the notions of Pareto optimality, core and stable sets can be defined. For simplicity of exposition, we formalize the last two notions with respect to a generic $\beta$–dominance relation, with $\beta \in \{\alpha, \alpha_1, \alpha_2\}$.

**Definition** 2.6 (Pareto optimal allocation) An **allocation** $x$ is said to be Pareto optimal (or, efficient) if it cannot be blocked by the grand coalition $N$ through another allocation $y$. That is, there does not exist another allocation $y$ such that:

$$U_i(y) > U_i(x), \text{ for every } i \in N .$$

**Definition** 2.7 ($\beta$-Core) Let $\beta \in \{\alpha, \alpha_1, \alpha_2\}$. An allocation $x$ is said to be a **$\beta$-core allocation** for the economy $E$ if it cannot be $\beta$–dominated by any coalition.
We will denote by $C^\beta(E)$ the set of the $\beta$–core allocations for the economy $E$.

By the implications among the dominance relations discussed above, it follows that:

$$C^{\alpha_2}(E) \subseteq C^{\alpha_1}(E) \subseteq C^{\alpha}(E).$$

We introduce now the notion of stable set for the $\beta$–dominance relation and study its general properties as well as its connections with the core notion.

**Definition** 2.8 (Internal and external stability; stable set) Let $\beta \in \{\alpha, \alpha_1, \alpha_2\}$.
A non-empty set $V$ of individually rational allocations is said to be:

- $\beta$–internally stable if the following condition holds:

$$\text{if } x \in V, \text{ then there is no } y \in V \text{ such that } y \succ_\beta x;$$

- $\beta$–externally stable if the following condition holds:

$$\text{if } x \in I \setminus V, \text{ then there is } y \in V \text{ such that } y \succ_\beta x;$$

- a (Von Neumann–Morgenstern) $\beta$-stable set if it is both $\beta$–internally stable and $\beta$–externally stable.

Notice that no relations among the notions of $\alpha$-stability, $\alpha_1$-stability and $\alpha_2$-stability can be inferred by the implications:

$$x \succ_\alpha y \Longrightarrow x \succ_{\alpha_1} y \Longrightarrow x \succ_{\alpha_2} y \,.$$

We can just derive that, for a given set $V$, the following implications hold true:

$$\alpha_2\text{–internal stability } \Longrightarrow \alpha_1\text{–internal stability } \Longrightarrow \alpha\text{–internal stability;}$$

$$\alpha\text{–external stability } \Longrightarrow \alpha_1\text{–external stability} \Longrightarrow \alpha_2\text{–external stability}\,.$$

As a consequence, the best one can aspire to is to prove that a non empty set $V$ of individually rational allocations is internally stable with respect to the weaker dominance relation and externally stable with respect to the stronger one.

Notice also that none of the dominance relations that we have introduced is transitive, since the blocking coalitions in two subsequent dominations may be different. However, transitivity occurs whenever the blocking coalition in the second domination is represented by the grand coalition $N$. As a consequence, allocations in every $\beta$–stable set are Pareto optimal, as stated in the next result.

**Proposition** 2.1 *Let $V$ be a $\beta$–stable set, where $\beta \in \{\alpha, \alpha_1, \alpha_2\}$. Then, every allocation in $V$ is Pareto optimal.*

*Proof.* Assume, by contradiction, that an allocation $x$ belongs to $V$ and is not Pareto optimal. Then, there would exists an allocation $t$ such that $U_i(t) > U_i(x)$ for each trader $i \in N$. Since $x$ is individually rational, $t$ is also individually rational. Moreover, by the internal stability, $t$ does not belong to $V$. Hence, by the external stability, there exists an allocation $y$ in $V$ such that $y \succ_\beta t$.
Independently of what definition is used, this implies that $y$ $\beta$–dominates $x$, which contradicts the internal stability of $V$. □

The following notion is central for our paper; it is inspired by the extreme $\alpha$-core (Yannelis (1991)) which, contrary to the core, accounts for the blocking power of just two extreme types of coalitions, the grand coalition $N$ and those formed by a single trader. The formal definition is as follows:

**Definition** 2.9 (EXTREME CORE) An allocation $x$ is said to be an extreme core allocation for the economy $E$ if it is individually rational and Pareto optimal.
We will denote by $C_e(E)$ the set of the extreme core allocations for the economy $E$.

Since in the previous notion the number of blocking coalitions reduces, the set of equilibrium allocations enlarges. However, it is worth noting that a comparison between the $\alpha$–core and the extreme core, which passes through the individual rationality of allocations, is not possible unless the economy involves just two traders. The situation is different for allocations in the $\alpha_1$–core, and consequently in the $\alpha_2$–core, which are also individually rational, so that a direct inclusion holds true in this case. On the other hand, it is easy to check that the set $C^\alpha(E) \cap I$ is $\alpha$–internally stable. Moreover, this set of allocations is included in every $\alpha$–stable set, as a consequence of the external stability.

Next result, whose proof is trivial and is omitted, addresses these points.

**Proposition** 2.2 *For any $\alpha$–stable set $V$, the following inclusions are true:*

$$C^\alpha(E) \cap I \subseteq V \subseteq C_e(E).$$

*For any $\alpha_1$–stable set $V_1$, the following inclusions are true:*

$$C^{\alpha_1}(E) \subseteq V_1 \subseteq C_e(E).$$

*For any $\alpha_2$–stable set $V_2$, the following inclusions are true:*

$$C^{\alpha_2}(E) \subseteq V_2 \subseteq C_e(E).$$

*Moreover, for a two person economy the previous inclusions are equivalences and $C^\alpha(E) \cap I = C^\alpha(E)$.*

As a consequence of Proposition 2.2, we can state that, whenever the set $C^\alpha(E) \cap I$ is also $\alpha$–externally stable, then it contains each stable set, otherwise the internal stability of the stable set would be contradicted. In this case, $C^\alpha(E) \cap I$ comes out to be the unique $\alpha$–stable set in the market. The same holds true for $C^{\alpha_1}(E)$ and $C^{\alpha_1}(E)$. On the other hand, in each of the previous dominance, this situation rarely occurs; indeed, it has been proved by Holly (1994) that the $\alpha$-core $C^\alpha(E)$ (and, consequently, $C^{\alpha_1}(E)$ and $C^{\alpha_2}(E)$) of an economy with more than two traders can be empty. The inclusions contained in Proposition 2.2 also state that, if the extreme core is $\alpha$ (respectively, $\alpha_1$ or $\alpha_2$)–stable, then it is the unique $\alpha$ (respectively, $\alpha_1$, $\alpha_2$)–stable set of the economy. For this reason, Section 3 is devoted to study the stability of the extreme core $C_e(E)$.

## 2.2 The notions of type

As anticipated in the Introduction, in this paper we consider the possibility of having types of agents among traders. As a general notation for modelling types, we will denote

by $k$ the number of types in the economy $E$ and by $T_i$ the set of all traders of type $i$, with $1 \leq i \leq k$.
It clearly holds that $k \leq n$, $|T_1| + \ldots + |T_k| = n$ and $\{T_1, \ldots, T_k\}$ forms a partition of the set $N$. When necessary, the double subscript $i, j$ will be used to name traders, where the first index $i \in \{1, \ldots, k\}$ denotes the type and the second subscript $j$ denote trader $j$ of type $i$. Moreover, given an allocation $x$, we will denote by $x^{T_i}$ the projection of $x$ onto $T_i$; that is, $x^{T_i} = (x_{ij})_{j \in T_i}$. The notation $x = (x^{T_1}, \ldots, x^{T_k})$ will be used to describe an allocation and, more in general, a consumption profile.
For what concerns the actual meaning of type, we shall refer to two possible definitions [5].
The first definition of type we consider is completely in line with the selfish framework: we say that two agents are of the same type if they have exactly the same attributes, that is the same initial endowment, the same consumption set as members of the grand coalition $N$ and the same (interdependent) utility function. That is:

**Definition** 2.10 (Type I) $\{T_1, \ldots, T_k\}$ is a partition of $N$ into traders of the same type, whenever for every $i \in \{1, \ldots, k\}$, $(i, j)$ and $(i, t) \in T_i$, the following three conditions hold:

**a.** $e_{ij} = e_{it}$.

**b.** $X_{ij}(N) = X_{it}(N)$.

**c.** $U_{ij}(x^{T_1}, \ldots, x^{T_k}) = U_{it}(x^{T_1}, \ldots, x^{T_k})$, for every consumption profile $x = (x^{T_1}, \ldots, x^{T_k})$.

The second definition of type is more in the spirit of interdependence; in this case, agents of the same type may have utility functions with different functional forms and, as a consequence, they may get different utility levels from the same consumption profile $x = (x^{T_1}, \ldots, x^{T_k})$. However, such utility functions have to exhibit some form of symmetry, as expressed in the last two conditions of next definition.

**Definition** 2.11 (Type II) $\{T_1, \ldots, T_k\}$ is a partition of $N$ into traders of the same type, whenever for every $i \in \{1, \ldots, k\}$, $(i, j)$ and $(i, t) \in T_i$, the following conditions hold:

**a.** $e_{ij} = e_{it}$.

**b.** $X_{ij}(N) = X_{it}(N)$.

[5] The first definition has been adopted in a model with externalities and replica economies by Velez (2016), while a notion close to our second definition has been adopted by Yi (1997) for games with externalities.

**c.** for every consumption bundle $x^{T_i} = (x_{i1}, \dots, x_{ij}, \dots, x_{it}, \dots, x_{is})$:

$$U_{ij}(x^{T_1}, \dots, x^{T_{i-1}}, x_{i1}, \dots, x_{ij}, \dots, x_{it}, \dots, x_{is}, x^{T_{i+1}}, \dots, x^{T_k}) = \tag{1}$$
$$= U_{it}(x^{T_1}, \dots, x^{T_{i-1}}, x_{i1}, \dots, x_{it}, \dots, x_{ij}, \dots, x_{is}, x^{T_{i+1}}, \dots, x^{T_k}) ,$$

where in the consumption profile in the second line bundles $x_{ij}$ and $x_{it}$ have been switched.

**d.** for every $l \in \{1, \dots, k\}$, $l \neq i$, for every $(l, r)$, $(l, v) \in T_l$, and for every consumption bundle $x^{T_l} = (x_{l1}, \dots, x_{lr}, \dots, x_{lv}, \dots, x_{lz})$:

$$U_{ij}(x^{T_1}, \dots, x^{T_{l-1}}, x_{l1}, \dots, x_{lr}, \dots, x_{lv}, \dots, x_{lz}, x^{T_{l+1}}, \dots, x^{T_k}) = \tag{2}$$
$$= U_{ij}(x^{T_1}, \dots, x^{T_{l-1}}, x_{l1}, \dots, x_{lv}, \dots, x_{lr}, \dots, x_{lz}, x^{T_{l+1}}, \dots, x^{T_k}) ,$$

where in the consumption profile in the second line bundles $x_{lr}$ and $x_{lv}$ have been switched.

Condition b. guarantees that, whenever we permute the consumption bundles of agents of a certain type in an allocation $x$, the new consumption profile meets the consumption set feasibility and, therefore, is itself an allocation.
Condition c. is an intra–type condition and its interpretation is clear: it states that the utility of two agents of the same type does not change when their bundles are switched whereas the consumption bundles of all other traders are kept unchanged. It is not difficult to check that equality (1) can be immediately extended to any consumption profile $x^{T_i^\sigma}$ where $\sigma$ is any permutation on the set $T_i$ exchanging $(i, j)$ and $(i, t)$ and $x^{T_i^\sigma}$ is the consumption profile obtained from $x^{T_i}$ after permuting traders of type $i$ according to $\sigma$.
Finally, condition d. expresses the externalities produced on type $i$ by a different type of traders: the utility level of each trader of type $i$ does not change when permuting the consumption bundles of two traders of a different type. Also in this case, one can easily extend equality (2) to any consumption bundle $x^{T_l^\sigma}$, where $\sigma$ is any permutation on $T_l$ and $x^{T_l^\sigma}$ is the consumption profile obtained from $x^{T_l}$ after permuting traders of type $l$ according to $\sigma$. Generally speaking, condition (2) implies a kind of anonymity of the externalities produced on a given type by the consumption of a different type: the preferences of traders of type $i$ depend on the consumptions of traders of a different type but the externality effects produced by this consumption on $i$ do not account for the identities of these traders.
Notice that the equality (1) embodies the following property of anonymity within each type $T_i$: the utility of trader $(i, j)$ does not change when its consumption bundle remains fixed and the other traders of the same type permute their commodities. This point is formalized by the next Proposition.

**Proposition** 2.3 *Let $x$ be an allocation, $(i,j)$ be a trader of $T_i$, $\sigma$ be any permutation on the set $T_i$ such that $\sigma(i,j)=(i,j)$. If condition* (1) *is satisfied, then it holds true that:*

$$U_{ij}(x^{T_1},\ldots,x^{T_i^\sigma},\ldots,x^{T_k})=U_{ij}(x^{T_1},\ldots,x^{T_i},\ldots,x^{T_k})$$

*where $T_i^\sigma=\sigma(T_i)$.*

*Proof.* We can suppose that $k>2$. If $\sigma$ is the identity permutation, the statement is trivial. Then, assume that $\sigma$ is not the identity and let $(i,t)$ be the first trader different from $(i,j)$ that is moved by $\sigma$ with $\sigma(i,t)=(i,s)$. Starting from the allocation $x$ and applying consecutively condition (1) to pairs $(i,j)$ and $(i,s)$, $(i,s)$ and $(i,t)$, $(i,t)$ and $(i,j)$, one gets that:

$$U_{ij}(x^{T_1},\ldots,x^{T_i},\ldots,x^{T_k})=U_{ij}(x^{T_1},\ldots,x^{T_i^{\sigma_1}},\ldots,x^{T_k})\,,$$

where $\sigma_1$ is the permutation on $T_i$ that switches $(i,t)$ and $(i,s)$ and keeps everything else unchanged and $x^{T_i^{\sigma_1}}$ is the allocation obtained by $x$ switching $x_{is}$ with $x_{it}$. If $\sigma=\sigma_1$, the proof is finished. If not, let $(i,z)$ be the second trader in $T_i$ which is moved by $\sigma$ and assume that $\sigma(i,z)=(i,v)$ $(z\neq i,t)$. Starting from the allocation $(x^{T_1},\ldots,x^{T_i^{\sigma_1}},\ldots,x^{T_k})$ and applying consecutively condition (1) to pairs $(i,j)$ and $(i,v)$, $(i,v)$ and $(i,z)$, $(i,z)$ and $(i,j)$, one gets that:

$$U_{ij}(x^{T_1},\ldots,x^{T_i^{\sigma_1}},\ldots,x^{T_k})=U_{ij}(x^{T_1},\ldots,x^{T_i^{\sigma_2}},\ldots,x^{T_k})\,,$$

where $\sigma_2$ is the permutation on $T_i$ that exchanges $(i,t)$ with $(i,s)$, and $(i,v)$ with $(i,z)$, and $x^{T_i^{\sigma_2}}$ is the allocation obtained by $x$ exchanging $x_{is}$ with $x_{it}$, and $x_{iv}$ with $x_{iz}$. By repeating this way of reasoning, in a finite number of steps the conclusion is reached. □

Due to Proposition 2.3, the following interpretation of condition (1) is possible: whenever a trader $(i,j)$ of type $i$ prefers the commodity bundle $a$ to $b$ under a given allocation, every other trader of the same type would also prefer $a$ to $b$, independently of what is the redistribution of the remaining commodity bundles within the type.

Next remark focuses on a technical comparison between the two definitions of type introduced above.

**Remark 2.2** We ask whether the two definitions of types stated before can be compared. The first part of such comparison is clear. Indeed, if two agents are of the same type according to the second definition, it can be the case that their utility functions are different and, as a consequence, they are not of the same type according to the first definition. Also, if two agents are of the same type according to the first definition, they are not in general of the same type according to the second definition. As an example, consider three agents of the same type according to the first definition, each endowed with the following utility function:

$$U_1(x_1, x_2, x_3) = U_2(x_1, x_2, x_3) = U_3(x_1, x_2, x_3) = 2x_1 + x_2 + 3x_3$$

Condition (1) does not hold true; indeed, for instance:

$$U_2(x_2, x_1, x_3) = 2x_2 + x_1 + 3x_3 \neq U_1(x_1, x_2, x_3).$$

However, condition (1) holds true for some specific functional forms $U$ characterized by total symmetry [6]. For instance, for the case of four agents within a type and just one commodity, we can consider the so called elementary symmetric polynomials:

$$\begin{aligned} U(x, y, z, t) &= x + y + z + t \\ U(x, y, z, t) &= x \cdot y \cdot z \cdot t \\ U(x, y, z, t) &= xy + xz + xt + yz + yt + zt \\ U(x, y, z, t) &= xyz + xyt + xzt + yzt \end{aligned}$$

Whenever all traders of a certain type have one of the previous utility functions, both conditions (1) and (2) hold and they can be considered of the same type also according to the second definition.

Two additional differences between the two definitions are worth noting.
In the second definition, contrary to what happens for the definition of "Type I", whenever $U_{ij}$ is considered as a function of just $x_{ij}$, the notion of types adopted in the selfish framework (see, Einy and Shitovitz, 2003) is easily recovered; in this case, condition (2) has no more implication.
Moreover, the notion of type–symmetric allocation considered in Einy and Shitovitz (2003) loses its meaning when one adopts the first definition of types, while it retains its meaning when the second definition is adopted and can be stated as follows.

**Definition** 2.12 (Type symmetric allocations) *Let $\{T_1, \ldots, T_k\}$ be a partition of $N$ into traders of Type II. An allocation $x = (x^{T_1}, \ldots, x^{T_k})$ is type–symmetric if for all $i \in \{1, \ldots, k\}$ and for every $j,\ t\ \in T_i$:*

$$U_{ij}(x) = U_{it}(x)\,.$$

[6] A function $f = f(x_1, \ldots, x_n)$ is totally symmetric if it is unchanged by any permutation of its variables.

Whenever $U_{ij}$ is considered as a function of just $x_{ij}$, the notion of type–symmetric allocation adopted in the selfish framework (see, Einy and Shitovitz, 2003) is recovered. Moreover, note that any equal–treatment allocation $x$, which allots the same bundle to all agents of the same type, is type–symmetric; indeed, as a consequence of condition (1) in the definition of Type II, it holds that for any $i = 1, \dots, k$ and for any $j, t \in T_i$:

$$U_{ij}(x) = U_{ij}(x^{T_1}, \dots, x_i, \dots, x_i, \dots, x^{T_k}) = U_{it}(x^{T_1}, \dots, x_i, \dots, x_i, \dots, x^{T_k}) = U_{it}(x) \,.$$

The following two results, which do not require any sort of assumptions and are based just on conditions (1) and (2) of Definition 2.11, are about type-symmetry; they will be useful in Section 3.2 in connection with the study of the external stability properties.

**Lemma** 2.1 (Permutations and types 1) *Let $x$ be a type–symmetric allocation, $\sigma$ be a permutation on the set $T_i$ and $T_i^\sigma = \sigma(T_i)$. Then, it holds true that for every $(i, j) \in T_i$:*

$$U_{ij}(x^{T_1}, \dots, x^{T_i^\sigma}, \dots, x^{T_k}) = U_{ij}(x^{T_1}, \dots, x^{T_i}, \dots, x^{T_k}) \,.$$

*Proof.* Starting with $U_{ij}(x^{T_1}, \dots, x^{T_i^\sigma}, \dots, x^{T_k})$ and applying condition (1) several times, we can state that there exists a trader $(i, t) \in T_i$ such that:

$$U_{ij}(x^{T_1}, \dots, x^{T_i^\sigma}, \dots, x^{T_k}) = U_{it}(x^{T_1}, \dots, x^{T_i}, \dots, x^{T_k}) \,.$$

Then, the conclusion easily follows from the type–symmetry of the allocation $x$. □

**Lemma** 2.2 (Permutations and types 2) *Let $x$ be a type–symmetric allocation. Then, the allocation obtained from $x$ after permuting consumption bundles within types is still type–symmetric.*

*Proof.* Let $\sigma_1, \dots, \sigma_k$ denote permutations over the sets $T_1, \dots, T_k$, respectively. Moreover, we denote by $x^{T_i^\sigma}$ the consumption profile for traders of type $i$ obtained from $x^{T_i}$ after permuting its consumption bundles according to $\sigma_i$; that is:

$$x^{T_i^\sigma} = (x_{\sigma_i(i1)}, \dots, x_{\sigma_i(is)}) \,.$$

Finally, the notation $x^\sigma$ stand for the allocation obtained from $x$ after permuting traders within each type through $\sigma_1, \dots, \sigma_k$, that is:

$$x^\sigma = (x^{T_1^\sigma}, \dots, x^{T_k^\sigma}) \,.$$

We want to show that the allocation $x^\sigma$ is type–symmetric. For this purpose, let $i \in \{1, \dots, k\}$ and $(i, j)$, $(i, t) \in T_i$.
By using conditions (1) and (2), the previous Proposition and the type–symmetry of the allocation $x$, the following chain of equalities holds true:

$$U_{ij}(x^\sigma) = U_{ij}(x^{T_1^\sigma}, \dots, x^{T_k^\sigma}) = U_{ij}(x^{T_1}, \dots, x^{T_i^\sigma}, \dots, x^{T_k}) = U_{ij}(x^{T_1}, \dots, x^{T_i}, \dots, x^{T_k}) =$$
$$= U_{it}(x^{T_1}, \dots, x^{T_i}, \dots, x^{T_k}) = U_{it}(x^{T_1}, \dots, x^{T_i^\sigma}, \dots, x^{T_k}) = U_{it}(x^{T_1^\sigma}, \dots, x^{T_i^\sigma}, \dots, x^{T_k^\sigma}) =$$
$$= U_{it}(x^\sigma)\,,$$

and the type–symmetry of $x^\sigma$ is thus proved. □

# 3 Existence and uniqueness of stable sets

In this section we face the problems of the existence and uniqueness of stable sets for each of the dominance relations introduced before. In particular, we investigate the internal and external stability of the set formed by all the individually rational, Pareto optimal allocations. For those cases where it makes sense, the property of type–symmetry is also added.
The Section is divided into two parts. In the first one, a framework with agents of "Type I" is analyzed. The second part covers the case of an economy populated by traders of "Type II". It also discusses a particular case which is intermediate between the selfish and the interdependent ones; precisely, we suppose that the utility function of each trader depends on the consumption bundles of all other traders of his type, that is, we still have interdependence but just within types, not across all traders in the market. In order to distinguish the latter model from the previous two, this sort of utility functions will be called partially interdependent.

## 3.1 The case of agents of Type I

In this Section we will focus on a model with agents of Type I. Since the utility function for agents of the same type is the same, the notation $U_i$ will be used instead of $U_{ij}$. Importantly, no restrictions are imposed on the number of traders in each type.

The assumptions which, differently combined, will be used are the following.

(A.1) (Continuity) For every $i \in N$, $U_i$ is continuous.

(A.2) (Boundary equivalence) Let $x$ be an allocation and let $Z$ be the set defined by $Z = \{j \in N : x_j$ has a zero component $\}$; then, for every $i \in N$:

$$U_i(x) = U_i(0^Z, x^{N \setminus Z}).$$

(A.3) (Strictly positive total endowment) $\sum_{i \in N} e_i \gg 0$.

(A.4) (Glove market assumption) For each type $i \in \{1, \ldots, k\}$, there exists a commodity $r_i \in \{1, \ldots, l\}$ such that $e_t^{r_i} = 0$ for every $t \neq i$ (where $e_t^{r_i}$ denotes the $r_i$-th component of the vector $e_t$).

Assumption (A.1) is standard while assumption (A.2) implies that the utility level for each agent is the same over allocations which give a boundary commodity bundle to each trader. For this reason, this assumption is referred to as the boundary equivalence assumption. Assumption $(A.4)$ is frequently used in connection with the study of stable sets. For a model with just two types of traders, two commodities and total initial endowment given by $(1, 1)$, it simply requires that one type is initially endowed with $(1, 0)$ and the other one with $(0, 1)$. For a more general case, it means that, although each commodity is present on the market at the beginning because of Assumption (A.3), each type of traders has a corner on some commodity. This assumption is sometimes referred to as the glove market assumption on the initial endowments.

**Remark 3.1** One may ask which classes of interdependent utility functions fulfill the set of assumptions stated before.
Assume that each agent $i \in N$ has a selfish utility function $u_i$ that only depends on his own consumption $x_i$ and that the interdependent utility function $U_i$ aggregates these individual selfish utilities for each $i$. The simplest example is one where agent $i$ cares about his own consumption and a weighted average of the other utilities according to the formula:

$$U_i = u_i + \frac{\beta_i}{n-1} \sum_{j \neq i} u_j$$

If $\beta_i$ is positive, than agent $i$ is altruistic or benevolent. On the contrary, the case of a negative $\beta_i$ refers to an envious or spiteful agent. If the selfish utilities $u_i$ are of the Cobb-Douglas type, it follows that the utility functions $U_i$ meet all the required assumptions.

**Remark 3.2** In the case of selfish preferences, the assumptions usually employed to get the existence and uniqueness of stable sets for exchange economies (see, Einy and Shitovitz

(2003) and Graziano et al. (2015)) are standard continuity, weakly strict monotonicity (that is, for every $x, y \in \mathbb{R}^{l \cdot s}_{+}$, $x \gg y$ implies $U_i(x) > U_i(y)$) and quasi–concavity of the utility functions. Moreover, one needs to require that for each commodity bundle $a$ that lies on the boundary of the consumption set $\mathbb{R}^{l}_{+}$, and for each trader $i$, it holds that $U_i(a) = U_i(0)$. This assumption, joint with monotonicity, implies that each trader prefers interior commodity bundles to the boundary ones.
Despite a comparison between the framework analyzed in this Section and the selfish one cannot be performed, we notice that the set of the assumptions is much slimmer here; in particular, neither monotonicity and quasi–concavity are needed nor any assumption concerning the consumption sets that would parallel the one used in Graziano at al. (2015) concerning the information sharing rule.

We now prove that, in an exchange economy $E$ whose agents exhibit interdependent preferences, the extreme core $C_e(E)$ is the unique stable set whenever the $\alpha$ or the $\alpha_1$-dominance relations are considered. The existence of stable sets appears of particular interest in the framework of interdependent utility functions since the more traditional notion of core has been proved to be empty even in models that satisfy the usual assumptions (see, Holly, 1994). The results that we are going to prove also show that the difference between an optimistic and a pessimistic attitude for a deviating coalition is not relevant when one deals with the $\alpha$-type dominance relation and traders of type I.
The following preliminary result is needed.

**Proposition** 3.1 (Pareto optimality and dominance) *Under Assumption (A.1), every individually rational allocation $z$ which is not Pareto optimal can be $\alpha$–dominated (equivalently, $\alpha_1$ or $\alpha_2$–dominated) by a Pareto optimal allocation through the grand coalition $N$.*

*Proof.* Let $z$ be an allocation which is individually rational but not Pareto optimal and $\delta$ a positive real number. Denote by $\mathcal{A}$ the set of all the allocations for the economy $E$ and consider the following set:

$$A = \{x \in \mathcal{A} : x \text{ is individually rational and } U_i(x) \geq U_i(z) + \delta, \forall\, i \in N\}.$$

The set $\mathcal{A}$ is non empty because it contains the initial endowment allocation $e$. The set $A$ is also non empty; indeed, since $z$ is not Pareto optimal, there exists an allocation $t$ such that $U_i(t) > U_i(z), \forall\, i \in N$. Moreover, the set $A$ is compact.
Define the function $\tilde{U}$ as follows:

$$\tilde{U}(x_1, \dots, x_n) = \sum_{i \in N} U_i(x).$$

By assumption (A.1), it is continuous on $A$.
Hence, $\tilde{U}$ has a maximal element on the set $A$. Let us denote it by $g$. It holds that $g$ is individually rational and $U_i(g) > U_i(z), \forall i \in N$.
We want to prove that $g$ is also Pareto optimal.
By way of contradiction, let us suppose that $g$ is not Pareto optimal. Then, there exists an allocation $\gamma$ such that:

$$U_i(\gamma) > U_i(g), \forall i \in N.$$

The allocation $\gamma$ belongs to $A$.
However, it holds that:

$$\tilde{U}(\gamma) = \sum_{i \in N} U_i(\gamma) > \sum_{i \in N} U_i(g) = \tilde{U}(g)\,,$$

and this contradicts the fact that $g$ is a maximal element for the function $\tilde{U}$ on the set $A$. Hence, we conclude that $g$ is Pareto optimal and this concludes the proof. □

As an immediate consequence, we obtain the following result.

**Corollary** 3.1 *Under Assumption (A.1), the extreme core $C_e(E)$ is non empty.*

We can now prove that the extreme core $C_e(E)$ is internally stable. It is worth noting that, contrary to the selfish case, no monotonicity assumption is needed in the proof.

**Theorem** 3.1 *Let the economy $E$ satisfy the Assumptions (A.2),(A.3) and (A.4). Then, the extreme core $C_e(E)$ is $\alpha_1-$ internally stable (and, therefore, also $\alpha-$ internally stable).*

*Proof.* By way of contradiction, let us assume that $C_e(E)$ is not $\alpha_1$–internally stable. Then, there exist two allocations $x$ and $y$ in $C_e(E)$ and a nonempty coalition $S$ such that $x \succ_{\alpha_1} y$; that is:

**a.** $x$ is an assignment for $S$;

**b.** $U_i(x^S, e^{N \setminus S}) > U_i(y)$, for all $i \in S$.

We first show that there exists a trader $\underline{i} \in S$ such that $x_{\underline{i}} \gg 0$. By way of contradiction, let us suppose that for every $i \in S$, $x_i$ has a zero component. Then, by assumption (A.2), it follows that for every $i \in S$ and for every allocation $z$:

$$U_i(z) = U_i(x^S, z^{N \setminus S}) = U_i(0^S, z^{N \setminus S}).$$

In particular, we can consider the case $z^{N\setminus S} = e^{N\setminus S}$. For this case, it holds that:

$$U_i(x^S, e^{N\setminus S}) = U_i(0^S, e^{N\setminus S}) = U_i(0_N) > U_i(y) \geq U_i(e) = U_i(0_N).$$

Hence, a contradiction is reached.
Therefore, we can conclude that there is at least one trader $i \in S$ which receives a strictly positive commodity bundle and, as a consequence, $\sum_{i \in S} x_i = \sum_{i \in S} e_i \gg 0$.
By Assumption (A.4), the coalition $S$ contains each type of trader. Therefore, the inequality $U_i(x^S, e^{N\setminus S}) > U_i(y)$ holds true for each type $i \in \{1, \ldots, k\}$.
The allocation $(x^S, e^{N\setminus S})$ is feasible for the grand coalition $N$ and a contradiction with the Pareto optimality of allocation $y$ is thus reached. □

Moving to the external stability, next theorem follows directly from the previous Proposition 3.1 and its proof is omitted.

**Theorem** 3.2 *Under Assumption (A.1), the extreme core $C_e(E)$ is $\alpha-$ externally stable (and, therefore, also $\alpha_1$ and $\alpha_2-$ externally stable).*

All the results contained in this section can be summarized by the next theorem that shows the existence and uniqueness of a stable set for the $\alpha-$ dominance relation. Since both sets for the $\alpha$ and the $\alpha_1$ dominance coincide with the extreme core, it also states that the difference between the optimistic and pessimistic attitude is not relevant under the assumptions stated above. Notice that, on the contrary, the $\alpha$-core and the $\alpha_1$-core may well be different (see Example 4 in Dufwenberg et al. (2011)).

**Theorem** 3.3 (EXISTENCE AND UNIQUENESS) *Let the economy $E$ satisfy the Assumptions (A.1)-(A.4). Then:*

1. *the extreme core $C_e(E)$ is $\alpha_1$-internally stable and $\alpha$-externally stable;*
2. *the extreme core $C_e(E)$ is $\alpha$-stable as well as $\alpha_1$-stable;*
3. *whenever a set $V$ exists which is $\alpha_1$-internally stable and $\alpha$-externally stable, then $V$ equals the extreme core;*
4. *whenever an $\alpha$-stable set $V$ exists, then $V$ equals the extreme core;*
5. *whenever an $\alpha_1$-stable set $V$ exists, then $V$ equals the extreme core;*
6. *whenever an $\alpha_2$-stable set $V$ exists, then $V$ equals the subset of the extreme core $C_e(E)$ formed by all the allocations that do not $\alpha_2$–dominate each other.*

We remark that the crossed dominance which appears in the first statement of the previous theorem can be interpreted in terms of different attitudes that coalitions may have with respect to the two criteria of internal and external stability. Generally speaking, the property of external stability is meant to exclude allocations from the stable set itself. In this light, with reference to the dominance relation adopted for the external stability, coalitions may be more conservative and select allocations by using the more stringent $\alpha$-criterion. On the other hand, coalitions may adopt less conservative views when deciding whether a deviation from a state into the solution set towards another state which is still inside the set is possible. This motivates the choice of the $\alpha_1$– dominance over the $\alpha$–dominance for the internal stability.

## 3.2 The case of agents of Type II

In this Section we focus on the second notion of agents' types, named "Type II", that differs from the one used in the previous section as regards the utility functions.
Under this change in the economic environment, the notion of type–symmetry matters; therefore, we move our focus from the extreme core $C_e(E)$ to the set $C_e^s(E)$, formed by all the individually rational, Pareto optimal, type–symmetric allocations and we analyze its stability when the economy $E$ is populated by $s$ traders of each type $i$, with $s > 1$.

The following set of assumptions will be used in the sequel.

(B.1) (Continuity) For every $(i, j) \in N$, $U_{ij}$ is continuous.

(B.2) (Type–wise strict quasi–concavity) For every $(i, j) \in N$, $U_{ij}$ is strictly quasi–concave with respect to each $x^{T_l}$, $l = 1, \ldots, k$.

(B.2)′ (Type–wise quasi–concavity) For every $(i, j) \in N$, $U_{ij}$ is quasi–concave with respect to each $x^{T_l}$, $l = 1, \ldots, k$.

(B.3) (Boundary equivalence) For every $(i, j) \in N$ and for every consumption profile $x$ over $N$, it holds that:

$$U_{ij}(x) = U_{ij}(0^Z, x^{N \setminus Z}) \, ,$$

where $Z = \{(s, t) \in N : x_{st} \text{ has a zero component}\}$.

(B.4) (Strictly positive total endowment) $\sum_{ij \in N} e_{ij} \gg 0$.

(B.5) (Glove market) For each type $i \in \{1, \ldots, k\}$, there exists a commodity $m_i \in \{1, \ldots, l\}$ such that $e_r^{m_i} = 0$ for every $r \neq i$ (where $e_r^{m_i}$ denotes the $m_i$-th component of the vector $e_r$).

(B.6) (Consumption sets and types) If coalition $S \subseteq N$ contains every type of agents, then $X_{ij}(S) = X_{ij}(N)$ for every $(i, j) \in S$.

Compared with the set of assumption used in the previous section, a (strict) quasi–concavity assumption on the interdependent utility functions is now added in the economic environment, along with a condition on the consumption sets.
Assumption (B.6) states that, for each trader, the consumption opportunities are the same in the grand coalition $N$ and in every other coalition containing every type of agents [7].
Two remarks about Assumptions (B.2) and (B.2)′ are in order.

**Remark 3.3** We first note that Assumption (B.2)′ is weaker than assuming the quasi–concavity of function $U_{ij}$. For instance, consider the function $f(x, y) = x^3 + x + y^3 + y$ which is quasi–concave with respect to both $x$ and $y$. However, $f$ is not quasi–concave. Indeed, note that both $f'_x > 0$ and $f'_y > 0$; moreover, the determinant of the following matrix $M$:

$$M = \begin{pmatrix} 0 & f'_x & f'_y \\ f'_x & f''_{xx} & f''_{xy} \\ f'_y & f''_{xy} & f''_{yy} \end{pmatrix}$$

is negative in $(1, 2)$.

**Remark 3.4** One may ask which examples of interdependent utility functions can fit with Assumption (B.2) or (B.2)′ when types are also taken into account. Here, we provide an example.
Suppose that every agent of type $i$, $i = 1, \ldots, k$, has a selfish utility function $u_i$. The interdependent utility that agent $(i, j) \in T_i$ receives by the consumption profile $x = (x^{T_1}, \ldots, x^{T_k})$ is given by a sum of three components: the selfish utility that trader $(i, j)$ obtains from his own bundle $x_{ij}$, an average of the utilities that every other agent $(i, t)$ of his same type gets from his own bundle $x_{it}$, an average of the utilities received by all other agents of types $r \neq i$, each type $r$ weighted with a coefficient $a_{ir}$ that, according to

[7] With reference to the asymmetric information framework, this assumption holds for all the customarily used information sharing rules, that is, the private, the fine and the coarse ones. More in general, it is satisfied whenever the consumption set of trader $i$ as a member of coalition $S$ depends just on the types not on the traders actually included in $S$.

its sign, measures the envy/altruism of type $i$ towards type $r$.
That is, the utility of agent $(i,j)$ with $i \in \{1,\dots,k\}$ is expressed by:

$$U_{ij}(x^{T_1},\dots,x^{T_k}) = u_i(x_{ij}) + \frac{1}{s-1}\sum_{t\in T_i, t\neq j} u_i(x_{it}) + \sum_{r\neq i} a_{ir}\sum_{t\in T_r}\frac{u_r(x_{rt})}{s}$$

where $s = |T_1| = \dots = |T_k|$.
It is easy to check that for the function $U_{ij}$ both conditions (1) and (2) defining agents of Type II hold.
Moreover, by assuming that every selfish utility function $u_l$, $l \in \{1,\dots,k\}$, is (strict) quasi–concave, it can be proved that $U_{ij}$ is (strict) quasi–concave with respect to each block $x^{T_1},\dots,x^{T_k}$ of variables provided that the coefficient $a_{ij}$ are nonnegative.
Let us first prove that $U_{ij}$ is quasi–concave with respect to $x^{T_i}$.
Let $A = (x^{T_1},\dots,x^{T_i},\dots,x^{T_k})$ and $B = (x^{T_1},\dots,\bar{x}^{T_i},\dots,x^{T_k})$. We want to show that, for every $\lambda \in [0,1]$:

$$U_{ij}(\lambda A + (1-\lambda)B) \geq \min\{U_{ij}(A), U_{ij}(B)\}\,.$$

It holds that:

$$\begin{aligned}
&U_{ij}(\lambda A + (1-\lambda)B) = u_i(\lambda x_{ij} + (1-\lambda)\bar{x}_{ij}) + \frac{1}{s-1}\sum_{t\in T_i, t\neq j} u_i(\lambda x_{it} + (1-\lambda)\bar{x}_{it}) + \\
&+ \sum_{r\neq i} a_{ir}\sum_{t\in T_r}\frac{u_r(x_{rt})}{s} \geq \min\{u_i(x_{ij}), u_i(\bar{x}_{ij})\} + \\
&+ \frac{1}{s-1}\left[\sum_{t\in T_i, t\neq j}\min\{u_i(x_{it}), u_i(\bar{x}_{it})\}\right] + \sum_{r\neq i} a_{ir}\sum_{t\in T_r}\frac{u_r(x_{rt})}{s} \geq \\
&\geq \min\{u_i(x_{ij}), u_i(\bar{x}_{ij})\} + \frac{1}{s-1}\left[\min\left\{\sum_{t\in T_i, t\neq j} u_i(x_{it}), \sum_{t\in T_i, t\neq j} u_i(\bar{x}_{it})\right\}\right] + \\
&+ \sum_{r\neq i} a_{ir}\sum_{t\in T_r}\frac{u_r(x_{rt})}{s} \geq \min\left\{u_i(x_{ij}) + \frac{\sum_{t\in T_i, t\neq j} u_i(x_{it})}{s-1}, u_i(\bar{x}_{ij}) + \frac{\sum_{t\in T_i, t\neq j} u_i(\bar{x}_{it})}{s-1}\right\} + \\
&+ \sum_{r\neq i} a_{ir}\sum_{t\in T_r}\frac{u_r(x_{rt})}{s} = \min\{U_{ij}(A), U_{ij}(B)\}\,,
\end{aligned}$$

that is what we had to prove.
Let us prove now that $U_{ij}$ is quasi–concave with respect to $x^{T_l}$, with $l \neq i$.
Let $C = (x^{T_1},\dots,x^{T_l},\dots,x^{T_k})$ and $D = (x^{T_1},\dots,\bar{x}^{T_l},\dots,x^{T_k})$. We want to show that, for every $\lambda \in [0,1]$:

$$U_{ij}(\lambda C + (1-\lambda)D) \geq \min\{U_{ij}(C), U_{ij}(D)\}\,.$$

It holds that:

$$U_{ij}(\lambda C+(1-\lambda)D)=u_i(x_{ij})+\frac{1}{s-1}\sum_{t\in T_i,t\neq j}u_i(x_{it})+\sum_{r\neq i,l}a_{ir}\sum_{t\in T_r}\frac{u_r(x_{rt})}{s}+$$

$$+a_{il}\sum_{t\in T_l}\frac{u_l(\lambda x_{lt}+(1-\lambda)\bar{x}_{lt})}{s}\geq$$

$$\geq u_i(x_{ij})+\frac{1}{s-1}\sum_{t\in T_i,t\neq j}u_i(x_{it})+\sum_{r\neq i,l}a_{ir}\sum_{t\in T_r}\frac{u_r(x_{rt})}{s}+a_{il}\sum_{t\in T_l}\frac{\min\{u_l(x_{lt}),u_l(\bar{x}_{lt})\}}{s}\geq$$

$$\geq u_i(x_{ij})+\frac{1}{s-1}\sum_{t\in T_i,t\neq j}u_i(x_{it})+\sum_{r\neq i,l}a_{ir}\sum_{t\in T_r}\frac{u_r(x_{rt})}{s}+\frac{a_{il}}{s}\min\left\{\sum_{t\in T_l}u_l(x_{lt}),\sum_{t\in T_l}u_l(\bar{x}_{lt})\right\}=$$

$$=\min\{U_{ij}(C),U_{ij}(D)\}\,.$$

We shall prove now that the set $C_e^s(E)$ formed by all Pareto optimal, type–symmetric, individually rational allocations satisfies a form of crossed stability; precisely, we will show that $C_e^s(E)$ is internally stable with respect to the $\alpha$–dominance relation and externally stable with respect to the $\alpha_2$–dominance relation.

The next two preliminary propositions will be used to prove the external stability of the set $C_e^s(E)$. It is worth noting that for both of them the notion of dominance involved is not relevant because only the grand coalition $N$ is allowed to block. Moreover, the second result is particularly important because it implies that the set $C_e^s(E)$ is nonempty. The proof of the next proposition follows the same line of Proposition 3.1 and is omitted.

**Proposition** 3.2 (Pareto optimality and dominance) *Under Assumption (B.1), every individually rational allocation $y$ which is not Pareto optimal can be $\alpha$–dominated (equivalently, $\alpha_1$–dominated and $\alpha_2$–dominated) by a Pareto optimal allocation through the grand coalition $N$.*

In the statement of the previous proposition, if we add the assumption that the allocation $y$ is type-symmetric, then we can prove that it can be dominated by an allocation which is not only Pareto optimal, but also type–symmetric, provided that quasi–concavity on the utility functions is also assumed. That is, the following result holds:

**Proposition** 3.3 (Pareto optimality, type–symmetry and dominance) *Under Assumptions (B.1) and (B.2)′, every individually rational, type–symmetric allocation $y$ which is not Pareto optimal can be $\alpha$–dominated (equivalently, $\alpha_1$–dominated and $\alpha_2$–dominated) by a type–symmetric, Pareto optimal allocation through the grand coalition $N$.*

*Proof.* Let $y$ be an allocation which is type–symmetric but not Pareto optimal. Since $y$ is not Pareto optimal, by Proposition 3.2, there exists a Pareto optimal allocation $h = (h^{T_1}, \dots, h^{T_k})$ that dominates $y$ through the grand coalition $N$, that is, for all $(i,j) \in N$:

$$U_{ij}(h) > U_{ij}(y)\,.$$

If $h$ is type–symmetric, the proof ends. Otherwise, we consider the allocation $\bar{h} = (\bar{h}^{T_1}, \dots, \bar{h}^{T_k})$ obtained from $h$ by averaging bundles over traders of the same type; it gives to each agent of type $i$ the same bundle $\bar{h}^{T_i} = \dfrac{1}{|T_i|} \sum_{j \in T_i} h_{ij}$, that is:

$$\bar{h}^{T_i} = \left( \frac{1}{|T_i|} \sum_{j \in T_i} h_{ij}, \dots, \frac{1}{|T_i|} \sum_{j \in T_i} h_{ij} \right)\,.$$

The allocation $\bar{h}$ is both feasible and type–symmetric.
We want to show that, for every $(i,j) \in N$:

$$U_{ij}(\bar{h}) > U_{ij}(y)\,.$$

To this aim, denote by $\sigma_1$ the permutation on each set $T_i$ which reindexes $(i,j)$ in $(i,j+1)$ where $j = 1, \dots, s-1$ and $(i,s)$ in $(i,1)$ and by $\sigma_l$, with $l = 1, \dots, s$, the same permutation $\sigma$ applied $l$ times. Moreover, for each $l = 1, \dots, s$, let $T_i^{\sigma_l}$ denote the array obtained from $T_i$ after $l$ permutations and $h_{\sigma_l}^{T_i}$ the corresponding consumption profile for type $T_i$.
Finally, let:

$$\begin{aligned} g^1 &= (h_{\sigma_1}^{T_1}, \dots, h_{\sigma_1}^{T_k})\,, \\ &\vdots \\ g^s &= (h_{\sigma_s}^{T_1}, \dots, h_{\sigma_s}^{T_k})\,. \end{aligned}$$

Note that:

$$\frac{1}{s} \sum_{l=1}^{s} g^l = \bar{h}\,.$$

Consider trader $(i,j) \in N$; by Assumption (B.2)$'$ we get that:

$$U_{ij}(\bar{h}) = U_{ij}\left( \frac{1}{s} \sum_{l=1}^{s} g^l \right) \geq \min\{U_{ij}(g^1), \dots, U_{ij}(g^s)\}\,. \tag{3}$$

For each $l = 1, \dots, s$, by condition (2) in the definition of agents' types, Proposition 2.1 and the type-symmetry of $y$, it holds that:

$$\begin{aligned} U_{ij}(g^l) &= U_{ij}(h_{\sigma_l}^{T_1}, \dots, h_{\sigma_l}^{T_i}, \dots, h_{\sigma_l}^{T_k}) = U_{ij}(h^{T_1}, \dots, h_{\sigma_l}^{T_i}, \dots, h^{T_k}) = \\ &= U_{it}(h^{T_1}, \dots, h_{\sigma_l}^{T_i}, \dots, h^{T_k}) = U_{it}(h) > U_{it}(y) = U_{ij}(y) \end{aligned} \tag{4}$$

By (3) and (4), we obtain that for each $(i,j) \in N$:

$$U_{ij}(\bar{h}) > U_{ij}(y)$$

If the allocation $\bar{h}$ is Pareto optimal, the proof is concluded. Otherwise, we consider a positive real number $\delta$ and the following set:

$$B = \{x \in \mathcal{A} : x \text{ is type-symmetric and } U_{ij}(x) \geq U_{ij}(\bar{h}) + \delta, \forall\,(i,j) \in N\}.$$

$B$ is non empty; indeed, since $\bar{h}$ is not Pareto optimal, there exists an allocation $x \in \mathcal{A}$ such that $U_{ij}(x) > U_{ij}(\bar{h}), \forall\,(i,j) \in N$. Since $\bar{h}$ is type–symmetric, as a consequence of what has just been proved, we can assume with no loss of generality that $x$ is also type-symmetric.
Moreover, the set $B$ is closed and, hence, compact.
Let $\overline{U}$ be the function defined on $B$ as in Proposition 3.2. By assumption (B.1), $\overline{U}$ is continuous on $B$ and, hence, it has a maximal element on the set $B$. Let us denote it by $g$. It holds that $g$ is type–symmetric and $U_{ij}(g) \geq U_{ij}(\bar{h}) + \delta, \forall\,(i,j) \in N$.
Moreover, following the same line of reasoning as in Proposition 3.2, we can prove that $g$ is also Pareto optimal.
Therefore, $g$ is a type–symmetric, Pareto optimal allocation which dominates $y$; that is:

$$U_{ij}(g) > U_{ij}(y), \text{ for every } (i,j) \in N$$

and this concludes the proof. $\square$

We prove now that the set $C_e^s(E)$ is internally stable with respect to the $\alpha$–dominance relation.

**Theorem** 3.4 (Internal stability) *Let the economy $E$ satisfy the Assumptions (B.3),(B.4) and (B.5). Then, the symmetric extreme core $C_e^s(E)$ is $\alpha$–internally stable.*

*Proof.* By way of contradiction, let $x$ and $y$ be two allocations in V such that $x \succ_\alpha y$. That is, there exists a coalition $S \subseteq N$ such that:

**a.** $x$ is an assignment for $S$;

**b.** $U_{ij}(x^S, z^{N\setminus S}) > U_{ij}(y)$, for all $(i,j) \in S$ and for all allocations $z$ such that $z^S = x^S$.

We first prove that there exists an agent $(i,j) \in S$ such that $x_{ij} \gg 0$. Indeed, if for every $(i,j) \in S$, $x_{ij}$ has a zero component, then by Assumption (B.3) we get that for every $(i,j) \in S$:

$$U_{ij}(x^S, z^{N\setminus S}) = U_{ij}(0^S, z^{N\setminus S})\,.$$

If we consider, in particular, the initial endowment as a redistribution among traders in the complementary coalition $N \setminus S$, we get that for every $(i,j) \in S$:

$$U_{ij}(x^S, z^{N\setminus S}) = U_{ij}(0^S, z^{N\setminus S}) = U_{ij}(0^S, e^{N\setminus S}) = U_{ij}(0^N)\,.$$

Moreover, since $y$ is individually rational and Assumptions (B.3) and (B.5) hold, $U_{ij}(y) \geq U_{ij}(e) = U_{ij}(0^N)$ and hence the inequality in condition b. cannot hold.
We can conclude that there exists an agent $(i,j) \in S$ such that $x_{ij} \gg 0$ and, therefore $\sum_{(i,j)\in S} x_{ij} \gg 0$.
As a consequence of Assumptions (B.4) and (B.5), we can thus infer that coalition $S$ contains every type of agents.
Consider now $x^{N\setminus S}$ itself as a physically feasible redistribution over the complementary coalition $N \setminus S$. By the type–symmetry of $x$ and $y$, condition b. can be rewritten as:

$$U_{ij}(x) > U_{ij}(y), \text{ for every } (i,j) \in N\,,$$

which contradicts the Pareto optimality for the allocation $y$. □

Finally, we show the property of external stability.

**Theorem** 3.5 (External Stability) *Let the economy $E$ satisfy the Assumptions (B.1), (B.2) and (B.6). Then, the set $C_e^s(E)$ is $\alpha_2$–externally stable.*

*Proof.* Let $y$ be an individually rational allocation that does not belong to $C_e^s(E)$. We have to find an allocation $x \in V$ and a non empty coalition $S \subseteq N$ such that:

**a.** $x$ is an assignment for $S$;

**b.** for all $i \in S$ and for an allocation $z$ with $z^S = x^S$, it holds that $U_i(x^S, z^{N\setminus S}) > U_i(y)$.

We distinguish three cases.
**First case.** Suppose that $y$ is type–symmetric but not Pareto optimal. Then by Proposition 3.3, it is dominated by a type-symmetric, Pareto optimal allocation $x$ through the grand coalition $N$. Since $x \in V$, the proof ends.
**Second case.** Suppose that $y$ is Pareto optimal but not type–symmetric. Since $y$ is assumed to be non-symmetric, there exist an agents' type $i \in \{1,\ldots,k\}$ and two agents $j,t \in T_i$ such that $U_{ij}(y) \neq U_{it}(y)$. With no loss of generality, we can assume that $i = 1$ and $U_{1j}(y) \geq U_{11}(y)$, for every $j \in \{1,\ldots,s\}$, with a strict inequality for at least one index $j$.
We can also assume that for every other type $i \neq 1$, it holds:

$$U_{ij}(y) \geq U_{i1}(y)\,,\ \forall (i,j) \in T_i\,.$$

Consider the type–symmetric allocation $\bar{y}$ obtained from $y$ and defined by:

$$\bar{y} = (\bar{y}^{T_1}, \ldots, \bar{y}^{T_k}) \,,$$

where $\bar{y}^{T_i}$ is the identical consumption bundle allotted to each trader of type $i$ given by $\bar{y}^{T_i} = \frac{1}{|T_i|} \sum_{(i,j)\in T_i} y_{ij}$, $i \in \{1, \ldots, k\}$.
We repeat now the same construction used for allocation $h$ in the proof of Proposition 3.3. That is, denote by $\sigma_1$ the permutation on each set $T_i$ which reindexes $(i, j)$ in $(i, j + 1)$ where $j = 1, \ldots, s-1$ and $(i, s)$ in $(i, 1)$ and by $\sigma_l$, with $l = 1, \ldots, s$, the same permutation $\sigma$ applied $l$ times. Moreover, for each $l = 1, \ldots, s$, let $T_i^{\sigma_l}$ denote the array obtained from $T_i$ after $l$ permutations and $y_{\sigma_l}^{T_i}$ the corresponding consumption profile.
Finally, let:

$$\begin{aligned} g^1 =& (y_{\sigma_1}^{T_1}, \ldots, y_{\sigma_1}^{T_k}) \,, \\ &\vdots \\ g^s =& (y_{\sigma_s}^{T_1}, \ldots, y_{\sigma_s}^{T_k}) \,. \end{aligned}$$

Note that:

$$\frac{1}{s} \sum_{l=1}^{s} g^l = \bar{y} \,.$$

By Assumption (B.2), it holds that:

$$U_{11}(\bar{y}) = U_{11}\left(\frac{1}{s}\sum_{l=1}^{s} g^l\right) > \frac{1}{s}\sum_{l=1}^{s} U_{11}(g^l). \tag{5}$$

For each $l = 1, \ldots, s$, there exists $j \neq 1$ and $(1, j) \in T_1$ such that:

$$U_{11}(g^l) = U_{1j}(y) \tag{6}$$

Since $U_{1j}(y) \geq U_{11}(y)$ for each $j \neq 1$ and with a strict inequality for at least one index, we get:

$$U_{11}(\bar{y}) > \frac{1}{s}\sum_{l=1}^{s} U_{11}(y) = U_{11}(y) \tag{7}$$

In the same way, as a consequence of strict quasi–concavity (B.2), for every type $i \neq 1$ we get:

$$U_{i1}(\bar{y}) > U_{i1}(y) \tag{8}$$

Consider the coalition $S = \{11, 21, \ldots, k1\}$. It satisfies conditions a. and b. through $\bar{y}$. Indeed, together with conditions (7) and (8), it is enough to note that the allocation $\bar{y}$ is an assignment for coalition $S$ and is also physically feasible for the counter coalition $N \setminus S$.

In particular, the consumption set feasibility for coalition $S$ is guaranteed by Assumption (B.6).
If $\bar{y}$ is Pareto optimal, the proof of the external stability of $C_e^s(E)$ ends.
If $\bar{y}$ is not Pareto optimal, then by Proposition 3.3 we can find an allocation $g$ such that it is type-symmetric, Pareto optimal and dominates $\bar{y}$ over the grand coalition $N$, that is: a. and b. are satisfied by $g$, $N$ and $\bar{y}$. Since $\bar{y}$, $S$ and $y$ satisfy a. and b., we get that also $g$, $S$ and $y$ satisfy the conditions and and this concludes the proof.
**Third case.** Suppose that $y$ is neither Pareto optimal nor type-symmetric. Since $y$ is not Pareto optimal, by Proposition 3.2, we can find a Pareto optimal allocation $h$ such that $h$ Pareto dominates $y$.
If $h$ is type-symmetric, then $h \in V$ and the proof ends.
If $h$ is not type-symmetric, by the previous case, we can find a type-symmetric and Pareto optimal allocation $g$ and a coalition $S$ such that $g$, $S$ and $h$ satisfy condition a. and b.: Therefore we can infer that $g$, $S$ and $y$ satisfy a. and b. and, since $g \in V$, this concludes the proof. □

The following theorem summarizes all the previous results.

**Theorem** 3.6 (Existence and Uniqueness) *Let the economy $E$ satisfy the Assumptions (B.1)-(B.6). Then:*

1. *the symmetric extreme core $C_e^s(E)$ is $\alpha$-internally stable and $\alpha_2$-externally stable;*
2. *whenever a set $V$ of symmetric allocations exists which is $\alpha$-externally stable and $\alpha_1$-internally stable, then $V$ equals the symmetric extreme core $C_e^s(E)$;*
3. *whenever a set $V$ of symmetric allocations exists which is $\alpha$-externally stable and $\alpha_2$-internally stable, then $V$ equals the symmetric extreme core $C_e^s(E)$;*
4. *whenever an $\alpha$-stable set $V$ of symmetric allocations exists, then $V$ equals the symmetric extreme core $C_e^s(E)$;*
5. *whenever an $\alpha_1$-stable set $V$ of symmetric allocations exists, then $V$ equals the subset of the symmetric extreme core $C_e^s(E)$ formed by all the allocations which do not $\alpha_1$–dominate each other;*
6. *whenever an $\alpha_2$-stable set $V$ of symmetric allocations exists, then $V$ equals the subset of the symmetric extreme core $C_e^s(E)$ formed by all the allocations which do not $\alpha_2$–dominate each other.*

Notice that, contrary to the framework of traders of Type I, here the uniqueness of the $\alpha$-stable set can be guaranteed but not the existence.

We conclude this section by analyzing a framework which is intermediate between the selfish and the interdependent ones. Precisely, we consider agents of Type II and suppose that the utility function of each trader $j$ of type $i$ just depends on the consumption bundles of all other traders of type $i$. That is, we still have interdependence but just within types not across all traders in the market.
In order to distinguish this model from the one analyzed so far, this sort of utility functions will be called partially interdependent preferences.
We note that, when there is just one trader for each type, this model coincides with one with selfish preferences; moreover, the partially interdependent framework can be considered as a particular case of that analyzed above, whenever the utility functions $U_{ij}$ just depend on $x^{T_i}$. The reason why it is considered on its own is two–fold and concerns the external stability of the set $C_e^s(E)$: when proving the external stability, in fact, the strict quasi–concavity can be replaced with strict monotonicity and quasi–concavity; moreover, a mechanism of redistribution among traders can be implemented which is not possible when utility functions are totally interdependent.
Formally, we adopt the notion of types provided by Definition 2.11, with two adjustments: condition (2) is not considered anymore while condition (1) is replaced as follows: for every $i = 1, \dots, k$ and for every consumption bundle $x^{T_i} = (x_{i1}, \dots, x_{ij}, \dots, x_{it}, \dots, x_{is})$, it holds that:

$$U_{ij}(x_{i1}, \dots, x_{ij}, \dots, x_{it}, \dots, x_{is}) = U_{it}(x_{i1}, \dots, x_{it}, \dots, x_{ij}, \dots, x_{is}). \tag{9}$$

In the next remark, we provide some examples of functions for which condition (9) holds true.

**Remark 3.5** When there are just two traders for each type, it is easy to find examples of functions for which condition (9) holds. For instance, in the case of just one commodity, we can consider the following two functions for traders of type 1:

$$U_{11}(x_{11}, x_{12}) = 2x_{11} + x_{12}$$
$$U_{12}(x_{11}, x_{12}) = x_{11} + 2x_{12}$$

On the other hand, for a framework with two commodities, distinguished by a superscript, we may consider the following utility functions which also identify traders of the same type according to condition (9):

$$U_{11}(x_{11}^1, x_{11}^2, x_{12}^1, x_{12}^2) = \sqrt{x_{11}^1 \cdot x_{11}^2} + \frac{x_{12}^1 + x_{12}^2}{2}$$
$$U_{12}(x_{11}^1, x_{11}^2, x_{12}^1, x_{12}^2) = \frac{x_{11}^1 + x_{11}^2}{2} + \sqrt{x_{12}^1 \cdot x_{12}^2}$$

Let us consider now a more complex context with four agents of type 1 and just one commodity. If traders are endowed with the following utility functions:

$$U_{11}(x_{11}, x_{12}, x_{13}, x_{14}) = x_{12} \cdot x_{13} \cdot x_{14} \,,$$
$$U_{12}(x_{11}, x_{12}, x_{13}, x_{14}) = x_{11} \cdot x_{13} \cdot x_{14} \,,$$
$$U_{13}(x_{11}, x_{12}, x_{13}, x_{14}) = x_{11} \cdot x_{12} \cdot x_{14} \,,$$
$$U_{14}(x_{11}, x_{12}, x_{13}, x_{14}) = x_{11} \cdot x_{12} \cdot x_{13} \,,$$

condition (9) is satisfied and, provided that their endowment is the same, all four traders can be considered of the same type.

For what concerns the notions of dominance relations given for partially interdependent preferences, two points have to be remarked.
First of all, some form of interdependence across types can still be detected in the blocking mechanism; indeed, all traders in $N \setminus S$ coordinate together their reaction to the deviation thus influencing, through $z$, the utility of each agent in $S$.
Second, we notice that, if the allocation to be blocked is type-symmetric, what really matters in a blocking coalition $S$ is the number of agents for each type not the actual identity of traders. More precisely, agents of the same type are interchangeable within a blocking coalition $S$ provided that the number of traders for each type remains the same. In fact, given condition (9) on the utility functions, consumption bundles in $x$ and in $z$ can be properly permutated within each type in such a way to guarantee a trader that is not originally included in $S$ a utility level grater than that provided by $y$. Let consider the following example to illustrate this point.

**Example** 3.1 Consider three types of traders and three traders for each type, that is:

$$N = \{11,\ 12,\ 13;\ 21,\ 22,\ 23;\ 31,\ 32,\ 33\}\,.$$

Suppose that $S = \{11,\ 12;\ 21\}$ blocks a given type–symmetric allocation $y$ through $x$. Focusing just on the first type, we have that:

$$U_{11}(x_{11}, x_{12}, z_{13}) > U_{11}(y^{T_1}),$$
$$U_{12}(x_{11}, x_{12}, z_{13}) > U_{12}(y^{T_1})\,.$$

If we substitute agent $(1,1)$ for agent $(1,3)$, for instance, we can allot the bundle $x_{11}$ to trader $(1,3)$ and the bundle $z_{13}$ to trader $(1,1)$ and we get:

$$U_{13}(z_{13}, x_{12}, x_{11}) = U_{11}(x_{11}, x_{12}, z_{13}) > U_{11}(y^{T_1}) = U_{13}(y^{T_1}).$$

Hence, also coalition $S' = \{12,\ 13;\ 21\}$ blocks the type-symmetric allocation $y$ through the assignment $x^{'}$ obtained from $x$ by properly permuting consumption bundles [8].

In the context of partially interdependent preferences we introduce the following new assumption.

(B.7) (Type–strict monotonicity) For every $i \in \{1, \dots, k\}$, for every consumption profiles $x^{T_i}$ and $y^{T_i}$ such that $x^{T_i} > y^{T_i}$, it holds that:

$$U_{ij}(x^{T_i}) > U_{ij}(y^{T_i}),\ \forall\,(i,j) \in T_i.$$

Moeover, given the form of the utility functions, the type–quasi concavity formulated in Assumption $(B.2)^{'}$ simply takes now the form of quasi–concavity.

The theorems about the internal stability of the symmetric extreme core $C_e^s(E)$ are identical, both in their statements and in their proofs, to those provided in the general case. On the contrary, we provide the proof of the external stability of the set $C_e^s(E)$ because the assumption of strict quasi–concavity is now replaced by quasi–concavity and strict monotonicity and, moreover, an interesting mechanism of redistribution among traders of different types is implemented in order to block an allocation $y$ which is Pareto optimal but not type–symmetric.

**Theorem** 3.7 (External Stability I) *Let the economy $E$ satisfy the Assumptions (B.1), $(B.2)^{'}$, (B.6) and (B.7). Then, the set $C_e^s(E)$ is $\alpha_2$– externally stable.*

*Proof.* Let $y$ be an individually rational allocation that does not belong to $C_e^s(E)$. We have to find an allocation $x \in C_e^s(E)$ and a coalition $S$ such that:

**a.** $x$ is an assignment for $S$;

**b.** $U_{ij}(z^{T_i}) > U_{ij}(y^{T_i})$, for every $(i,j) \in S$, for an allocation $z$ such that $z^S = x^S$.

[8] We also remark that such interchangeability of traders of the same type within a blocking coalition does not apply anymore for a model with agents of Type I.

We distinguish three cases.

**First case.** Suppose that $y$ is type–symmetric but not Pareto optimal. Then it is dominated by a type-symmetric, Pareto optimal allocation $x$ through the grand coalition $N$. Since $x \in C_e^s(E)$, the proof ends.

**Second case.** Suppose that $y$ is Pareto optimal but not type–symmetric. Since $y$ is assumed to be non-symmetric, there exist an agents' type $i \in \{1, \ldots, k\}$ and two agents $j, t \in T_i$ such that $U_{ij}(y^{T_i}) \neq U_{it}(y^{T_i})$. Without loss of generality, we can assume that $i = 1$ and $U_{1j}(y^{T_1}) \geq U_{11}(y^{T_1})$, for every $j \in \{1, \ldots, s\}$, with a strict inequality for at least one index $j$.

Consider the type-symmetric allocation $\bar{y}$ defined by:

$$\bar{y} = (\bar{y}^{T_1}, \ldots, \bar{y}^{T_k})$$

which allots to each trader of type $i$ the same consumption bundle given by $\bar{y}^{T_i} = \dfrac{1}{|T_i|} \sum_{j \in T_i} y_{ij}$, $i \in \{1, \ldots, k\}$.

By Assumption $(B.2)'$, it holds that:

$$U_{11}(\bar{y}^{T_1}) > U_{11}(y^{T_1}).$$

By the continuity Assumption (B.1), we can pick some $0 < \varepsilon < 1$ such that:

$$U_{11}(\varepsilon \bar{y}^{T_1}) > U_{11}(y^{T_1}) .$$

Consider the coalition $S = \{11, 21, \ldots, k1\}$ and the allocation $\bar{g}$ defined as follows:

$$\bar{g}_{ij} = \begin{cases} \varepsilon \bar{y}^{T_1} \,, & \text{if } (i,j) \in T_1; \\ \\ \bar{y}^{T_i} + \dfrac{1-\varepsilon}{k-1} \bar{y}^{T_1}, & \text{if } (i,j) \in T_i \,, i \neq 1. \end{cases}$$

Note that the allocation $g$ is feasible for both the coalition $S$ and the grand coalition $N$. We want to show that $S$ can dominate the allocation $y$ through $\bar{g}$ in the sense of condition a. and b.

Indeed, it clearly holds that:

$$U_{11}(\bar{g}^{T_1}) = U_{11}(\varepsilon \bar{y}^{T_1}) > U_{11}(y^{T_1}) .$$

Moreover, by Assumptions (B.7) and $(B.2)'$, for every $l \in \{2, 3, \ldots, k\}$:

$$U_{l1}(\bar{g}^{T_l}) > U_{l1}(\bar{y}^{T_l}) \geq U_{l1}(y^{T_l}) .$$

If $\bar{g}$ is Pareto optimal, the proof of the external stability of $C_e^s(E)$ ends.
If $\bar{g}$ is not Pareto optimal, then by Proposition 3.3 we can find an allocation $\tilde{g}$ such that it is type-symmetric, Pareto optimal and dominates $\bar{g}$ over the grand coalition $N$, that is:

$$\tilde{g} \succ_N \bar{g}$$

We can conclude that

$$\tilde{g} \succ_S y$$

and this concludes the proof.
**Third case.** Suppose that $y$ is neither Pareto optimal nor type-symmetric. Since $y$ is not Pareto optimal, by Proposition 3.2, we can find a Pareto optimal allocation $h$ such that $h \succ_{\alpha_1}^N y$.
If $h$ is type-symmetric, then $h \in C_e^s(E)$ and the proof ends.
If $h$ is not type-symmetric, by the previous case, we can find a type-symmetric and Pareto optimal allocation $\bar{g}$ and a coalition $S$ such that $S$ dominates $h$ via $g$ as in conditions a. and b. Since $h$ Pareto dominates $y$, we can infer the conclusion. □

# 4 A final remark and conclusions

We conclude by noticing that the way consumption sets are modelled in our paper easily includes an asymmetric information setting with a sharing rule. Indeed, consider a framework where there is uncertainty over the states of nature: the exogenous uncertainty and the asymmetric information initially possessed by traders are formulated respectively in terms of a measurable space $(\Omega, \mathcal{F})$, with $\Omega$ denoting a finite set of $k$ states of nature and the field $\mathcal{F}$ representing the set of all possible events, and of measurable partitions $P_i$ of $\Omega$ [9]. At $\tau = 0$ agents make contracts that may be contingent on the realized state of nature at period $\tau = 1$ when each trader receives his private information: after receiving it, he is not necessarily able to distinguish which state of nature $\omega \in \Omega$ actually occurs but he can observe the element of his partition $P_i$ that contains such a state. In this period traders receive their endowment of physical resources and agreements are carried out.
Given a partition $P$ of $\Omega$, a commodity bundle $x = (x(\omega))_{\omega \in \Omega} \in (\mathbb{R}_+^l)^\Omega$ is said to be $P$–measurable when it is constant over the elements of the partition $P$.
The state-dependent initial endowment of physical resources for each agent $i$ is the measurable commodity bundle given by $e_i : \Omega \longrightarrow \mathbb{R}_+^l$.

[9]Mathematically, a partition of a finite set $\Omega$ is a division of that set into a collection of nonempty, pairwise disjoint, and collectively exhaustive subsets of $\Omega$.

Starting from these primitive data, a sharing rule (Allen, 2006) can be defined in order to describe how the initial information of each trader potentially varies when he joins different coalitions and share his information.
If $\mathcal{P}$ denotes the set of all possible partitions of $\Omega$, an ***information sharing rule for a coalition*** $S \subseteq N$ is a mapping:

$$\Gamma^S : \underbrace{\mathcal{P} \times \ldots \times \mathcal{P}}_{|S|-\text{times}} \longrightarrow \underbrace{\mathcal{P} \times \ldots \times \mathcal{P}}_{|S|-\text{times}}$$

with the property that if $|S| = 1$[10], then $\Gamma^S = Id$ where $Id$ denotes the identity map.
An ***information sharing rule*** is a collection $\Gamma = (\Gamma^S)_{S \subseteq N}$ of $2^n$ information sharing rules, one for every possible coalition.
To shorten notation, for every $i \in S$ we will denote by $\Gamma_i^S$ the information that trader $i$ has within coalition $S$.
The definition above is extremely general and the information $\Gamma_i^S$ that each member of the coalition $S$ has is not related to his initial information $P_i$.
The sharing rule $\Gamma$ influences the consumption possibilities for each trader in all coalitions he joins; in fact, as shown by next definitions, each trader $i$ as a member of $N$ (respectively, $S$) can only consider consumption bundles that are measurable with respect to the partition $\Gamma_i^N$ (respectively, $\Gamma_i^S$).
An **allocation** under the information sharing rule $\Gamma$ is a vector $x = (x_i)_{i \in N}$ with $x_i \in (\mathbb{R}_+^l)^k$ such that:

**i)** $x_i$ is $\Gamma_i^N$ - measurable, for every $i \in N$ (*informational feasibility*);

**ii)** $\sum_{i \in N} x_i(\omega) = \sum_{i \in N} e_i(\omega),\ \forall\, \omega\ \in S$ (*physical feasibility*).

An **assignment** for coalition $S$ under the information sharing rule $\Gamma$ is a vector $x = (x_i)_{i \in S}$ with $x_i \in (\mathbb{R}_+^l)^k$ such that:

**i)** $x_i$ is $\Gamma_i^S$ - measurable, for every $i \in S$ (*informational feasibility*);

**ii)** $\sum_{i \in S} x_i(\omega) = \sum_{i \in S} e_i(\omega),\ \forall\, \omega\ \in S$ (*physical feasibility*).

That is, in such a framework the consumption set of each trader $i \in N$ depends and varies according to what coalition he joins.
Our final comment deals with the question whether the allocations contained in a stable set $V$ in the presence of asymmetric information and information sharing rules are incentive compatible. This issue is relevant because incentive compatibility somehow ensures

[10]The notation $|S|$ stands for the cardinality of the set $S$.

in its turn a form of stability. The coalitional incentive compatibility of stable sets in the case of selfish models follows from Proposition 1 in Graziano et al. (2015). A similar result can be proved in the case of interdependent preferences analyzed in this paper (see also Pesce and Yannelis (2012)).

To summarize, in this paper we consider an exchange economy where traders do not care only about their own consumption but have interdependent preferences. For these economies, we introduce the notion of stable sets with externalities and address the problem of their existence. Several frameworks are distinguished based on two elements: a. the notion of dominance between two allocations; b. the notion of agents' types. As regards the first element, different definitions are considered that differ in the way agents outside of a blocking coalition react to a deviation; as to the second element, we focus on two definitions of types such that the utility function of each agent exhibits different degrees of interdependence towards the other individuals in the market. In each of the resulting frameworks, we study the internal and external stability of the set of individually rational, Pareto optimal, possibly type–symmetric allocations. The existence of stable sets with externalities is particularly relevant for exchange economies because the more traditional notion of core is frequently empty in such frameworks. For this reason, the notions we introduce in this paper can hopefully find applications in general equilibrium as well as game theory models.

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