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Basis Expansions in Applied Mathematics

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Contents

Chapter 1

Introduction and Main Results

Basis expansions are an extremely useful tool in applied mathematics. By using them, we can express a function representing a physical quantity as a linear combination of simpler "modules" with well-known properties. They are particularly useful for the applications described in this thesis. Perhaps the best known expansion of this type is the Fourier series of a periodic function, as decomposition into the infinite sum of simple sinusoidal and cosinusoidal elements, originally proposed by Fourier to study heat transfer. This dissertation employs some mathematical tools on problems taken from various areas of Engineering, exploiting their expansion properties:

- non-integer bases, whose study was started by Rényi and Parry in the 1950s, and subsequently deepened by a group of Hungarian mathematicians led by Paul Erdös. In [116] and [117], A.C. Lai, P. Loreti and myself applied non-integer bases to mathematical models in Robotics;

- orthogonal polynomials whose topic, historically, probably originated from the Legendre polynomials, firstly employed in the determination of the force of attraction exerted by solids of revolution. In [186] A. M. Bersani and myself introduced a class of orthogonal polynomials, which follows the same recursive rule of the well-known Lucas-Lehmer integer sequence and, in [189], we applied it to the solution of Love's problem, related to the electrostatic field generated by two circular co-axial conducting disks;

- orthonormal bases, that are introduced in the earliest college courses, and the Riesz bases. These latter, named after the Hungarian mathematician Frigyes Riesz (1880-1956), represent a generalization of the orthonormal bases. In [11], [122] and [188], A. Avantaggiati, P. Loreti and myself considered exponential $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ and sinc $\{\mathrm{sinc}(t-\lambda_n)\}_{n\in\mathbb{Z}}$ Riesz basis (with $\lambda_n \in \mathbb{C}$).

We can consider expansions of *real numbers in non-integer bases*, as an "elementary" case of basis expansion which, however, contains well-known difficult theoretical problems. Problems related to the expansions of real numbers in non-integer bases have been systematically studied since the late 1950s, starting with the seminal works by Rényi [159] and Parry [149]. Given a complex number λ greater than 1 in modulus and a possibly infinite set $A \subset \mathbb{C}$ we say that $z \in \mathbb{C}$ is representable in base λ and with alphabet A if there exists a sequence $\{z_j\}_{j\geq 1}$ of digits of A such that

$$z = \sum_{j=1}^{\infty} \frac{z_j}{\lambda^j}.$$

A digit sequence $\{z_j\}_{j\geq 1}$ satisfying the above equality is called expansion of z in base λ and with alphabet A. It is well-known that coplanar rotations, like the ones performed by each finger of our hand, can be read as products on the complex plane. Therefore to perform infinite rotations and scalings corresponds to consider complex-based power series and, consequently, expansions in non-integer bases. This suggests us the relation between robotics (planar manipulators, robotic hands etc.) and theory of expansions in non-integer bases, as initially guessed in [45]. We will deal with this topic in Chapter 2.

Chapter 2 relies also on the theory of *Iterated Function Systems* [109], which turns out to be one of the most common ways for generating fractals.

Definition 1.0.1. An iterated function system, or IFS, on a metric space $\mathbb{X}$ is a finite collection of mappings $w_i : \mathbb{X} \to \mathbb{X}$, $i = 1, 2, \ldots, N$, which are usually contractive. The contractivity of the IFS is the number $c := \max_i c_i$, where c_i is the contractivity of w_i.

Let us consider, for instance, the celebrated Cantor set. In the classical construction, we start with the unit interval $I_0 = [0, 1]$. The first steps consists in deleting the open middle subinterval $(1/3, 2/3)$ from the interval $[0, 1]$, leaving a subset I_1 composed by two line segments: $I_1 = [0, 1/3] \cup [2/3, 1]$. Next, the open middle third of each of these remaining segments is deleted, leaving four line segments: $[0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$. This transformation is iterated ad infinitum, where indicating with a the generic minimum of every subinterval at the n-th step, the subinterval $\left[a, a + \frac{1}{3^n}\right]$ is transformed into

$$\bar{I}^a_{n+1} \cup \bar{\bar{I}}^a_{n+1} = \left[a, a + \frac{1}{3^{n+1}}\right] \cup \left[a + \frac{2}{3^{n+1}}, a + \frac{1}{3^n}\right],$$

at the $(n+1)$-th step. Calling I_n the subset obtained at the n-th step, the Cantor set is defined as the intersection of all the I_n, i.e. $\mathcal{C} := \bigcap_{n=1}^{\infty} I_n$. One of the most remarkable properties of $\mathcal{C}$ is that it is composed of smaller pieces, each of which is an exactly scaled copy of $\mathcal{C}$. To be slightly more precise, considering the generic subinterval $I^a_n = [a, b] \subset I_n$, such that $b - a = \frac{1}{3^n}$, and defining $w_0, w_1 : \mathbb{R} \to \mathbb{R}$ by

$$w_0(x) = \frac{1}{3}(x + 2a)\,; \quad w_1(x) = \frac{1}{3}(x + 2b)\,,$$

we see that $w_0\left(I^a_n\right) = \bar{I}^a_{n+1}$, $w_1\left(I^a_n\right) = \bar{\bar{I}}^a_{n+1}$.

Robotics has profound cultural roots [169]. Over the course of centuries, human beings have constantly attempted to seek for substitutes that would be able to mimic their behaviour in the various instances of interaction with the surrounding environment. On the other hand, one of human beings' greatest ambitions has been to give life to their artifacts. Just think, for example, to the legend of the Titan Prometheus, to the giant Talus and to Frankenstein in modern times. The word *robot* comes from the Czech *robota*, which has the meaning of "heavy work" or "forced labor". In the 1940s, the image of the robot as a mechanical artifact takes hold, especially thanks to the imagination of Isaac Asimov, who introduces the term *robotics* as the science devoted to the study of robots which was based on three fundamental laws [169]:

1. A robot can not injure a human being or, through inaction, allow a human being to come to harm.
2. A robot must obey orders given it by human beings except when such orders would conflict with the First Law.
3. A robot must protect its own existence as long as such a protection does not conflict with the First or Second Law.

The full scope of robotics lies at the intersection of mechanics, electronics, signal processing, control engineering, computing and mathematical modelling.

Mechanics and signal processing are other branches of application of basis expansions investigated here. For these topics a "keyword" is the term *orthogonality*.

The topic of **orthogonal polynomials** has its origin in the XIXth century theories of continued fractions and the moment problem. A family of polynomials $p_n(x)$ for $n = 0, 1, 2, \ldots$ in the interval $[a, b]$, where n indicates the degree of polynomial, is called a "sequence of orthogonal polynomials" in the interval $[a, b]$ with respect to the weight function $w(x)$ (which is positive in the interval $[a, b]$) if:

$$\int_a^b w(x)p_n(x)p_m(x)\,dx = 0 \qquad \forall n, m = 0, 1, 2, \ldots \qquad \text{with } n \neq m.$$

Classical orthogonal polynomials, such as those of Legendre, Laguerre and Hermite, but also Chebyshev, Krawtchouk and others polynomials, have found widespread use in all areas of science and engineering. Typically, more complicated functions are expanded with respect to basis functions like orthogonal polynomials [80].

As we know, that of "basis", is a more general and abstract concept that is not limited to functions and polynomials. Every student in mathematics learns about bases in vector spaces, allowing one to represent each element in a unique way. In fact, for every $f \in V$ and V finite dimensional vector space, if $\{e_k\}_{k=1}^m$ in V is a basis for V, there exist unique scalar coefficients $\{c_k\}_{k=1}^m$ such that

$$f = \sum_{k=1}^{m} c_k e_k$$

If space V is equipped with an inner product $\langle \cdot, \cdot \rangle$ and, in addition, $\{e_k\}_{k=1}^m$ is an **orthonormal basis**, then one can easily find the expression of scalar coefficients $\{c_k\}_{k=1}^m$. In fact:

$$\langle f, e_j \rangle = \langle \sum_{k=1}^{m} c_k e_k, e_j \rangle = \sum_{k=1}^{m} c_k \underbrace{\langle e_k, e_j \rangle}_{\neq 0 \Leftrightarrow j=k} = c_j \;\Rightarrow\; f = \sum_{k=1}^{m} \langle f, e_k \rangle e_k$$

Orthonormal bases are widely used in mathematics as well as in physics, signal processing, and many other areas where one needs to represent functions in terms of bases. The following result characterizes all orthonormal bases for a separable infinite-dimensional Hilbert space $\mathcal{H}$, starting with one orthonormal basis.

Let $\{e_k\}_{k=1}^\infty$ be an orthonormal basis for $\mathcal{H}$. Then the orthonormal bases for $\mathcal{H}$ are precisely the sets $\{Ue_k\}_{k=1}^\infty$, where $U : \mathcal{H} \to \mathcal{H}$ is a unitary operator.

However, there are some systems that work well even without an important property as the orthogonality:

1) The definition of **a Riesz basis** - which is a little more general than orthonormal basis - appears by weakening the condition of the operator U (see [46], [199]):

Definition 1.0.2. A Riesz basis for $\mathcal{H}$ is a family of vectors of the form $\{Ue_k\}_{k=1}^{\infty}$, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$ and $U : \mathcal{H} \to \mathcal{H}$ is a bounded bijective operator.

2) A **frame** is a sequence of vectors in a Hilbert space, not necessarily orthogonal to each other, satisfying certain inequalities. Frames are a generalization of the basis concept and among the reasons for their widespread use in the last twenty years we can read the following motivations [46]. The main point is the missing flexibility: the conditions for being a basis are so strong that even a slight modification of a basis might destroy the basis property. One reason is that, if $\{x_n\}$ is a basis for a Banach space X, each element x of X is a linear combination (even infinite) of the basis element,

$$x = \sum c_n x_n \tag{1.0.1}$$

i.e., the so-called expansion property of $\{x_n\}$. Now, if y is an arbitrary element of X, then $\{x_n\} \cup \{y\}$ is not a basis, despite the fact that each $x \in X$ has representations of the form

$$x = \sum c_n x_n + \tilde{c}\, y \tag{1.0.2}$$

This means that the basis property is destroyed when an arbitrary nonempty collection of vectors is added to $\{x_n\}$, but the expansion property is preserved. Is this a frame? At first glance, the above construction might appear artificial: why would one like to add elements to a basis? One reason is that we gain some freedom: the coefficients in (1.0.1) are unique, but in (1.0.2) we can choose among several options. The abstract results of frames theory (that overcomes this limit of the bases) can be applied in signal processing: for example, if x is the *signal*, the additional term in the (1.0.2) can be seen as noise term.

Another annoying fact about bases is their lack of stability against applications of operators. If, for example, $\{x_n\}$ is an orthonormal basis, then only very special operators (the unitary ones) U will make $\{Ux_n\}$ an orthonormal basis. If $\{x_n\}$ is a basis, then we need U to be a bounded bijective operator in order for $\{Ux_n\}$ to be a basis. Frames are considerably more stable than bases: application of just a bounded surjective operator will preserve the frame property, as we can see in [46], [199] or in the review article [39].

This dissertation is devoted to the generalization and development of studies described in articles published by the author in collaboration with others, and based on the expansion property of some systems: non-integer bases, orthogonal polynomials, orthonormal bases, Riesz bases. Frames - as generalization of Riesz bases and as example of non-orthogonal systems - are only approached here, and they represent a research topic that the author will follow in the future.

We now proceed to outline the dissertation and its main results.

The robots proposed in Chapter 2 involve the Fibonacci sequence. We develop a more general technique which is based on recursively generated sequences and, among them, the

Fibonacci sequence is a classic example. The approach based on the Fibonacci sequence in robot models is motivated by the ubiquitous presence of Fibonacci numbers in nature (see [17] and [194]) and, in particular, in human limbs [148].

We recall that the k-th Fibonacci number, f_k, is defined by means of

$$\begin{cases} f_0 = f_1 = 1; \\ f_{k+2} = f_{k+1} + f_k \quad k \geq 0. \end{cases} \tag{1.0.3}$$

The Fibonacci sequence was first defined in 1202 by the Italian mathematician Leonardo of Pisa, nicknamed Filius Bonacci or Fibonacci. In Chapter 2 we study two robot models. These robots belong to the class of so-called *macroscopically-serial hyper-redundant manipulators* — the term was first introduced in [43] — which are planar manipulators with rigid links and with an arbitrarily large number of degrees of freedom.

The first model studied here is the model of a robot finger. A configuration of a finger is the sequence $(\mathbf{x}_k)_{k=0}^K \subset \mathbb{R}^3$ of its junctions. The configurations of every finger are ruled by two phalanx-at-phalanx motions: extension and rotation. In particular, the length of k-th phalanx of the finger is either 0 or $\dfrac{f_k}{q^k}$. Parameter q is a fixed ratio: this choice is ruled by a binary control we denote by using the symbol u_k, so that the length l_k of the k-th phalanx is

$$l_k := \| \mathbf{x}_k - \mathbf{x}_{k-1} \| = \frac{u_k f_k}{q^k}.$$

Doing so we will introduce a robot hand model composed by an arbitrarily large number of hyper-redundant binary planar manipulators, where the length of each link scales according to the Fibonacci sequence. Let

$$R_\infty := \left\{ \sum_{k=0}^{\infty} \frac{u_k f_k}{q^k} e^{-i\omega \sum_{j=0}^{k} v_j} | (v_j), (u_j) \in \{0,1\}^\infty \right\}$$

be the *asymptotic reachable workspace* and let $co(R_\infty)$ be the convex hull of R_∞; in Chapter 2 we investigate the reachable workspace and its convex hull. An example of convex hull is depicted in Figure 1.1, for $\omega = \pi/3$.

The second model described in Chapter 2 is the model of a planar hyper-redundant manipulator that is analogous in morphology to robotic snakes and tentacles, based on a discrete linear dynamical system involving the Fibonacci sequence [117]. The hyper-redundant manipulator is controlled by a sequence of couples of discrete actuators on the junctions, ruling both the length and the orientation of every link.

The first main results for this model are Theorem 2.2.3 and Theorem 2.2.8; they deal with some asymptotic controllability properties of the manipulator. The investigation begins with the study of the *total length* of the manipulator, $L(\mathbf{u}) := \sum_{n=0}^{\infty} \frac{u_n f_n}{q^n}$, under the condition $q > \varphi$ for the convergence of the above described series, where $\varphi = (1+\sqrt{5})/2$ is the Golden Ratio. Theorem 2.2.3 is a first investigation of the behaviour of the set of possible total lengths $L_{\infty,q} := \{L(\mathbf{u}) \mid \mathbf{u} \in \{0,1\}^\infty\}$ as $q \to \infty$. It states that we can arbitrarily set the length of manipulator within the range $[0, L(\mathbf{1})]$ (where we have set $\mathbf{1} := (1,1,\ldots,1,...)$) if and only if the scaling ratio q belongs to the range $(\varphi, 1+\sqrt{3}]$. Theorem 2.2.3 can be employed to prove sufficient conditions for the *local asymptotic controllability* of the control system underlying the model (see

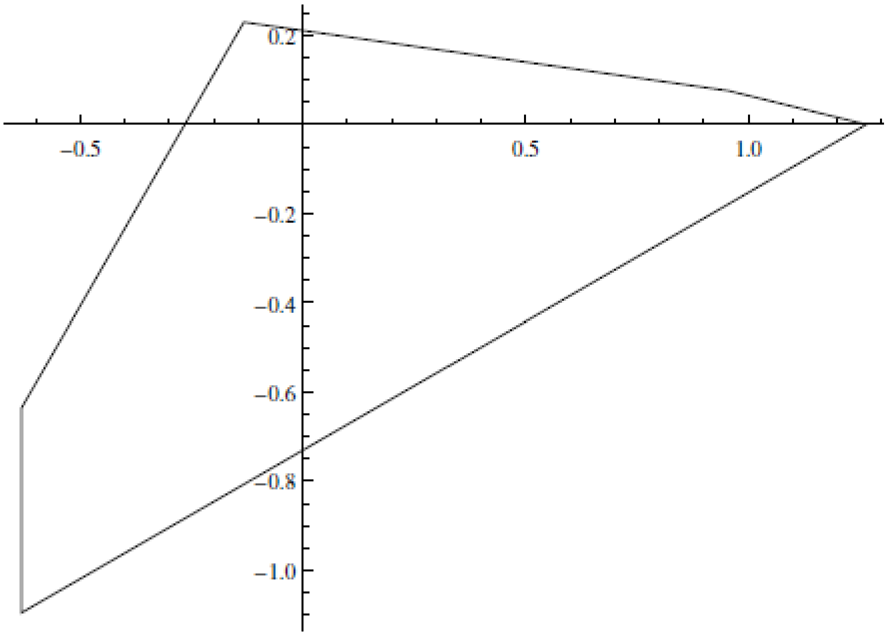


Figure 1.1: Convex hull of R_∞ with $q = \varphi + 1$, where φ is the Golden Mean.

Theorem 2.2.8), that is the possibility of placing the end effector of the manipulator arbitrarily close to any point belonging to a sufficiently small neighborhood of the origin. Theorem 2.2.8 states that if q belongs to a certain range, then the asymptotic reachable workspace contains a neighborhood of the origin.

The third main result for the second model described in Chapter 2 concerns the characterization of $L_{\infty,q,\omega}$ and the set of full-rotation configurations in terms of the attractor of a suitable Iterated Function System (IFS). This approach gives access to well-established results in fractal geometry in order to further investigate the topological properties of the reachable workspace, and to use known efficient algorithms for the generation of self-similar sets (e.g. Random Iteration Algorithm) to have a numerical approximation of the asymptotic reachable set.

Chapter 3 is devoted to study the stability of exponential Riesz bases $\{e^{i\lambda_n t}\}$ for $\lambda_n \in \mathbb{C}$, and of the cardinal sine sequences $\{\mathrm{sinc}(x - \lambda_n)\}_{n\in\mathbb{Z}}$ for $\lambda_n \in \mathbb{R}$. In Section 3.1 we study the exponential Riesz bases $\{e^{i\lambda_n t}\}$ for $\lambda_n \in \mathbb{C}$, recalling that, when $\lambda_n \in \mathbb{R}$, exponential Riesz bases are stable in the sense that a small perturbation of a Riesz basis produces a Riesz basis; it is proved by Paley and Wiener ([199] and [147]). The celebrated theorem by M. I. Kadec shows that 1/4 is the stability bound for $\{e^{i\lambda_n t}\}$ on $L^2[-\pi, \pi]$, where $\lambda_n \in \mathbb{R}$:

If $\{\lambda_n\}_{n\in\mathbb{Z}}$ is a sequence of real numbers for which

$$|\lambda_n - n| \leqq L < \frac{1}{4}, \qquad n = 0, \pm 1, \pm 2, \ldots,$$

then the system $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ satisfies the Paley-Wiener criterion and so forms a Riesz basis for $L^2[-\pi, \pi]$.

Indeed, Kadec's 1/4-Theorem applies for sequences of real numbers. Duffin and Eachus [63] showed that the Paley - Wiener criterion is satisfied whenever the sequences are complex and

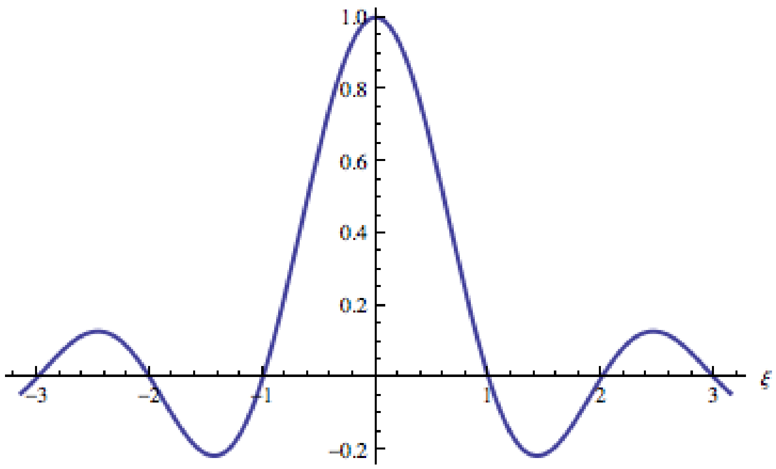


Figure 1.2: Graph of sinc x.

$\frac{\log 2}{\pi}$ is a stability bound. A consequence of Theorem 3.1.3, introduced in Section 3.1, is that, thanks to a limitation on the imaginary part of λ_n, the constant $\frac{\log 2}{\pi}$ can be replaced by 1/4. Section 3.2 is devoted to prove, in the spirit of Kadec's 1/4-Theorem, a stability result for a cardinal sine sequence $\{\mathrm{sinc}(x-\lambda_n)\}_{n\in\mathbb{Z}}$ for $\lambda_n \in \mathbb{R}$, where the *sinc-function*, whose graph is shown in Figure 1.2, is defined by

$$\mathrm{sinc}(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x} & x \neq 0, \\ 1 & x = 0\,. \end{cases} \tag{1.0.4}$$

We denote $L^2(-\infty,+\infty)$ the Hilbert space of real functions that are square integrable in Lebesgue's sense:

$$L^2(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R}\,\text{measurable function} \,\middle|\, \int_{-\infty}^{+\infty} |f(x)|^2 dx < +\infty \right\}.$$

Introducing in $[-\pi,\pi]$ the scalar product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)g(x)dx\,,$$

the L^2-norm on $[-\pi,\pi]$ is defined as

$$||f|| = \sqrt{\langle f, f\rangle}\,.$$

Given $f \in L^2(\mathbb{R})$ we denote by $\hat{f}$ the Fourier transform of f,

$$\hat{f}(\omega) = \mathcal{F}(f)(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)e^{-i\omega x}dx.$$

In Section 3.2 we show that Kadec's bound 1/4 is still the stability bound for the sinc basis on the Paley-Wiener space PW_π, which is also known as the space of "bandlimited functions" and is also characterized by an orthonormal basis consisting of functions $\{\mathrm{sinc}(x-n)\}_{n\in\mathbb{Z}}$. A function $f \in L^2(\mathbb{R})$ is *bandlimited* on $[-\Omega,\Omega]$ if $\mathrm{supp}(\hat{f}) \subseteq [-\Omega,\Omega]$ (that is, $\hat{f}(\xi) = 0$ for almost every $|\xi| > \Omega$). A definition of the Paley-Wiener space can be found in [90]:

Definition 1.0.3 ([90], p. 270)**.** The *Paley-Wiener space* PW_π is the space of functions in $L^2(\mathbb{R})$ whose Fourier transforms are supported within the interval $[-\pi, \pi]$:

$$PW_\pi = \left\{ f \in L^2(\mathbb{R}) \,:\, \mathrm{supp}(\hat{f}) \subseteq [-\pi, \pi] \right\} .$$

Chapter 4 introduces a new sequence of polynomials, which follow the same recursive rule of the well-known Lucas-Lehmer integer sequence.

$$L_n(x) = L_{n-1}(x)^2 - 2\,; \qquad\qquad L_0(x) = x\ . \tag{1.0.5}$$

Lucas-Lehmer polynomials are related to the Chebyshev polynomials of the first and second kind. As we know [161, 18, 79], the Chebyshev polynomials of first and second kind ($T_n(x)$ and $U_n(x)$) satisfy the recurrence relations

$$\begin{cases} T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) & n \geq 2 \\ T_0(x) = 1, \quad T_1(x) = x \end{cases}$$

and

$$\begin{cases} U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x) & n \geq 2 \\ U_0(x) = 1, \quad U_1(x) = 2x\,, \end{cases}$$

respectively. We will prove, among other properties, that

$$L_n(x) = 2\ T_{2^{n-1}}\left(\frac{x^2}{2} - 1\right) \quad \text{and} \quad \prod_{i=1}^{n} L_i(x) = U_{2^n-1}\left(\frac{x^2}{2} - 1\right).$$

In Section 4.2 we reinvestigate the structure of the solution of a well-known Love's problem, related to the electrostatic field generated by two circular co-axial conducting disks, in terms of orthogonal polynomial expansions, enlightening the role of the Lucas-Lehmer polynomials, introduced in Section 4.1. We also show that the solution can be expanded more conveniently with respect to a Riesz basis obtained starting from Chebyshev polynomials. In Section 4.3 we discuss some relations between zeros of Lucas-Lehmer polynomials and Gray code. Gray code is a particular binary code which is widely used in Informatics. Given a binary code, we say that its **order** is the number of bits with which the code is built, while its **length** is the number of strings that compose it. The Gray code [78, 137] is a binary code of order n and length 2^n.

Let us consider the code for $n-1$ bits which is formed by binary strings

$$\begin{aligned} &g_{n-1,1} \\ &\dots \\ &g_{n-1,2^{n-1}-1} \\ &g_{n-1,2^{n-1}}\ . \end{aligned} \tag{1.0.6}$$

The Gray code is an ordered code such that two successive code word have Hamming distance 1. In this sense, the following is a particular Gray code that can be generated in this way:

$$\begin{array}{l}
0g_{n-1,1} \\
\ldots \\
0g_{n-1,2^{n-1}-1} \\
0g_{n-1,2^{n-1}} \\
1g_{n-1,2^{n-1}} \\
1g_{n-1,2^{n-1}-1} \\
\ldots \\
1g_{n-1,1}
\end{array} \tag{1.0.7}$$

Just as an example, we have: for $n = 1$: $g_{1,1} = 0$; $g_{1,2} = 1$; for $n = 2$: $g_{2,1} = 00$; $g_{2,2} = 01$; $g_{2,3} = 11$; $g_{2,4} = 10$; for $n = 3$: $g_{3,1} = 000$; $g_{3,2} = 001$; $g_{3,3} = 011$; $g_{3,4} = 010$; $g_{3,5} = 110$; $g_{3,6} = 111$; $g_{3,7} = 101$; $g_{3,8} = 100$; and so on. We apply this binary law to the study of nested square roots of 2 expressed by (4.1.2), associating bits 0 and 1 to $\oplus$ and $\ominus$ signs in the nested form. This gives the possibility to obtain an ordering for the zeros of Lucas-Lehmer polynomials, which assume the form of nested square roots of 2 expressed by (4.1.2).

In Section 4.4 we obtain π as the limit of a sequence related to the zeros of the class of polynomials $L_n(x)$ discussed in previous sections. The results obtained here are based on the placement of the zeros of the polynomials $L_n(x)$. Since zeros have a structure of nested radicals, in this way we can build infinite sequences of nested radicals converging to π.

The need of understanding the properties of π and the need of computing its value in a more and more precise way, since the origins of the mathematical thinking, has challenged many mathematicians along more than three millennia [19, 69]. As observed by J. M. Borwein in [34] (p. 543), "*One motivation for computations of π was very much in the spirit of modern experimental mathematics: to see if the decimal expansion of π repeats, which would mean that π is the ratio of two integers (i.e., rational), or to recognize π as algebraic — the root of a polynomial with integer coefficients — and later to look at digit distribution.*".

In 1882 von Lindemann showed the transcendence (and *a fortiori* the irrationality) of π. In the meantime, many mathematicians continued to discover several sequences and series converging to π. The recent literature ([13], [32], [33], [151]), after centuries devoted to the search of elegant formulas and to the study of the irrationality of π, focused mainly on the search for rapidly converging formulas.

The famous Indian mathematician Ramanujan determined several sequences converging to π very rapidly. In particular, it is noteworthy to cite 17 different extraordinary series, converging very rapidly to $1/\pi$, [27] (p. 352-354), [154]. Here we report one of the most intriguing:

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26390k)}{(k!)^4 396^{4k}} \ . \tag{1.0.8}$$

The "spirit of the modern experimental Mathematics" [32, 33] was built mainly by means of the birth, growth and development of the computer technologies in the 1950s and of the discovery of more and more advanced and efficient algorithms apt to perform highly precise arithmetic computations. Today we know millions digits of π value. Soon the challenge was shifted to more and more rapid and efficient algorithms. The advent of computers in the XXth century led to an increased rate of π calculation records. But major improvements were obtained by means of new extraordinarily efficient algorithms. For example, in 1965 it was shown that an optimal algorithm used to compute the so-called *Fast Fourier Transform* (FFT), could be adapted to perform arithmetics on huge integer numbers more rapidly than with previous algorithms and with less computational costs [30] [32].

The last quarter of the XXth century was characterized by the discovery of several algorithms with reduced operational complexity. These algorithms require the complete execution of multiplications, divisions, extractions of square roots, which need large scale FFT operations, implying the usage of huge memory and heavy parallel computing ([10] and [34]). The formulas of BBP type [13] (named after Bailey-Borwein-Plouffe) are more recent. It is however clear that a reason for the modern computations of π resides in the goal of taking advantage of the impressive computation power of modern computers.

We are aware that the rate of convergence of our sequences, introduced in Section 4.4, is slower than (1.0.8) and other more recent series [151]. However, determining the computational costs and the convergence rates of the sequences converging to π here introduced is beyond the scopes of this doctoral thesis.

Chapter 2

Robot's mathematical models based on Fibonacci sequence.

Fibonacci numbers attracted the interest of researchers due to their fascinating algebraic properties (e.g. the relation with Golden Mean) and due of their recurrence in natural phenomena. Examples of relations with Fibonacci sequence can be found in the branches of trees, in the arrangement of sunflowers seeds and, most interestingly for our model, in some human anatomic proportions (see [86]). The approach followed here is motivated by the ubiquitous presence of Fibonacci numbers in nature (see [17] and [194]) and, in particular, in human limbs [148].

The robots proposed in this Chapter are planar manipulators with rigid links and with an arbitrarily large number of degrees of freedom, i.e., they belong to the class of so-called *macroscopically-serial hyper-redundant manipulators* – the term was first introduced in [43], and they involve the Fibonacci sequence. Hyper-redundant architecture was intensively studied back to the late 1960s, when the first prototype of hyper-redundant robot arm was built [8]. The interest of researchers in devices with redundant controls is motivated by their ability to avoid obstacles and to perform new forms of locomotion and grasping – see for instance [14], [36] and [44].

A large number of papers were devoted in the literature to both continuously and discretely controlled hyper-redundant manipulators. Our approach, based on discrete actuators, is motivated by their precision with low cost compared to actuators with continuous range-of-motion. Moreover the resulting discrete space of configurations reduces the cost of position sensors and feedbacks. In [64] the inverse kinematics of discrete hyper-redundant manipulators is investigated. Throughout the analysis of the reachable workspace (and in particular of the density of its points) an algorithm solving the inverse kinematics problem in linear time with respect the number of actuators is introduced. In general the number of points of the reachable workspace increases exponentially, the computational cost on the optimization of the density distribution of the workspace is investigated in [120]. Note that the concept of a binary tree describing all the possible configurations underlies above mentioned approaches, in our method the self-similar structure of such a tree gives access to well-established results on fractal geometry and iterated function systems theory. Robotic devices with a similar fractal structure are described in [133]. Other approaches to the investigation of the reachable workspace include those based on harmonic analysis [41], and Fast Fourier Transform [191]. We also refer to [42] for a description of the geometry of the reachable workspace. The control of the rotation at every joint is a common feature of all above mentioned manipulators. The study of a control ruling the extension of every link has twofold applications. In one hand it can be physically implemented by means of telescopic links, that are particularly efficient in constrained workspaces (see [3]). On the other hand, our models can be considered discrete approximations of continuous snake-like manipulators - see for instance the approach in [7] to the discretization of a continuous curve and its applications to snake-like robots.

The aim of Section 2.1 is to give a model of robot hand whose links scale according to Fibonacci sequence as introduced in [116], and to develop a theoretical background (related to the theory of iterated function systems) in order to study some geometrical features of a such a manipulator. Self-similarity of configurations and an arbitrarily large number of fingers (including the opposable thumb) and phalanxes are the main features. Binary controls rule the dynamics of the hand, in particular the extension and the rotation of each phalanx. We assume that each finger moves on a plane; every plane is assumed to be parallel to the others, excepting the thumb and the index finger, that belong to the same plane. A discrete dynamical system models the position of the extremal junction of every finger. A *configuration* is a sequence of states of the system corresponding to a particular choice for the controls, while the union of all the possible states of the system is named *reachable workspace for the finger*. The closure of the reachable workspace is named *asymptotic reachable workspace*. Our model includes two binary control parameters on every phalanx of every finger of the robot hand. The first control parameter rules the length of the k-th phalanx, that can be either 0 or $f_k q^{-k}$, where f_k is the k-th Fibonacci number and q is a fixed scaling ratio, while the other control rules the angle between the current phalanx and the previous one. Such an angle can be either π, namely the phalanx is consecutive to the previous, or a fixed angle $\pi - \omega \in (0, \pi)$. The structure of the finger ensures the set of possible configurations to be the projection of a particular self-similar set. We also establish a connection between our model and the theory of iterated function systems. This yields several results describing the reachable workspace and some conditions on the parameters in order to avoid self-intersecting configurations.

The aim of Section 2.2 is to give a model of a planar hyper-redundant manipulator as introduced in [117], that is analogous in morphology to robotic snakes and tentacles, based on a discrete linear dynamical system involving the Fibonacci sequence. The hyper-redundant manipulator is controlled by a sequence of couples of discrete actuators on the junctions, ruling both the length and orientation of every link. Crucial as it is, the effective control of hyper-redundant manipulator is difficult for its redundancy; see, for example [121]. For instance, the number of points of the reachable workspace increases exponentially with the number of degrees of freedom. In Section 2.2, we employ the self-similarity of the Fibonacci sequence in order to provide alternative techniques of investigation of the reachable workspace based on combinatorics and on fractal geometry.

The purpose of the Section is to provide a theoretical background suitable for applications to inverse kinematic problems in a fashion like [64]. Furthermore, in [111] the design of a manipulator modeling human arm and with link lengths following the Fibonacci sequence, provides a method for the self-collision avoidance problem. We believe that analogous geometrical properties can be extended to manipulators which are inspired by other biological forms, through the self-similarity induced by Fibonacci numbers.

Hyper-redundant manipulators considered here are planar manipulators. This is only a first step in exploring an approach that, to the best of our knowledge, could add novelty to the existing literature in this field; therefore, for future work, its extension to the 3D case represents a natural progress of this work.

We also mention that the workspaces of planar manipulators in the above cited papers (e.g. [64]) are quite different from those depicted here. This is mainly due to the fact that we represent only a subset of the workspace, corresponding to the particular subclass of *full-rotation configurations* whose relation with fractal geometry is the most striking. Furthermore, unlike above mentioned works, our robotic device has a telescopic structure modeled by the possibility of ruling not only the orientation but also the length of each link: we believe this additional feature to possibly affect the shape of the workspace.

The theoretical background relies on the theory of Iterated Function Systems – see [70] for a general introduction on the topic. The approach proposed here is inspired by the relation between robotics and theory of expansions in non-integer bases, that was first introduced in [45] and later applied to planar manipulators in [113], [114], [115], [116] and [117]. For an overview on the expansions in non-integer bases we refer to the Rényi's seminal paper [159] and to the papers [149] and [68]. For the geometrical aspects of the expansions in complex base, namely the arguments that are more related to problem studied here, we refer to the papers [81], [82], [98] and [105]. The techniques developed here in order to study the full-rotation configuration generalize previous results in [112].

2.1 A model for robotic hand based on Fibonacci sequence.

In our model the robot hand is composed by H fingers, every finger has an arbitrary number of phalanxes. We assume junctions and phalanxes of each finger to be thin, so to be respectively approximated with their middle axes and barycentres and we also assume the junctions of every finger to be coplanar. Inspired by the human hand, we set the fingers of our robot as follows: the first two fingers are coplanar and they have in common their first junction (they are our robotic version of the thumb and the index finger of the human hand) while the remaining $H-1$ fingers belong to parallel planes. By choosing an appropriate coordinate system $oxyz$ we may assume that the the first two fingers belong to the plane $p^{(1)} : z = 0$ while, for $h \geq 2$, h-th finger belongs to the plane $p^{(h)} : z = z_0^{(h)}$ for some $z_0^{(2)}, \ldots, z_0^{(H)} \in \mathbb{R}$.

We now describe in more detail the model of a robot finger. A configuration of a finger is the sequence $(\mathbf{x}_k)_{k=0}^{K} \subset \mathbb{R}^3$ of its junctions. The configurations of every finger are ruled by two phalanx-at-phalanx motions: extension and rotation. In particular, the length of k-th phalanx of the finger is either 0 or $\dfrac{f_k}{q^k}$, where f_k is the k-th fibonacci number, namely

$$\begin{cases} f_0 = f_1 = 1; \\ f_{k+2} = f_{k+1} + f_k \quad k \geq 0. \end{cases} \tag{2.1.1}$$

while q is a fixed ratio: this choice is ruled by a binary control we denote by using the symbol u_k, so that the length l_k of the k-th phalanx is

$$l_k := \| \mathbf{x}_k - \mathbf{x}_{k-1} \| = \frac{u_k f_k}{q^k}.$$

As all phalanxes of a finger belong to the same plane, say p, in order to describe the angle between two consecutive phalanxes, say the $k-1$-th and the k-th phalanx, we just need to consider a one-dimensional parameter, ω_k. Each phalanx can lay on the same line as the former or it can form with it a fixed planar angle $\omega \in (0, \pi)$, whose vertex is the $k-1$-th junction. In other words, two consecutive phalanxes form either the angle π or $\pi - \omega$. By introducing the binary control v_k we have that the angle between the $k-1$-th and k-th phalanx is $\pi - \omega_k$, where

$$\omega_k = v_k \omega.$$

To describe the kinematics of the finger we adopt the Denavit-Hartenberg (DH) convention. To this end, first of all recall that our base coordinate frame $oxyz$ is such that oxy is parallel to p (hence to every plane $p^{(h)}$) and we consider the finger coordinate frame $o_I x_I y_I z_I$ associated to the 4×4 homogeneous transform

$$A_I = \begin{pmatrix} \cos\omega_I & -\sin\omega_I & 0 & x_I \\ \sin\omega_I & \cos\omega_I & 0 & y_I \\ 0 & 0 & 1 & z_0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for some $\omega_I \in [0, 2\pi)$. In particular if $\mathbf{x}$ and $\mathbf{x}_0$ are respectively coordinates of a point with respect to $oxyz$ and $o_I x_I y_I z_I$ then

$$\begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = A_I \begin{pmatrix} \mathbf{x}_0 \\ 1 \end{pmatrix}.$$

Remark 2.1.1. When only one finger is considered one may assume the base coordinate frame to coincide with the finger coordinate frame: this reduces A_I to the identity and it could be omitted it in the model. The need of a coordinate frame for the finger rises when more than one finger, especially in the case of co-planar, opposable fingers, is considered.

Now, the (DH) method consists in attaching to every phalanx, say the k-th phalanx, a coordinate frame $o_k x_k y_k z_k$, so that $\mathbf{x}_k$ coincides with o_k and $\mathbf{x}_k - \mathbf{x}_{k-1}$ is parallel to $o_k x_k$. Note that the coordinates of $\mathbf{x}_{k+1}$ with respect to $o_k x_k y_k z_k$ are $(\frac{u_{k+1} f_{k+1}}{q^{k+1}} \cos\omega_{k+1}, \frac{u_{k+1} f_{k+1}}{q^{k+1}} \sin\omega_{k+1}, 0)$.

Since we are considering a planar manipulator, for every $k > 1$ the geometric relation between the coordinate systems the $k-1$-th and the k-th phalanx is expressed by the matrix

$$A_k := \begin{pmatrix} \cos\omega_k & -\sin\omega_k & 0 & \frac{u_k f_k}{q^k} \cos\omega_k \\ \sin\omega_k & \cos\omega_k & 0 & -\frac{u_k f_k}{q^k} \sin\omega_k \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where the rotation matrix

$$\begin{pmatrix} \cos\omega_k & -\sin\omega_k & 0 \\ \sin\omega_k & \cos\omega_k & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

represents the rotation of the coordinate frame $o_k x_k y_k z_k$ with respect to $o_{k-1} x_{k-1} y_{k-1} z_{k-1}$ and the vector $\left(\frac{u_k f_k}{q^k} \cos\omega_k, -\frac{u_k f_k}{q^k} \sin\omega_k, 0\right)$ represent the position of o_k with respect to $o_{k-1} x_{k-1} y_{k-1} z_{k-1}$.

Set

$$T_k := A_I \prod_{j=0}^{k} A_j$$

By definition T_k is the composition the transforms $A_I, A_0, ..., A_k$ and, consequently, it represents the relation between the base coordinate frame $oxyz$ and $o_k x_k y_k z_k$. In particular

$$T_k = \begin{pmatrix} R_k & P_k \\ 0 & 1 \end{pmatrix}$$

where R_k is a 3×3 rotation matrix and the entries of the vector P_k are the coordinates of $o_k (= x_k)$ in the reference system $oxyz$. Expliciting T_k one has

$$R_k = \begin{pmatrix} \cos\left(\omega_I + \sum_{j=0}^{k} \omega_j\right) & -\sin\left(\omega_I + \sum_{j=0}^{k} \omega_j\right) & 0 \\ \sin\left(\omega_I + \sum_{j=0}^{k} \omega_j\right) & \cos\left(\omega_I + \sum_{j=0}^{k} \omega_j\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$P_k = P_I + \sum_{j=0}^{k} R_j \begin{pmatrix} \frac{u_j f_j}{q^j} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x_I + \sum_{j=0}^{k} \frac{u_j f_j}{q^j} \cos\left(\sum_{n=0}^{j} \omega_n\right) \\ y_I - \sum_{j=0}^{k} \frac{u_j f_j}{q^j} \sin\left(\sum_{n=0}^{j} \omega_n\right) \\ z_I \end{pmatrix}$$

Then, for every $k \geq 0$,

$$\begin{pmatrix} x_k \\ y_k \\ z_k \end{pmatrix} = \begin{pmatrix} x_I + \sum_{j=0}^{k} \frac{u_j f_j}{q^j} \cos\left(\sum_{n=1}^{j} \omega_n\right) \\ y_I - \sum_{j=0}^{k} \frac{u_j f_j}{q^j} \sin\left(\sum_{n=1}^{j} \omega_n\right) \\ z_I \end{pmatrix} \tag{2.1.2}$$

2.1.1 Characterization of the Reachable Workspace via Iterated Function Systems

We fix as initial state $(x_I, y_I, z_I) = (0, 0, 0)$ and assume $\omega_I = 0$. By employing the isometry between $\mathbb{R}^2$ and $\mathbb{C}$ and by considering that our manipulator is essentially planar, we may rewrite (2.1.2) as

$$\begin{cases} x_k = \sum_{j=0}^{k} \frac{u_j f_j}{q^j} e^{-i\omega \sum_{n=0}^{j} v_n} \\ x_I = 0. \end{cases} \tag{2.1.3}$$

We aim to study the *asymptotic reachable workspace*

$$R_\infty := \left\{ \sum_{k=0}^{\infty} \frac{u_k f_k}{q^k} e^{-i\omega \sum_{j=0}^{k} v_j} | (v_j), (u_j) \in \{0, 1\}^\infty \right\}$$

In order to have a more compact notation, infinite binary (control) sequences (u_j) and (v_j) are equivalently denoted by $\mathbf{u}$ and $\mathbf{v}$, respectively. We set

$$x(\mathbf{u}, \mathbf{v}) := \sum_{k=0}^{\infty} \frac{u_k f_k}{q^k} e^{-i\omega \sum_{j=0}^{k} v_j}$$

and we define the *shift operator* on R_∞

$$\sigma : x(\mathbf{u}, \mathbf{v}) \mapsto x(\sigma(\mathbf{u}), \sigma(\mathbf{v}))$$

so that if $x = x(\mathbf{u}, \mathbf{v})$ then

$$\sigma(x) = \sum_{k=0}^{\infty} \frac{u_{k+1} f_k}{q^k} e^{-i\omega \sum_{j=0}^{k} v_{j+1}}.$$

Finally we define the auxiliary set

$$Q_\infty = \left\{ (x, \sigma(x)) \mid x = x(\mathbf{u}, \mathbf{v});\ \mathbf{u},\ \mathbf{v} \in \{0,1\}^\infty \right\}.$$

Note that $Q_\infty \in R_\infty \times R_\infty$ and $\pi(Q_\infty) = R_\infty$ where $\pi : \mathbb{C}^2 \to \mathbb{C}$ denotes the projection of a bidimensional complex vector on its first component.

We characterize Q_∞ and, consequently, R_∞ via the linear maps $F_{00}, F_{10}, F_{01}, F_{11} : \mathbb{C}^2 \to \mathbb{C}^2$ defined as follows

$$F_{uv}(z) = e^{-i\omega v} \left(A_q z + \begin{pmatrix} u \\ 0 \end{pmatrix} \right) \quad \text{for } u, v \in \{0,1\}$$

where $z \in \mathbb{C}^2$ and

$$A_q := \begin{pmatrix} \frac{1}{q} & \frac{1}{q^2} \\ 1 & 0 \end{pmatrix}.$$

In order to describe the action of F_{uv}'s on Q_∞, for any $\mathbf{u}, \mathbf{v} \in \{0,1\}^\infty$ set $\bar{\mathbf{u}}(u) := u\mathbf{u}$ and $\bar{\mathbf{v}}(v) := v\mathbf{v}$. In other words

$$\bar{u}_k(u) = \begin{cases} u & \text{if } k = 0 \\ u_{k-1} & \text{otherwise} \end{cases}, \quad \bar{v}_k(v) = \begin{cases} v & \text{if } k = 0 \\ v_{k-1} & \text{otherwise} \end{cases}.$$

Lemma 2.1.2. *Let $\mathbf{u}, \mathbf{v} \in \{0,1\}^\infty$, $u, v \in \{0,1\}$. Set $x = x(\mathbf{u}, \mathbf{v})$ and $\bar{x} = x(\bar{\mathbf{u}}(u), \bar{\mathbf{v}}(v))$. One has*

$$F_{uv}(x, \sigma(x)) = (\bar{x}, \sigma(\bar{x})) = (\bar{x}, x). \tag{2.1.4}$$

Remark 2.1.3. F_{uv} acts on $x(\mathbf{u}, \mathbf{v})$ by prepending to the control sequences $\mathbf{u}$ and $\mathbf{v}$ the controls u and v. Lemma 2.1.2 also implies that $F_{uv}(Q_\infty) \subset Q_\infty$ for every $u, v \in \{0,1\}$.

Proof of Lemma 2.1.2. By definition of F_{uv} and of σ, and recalling $\sigma(\bar{\mathbf{u}}(u)) = \mathbf{u}$ and $\sigma(\bar{\mathbf{v}}(v)) = \mathbf{v}$, one has

$$F_{uv}((x, \sigma(x)) = \left(e^{-i\omega v} \left(\frac{1}{q} x + \frac{1}{q^2} \sigma(x) + u \right), \sigma(\bar{x}) \right)$$

Then it is left to prove

$$e^{-i\omega v} \left(\frac{1}{q} x + \frac{1}{q^2} \sigma(x) + u \right) = \bar{x}.$$

Recalling $f_0 = f_1$, one has that $e^{-i\omega v}\left(\frac{1}{q}x + \frac{1}{q^2}\sigma(x) + u\right)$ is equal to

$$
\begin{aligned}
& ue^{-i\omega v} + \sum_{k=0}^{\infty} \frac{u_k f_k}{q^{k+1}} e^{-i\omega(\sum_{j=0}^{k} v_j + v)} + \sum_{k=0}^{\infty} \frac{u_{k+1} f_k}{q^{k+2}} e^{-i\omega(\sum_{j=0}^{k} v_j + v)} \\
&= ue^{-i\omega v} + \frac{u_0 f_0}{q} e^{-i\omega(v_0+v)} + \sum_{k=1}^{\infty} \frac{u_k f_{k+1}}{q^{k+1}} e^{-i\omega(\sum_{j=0}^{k} v_j + v)} \\
&= uf_0 e^{-i\omega v} + \frac{u_0 f_1}{q} e^{-i\omega(v_0+v)} + \sum_{k=1}^{\infty} \frac{u_k f_{k+1}}{q^{k+1}} e^{-i\omega(\sum_{j=0}^{k} v_j + v)} \\
&= {\frac{\bar{u}_0 f_0}{e}}^{-i\omega \bar{v}_0} + \frac{\bar{u}_1 f_1}{q} e^{-i\omega \bar{v}_1} + \sum_{k=2}^{\infty} \frac{\bar{u}_k f_k}{q^k} e^{-i\omega \sum_{j=0}^{k} \bar{v}_j} \\
&= \sum_{k=0}^{\infty} \frac{\bar{u}_k f_k}{q^k} e^{-i\omega \sum_{j=0}^{k} \bar{v}_j} = \bar{x}.
\end{aligned}
$$

□

Before stating next result, we assume that the scaling ratio q is greater than the Golden Mean.

Proposition 2.1.4. *Q_∞ is the unique compact subset of $\mathbb{C}^2$ satisfying*

$$\bigcup_{u,v\in\{0,1\}} F_{uv}(Q_\infty) = Q_\infty. \tag{2.1.5}$$

Proof. First of all we show (2.1.5) by double inclusion. The inclusion $\subseteq$ directly follows by Lemma 2.1.2 – see also Remark 2.1.3. Thus it suffices to show that for every $\bar{x} \in R_\infty$ there exist $x \in R_\infty$ and $u, v \in \{0,1\}$ such that

$$F_{uv}(x, \sigma(x)) = (\bar{x}, \sigma(\bar{x})).$$

Let $\bar{\mathbf{u}}, \bar{\mathbf{v}} \in \{0,1\}^\infty$ be a couple of the control sequences satisfying $\mathbf{x} = x(\mathbf{u}, \mathbf{v})$. Then, again by Lemma 2.1.2,

$$F_{\bar{u}_1\bar{v}_1}(\sigma(\bar{x})), \sigma(\sigma(\bar{x})))) = (\bar{x}, \sigma(\bar{x})).$$

Since R_∞ is closed with respect to σ, then $x \doteq \sigma(\bar{x}) \in R_\infty$ and this completes the proof of (2.1.5).

Now, let us prove the uniqueness of Q_∞. First of all we note that for every $u, v \in \{0,1\}$, F_{uv} is a linear map and consider its spectral radius $R(q)$. $R(q)$ is hence the greatest modulus of the eigenvalues of A_q. If $q > \varphi$, where φ is the Golden Mean, then $R(q) = \frac{\sqrt{5}q+q}{2q^2} < 1$. Consequently the induced norm of A_q^k

$$||A_q^k|| := \max_{z\in\mathbb{C}^2, z\neq(0,0)-} ||A_q^k z|| \to 0 \quad \text{as } k \to \infty.$$

Then there exists k_q such that if $k \geq k_q$ then F_{uv}^k is a contraction. Since the quantity k_q is independent on u and v, one has that any concatenation of length $k \geq k_q$ of F_{uv}'s, say

$$G_{\mathbf{u}^k,\mathbf{v}^k} \doteq F_{u_1^k v_1^k} \circ \cdots \circ F_{u_k^k v_k^k},$$

is a contraction. Consequently one may consider the *Hutchinson operator*

$$\mathcal{G}(\cdot) := \bigcup_{\mathbf{u}^k,\mathbf{v}^k \in \{0,1\}^k} G_{\mathbf{u}^k,\mathbf{v}^k}(\cdot)$$

and deduce by (2.1.5)

$$\mathcal{G}(Q_\infty) = Q_\infty. \tag{2.1.6}$$

Since $\mathcal{G}$ is generated by a finite set of contractive maps, namely by an Iterated Function System, then

$$Q_\infty \text{ is the only compact subset of } \mathbb{C}^2 \text{ enjoying (2.1.6)} \tag{U}$$

In order to seek a contradiction, assume now that there exists a compact set $X \subset \mathbb{C}^2$ differen than Q_∞ satisfying (2.1.5). Then X also satisfies (2.1.6): the uniqueness condition (U) provides the required contradiction and concludes the proof. □

2.1.2 Characterization of the convex hull of the Reachable Workspace

Through this section we employ Proposition 2.1.4 in order to characterize the $co(R_\infty)$, co denoting the convex hull of a set. We begin by the following general fact.

Lemma 2.1.5. *Let $\{F_1, \dots, F_H\}$ be a finite set of linear maps on a metric space X and assume that there exists and it is unique a compact set Q satisfying*

$$\mathcal{F}(Q) := \bigcup_{h=1}^{H} F_h(Q) = Q.$$

If

$$\mathcal{F}(Y) \subseteq Y \tag{2.1.7}$$

for some $Y \subset X$ then

$$Q \subseteq Y\,. \tag{2.1.8}$$

Proof. By iterating (2.1.7) for one has for every k

$$Y \supseteq \mathcal{F}(Y) \supseteq \mathcal{F}^2(Y) \supseteq \cdots \supseteq \mathcal{F}^k(Y)$$

then as $k \to \infty$, the set sequence $\mathcal{F}^k(Y)$ converges to a set $\bar{Y}$ satisfying

$$\mathcal{F}(\bar{Y}) = \bar{Y} \subseteq Y.$$

By the uniqueness of Q one has $\bar{Y} = Q$ and this completes the proof. □

Theorem 2.1.6. *Following the notations of previous Section, let $V \subset Q_\infty$ be such that*

$$\mathcal{F}(V) := \bigcup_{u,v \in \{0,1\}} F_{uv}(V) \subseteq co(V)\,. \tag{2.1.9}$$

Then

$$co(R_\infty) = co(\pi(V)).$$

Proof. The linearity of F_{uv}'s and (2.1.14) imply

$$\mathcal{F}(co(V)) \subseteq co(V). \tag{2.1.10}$$

This together with Proposition 2.1.4, implies that we may apply Lemma 2.1.5 to Q_∞ and $Y = co(V)$ and deduce $Q_\infty \subseteq co(V)$. By assumption we also have $V \subset Q_\infty$, then $co(Q_\infty) = co(V)$. The claim hence follows by the fact that $R_\infty = \pi(Q_\infty)$ and that projection is a convex map. □

Next result gives a more operative description of $co(R_\infty)$.

Theorem 2.1.7. *Let W be a compact subset of Q_∞. If*

$$\pi(\mathcal{F}(W)) \subseteq \pi(co(W)) \tag{2.1.11}$$

then

$$co(R_\infty) = co(\pi(W)).$$

Proof. We show the claim by double inclusion. The inclusion $\supseteq$ is trivial, since we assumed $W \subseteq Q_\infty$ and, consequently $\pi(W) \subseteq R_\infty$. Now we show by induction that if (2.1.12) holds then for every k

$$\pi(\mathcal{F}^k(W)) \subseteq \pi(co(W)). \tag{2.1.12}$$

The case $k = 1$ is given by (2.1.12) itself. We then assume as inductive hypothesis

$$\pi(\mathcal{F}^{k-1}(W)) \subseteq \pi(co(W))$$

so that we get for every $(\hat{w}, \sigma(\hat{w})) \in W$

$$\mathcal{F}^k(\hat{w}) = \mathcal{F}(w, \sigma(w)) \quad \text{with } w \in \pi(co(W))$$

In particular, $w = \sum_k \lambda_k w_k$ for some $w_k \in \pi(W)$ and some convex combinators λ_k. Since $W \subseteq Q_\infty$, if $w_k \in \pi(W)$ then $(w_k, \sigma(w_k)) \in W$. Then, by (2.1.12)

$$\pi(\mathcal{F}^k(\hat{w})) = \sum_k \lambda_k \pi(\mathcal{F}(w_k, \sigma(w_k))) \subseteq co(\pi(W)). \tag{2.1.13}$$

Now, note that $\mathcal{F}^k(W)$ is a non-decreasing sequence of compact sets, consequently as $k \to \infty$ it tends to some compact set $\bar{W}$ satisfying $\mathcal{F}(\bar{W}) = \bar{W}$. By Proposition 2.1.4 we get $\bar{W} = Q_\infty$. Consequently

$$R_\infty = \pi(Q_\infty) = \lim_{k\to\infty} \pi(\mathcal{F}^k(W)) \subseteq \pi(co(W)) \tag{2.1.14}$$

The claim follows by noting that above inclusion implies $co(R_\infty) \subseteq \pi(co(W))$. □

Explicit description of $co(R_\infty)$ in a particular case

In [116], we have considered the case $\omega = \pi/3$ and we have shown that $co(R_\infty)$ is a polygon whose vertices are

$$\begin{array}{ll} \mathbf{v}_1 := \sum_{k=0}^{\infty} \frac{f_k}{q^k}; & \mathbf{v}_2 := e^{-i\omega} \sum_{k=0}^{\infty} \frac{f_k}{q^k}; \\ \mathbf{v}_3 := e^{-i\omega} + e^{-i2\omega} \sum_{k=1}^{\infty} \frac{f_k}{q^k}; & \mathbf{v}_4 := e^{-i2\omega} \sum_{k=1}^{\infty} \frac{f_k}{q^k}; \\ \mathbf{v}_5 := 1 + e^{-i2\omega} \sum_{k=2}^{\infty} \frac{f_k}{q^k}. & \end{array}$$

See Figure 1.1 in Chapter 1.

We have applied Theorem 2.1.7, introducing the symbols $\mathbf{0}$ and $\mathbf{1}$ to denote infinite sequences of 0's and 1's, respectively, and to note that

$$\begin{array}{ll} \mathbf{v}_1 = x(\mathbf{1}, \mathbf{0}); & \mathbf{v}_2 = x(\mathbf{1}, 1\mathbf{0}); \\ \mathbf{v}_3 = x(\mathbf{1}, 11\mathbf{0}); & \mathbf{v}_4 = x(0\mathbf{1}, 11\mathbf{0}); \\ \mathbf{v}_5 = x(10\mathbf{1}, 011\mathbf{0}). & \end{array}$$

So that, recalling the definition $\sigma(x(\mathbf{u}, \mathbf{v})) = x(\sigma(\mathbf{u}), \sigma(\mathbf{v}))$ where $\sigma(\mathbf{u})$ denotes the unit shift of $\mathbf{u}$, one gets

$$\begin{array}{ll} \sigma(\mathbf{v}_1) = \mathbf{v}_1 & \sigma(\mathbf{v}_2) = \mathbf{v}_1; \\ \sigma(\mathbf{v}_3) = \sigma(\mathbf{v}_4) = \mathbf{v}_2; & \sigma(\mathbf{v}_5) = \mathbf{v}_4. \end{array}$$

Let $W = \{(\mathbf{v}_h, \sigma(\mathbf{v}_h)) \mid h = 1, \ldots, 5\}$. By Theorem 2.1.7 one has

$$co(R_\infty) = co(\{\mathbf{v}_h \mid h = 1, \ldots, 5\})$$

if for every $h = 1, \ldots, 5$ and for every $u, v \in \{0, 1\}$

$$\pi(F_{uv}(\mathbf{v}_h, \sigma(\mathbf{v}_h))) \in \pi(co(W)). \tag{2.1.15}$$

In [116], we have shown above inclusion by distinguishing the cases $h = 1, \ldots, 5$, observing that $(0, 0) \in co(\pi(V))$. Consequently, we have used the fact that if $\mathbf{z} \in V$ then for every $c \in [0, 1]$, $c\mathbf{z} \in co(V)$. For a more detailed discussion, see [116].

2.2 A Fibonacci control system with application to hyper-redundant manipulators

A discrete dynamical system models the position of the extremal junction of the manipulator. The model includes two binary control parameters on every link. The first control parameter, denoted u_n, rules the length of the n-th link $l_n := u_n f_n q^{-n}$, where f_n is the n-th Fibonacci number and q is a constant scaling ratio. The other control, v_n, rules the angle between the current link and its predecessor, denoted $\omega_n := (\pi - \omega)v_n$, where ω a fixed angle in $(0, \pi)$. Therefore when $v_n = 0$, the n-th link is collinear with its predecessor, and when $v_n = 1$, it forms a fixed angle $\pi - \omega \in (0, \pi)$ with the $n-1$-th link. In Section 2.2.1 we show that, under these assumptions, the position of the n-th junction, $x_n(\mathbf{u}, \mathbf{v})$ is ruled by the relation

$$x_n(\mathbf{u}, \mathbf{v}) = x_{n-1}(\mathbf{u}, \mathbf{v}) + u_n \frac{f_n}{q^n} e^{-i\omega \sum_{h=0}^{n} v_h} \tag{2.2.1}$$

where $\mathbf{u} = (u_j)$, $\mathbf{v} = (v_j) \in \{0, 1\}^\infty$. By assuming that the n-th junction is positioned at time n (namely by reading the index n as a discrete time variable) above equation may be reinterpreted as a discrete control system, whose trajectories model the configurations of the manipulator. This is a stationary problem: indeed, at this stage of the investigation we are interested on the reachable workspace of the manipulator (namely a static feature of robot) rather than its kinematics. In this setting, if the number of the links is finite, say it is equal to N, then the position of the end effector of the manipulator (i.e., the position of its extremal junction) is represented by $x_N(\mathbf{u}, \mathbf{v})$. We call *reachable workspace* the set

$$W_{N,q,\omega} := \{x_N(\mathbf{u}, \mathbf{v}) \mid \mathbf{u}, \mathbf{v} \in \{0,1\}^N\}.$$

By allowing an infinite number of links, we also may introduce the definition of the *asymptotic reachable workspace*

$$W_{\infty,q,\omega} := \{\lim_{N\to\infty} x_N(\mathbf{u}, \mathbf{v}) \mid \mathbf{u}, \mathbf{v} \in \{0,1\}^\infty\}.$$

The main results, Theorem 2.2.3 and Theorem 2.2.8, deal with some asymptotic controllability properties of the manipulator.

Indeed, the investigation begins with the study of the quantity

$$L(\mathbf{u}) := \sum_{n=0}^{\infty} \frac{u_n f_n}{q^n}$$

called the *total length* of the manipulator[1]. First of all we notice that the condition $q > \varphi$, where $\varphi = (1+\sqrt{5})/2$ is the Golden Ratio, ensures the convergence of above series. Theorem 2.2.3 is a first investigation of the behaviour of the set of possible total lengths

$$L_{\infty,q} := \{L(\mathbf{u}) \mid \mathbf{u} \in \{0,1\}^\infty\}$$

as $q \to \infty$. In particular we show that if q is less than or equal to $1+\sqrt{3}$ then $L_{\infty,q}$ is an interval. This estimate is sharp, indeed we shall also prove that when $q > 1+\sqrt{3}$ then $L_{\infty,q}$ is a disconnected set. In other words, Theorem 2.2.3 states that we can arbitrarily set the length of manipulator within the range $[0, L(\mathbf{1})]$ (where we have set $\mathbf{1} := (1, 1, \dots, 1, ...)$) if and only if the scaling ratio q belongs to the range $(\varphi, 1+\sqrt{3}]$. The proof of Theorem 2.2.3 is constructive and an explicit algorithm is given.

Theorem 2.2.3 turns out to be also a useful tool in order to prove sufficient conditions for the *local asymptotic controllability* of the control system underlying the model (see Theorem 2.2.8), that is the possibility of placing the end effector of the manipulator arbitrarily close to any point belonging to a sufficiently small neighborhood of the origin. More precisely, Theorem 2.2.8 states that, under some technical assumptions (namely we assume that the maximal rotation angle ω is of the form $2d\pi/p$ for some $d, p \in \mathbb{N}$), if q belongs to a certain range, then the asymptotic reachable workspace contains a neighborhood of the origin[2].

The approach in the investigation of $L_{\infty,q}$ and $R_{\infty,q,\omega}$, the latter defined as

$$R_{\infty,q,\omega} := \left\{ \sum_{k=0}^{\infty} u_k \frac{f_k}{q^k e^{i\omega k}} \mid \mathbf{u} \in \{0,1\}^\infty \right\},$$

strongly relies on the particular choice of the lengths of the links, $l_n(u_n) := u_n f_n q^{-n}$, and in particular, on the fact that, fixing $\mathbf{u} = (u_n)$ the "backward" sequence $\bar{L}_n(\mathbf{u}) = \sum_{j=0}^{n} l_j(u_{n-j})$

[1]Notice that $L(\mathbf{u}) = L(\mathbf{u}, \mathbf{v})$ for all $\mathbf{v} \in \{0,1\}^\infty$ where $L(\mathbf{u}, \mathbf{v}) := \sum_{n=1}^{\infty} |x_n(\mathbf{u}, \mathbf{v}) - x_{n-1}(\mathbf{u}, \mathbf{v})|$

[2]Actually, we prove that such a neighborhood is indeed a polygon which is symmetric with respect to the origin.

satisfies the recursive, contractive relation

$$\bar{L}_{n+1}(\mathbf{u}) = \frac{u_n + \bar{L}_n(\mathbf{u})}{q} + \frac{\bar{L}_{n-1}(\mathbf{u})}{q^2}. \tag{2.2.2}$$

A suitable generalization of (2.2.2) is interpreted as a discrete control dynamical system, the *Fibonacci control system*, which is investigated by means of combinatorial arguments.

We then use a generalization of above approach in order to study a suitable subset of $R_{\infty,q,\omega}$, the set of *full-rotation configurations* (namely the configurations corresponding to the choice $\mathbf{v} = \mathbf{1}$). This approach is motivated by the fact that the full-rotation configurations satisfy a contractive, recursive relation similar to (2.2.2).

The third main result of the Section concerns the characterization of $L_{\infty,q,\omega}$ and the set of full-rotation configurations in terms of the attractor of a suitable Iterated Function System (IFS). This approach gives access to well-established results in fractal geometry in order to further investigate the topological properties of the reachable workspace, and to use known efficient algorithms for the generation of self-similar sets (e.g. Random Iteration Algorithm) to have a numerical approximation of the asymptotic reachable set.

In what follows we show some numerical simulations approximating the asymptotic reachable set associated with full-rotation configurations. However a deeper exploitition of these potential applications is beyond the purposes of present work.

We finally remark that for all $N \geq 0$ we have the inclusion $W_{N,q,\omega} \subset W_{\infty,q,\omega}$ and, consequently, the Hausdorff distance between $W_{N,q,\omega}$ and $W_{\infty,q,\omega}$ satisfies

$$\begin{aligned} d_H(W_{N,q,\omega}, W_{\infty,q,\omega}) &= \sup_{x_\infty \in W_{\infty,q,\omega}} \inf_{x_N \in W_{N,q,\omega}} |x_\infty - x_N| \\ &\leq \sum_{k=N+1}^{\infty} \frac{f_k}{q^k} \leq \frac{q}{q^N(q^2 - q - 1)}. \end{aligned}$$

The above relation establishes a global error estimate for the approximation of $W_{\infty,q,\omega}$ with $W_{N,q,\omega}$, hence every above mentioned asymptotic controllability property is inherited by a practical implementable manipulator with a finite number of links N by paying an explicitly given, exponential decaying cost in terms of precision.

In Section 2.2.1 we introduce the model and we state the main results on the density of the reachable workspace. The remaining part of the section is devoted to the analysis of the dynamical system underlying the model. Section 2.2.2 includes the introduction of such Fibonacci control system and to its preliminary properties. In Section 2.2.2 and Section 2.2.2 we establish some properties of reachability and local controllability. Finally in Section 2.2.3 we establish a relation with the theory of Iterated Function Systems and we point out some parallelisms with classical expansions in non-integer bases.

2.2.1 A model for a snake-like robot.

Throughout this section we introduce a model for a snake-like robot. We assume links and junctions to be thin, so to be respectively approximated with their middle axes and barycentres. We also assume axes and barycentres to be coplanar and, by employing the isometry between $\mathbb{R}^2$, we use the symbols $x_0, x_1, ..., x_n \in \mathbb{C}$ to denote the position of the barycentres of the junctions, therefore the length l_n of the n-th link is

$$l_n = |x_n - x_{n-1}|. \tag{2.2.3}$$

We assume l_n to be ruled by a binary control u_n, and in particular,

$$l_n := u_n \frac{f_n}{q^n}. \tag{2.2.4}$$

where (f_n) is Fibonacci sequence, namely $f_0 = f_1 := 1$ and $f_{n+2} = f_{n+1} + f_n$ for all $n \geq 0$.

Now, consider the quantity

$$L(\mathbf{u}) = \sum_{n=0}^{\infty} l_n(u_n) \quad \text{with } \mathbf{u} = (u_n) \in \{0,1\}^\infty$$

representing the total length of the configuration of the snake-like robot corresponding to the control $\mathbf{u}$.

Remark 2.2.1. In order to simplify subsequent notations we fix as the base of the manipulator the point $x_{-1} = 0$, so that the 0-th link is well defined and it may be of length either 0 or 1.

We also define the quantity

$$S(q, h, p) := \sum_{k=0}^{\infty} \frac{f_{pk+h}}{q^{pk}}. \tag{2.2.5}$$

The most general form of this definition will be used only in Section 2.2.2. At this stage, it is useful to introduce for brevity the notation

$$S(q) := S(q, 0, 1) = \sum_{n=0}^{\infty} \frac{f_n}{q^n} = \begin{cases} \dfrac{q^2}{q^2 - q - 1} & \text{if } q > \varphi; \\ \\ +\infty & \text{if } q \in (0, \varphi] \end{cases} \tag{2.2.6}$$

where $\varphi := \frac{1+\sqrt{5}}{2}$ denotes the Golden Mean.

Remark 2.2.2. If $q > \varphi$ then for every $\mathbf{u} \in \{0,1\}^\infty$, one has $L(\mathbf{u}) \in [L(\mathbf{0}), L(\mathbf{1})] = [0, S(q)]$.

In what follows we show that if the scaling ratio q belongs to a fixed interval and if we allow the number of links to be infinite, then we may constraint the total length of the snake-like robot $L(\mathbf{u})$ to be any value in the interval $[0, S(q)]$.

Theorem 2.2.3 ([117]). *If $q \in (\varphi, 1+\sqrt{3}]$ then for every $\bar{L} \in [0, S(q)]$ there exists a binary control sequence $\mathbf{u} \in \{0,1\}^\infty$ such that*

$$L(\mathbf{u}) = \bar{L}.$$

Remark 2.2.4. The proof of Theorem 2.2.3 is postponed to Section 2.2.2 below.

We now continue the building of the model. In view of (2.2.3), if $x_0 = 0$ one has for every n

$$x_n(\mathbf{u}) = \sum_{k=0}^{n} u_k \frac{f_k}{q^k e^{i\omega_k}}, \tag{2.2.7}$$

where $-\omega_k \in (-\pi, \pi]$ is the argument of $x_k - x_{k-1}$ for $k = 1, \ldots, n$ and, consequently, it represents the orientation of the k-th link with respect to the global reference system given by the real and imaginary axes.

Example 2.2.5. If the angle between two consecutive links is constantly equal to $\pi - \omega \in [0, 2\pi)$, then $\omega_n = n\omega \mod (-\pi, \pi]$.

So far we introduced a control sequence ruling the length of each link. We now endow the model with another binary control sequence $\mathbf{v} = (v_n)$, ruling the angle between two consecutive links. In the model, the angle between two consecutive links is either π or $\pi - \omega$ for some fixed $\omega \in (0, \pi)$. If $v_n = 0$ then the angle between the $n-1$-th link and the n-th link is π, while if $v_n = 1$ then the angle between the $n-1$-th link and the n-th link is $\pi - \omega$ so that

$$v_n = \begin{cases} 1 & \textit{rotation of the angle } \omega \textit{ of the } n\textit{-th link;} \\ 0 & \textit{no rotation.} \end{cases} \tag{2.2.8}$$

We notice that, under these assumptions, $\omega_n = \omega_n(\mathbf{v})$ in (2.2.7) is indeed a controlled quantity, while $L(\mathbf{u})$ is yet independent from $\mathbf{v}$.

Proposition 2.2.6. *Let $n \geq 0$ and $u_j = 1$ and $v_j \in \{0,1\}$ for $j = 1, ..., n$. Then*

$$\omega_n = \sum_{j=0}^{n} v_j \omega \mod (-\pi, \pi] \tag{2.2.9}$$

Proof. We adopt the notation $Arg(z) \in (-\pi, \pi]$ to represent the principal value of the argument function $arg(z)$. In view of (2.2.7)

$$w_{n+1} = -\mathrm{Arg}(x_{n+1}(\mathbf{u}) - x_n(\mathbf{u})). \tag{2.2.10}$$

On the other hand, x_n is the vertex of the angle between the n-th link and the $n+1$-th link, therefore we have the relations

$$\mathrm{Arg}(x_{n+1}(\mathbf{u}) - x_n(\mathbf{u})) - \mathrm{Arg}(x_{n-1}(\mathbf{u}) - x_n(\mathbf{u})) \mod (-\pi, \pi] = -v_{n+1}\omega \tag{2.2.11}$$

By a comparison between (2.2.10) and (2.2.11) we get

$$w_{n+1} = w_n + v_{n+1}\omega \mod (-\pi, \pi]. \tag{2.2.12}$$

and, consequently, the claim. $\square$

Remark 2.2.7. We notice that if $u_n = 0$ then any choice of $\omega_n(\mathbf{v})$ satisfies 2.2.7. So, if the link is not extended, the rotation of the angle is meant as a rotation of the reference frame of the link.

For example, if $v_n = v_{n+1} = u_{n-1} = u_{n+1} = 1$ and $u_n = 0$, one has that $x_{n-1} = x_n$ but the angle formed by the $n-1$-th junction and the $n+1$-th junction is $\pi - 2\omega$.

In view of Proposition 2.2.6 and of above Remark, we set $\omega_n(\mathbf{v}) := \sum_{j=0}^{n} v_j \omega$, so that the complete control system for the joints of manipulator reads:

$$x_n(\mathbf{u}, \mathbf{v}) = \sum_{k=0}^{n} u_k \frac{f_k}{q^k} e^{-i\omega \sum_{j=0}^{k} v_j}. \tag{2.2.13}$$

The second main result describes the topology of the asymptotic reachable workspace when the rotation angle ω is rational with respect to π, namely it satisfies $\omega = 2\pi\frac{d}{p}$ for some $d, p \in \mathbb{N}$. One has a local controllability result when the scaling ratio q is lower than a threshold depending on p, that we denote $q(p)$. In particular $q(p)$ is defined as the greatest real solution of the equation

$$\sum_{k=0}^{\infty} \frac{f_{pk}}{q^{pk}} = 2.$$

In Section 2.2.2 below we give a closed formula for $q(p)$.

Theorem 2.2.8 ([117])**.** *If $\omega = 2\pi\frac{d}{p}$ for some $d, p \in \mathbb{N}$ and if $q \in (\varphi, q(p)]$ then the asymptotic reachable workspace*

$$W_{\infty,q,\omega} := \left\{ \lim_{n\to\infty} x_n(\mathbf{u}, \mathbf{v}) \mid \mathbf{u}, \mathbf{v} \in \{0,1\}^{\infty} \right\}$$

contains a neighborhood of the origin.

The proof of Theorem 2.2.8 is postponed to Section 2.2.2 below.

2.2.2 A Fibonacci control system

Throughout this section we introduce an auxiliary control system, that we call *Fibonacci control system* and we study its asymptotic reachable set.

We shall see that the reachability properties of the Fibonacci control system are somehow inherited by manipulator (modeled in previous section as the sequence of junctions $x_n(\mathbf{u}, \mathbf{v})$) and that this relation provides an indirect proof of Theorem 2.2.3 and Theorem 2.2.8.

In order to gradually introduce Fibonacci control system, we begin with some remarks on particular configurations of $x(\mathbf{u}, \mathbf{v})$.

We notice that for every $\mathbf{u}$

$$x(\mathbf{u}, \mathbf{0}) = \sum_{k=0}^{\infty} u_k \frac{f_k}{q^k} = L(\mathbf{u})$$

and

$$x(\mathbf{u}, \mathbf{1}) = \sum_{k=0}^{\infty} u_k \frac{f_k}{q^k e^{i\omega k}} = \sum_{k=0}^{\infty} u_k \frac{f_k}{z^k}, \quad \text{where } z = qe^{i\omega}.$$

Then both Theorem 2.2.3 and Theorem 2.2.8 are related to the study of the set

$$R_\infty(z) := \left\{ \sum_{k=0}^{\infty} u_k \frac{f_k}{z^k} \mid u_k \in \{0,1\} \right\}.$$

Indeed

$$L_\infty(q) = \{L(\mathbf{u}) \mid \mathbf{u} \in \{0,1\}^\infty\} = R_\infty(q)$$

and

$$W_{\infty,q,\omega} \supseteq \{x(\mathbf{u}, \mathbf{1}) \mid \mathbf{u} \in \{0,1\}^\infty\} = R_\infty(qe^{i\omega})$$

In particular, the relation with Theorem 2.2.8 becomes clear by noticing that if we are able to show that $R_\infty(qe^{i\omega})$ is a neighborhood of the origin then the claim of Theorem 2.2.8 follows.

Remark 2.2.9. Notice that if $|z| > \varphi$ then $R(z)$ is well defined and it is a compact set. Indeed one has

$$\lim_{n\to\infty} |\sum_{k=n}^{\infty} u_k \frac{f_k}{z^k}| \leq \lim_{n\to\infty} \sum_{k=0}^{n} |\frac{f_k}{z^k}| \leq \lim_{n\to\infty} \sum_{k=n}^{\infty} \frac{\varphi^{k-1}}{|z|^k} = 0.$$

(for the proof of the estimate $f_k \leq \varphi^{k-1}$ see Proposition 2.2.29 below) and, consequently, the convergence of the series $\sum_{k=0}^{\infty} u_k \frac{f_k}{z^k}$. Furthermore one has

$$|\sum_{k=0}^{\infty} u_k \frac{f_k}{z^k}| \leq \varphi^{-1} \left(1 + \frac{1}{1 - \varphi/|z|}\right)$$

thus $R(z)$ is a bounded set. Finally $R(z)$ is closed by the continuity of the map

$$\mathbf{u} \mapsto \sum_{k=0}^{\infty} u_k \frac{f_k}{z^k}$$

with respect to the topology on infinite sequences induced by the distance $d(\mathbf{u}, \mathbf{v}) = 2^{-\min\{k | u_k \neq v_k\}}$.

In view of above reasoning, in what follows we shall focus on the study of $R_\infty(z)$, by constructing the theoretical background necessary to prove Theorem 2.2.3 and Theorem 2.2.8 and by investigating further properties of $R_\infty(z)$.

We finally introduce the Fibonacci control system

$$\begin{cases} \bar{x}_0(\mathbf{u}) = u_0 \\ \bar{x}_1(\mathbf{u}) = u_1 + \dfrac{u_0}{z} \\ \bar{x}_{n+2}(\mathbf{u}) = u_{n+2} + \dfrac{\bar{x}_{n+1}(\mathbf{u})}{z} + \dfrac{\bar{x}_n(\mathbf{u})}{z^2} \quad \text{for } n \geq 0\,, \end{cases} \tag{F}$$

so that $\bar{x}_n(\mathbf{u})$ is the (discrete) trajectory corresponding to the control $\mathbf{u} \in \{0,1\}^\infty$.

Remark 2.2.10. The first terms of $\bar{x}_n(\mathbf{u})$ are

$$u_0,\ u_1 + \frac{u_0}{z},\ u_2 + \frac{u_1}{z} + \frac{2u_0}{z^2},\ u_3 + \frac{u_2}{z} + \frac{2u_1}{z^2} + \frac{3u_0}{z^3},\ \ldots .$$

In general, by an inductive argument it is possible to prove that for each $n \in \mathbb{N}$

$$\bar{x}_n(\mathbf{u}) = \sum_{k=0}^{n} \frac{f_k}{z^k} u_{n-k}.$$

We finally point out that the above equality implies that $R_\infty(z)$ contains $\{\bar{x}_n(\mathbf{u}) \mid \mathbf{u} \in \{0,1\}^\infty,\ n \in \mathbb{N}\}$, i.e. the reachable set of the system (F).

Asymptotical reachable set in real case

Throughout this section we consider a real number $q > \varphi$ and we show that $R_\infty(q) = [0, S(q)]$ if and only if $q \leq 1 + \sqrt{3}$ (namely we prove Theorem 2.2.3[3]). For brevity, we specialize the definition of $S(q, h, p)$ given in (2.2.5) as follows:

$$S(q,h) := \sum_{k=0}^{\infty} \frac{f_{h+k}}{q^k} = \frac{q^2 f_h + q f_{h-1}}{q^2 - q - 1} \tag{2.2.14}$$

Last equality can be proved by a simple inductive argument. We also shall use the following recursive relation

$$S(q,h) = q(S(q,h-1) - f_{h-1}). \tag{2.2.15}$$

Finally note that $S(q, 0) = S(q)$.

Lemma 2.2.11. *Let $q > \varphi$. Then*

$$f_h \leq \frac{S(q,h+1)}{q} \quad \textit{for every } h \tag{2.2.16}$$

if and only if $q \leq 1 + \sqrt{3}$.

Proof. To prove the only if part we seek a contradiction by assuming $q > 1+\sqrt{3}$ and by showing that (2.2.16) fails for some h. In particular it is immediate to check that when $h = 0$ one has

$$1 > \frac{q+1}{q^2 - q - 1}\,. \tag{2.2.17}$$

Now, to show the if part we notice that

$$q \leq \frac{1}{2}\left(\frac{f_{h+1}}{f_h} + 1\right) + \sqrt{\frac{1}{4}\left(\frac{f_{h+1}}{f_h} + 1\right)^2 + 2} \quad \text{for every } h.$$

Since, for every h

$$\frac{f_{h+1}}{f_h} \geq 1 = \frac{f_1}{f_0}$$

[3]Indeed the claim immediately follows by recalling the equality $\{L(\mathbf{u}) \mid \mathbf{u} \in \{0,1\}^\infty\} = R_\infty(q)$

it follows that $q \leq 1+\sqrt{3}$.

$\square$

Theorem 2.2.12 ([117])**.** *Let $q \leq 1+\sqrt{3}$ and $x \in [0, S(q,0)]$ and consider the sequences (r_h) and (u_h) defined by*

$$\begin{cases} r_0 = x; \\ u_h = \begin{cases} 1 & \text{if } r_h \in [f_h, S(q,h)] \\ 0 & \text{otherwise} \end{cases} \\ r_{h+1} = q(r_h - u_h f_h) \end{cases} \tag{2.2.18}$$

Then

$$x = \sum_{k=0}^{\infty} \frac{f_k}{q^k} u_k \tag{2.2.19}$$

and, consequently, $R_\infty(q) = [0, S(q,0)]$. Moreover if $q > 1+\sqrt{3}$ then $R_\infty \subsetneq [0, S(q,0)]$.

Proof. Fix $x \in [0, S(q,0)]$ and first of all note that

$$x = \sum_{k=0}^{h} \frac{f_k}{q^k} u_k + \frac{r_{h+1}}{q^{h+1}} \quad \text{for all } h \geq 0. \tag{2.2.20}$$

Indeed above equality can be shown by induction on h. For $h = 0$ one has $r_1 = q(x - u_0 f_0)$ and consequently $x = f_0 u_0 + r_1/q$. Assume now (2.2.20) as inductive hypothesis. Then

$$r_{h+2} = q^{h+2} \left(x - \sum_{k=0}^{h} \frac{f_k}{q^k} u_k \right) - q f_{h+1} u_{h+1}$$

and, consequently,

$$x = \sum_{k=0}^{h+1} \frac{f_k}{q^k} u_k + \frac{r_{h+2}}{q^{h+2}}.$$

Now we claim that if $q \leq 1+\sqrt{3}$ then

$$r_h \in [0, S(q,h)] \quad \text{for every } h. \tag{2.2.21}$$

We show the above inclusion by induction. If $h = 0$ then the claim follows by the definition of r_0 and by the fact that $x \in [0, S(q,0)]$. Assume now (2.2.21) as inductive hypothesis. One has $r_h \in [0, S(q,h)] = [0, f_h) \cup [f_h, S(q,h)]$. If $r_h \in [0, f_h)$ then $r_{h+1} = q r_h \in [0, q f_h] \subseteq [0, S(q, h+1)]$ - where the last inclusion follows by Lemma 2.2.11. If otherwise $r_h \in [f_h, S(q,h)]$ then $r_{h+1} = q(r_h - f_h) \subseteq [0, q(S(q,h) - f_h)] = [0, S(q, h+1)]$ - see (2.2.15).

Recalling $f_n \sim \varphi^n$ as $n \to \infty$, one has

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{f_k}{q^k} u_k = \lim_{h \to \infty} \sum_{k=0}^{h-1} \frac{f_k}{q^k} u_k &\overset{(2.2.20)}{=} x - \lim_{h \to \infty} \frac{r_h}{q^h} \overset{(2.2.21)}{\geq} x - \lim_{h \to \infty} \frac{S(q,h)}{q^h} \\ &\overset{(2.2.14)}{=} x - \lim_{h \to \infty} \frac{q^2 f_{h+1} + q f_h}{q^h (q^2 - q - 1)} = x. \end{aligned}$$

On the other hand

$$\sum_{k=0}^{\infty} \frac{f_k}{q^k} u_k = x - \lim_{h \to \infty} \frac{r_h}{q^h} \leq x$$

and this proves (2.2.19). It follows by the arbitrariness of x that if $q \leq 1 + \sqrt{3}$ then $R_\infty = [0, S(q, 0)]$.

Finally assume $q > 1 + \sqrt{3}$. By Lemma 2.2.11 there exists $x \in (S(q, 1)/q, f_1)$. In order to find a contradiction, assume $x \in R_\infty$. Then

$$x = u_0 f_0 + \frac{1}{q} \sum_{k=0}^{\infty} \frac{f_{k+1}}{q^k} u_{k+1}$$

Note that $u_0 \neq 1$ because $x < f_1 = 1$. Then $u_0 = 0$ and

$$x = \frac{1}{q} \sum_{k=0}^{\infty} \frac{f_{k+1}}{q^k} u_{k+1} \leq \frac{1}{q} \sum_{k=0}^{\infty} \frac{f_{k+1}}{q^k} = \frac{S(q, 1)}{q}$$

but this contradicts $x \in (S(1, q)/q, f_1)$. Then $x \in [0, S(q, 0)] \setminus R_\infty$ and this concludes the proof. $\square$

Asymptotical reachable set in complex case

Throughtout this section we investigate $R_\infty(z)$ with $z = qe^{i\omega}$ and $\omega = \frac{d}{p} 2\pi$; $d, p \in \mathbb{N}$. First of all we notice that $z^p = q^p$ and consequently

$$\sum_{k=0}^{\infty} u_k \frac{f_k}{z^k} = \sum_{h=0}^{p-1} z^{-h} \sum_{k=0}^{\infty} u_{pk+h} \frac{f_{pk+h}}{q^{pk}}. \tag{2.2.22}$$

Above equality implies that if $p \geq 2$ and if

$$R_\infty^h := \left\{ \sum_{k=0}^{\infty} \frac{u_{pk+h} f_{pk+h}}{q^{pk}} \mid u_{pk+h} \in \{0, 1\} \right\}$$

is an interval (and not a disconnected set) then

$$R_\infty(z) = \left\{ \sum_{k=0}^{\infty} \frac{f_k}{z^k} u_k \mid u_k \in \{0, 1\} \right\} = \sum_{j=0}^{p-1} z^{-h} R_\infty^h$$

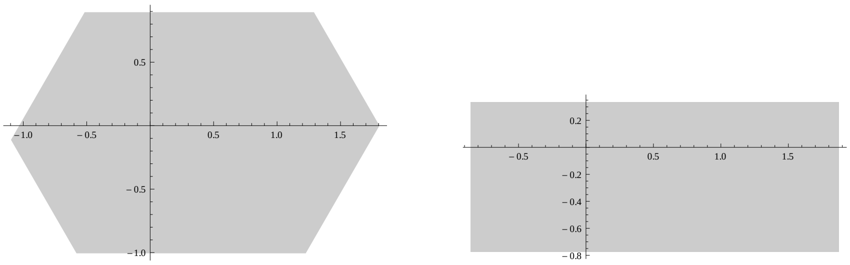
is a polygon containing the origin in its interior - note that $\min R_\infty^h = 0$. In what follows we show that if q is small enough, then such a local controllability condition is satisfied.

By definition 2.2.5, so that $R_\infty^h \subset [0, S(q, h, p)]$ for every $h = 0, \ldots, p-1$ and from simple inductive arguments, we have the following recursive relation

$$S(q, h, p) = f_{h-1} S(q, 1, p) + f_{h-2} S(q, 0, p) \tag{2.2.23}$$

Moreover one has

$$S(q, p, p) = q^p (S(q, 0, p) - f_0) \tag{2.2.24}$$



(a) $R_\infty(2e^{i2\pi/3})$. (b) $R_\infty(2e^{i\pi/2})$.

Figure 2.1: By Theorem 2.2.18, $R_\infty(2e^{i2\pi/p})$ with $p = 3, 4$ is a polygon.

$$S(q, p+1, p) = q^p(S(q, 1, p) - f_1) \tag{2.2.25}$$

and, more generally,

$$S(q, p+h, p) = q^p(S(q, h, p) - f_h). \tag{2.2.26}$$

Example 2.2.13. Let $q = 2$ and $p = 4$. In view of (2.2.23),

$$\begin{aligned}
R^0_\infty &\subseteq [0, S(2, 0, 4)]\\
R^1_\infty &\subseteq [0, S(2, 1, 4)]\\
R^2_\infty &\subseteq [0, S(2, 0, 4) + S(2, 1, 4)]\\
R^3_\infty &\subseteq [0, S(2, 0, 4) + 2S(2, 1, 4)].
\end{aligned}$$

See Section 2.2.2 for the explicit calculation of $S(q, h, p)$. In Theorem 2.2.18 below, we show that above inclusions are actually equalities, so that

$$R_\infty = R^0_\infty - \frac{i}{2}R^1_\infty - \frac{1}{4}R^2_\infty + \frac{i}{8}R^3_\infty$$

is a rectangle in the complex plane - see Figure 2.1.

Lemma 2.2.14. *If $q \le q(p)$ then for every $h \in \mathbb{N}$*

$$S(q, p, p+h) \ge q^p f_h. \tag{2.2.27}$$

Proof. The case $h = 0$ follows by the definition of $q(p)$ and by (2.2.24). If $h = 1$ then

$$S(q, p, p+1) \ge S(q, p, p) \ge q^p f_0 = q^p f_1.$$

Fix now $h \ge 2$ and now (2.2.27) as inductive hypothesis for every integer lower than h. It follows by (2.2.23)

$$S(q, p, p+h) = f_{h-1}S(q, 1, p) + f_{h-2}S(q, 0, p) \ge 2(f_{h-1} + f_{h-2}) = 2f_h$$

therefore, by (2.2.26), we finally get

$$S(q, p, p+h) = q^p(S(q, h, p) - f_h) \ge q^p f_h.$$

□

Finally let us define $q(p)$ as the greatest solution of the equation

$$S(q, 0, p) = 2f_0 = 2$$

Note that if $q \le q(p)$ then $S(q, 0, p) \ge 2$.

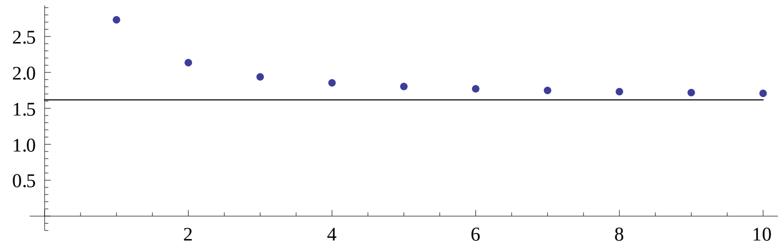


Figure 2.2: $q(p)$ for $p = 1, \ldots, 10$. Note that $q(p)$ tends to φ as $p \to \infty$. Indeed it suffices to recall $f_p \sim \varphi^p$ to have $\lim_{p\to\infty} q(p)/\varphi = 1$.

Remark 2.2.15. The value $q(p)$ is explicitly calculated in Section 2.2.2 below. Among other results, we shall show

$$q(p) = \begin{cases} \left(\frac{1}{2}(f_{p-2} + 2f_p) + \frac{1}{2}\sqrt{(f_{p-2} + 2f_p)^2 - 8}\right)^{\frac{1}{p}} & p \text{ even;} \\ \left(\frac{1}{2}(f_{p-2} + 2f_p) + \frac{1}{2}\sqrt{(f_{p-2} + 2f_p)^2 + 8}\right)^{\frac{1}{p}} & p \text{ odd.} \end{cases} \tag{2.2.28}$$

We notice that above equality implies $q(p) \sim f(p)^{1/p} \sim \varphi$ as $p \to \infty$.

Example 2.2.16. $q(1) = 1+\sqrt{3}$, $q(2) = \sqrt{\frac{1}{2}\left(5 + \sqrt{17}\right)}$, $q(3) = \sqrt[3]{\frac{1}{2}\left(7 + \sqrt{57}\right)}$, $q(4) = \sqrt[4]{6 + \sqrt{34}}$.

Lemma 2.2.17. *Let $p, h \in \mathbb{N}$ and let $q \leq q(p)$. For $x \in [0, S(q, h, p)]$ consider the sequences (r_n) and (u_n) defined by*

$$\begin{cases} r_0 = x; \\ u_n = \begin{cases} 1 & \text{if } r_n \in [f_n, S(q, np + h)] \\ 0 & \text{otherwise} \end{cases} \\ r_{n+1} = q^p(r_n - u_n f_{np+h}). \end{cases} \tag{2.2.29}$$

Then

$$x = \sum_{k=0}^{\infty} \frac{f_{pk+h}}{q^{pk}} u_k \tag{2.2.30}$$

and, consequently, $R_\infty^h = [0, S(q, 0, p)]$. Moreover if $q > q(p)$ then $R_\infty \subsetneq [0, S(q, 0, p)]$.

Proof. Fix $h \in \mathbb{N}$ and $x \in [0, S(q, 0, p)]$. First of all note that

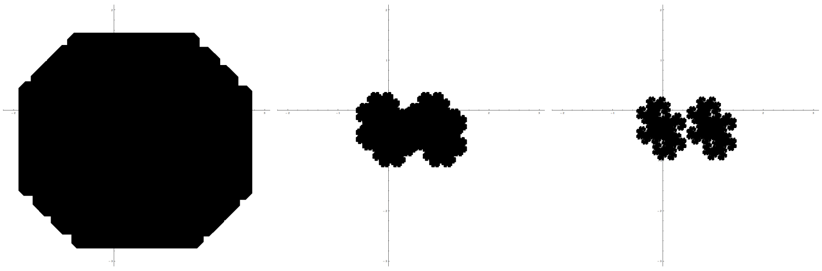
$$x = \sum_{k=0}^{n} \frac{f_{pk+h}}{q^{pk}} u_k + \frac{r_{n+1}}{q^{p(n+1}} \qquad \text{for all } n. \tag{2.2.31}$$

Indeed for $h = 0$ one has $r_1 = q^p(x - u_0 f_h)$ and consequently $x = f_h u_0 + r_1/q^p$. Assume now (2.2.31) as inductive hypothesis. Then

$$\begin{aligned} r_{n+2} &= q^p(r_{n+1} - u_{n+1} f_{p(n+1)+h}) \\ &= q^{p(n+2)} \left(x - \sum_{k=0}^{n} \frac{f_{kp+h}}{q^{pk}} u_k \right) - q^{p(n+2)} f_{p(n+1)+h} u_{n+1} \end{aligned}$$

and, consequently,

$$x = \sum_{k=0}^{n+1} \frac{f_{kp+h}}{q^{kp}} u_k + \frac{r_{h+2}}{q^{p(n+2)}}.$$



(a) $h = 0$ (b) $h = 0.4$ (c) $h = 0.5$

Figure 2.3: Approximations of $R_\infty(z)$ with $z = (q(p) + h)e^{i\pi/4}$ and $h = 0, 0.4, 0.5$. Note that, by Theorem 2.2.18, if $h = 0$ then $R_\infty(z)$ is indeed an octagon. See Section 2.2.3 and, in particular, Remark 2.2.27 for a description of the approximation techniques.

Now, we claim that for every n if $q \leq q(p)$ then

$$r_n \in [0, S(q, pn + h, p)]. \tag{2.2.32}$$

We show the above inclusion by induction. If $h = 0$ then the claim follows by the definition of r_0 and by the fact that $x \in [0, S(q, h, p)]$. Assume now (2.2.32) as inductive hypothesis. One has $r_n \in [0, S(q, pn + h, p)] = [0, f_{pn+h}) \cup [f_{pn+h}, S(q, pn + h, p)]$. If $r_n \in [0, f_{pn+h})$ then $r_{n+1} = q^p r_n \in [0, q^p f_{pn+h}] \subseteq [0, S(q, (n+1)p+h, p)]$ - where the last inclusion follows by Lemma 2.2.14. If otherwise $r_n \in [f_{pn+h}, S(q, pn + h, p)]$ then $r_{n+1} = q^p(r_n - f_{np+h}) \subseteq [0, q(S(q, pn + h, p) - f_{pn+h})] = [0, S(q, p(n + 1) + h, p)]$ - see (2.2.26).

Recalling $f_n \sim \varphi^n$ as $n \to \infty$, one has

$$\sum_{k=0}^{\infty} \frac{f_{pk+h}}{q^{pk}} u_k = \lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{f_{pk+h}}{q^{pk}} u_k \overset{(2.2.31)}{=} x - \lim_{n\to\infty} \frac{r_n}{q^{pn}} \overset{(2.2.32)}{\geq} x - \lim_{n\to\infty} \frac{S(q, pn + h, p)}{q^{pn}}$$
$$\overset{(2.2.23)}{=} x - \lim_{n\to\infty} \frac{f_{pn+h-1} S(q, 1, p) + f_{pn+h-2} S(q, 0, p)}{q^{ph}} = x.$$

On the other hand

$$\sum_{k=0}^{\infty} \frac{f_{pk+h}}{q^{pk}} u_k = x - \lim_{n\to\infty} \frac{r_n}{q^{pn}} \leq x$$

and this proves (2.2.30). It follows by the arbitrariness of x that if $q \leq q(p)$ then $R^h_\infty = [0, S(q, 0, p)]$. Finally assume $q > q(p)$. By Lemma 2.2.11 there exists $x \in (S(q, h, p)/q^p, f_h)$. In order to find a contradiction, assume $x \in R^h_\infty$. Then

$$x = u_0 f_h + \frac{1}{q^p} \sum_{k=0}^{\infty} \frac{f_{p(k+1)+h}}{q^{pk}} u_{k+1}$$

Note that $u_0 \neq 1$ because $x < f_h$. Then $u_0 = 0$ and

$$x = \frac{1}{q^p} \sum_{k=0}^{\infty} \frac{f_{p(k+1)+h}}{q^{pk}} u_{k+1} \leq \frac{1}{q^p} \sum_{k=0}^{\infty} \frac{f_{p(k+1)}}{q^{pk}} = \frac{S(q, h, p)}{q^p}$$

but this contradicts $x \in (S(q, h, p)/q^p, f_h)$. Then $x \in [0, S(q, 0, p)] \setminus R^h_\infty$ and this concludes the proof. $\square$

Theorem 2.2.18 ([117]). *If $\varphi < |z| \leq q(p)$ then $R_\infty(z)$ is a polygon on the complex plane containing the origin.*

Proof. It follows by Lemma 2.2.17 and by

$$R_\infty(z) = \left\{ \sum_{k=0}^{\infty} \frac{f_k}{z^k} u_k \mid u_k \in \{0,1\} \right\} = \sum_{h=0}^{p-1} z^{-h} R_\infty^h.$$

□

Proof of Theorem 2.2.8

Theorem 2.2.8 immediately follows by

$$R_\infty(qe^{i\omega}) = \{x(\mathbf{u}, \mathbf{1}) \mid \mathbf{u} \in \{0,1\}^\infty\} \subset W_{\infty,q,\omega}$$

and by Theorem 2.2.18.

An explicit formula for $q(p)$

By a comparison between (2.2.23),(2.2.24) and (2.2.25), $S(q,0,p)$ and $S(q,1,p)$ are solution of the following system of equations

$$\begin{cases} q^p(S(q,0,p) - f_0) = f_{p-1}S(q,1,p) + f_{p-2}S(q,0,p) \\ q^p(S(q,1,p) - f_1) = f_p S(q,1,p) + f_{p-1}S(q,0,p) \end{cases} \tag{2.2.33}$$

$$\begin{cases} (q^p - f_{p-2})S(q,0,p) - f_{p-1}S(q,1,p) = f_0 q^p \\ -f_{p-1}\ S(q,0,p) + (q^p - f_p)S(q,1,p) = f_1 q^p \end{cases} \tag{2.2.34}$$

whose solution is

$$S(q,0,p) = \frac{\begin{vmatrix} f_0 q^p & -f_{p-1} \\ f_1 q^p & q^p - f_p \end{vmatrix}}{\begin{vmatrix} q^p - f_{p-2} & -f_{p-1} \\ -f_{p-1} & q^p - f_p \end{vmatrix}}, \tag{2.2.35}$$

$$S(q,1,p) = \frac{\begin{vmatrix} q^p - f_{p-2} & f_0 q^p \\ -f_{p-1} & f_1 q^p \end{vmatrix}}{\begin{vmatrix} q^p - f_{p-2} & -f_{p-1} \\ -f_{p-1} & q^p - f_p \end{vmatrix}}. \tag{2.2.36}$$

We now show that the solutions in (2.2.35) and (2.2.36) are well defined.

Proposition 2.2.19. *Let*

$$\Delta_p(q) := \begin{vmatrix} q^p - f_{p-2} & -f_{p-1} \\ -f_{p-1} & q^p - f_p \end{vmatrix} = (q^p - f_{p-2})(q^p - f_p) - f_{p-1}^2$$

Then

$$\Delta_p(q) = q^{2p} - (f_{p-2} + f_p)q^p + (-1)^p \tag{2.2.37}$$

and the real roots of $\Delta_p(q)$ are $\pm\varphi$ and $\pm(\varphi - 1)$ if p is even and $-\varphi$ and $\varphi - 1$ if p is odd. In particular if $q > \varphi$ then $\Delta_p \neq 0$.

Proof. The equality in (2.2.37) follows by Cassini identity for $p \geq 2$

$$f_{p-2}f_p - f_{p-1}^2 = (-1)^p.$$

Now, we notice that $\Delta_p(q) = 0$ if and only if

$$\begin{cases} z = q^p \\ z^2 - (f_{p-2} + f_p)z + (-1)^p = 0. \end{cases}$$

We first discuss the case of an even p. When p is even then $\Delta_p(q)$ has exactly 4 real solutions

$$\begin{aligned} q_{1,2}^{even} &= \pm \sqrt[p]{\frac{1}{2}(f_{p-2} + f_p) - \frac{1}{2}\sqrt{(f_{p-2} + f_p)^2 + 4}}, \\ q_{3,4}^{even} &= \pm \sqrt[p]{\frac{1}{2}(f_{p-2} + f_p) + \frac{1}{2}\sqrt{(f_{p-2} + f_p)^2 + 4}}. \end{aligned}$$

Now, for every $p \in \mathbb{N}$ one has that the Golden Mean φ satisfies

$$\varphi^p = f_{p-1}\varphi + f_{p-2}$$

and, consequently,

$$\begin{aligned} \varphi^{2p} &= (f_{p-1}\varphi + f_{p-2})^2 \\ &= f_{p-1}^2\varphi^2 + 2f_{p-1}f_{p-2}\varphi + f_{p-2}^2 \\ &= (f_{p-1}^2 + 2f_{p-1}f_{p-2})\varphi + f_{p-1}^2 + f_{p-2}^2. \end{aligned}$$

This, together with $\Delta(q) = \Delta(-q)$ and Cassini identity, implies

$$\Delta_p(\varphi) = \Delta_p(-\varphi) = f_{p-1}(f_{p-1} + f_{p-2} - f_p)\varphi + f_{p-1}^2 - f_p f_{p-2} + 1 = 0.$$

Moreover, since $\varphi - 1 = 1/\varphi$ and $\Delta(q) = \Delta(-q)$,

$$\Delta_p(\varphi - 1) = \Delta_p(1 - \varphi) = \Delta_p(1/\varphi) = \frac{\Delta_p(\varphi)}{\varphi^{2p}} = 0.$$

This concludes the proof for the even case.
Now, if p is odd then $\Delta_p(q) = 0$ has exactly 2 real solutions

$$q_{1,2}^{odd} = \sqrt[p]{\frac{1}{2}(f_{p-2} + f_p) - \frac{1}{2}\sqrt{(f_{p-2} + f_p)^2 - 4}}.$$

Again by Cassini identity

$$\begin{aligned} \Delta_p(\varphi) &= \varphi^{2p} - (f_{p-2} + f_p)\varphi + 1 \\ &= f_{p-1}(f_{p-1} + f_{p-2} - f_p)\varphi + f_{p-1}^2 - f_p f_{p-2} - (-1)^p = 0. \end{aligned}$$

Since $1 - \varphi = -1/\varphi$ we finally obtain

$$\Delta_p(1 - \varphi) = \Delta_p(-1/\varphi) = -\frac{\Delta_p(\varphi)}{\varphi^{2p}} = 0.$$

□

Example 2.2.20. For $p = 1$ we already showed

$$S(q,0,1) = S(q) = \frac{q^2}{q^2 - q - 1} \quad S(q,1,1) = S(q) = \frac{q^2 + q}{q^2 - q - 1}.$$

For $p = 2$, namely when $z = -q$,

$$S(q,0,2) = \frac{q^2(q^2-1)}{q^4 - 3q^2 + 1} \quad S(q,1,2) = \frac{q^4}{q^4 - 3q^2 + 1}.$$

For $p = 3$, namely when z is a rescaled cubic root of unity,

$$S(q,0,3) = \frac{q^3\,(q^3-1)}{q^6 - 4q^3 - 1} \quad S(q,1,3) = \frac{q^6 + q^3}{q^6 - 4q^3 - 1}.$$

For $p = 4$

$$S(q,0,4) = \frac{q^4(q^4-2)}{q^8 - 7q^4 + 1} \quad S(q,1,4) = \frac{q^8 + q^4}{q^8 - 7q^4 + 1}.$$

We now give a closed formula for $q(p)$.

Proposition 2.2.21. *For every $p \in \mathbb{N}$*

$$q(p) = \begin{cases} \left(\frac{1}{2}(f_{p-2} + 2f_p) + \frac{1}{2}\sqrt{(f_{p-2} + 2f_p)^2 - 8}\right)^{\frac{1}{p}} & p \ \textit{even}; \\ \left(\frac{1}{2}(f_{p-2} + 2f_p) + \frac{1}{2}\sqrt{(f_{p-2} + 2f_p)^2 + 8}\right)^{\frac{1}{p}} & p \ \textit{odd}. \end{cases} \tag{2.2.38}$$

Proof. We recall that $q(p)$ is defined as the greatest solution of $\sum_{k=0}^{\infty} \frac{f_{kp}}{q^{kp}} = 2$ namely of

$$S(q,0,p) = \frac{q^{2p} - f_{p-2}q^p}{q^{2p} - (f_{p-2} + f_p)q^p + (-1)^p} = 2.$$

Solving above equation one gets

$$q^{2p} + (-f_{p-2} - 2f_p)q^p + 2(-1)^p = 0$$

and finally (2.2.38). □

2.2.3 A characterization of the reachable set via Iterated Function Systems

Throughtout this section we characterize $R_\infty(q)$, with $q \in \mathbb{R}, q > \varphi$, as a projection on $\mathbb{R}$ of the attractor of a (linear) Iterated Function System defined on $\mathbb{R}^2$.

2.2.4 Some basic facts about IFSs

An iterated function system (IFS) is a set of contractive functions $G_j : X \to X$, where $(X, \mathbf{d})$ is a metric space. We recall that a function if for every $x, y \in X$

$$d(f(x), f(y)) < c \cdot d(x,y)$$

for some $c < 1$. In [97] Hutchinson showed that every finite IFS, namely every IFS with finitely many contractions, admits a unique non-empty compact fixed point Q with respect to the Hutchinson operator

$$\mathcal{G} : S \mapsto \bigcup_{j=1}^{J} G_j(S).$$

Moreover for every non-empty compact set $S \subseteq \mathbb{C}$

$$\lim_{k\to\infty} \mathcal{G}^k(S) = Q.$$

The *attractor* Q is a self-similar set and it is the only bounded set satisfying $\mathcal{F}(Q) = Q$.

2.2.5 The reachable set is a projection of the attractor of an IFS

Let $q > \varphi$, $\mathbf{v} \in \mathbb{R}^2$ and consider the linear map from $\mathbb{R}^2$ onto itself

$$F_{q,\mathbf{v}}(\bar{\mathbf{x}}) = \mathbf{v} + A(q)\bar{\mathbf{x}}$$

where

$$A(q) = \begin{pmatrix} \frac{1}{q} & \frac{1}{q^2} \\ 1 & 0 \end{pmatrix}.$$

We notice that if $\bar{x}(\mathbf{u})$ is a trajectory of the system (F) with $z = q$, then

$$\begin{pmatrix} \bar{x}_{n+2}(\mathbf{u}) \\ \bar{x}_{n+1}(\mathbf{u}) \end{pmatrix} = \begin{pmatrix} u_{n+2} \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{q} & \frac{1}{q^2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \bar{x}_{n+1}(\mathbf{u}) \\ \bar{x}_n(\mathbf{u}) \end{pmatrix}$$

namely

$$(\bar{x}_{n+2}(\mathbf{u}), \bar{x}_{n+1}(\mathbf{u}))^T = F_{q,(u_{n+2},0)}(\bar{x}_{n+1}(\mathbf{u}), \bar{x}_n(\mathbf{u}))^T). \tag{2.2.39}$$

We now introduce the concept at the base of the symbolic dynamics, which is a particular application from $\mathbf{u} \in \{0,1\}^\infty$ into itself that iterates in a natural way.

Definition 2.2.22. The application $\sigma : \{0,1\}^\infty \to \{0,1\}^\infty$ defined by

$$\sigma(\mathbf{u}) = \sigma(u_0, u_1, u_2, ...) = (u_1, u_2, ...) \tag{2.2.40}$$

it is said *unit shift.*

Set $x(\mathbf{u}) := x(\mathbf{u}, \mathbf{0}) = \sum_{k=0}^{\infty} \frac{f_k}{q^k} u_k$ (see Definition (2.2.13)) and define

$$\begin{aligned} Q_\infty &:= \{(x(\mathbf{u}), x(\sigma(\mathbf{u})) \mid \mathbf{u} \in \{0,1\}^\infty\} \\ &= \left\{ \left(\sum_{k=0}^{\infty} \frac{f_k}{q^k} u_k, \sum_{k=0}^{\infty} \frac{f_k}{q^k} u_{k+1} \right) \mid \mathbf{u} \in \{0,1\}^\infty \right\}. \end{aligned}$$

Proposition 2.2.23. *For every $q > \varphi$*

$$\bigcup_{u\in\{0,1\}} F_{q,(u,0)}(Q_\infty) = Q_\infty.$$

Proof. Let $\mathbf{u} = (u_0, u_1, \dots) \in \{0,1\}^\infty$. One has

$$F_{q,(u_0,0)}(x(\sigma(\mathbf{u})), x(\sigma^2(\mathbf{u}))) = (x(\mathbf{u}), x(\sigma(\mathbf{u})))$$

and this implies $Q_\infty \subseteq \bigcup_{u\in\{0,1\}} F_{q,(u,0)}(Q_\infty)$. Now let $u \in \{0,1\}$ and $\mathbf{d} \in \{0,1\}^\infty$. Define $\mathbf{u} = (u, \mathbf{d}) = (u, d_0, d_1, \dots)$ and note that $\sigma(\mathbf{u}) = \mathbf{d}$. One has

$$F_{q,(u,0)}(x(\mathbf{d}), x(\sigma(\mathbf{d})) = (x(\mathbf{u}), x(\mathbf{d})) = (x(\mathbf{u}), x(\sigma(\mathbf{u})))$$

and this implies the inclusion $\bigcup_{u\in\{0,1\}} F_{q,(u,0)}(Q_\infty) \subseteq Q_\infty$. □

Note that in general $F_{q,\mathbf{v}}$ is not a contractive map. However the spectral radius of $A(q)$, say $\rho(q)$, satisfies

$$\rho(q) = \frac{\varphi}{q} < 1 \quad \text{for every } q > \varphi$$

Then

$$\lim_{k\to\infty} A^k(q) = 0.$$

In particular there exists $k(q)$ such that for every $k \geq k(q)$

$$||A^k(q)|| := \max_{\mathbf{x}\neq(0,0)} \frac{||A^k(q)\mathbf{x}||}{||\mathbf{x}||} < 1.$$

Example 2.2.24. Let $k = 2$. One has

$$||A^2(q)||_2 = \frac{q^4 + 5q^2 + 1}{q^6}$$

- see Section 2.2.6 below for a detailed computation of $||A^k(q)||$. Therefore $||A^2(q)|| < 1$ if and only if

$$q^6 - q^4 - 5q^2 - 1 > 0,$$

namely $k(q) = 2$ for every $q > \bar{q} \simeq 1.69299$ where $\bar{q}$ is the unique positive solution of equation $q^6 - q^4 - 5q^2 - 1 = 0$.

Now, for every binary sequence of length k, say $\mathbf{u}_k$, define the vector

$$\mathbf{v}(\mathbf{u}_k) := \sum_{h=0}^{k-1} A^h(q) \begin{pmatrix} u_{k+1-h} \\ 0 \end{pmatrix}.$$

and for every k, the vector function:

$$G_{q,\mathbf{u}_k}(\mathbf{x}) = \mathbf{v}(\mathbf{u}_k) + A^k(q)\mathbf{x} = \sum_{h=0}^{k-1} A^h(q) \begin{pmatrix} u_{k+1-h} \\ 0 \end{pmatrix} + A^k(q)\mathbf{x}.$$

One has that for $k = 1$

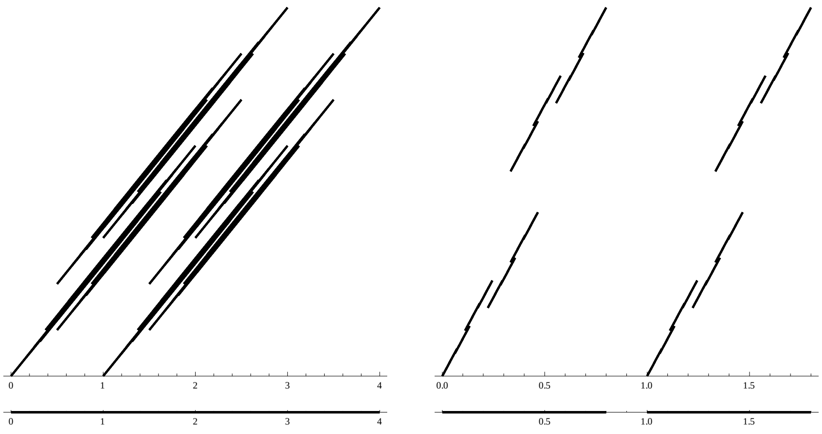
$$G_{q,\mathbf{u}_1} = F_{q,(u_2,0)} \tag{2.2.41}$$

and, more generally,

$$G_{q,\mathbf{u}_k} = F_{q,(u_{k+1},0)} \circ F_{q,(u_k,0)} \circ \dots \circ F_{q,(u_2,0)}. \tag{2.2.42}$$

Remark 2.2.25. If $\mathbf{u}_k = (u_{n+2}, \cdots, u_{n+1+k})$ then

$$(x_{n+1+k}, x_{n+k})^T = G_{q,\mathbf{u}_k}(x_{n+1}, x_n)^T.$$



(a) $q = 2$ (b) $q = 3$

Figure 2.4: An approximation of $Q_\infty(q)$, with $q = 2, 3$, and of its projection on x-axis $R_\infty(q)$. It is obtained by 4 iterations of the IFS $\mathcal{G}_{q,2}$ with initial datum $[0, S(q)] \times [0, S(q)]$.

Theorem 2.2.26 ([117])**.** *For $k \geq k(q)$ and for every $\mathbf{u}_k \in \{0,1\}^k$ the map $G_{q,\mathbf{u}_k}$ is a contraction and*

$$\bigcup_{\mathbf{u}_k \in \{0,1\}^k} G_{q,\mathbf{u}_k}(Q_\infty) = Q_\infty. \tag{2.2.43}$$

Moreover $Q_\infty(q)$ is the attractor of a two-dimensional linear Iterated Function System (IFS)

$$\mathcal{G}_{q,k} := \{G_{q,\mathbf{u}_k} \mid \mathbf{u}_k \in \{0,1\}^k\},$$

namely for every compact set $X \subset \mathbb{R}^2$ one has

$$\lim_{n\to\infty} \mathcal{G}^n_{q,k}(X) = Q_\infty(q).$$

Proof. By the definition of $k(q)$, for each $\mathbf{u}_k \in \{0,1\}^k$, $G_{q,\mathbf{u}_k}$ is a contractive map. The equality (2.2.43) follows by Proposition 2.2.23 and by (2.2.42). The second part of the statement follows by the fact that in general the unique invariant compact set of an IFS is also an attractor, see for instance [70]. □

Remark 2.2.27 (Some remarks on the approximation of R_∞ in the complex case.). Theorem 2.2.26 gives an operative way to approximate $Q_\infty(q)$ and, consequently, $R_\infty(q)$, see Figure 2.4. Above reasonings apply when considering as a base a complex number $z = qe^{i\omega}$, so that $Q_\infty(z) \subset \mathbb{C} \times \mathbb{C}$. Note that

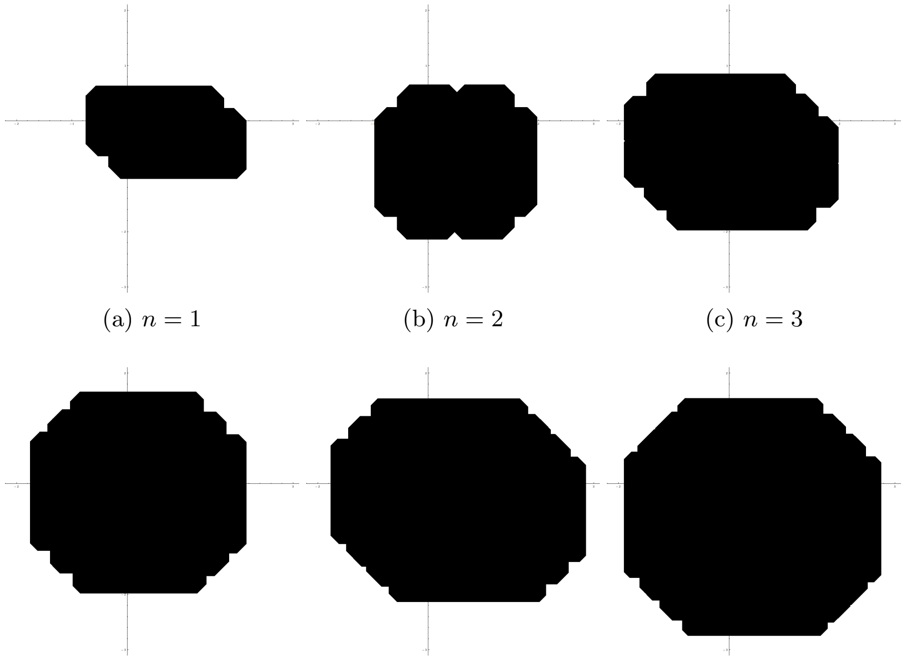
$$Q_\infty(z) \subset H(z) := \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} \mid \max\{|\Re(z_h)|, |\Im(z_h)|\} \leq S(|z|),\ h = 1, 2\}$$

and $\lim_{n\to\infty} \mathcal{G}^n_{z,k}(H(z)) = Q_\infty(z)$. Then one may approximate $Q_\infty(z)$ by iteratively applying $\mathcal{G}_{z,k}$ to $H(z)$. To this end, it is possible to employ the isometry between $\mathbb{C}$ and $\mathbb{R}^2$ in order to set the problem on $\mathbb{R}^4$. Then the real-valued counterpart of $H(z)$ is the hypercube

$$\tilde{H}(z) := \{\mathbf{x} \in \mathbb{R}^4 \mid |\mathbf{x}|_{\max} \leq S(|z|)\}$$

while we denote by $\tilde{\mathcal{G}}_{z,k}$ and by $\tilde{G}_{z,\mathbf{u}}$ the real-valued counterparts of $\mathcal{G}_{z,k}$ and of $G_{z,\mathbf{u}}$, respectively, so that

$$\mathcal{G}^n_{z,k}(x) = \bigcup_{\mathbf{u} \in \{0,1\}^{nk}} G_{z,\mathbf{u}}(x).$$



(a) $n = 1$ (b) $n = 2$ (c) $n = 3$

(d) $n = 4$ (e) $n = 5$ (f) $n = 6$

Figure 2.5: Various iterations of $\tilde{\mathcal{G}}^n_{z,k}(\tilde{H}(z))$ with $z = q(8)e^{i\pi/4}$.

We then may get a bidimensional representation of an approximation of $R_\infty(z)$ by projecting $\tilde{G}^n_{z,k}(\tilde{H}(z))$ on $\mathbb{R}^2$. However this yields some complexity issues in numerical simulations. Indeed a brute force attack consists in applying $\tilde{\mathcal{G}}^n_{z,k}$ to a four-dimensional grid rastering $\tilde{H}(z)$ and then projecting the result on $\mathbb{R}^2$. Thus the generation of an image with5 $N \times N$ pixels involves the computation of $2^{kn}N^4$ points.

In order to restrain the computational cost, we employed the geometric properties of $\tilde{G}_{q,\mathbf{u}_k}$. Indeed for every $\mathbf{u}$, $\tilde{G}_{z,\mathbf{u}}$ is an affine map, thus it preserves parallelism and convexity. In view of these properties we considered only the 16 vertices of $\tilde{H}(z)$, say $\mathbf{x}_j$, with $j = 1, \dots, 16$. Our method consists in computing the $\tilde{G}_{z,\mathbf{u}}(\mathbf{x}_j)$'s separately, in projecting the result (namely 2^{kn} points) on $\mathbb{R}^2$ and finally on computing their convex hull, employing the fact that this projection, say π, preserves convexity, too. In other words we employed the identity

$$\pi(\tilde{G}_{z,\mathbf{u}}(\tilde{H}(z))) = \pi(\tilde{G}_{z,\mathbf{u}}(co(\{\mathbf{x}_j\}))) = co(\pi(\tilde{G}_{z,\mathbf{u}}(\mathbf{x}_j))),$$

so that

$$\tilde{G}^n_{z,k}(\tilde{H}(z)) = \bigcup_{\mathbf{u}\in\{0,1\}^{kn}} co(\pi(\tilde{G}_{z,\mathbf{u}}(\mathbf{x}_j))).$$

With this method we need to compute $2^{kn} \cdot 16$ points and we may possibly store the result on a vectorial format, instead of a raster one. See Figure 2.6 and Figure 2.5 for some examples.

Remark 2.2.28 (Some remarks on the analogies with expansions in non-integer bases). We notice that the $G_{q,\mathbf{u}_k}$'s share the same scaling factor, $A^k(q)$, and they differ for the translation component $\mathbf{v}(\mathbf{u}_k)$. A similar structure also emerges for the one-step recursion case, generating power series with coefficients in $\{0, 1\}$. Indeed

$$\tilde{x}_n = \sum_{k=0}^{n} \frac{u_{n-k}}{q^k} \quad \Leftrightarrow \quad \begin{cases} \tilde{x}_0 = u_0 \\ \tilde{x}_{n+1} = u_{n+1} + \frac{\tilde{x}_n}{q}. \end{cases} \tag{2.2.44}$$

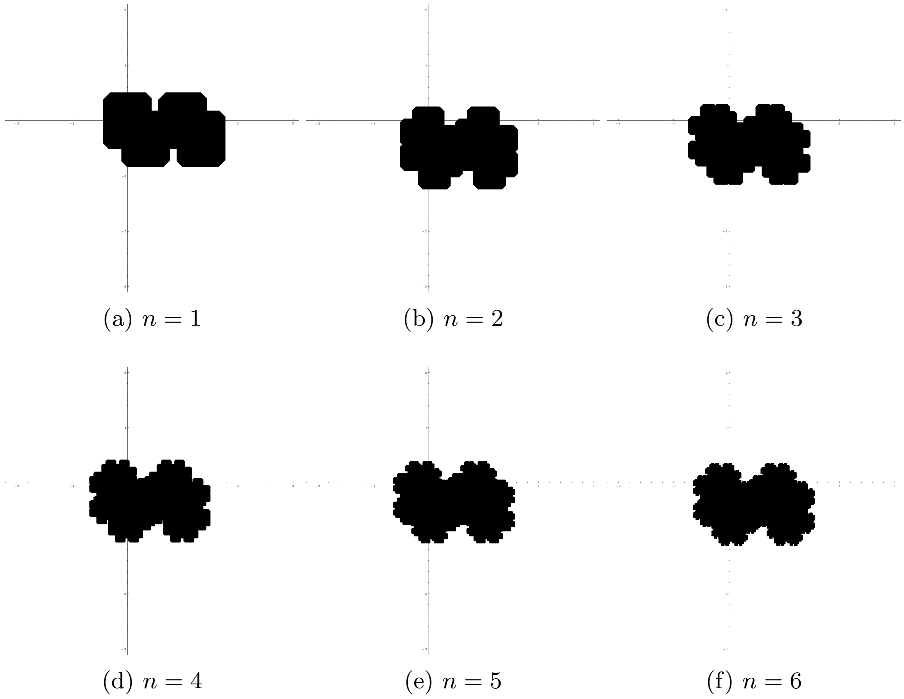


Figure 2.6: Various iterations of $\tilde{\mathcal{G}}^n_{z,2}(\tilde{H}(z))$ with $z = (q(8) + 0.3)e^{i\pi/4}$. Notice the similarity with the twin-dragon curve, generated by expansions in complex base with argument again $\pi/4$.

and setting

$$\tilde{R}_\infty(q) := \left\{ \sum_{k=0}^{\infty} \frac{u_k}{q^k} \mid u_k \in \{0,1\} \right\}$$

one has that

$$\tilde{R}_\infty(q) = \bigcup_{u \in \{0,1\}} \tilde{G}_{q,c}(\tilde{R}_\infty)$$

where

$$\tilde{G}_{q,u}(\tilde{x}) = u + \frac{\tilde{x}}{q}.$$

The differences and analogies between the two systems can be summarized as follows

1. both systems are related to power series;
2. $\tilde{R}_\infty(q)$ can be generated by a one-step recursive algorithm and it is the attractor of a one-dimensional IFS, the radius of convergence is 1. The buffer needed (i.e. the number of digits the IFS depends on) is constantly equal to 1;
3. $R_\infty(q)$ can be generated by a two-steps recursive algorithm and it is the attractor of a two-dimensional IFS, the radius of convergence is φ. The buffer needed, $k(q)$, depends on q and it goes to infinity as q tends to φ from above.

2.2.6 A sufficient contractivity condition

In what follows we provide an upper estimate for $k(q)$.

Proposition 2.2.29 (An upper estimate for the Fibonacci sequence)**.** *For every $n \in \mathbb{N}$*

$$f_{n+1} \leq \varphi^n.$$

Proof. By induction on n. First, as base cases, we will consider the cases when $n = 1$ and $n = 2$. Note that $1 < \varphi < 2$. By adding 1 to each term in the inequality, we obtain $2 < \varphi + 1 < 3$. The two inequalities together yield

$$1 < \varphi < 2 < \varphi + 1 < 3.$$

Using the relation $\varphi + 1 = \varphi^2$ and the first few Fibonacci numbers, we can rewrite this as

$$f_2 < \varphi < f_3 < \varphi^2 < f_4$$

which shows that the statement is true for $n = 1$ and $n = 2$. Now, as the induction hypothesis, suppose that $f_{i+1} < \varphi_i < f_{i+2}$ for all i such that $0 \leq i \leq k + 1$.

$$f_{k+2} < \varphi^{k+1} < f_{k+3}$$

and

$$f_{k+1} < \varphi^k < f_{k+2}.$$

Adding each term of the two inequalities, we obtain

$$f_{k+2} + f_{k+1} < \varphi^{k+1} + \varphi^k < f_{k+3} + f_{k+2}.$$

Using the relation $\varphi^{k+1} + \varphi^k = \varphi^{k+2}$ and the first few Fibonacci numbers, we can rewrite this inequality as

$$f_{k+3} < \varphi^{k+2} < f_{k+4}$$

which shows that the inequality holds for $n = k + 2$. □

Lemma 2.2.30 (Explicit computation of $A^k(q)$)**.** *For every $q > \varphi$ and for every $k \in \mathbb{N}$*

$$A^k(q) = \frac{1}{q^{k+1}} \begin{pmatrix} f_{k+1}q & f_k \\ f_k q^2 & f_{k-1}q \end{pmatrix}. \tag{2.2.45}$$

Proof. By induction on k. Base step, $k = 1$, is trivially satisfied. Assume now (2.2.45) as inductive hypothesis. For $k + 1$ we have

$$A^{k+1}(q) = A^k(q)A(q) = \frac{1}{q^{k+2}} \begin{pmatrix} (f_{k+1} + f_k)q & f_{k+1} \\ (f_k + f_{k-1})q^2 & f_k \end{pmatrix} = \frac{1}{q^{k+2}} \begin{pmatrix} f_{k+2}q & f_{k+1} \\ f_{k+1}q^2 & f_k q \end{pmatrix}.$$

and this concludes the proof. □

Proposition 2.2.31. *For every $q > \varphi$*

$$k(q) \leq \frac{\ln\left(\frac{1}{\varphi^2 q^2}(q^4 + 3q^2 + 1)\right)}{2\left(\ln q - \ln \varphi\right)}. \tag{2.2.46}$$

Proof. Fix k and set

$$B(q) := \begin{pmatrix} f_{k+1}q & f_k \\ f_k q^2 & f_{k-1}q \end{pmatrix}$$

so that, by Lemma 2.2.30, one has

$$A^k(q) = \frac{1}{q^{k+1}} B(q).$$

Denote by $\lambda_{max}(A)$ the greatest eigenvalue of A in modulus. One has that the matrix norm consistent with Euclidean norm satisfies the following identity

$$||A|| := \max_{\mathbf{x} \neq (0,0)} ||A\mathbf{x}|| = \sqrt{\lambda_{max}(A^T A)}.$$

Then

$$||A^k(q)|| = \frac{||B(q)||}{q^{k+1}} = \sqrt{\lambda_{max}(B^T(q)B(q))}.$$

The product matrix $B^T(q)B(q)$ has the form:

$$B^T(q)B(q) = \begin{pmatrix} f_{k+1}^2 q^2 + f_k^2 q^4 & f_k f_{k+1} q + f_k f_{k-1} q^3 \\ f_k f_{k+1} q + f_k f_{k-1} q^3 & f_k^2 + f_{k-1}^2 q^2 \end{pmatrix}.$$

The characteristic polynomial $p(\lambda)$ associated to $B^T(q)B(q)$ is hence

$$\begin{aligned} p(\lambda) = \lambda^2 - \lambda \left(f_{k+1}^2 q^2 + f_k^2 q^4 + f_k^2 + f_{k-1}^2 q^2 \right) + \\ + q^4 \left(f_{k-1}^2 f_{k+1}^2 + f_k^4 - 2 f_{k+1} f_{k-1} f_k^2 \right). \end{aligned}$$

The free term of characteristic polynomial is linked to algebraic identities involving the Fibonacci numbers,

$$f_{k-1}^2 f_{k+1}^2 + f_k^4 - 2 f_{k+1} f_{k-1} f_k^2 = 1.$$

In fact

$$f_{k-1}^2 f_{k+1}^2 + f_k^4 - 2 f_{k+1} f_{k-1} f_k^2 = \left(f_k^2 - f_{k-1} f_{k+1} \right)^2$$

involving Cassini's identity

$$f_{n-1} f_{n+1} - f_n^2 = (-1)^{n+1},$$

Then, the characteristic polynomial becomes:

$$p(\lambda) = \lambda^2 - \lambda \left(f_{k+1}^2 q^2 + f_k^2 q^4 + f_k^2 + f_{k-1}^2 q^2 \right) + q^4.$$

Set $\bar{\lambda}_{max} = f_k^2 q^4 + (f_{k+1}^2 + f_{k-1}^2) q^2 + f_k^2$ and note that

$$\lambda_{max}(B^T(q)B(q)) = \frac{1}{2} \left(\bar{\lambda}_{max} + \sqrt{\bar{\lambda}_{max}^2 - 4q^2} \right) \leq \bar{\lambda}_{max}.$$

Furthermore by Proposition 2.2.29 we have

$$\bar{\lambda}_{max} \leq \varphi^{2k-2} q^4 + (\varphi^{2k} + \varphi^{2k-4}) q^2 + \varphi^{2k-2} = \varphi^{2k-2}(q^4 + 3q^2 + 1)$$

and finally

$$||A^k(q)|| = \frac{\lambda_{max}}{q^{2k+2}} \leq \frac{\bar{\lambda}_{max}}{q^{2k+2}} \leq \frac{\varphi^{2k-2}}{q^{2k+2}} (q^4 + 3q^2 + 1).$$

Consequently if

$$\frac{\varphi^{2k-2}}{q^{2k+2}} (q^4 + 3q^2 + 1) < 1$$

then $||A^k(q)|| < 1$. To solve above inequality with respect to k we apply the logarithm, requiring that the final report is less than 0:

$$2k \ln \left(\frac{\varphi}{q} \right) + \ln \left(\frac{1}{\varphi^2 q^2} (q^4 + 3q^2 + 1) \right) < 0.$$

We finally obtain that if

$$k > \frac{\ln\left(\frac{1}{\varphi^2 q^2}(q^4+3q^2+1)\right)}{2(\ln q - \ln\varphi)}$$

then $||A^k(q)|| < 1$ and hence the claim. $\square$

It is well-known that f_k is the closest integer to $\frac{\varphi^k}{\sqrt{5}}$. Therefore it can be found by rounding in terms of the nearest integer function: $f_k = \left[\frac{\varphi^k}{\sqrt{5}}\right]$, $k \geq 0$. That gives a very sharp inequality. In fact, if k is an even number, then $f_k = \left[\frac{\varphi^k}{\sqrt{5}}\right] < \frac{\varphi^k}{\sqrt{5}}$ i.e. $f_{2k} = \left[\frac{\varphi^{2k}}{\sqrt{5}}\right] < \frac{\varphi^{2k}}{\sqrt{5}}$. We notice that $\frac{\varphi^k}{\sqrt{5}} < \varphi^{k-1}$. By the same procedure applied previously, we get

$$\bar{\lambda}_{max} \leq \frac{q^2}{5}\left(\varphi^{2k-2}+\varphi^{2k+2}\right) + \frac{\varphi^{2k}}{5}(q^4+1).$$

We have

$$\begin{aligned} &\tfrac{\bar{\lambda}_{max}}{q^{2k+2}} \leq \left(\tfrac{\varphi}{q}\right)^{2k} \tfrac{1}{5q^2}\left(\tfrac{q^2}{\varphi^2}+q^2\varphi^2+q^4+1\right) \leq 1 \\ &\Leftrightarrow 2k\ln\left(\tfrac{\varphi}{q}\right) + \ln\left(\tfrac{1}{5\varphi^2}+\tfrac{\varphi^2}{5}+\tfrac{q^2}{5}+\tfrac{1}{5q^2}\right) \leq 0 \end{aligned}$$

whence

$$k \geq \frac{\ln\left(\frac{1}{5\varphi^2}+\frac{\varphi^2}{5}+\frac{q^2}{5}+\frac{1}{5q^2}\right)}{2(\ln q - \ln\varphi)} \quad (q > \varphi) \tag{2.2.47}$$

for k even.

Remark 2.2.32. Now we want to compare the values of $k(q)$, and suppose that $k(q)$ of (2.2.47) is greater than (2.2.46).

$$\frac{\ln\left(\frac{1}{5\varphi^2}+\frac{\varphi^2}{5}+\frac{q^2}{5}+\frac{1}{5q^2}\right)}{2(\ln q - \ln\varphi)} > \frac{\ln\left(1+\frac{1}{\varphi^4}+\frac{q^2}{\varphi^2}+\frac{1}{q^2\varphi^2}\right)}{2(\ln q - \ln\varphi)}$$

i.e.

$$q^4(\varphi^4-5\varphi^2)+q^2(\varphi^2+\varphi^6-5\varphi^4-5)-5\varphi^2+\varphi^4 > 0$$

which doesn't admit solution. Then

$$\frac{\ln\left(\frac{1}{5\varphi^2}+\frac{\varphi^2}{5}+\frac{q^2}{5}+\frac{1}{5q^2}\right)}{2(\ln q - \ln\varphi)} < \frac{\ln\left(1+\frac{1}{\varphi^4}+\frac{q^2}{\varphi^2}+\frac{1}{q^2\varphi^2}\right)}{2(\ln q - \ln\varphi)}.$$

2.3 Conclusions and perspectives

In Section 2.1 introduced a robot hand model composed by an arbitrarily large number of hyper-redundant binary planar manipulators. The length of each link scales according to the Fibonacci sequence. Our assumptions (e.g. binary controls, kinematic redundancy, planar motion...) have twofold motivations. In one hand they facilitate the development of a theory relating fractal geometry and automatic control. On the other hand they appear validated by practical motivations in a wide literature. We described the kinematics of each finger by giving an explicit formula for the position of the end-effectors. We then addressed the investigation of the reachable workspace, by characterizing it as a projection of the attractor of a suitable IFS (Section 2.1.1). The relation with iteration function systems also allows to describe the convex hull of the reachable workspace: this technique is finally applied to the explicit characterization in a particular case.

In Section 2.2 we studied the workspace of a hyper-redundant manipulator, modeling a snake-like robot with links decaying as a scaled Fibonacci sequence. We give a formal proof of the results, highlighted by numerical simulations based on a fractal geometry approach. The main novelty of the Section consists in the exploitation of self-similar structure (induced by the dependence on the Fibonacci sequence) for a combinatoric study of the reachable workspace. We finally notice that, by the arbitrariness of the number of links, the asymptotic properties of the model (e.g. the possibility of setting an arbitrary global length for the manipulator) extend by approximation to the case with a finite number of links with arbitrary small tolerance.

Possible developments of the present work include the search for solutions for inverse kinematic problems in a fashion like [121], and for obstacle avoidance algorithms similar to those presented in the Chirikjian and Burdick's seminal report [43].

The results in the present chapter extend techniques previously developed in [113], [114] and [115] for the case of links with a constant ratio. The several explicit results obtained also in this more complicated case suggest that the relation with IFSs is a deep connection and a powerful theoretical tool for the investigation of automatic control. In this chapter we studied the purely discrete case in order to give closed formulae and to emphasize the relation with IFSs. However we plan to investigate the continuous case in a future work. The issues concerning the practical implementation of our models are beyond the purposes of the present chapter; but of course it would be interesting to establish the link between the theoretical approach and its application. Other open problems include a tuning of parameters in order to avoid self-intersecting configurations, and include grasping algorithms and optimal control strategies.

Chapter 3

Theorems for Exponential and Sinc Bases.

It is well known that the system $\{e^{int}\}_{n\in\mathbb{Z}}$ is an orthonormal and a Riesz basis for $L^2(-\pi,\pi)$. The question on the *stability* of the exponential system consists in asking whether expansion property of the system is still valid if we replace n with a *perturbation* λ_n. Kadec's theorem is a classical and famous result which gives a criterion for the nodes $\{\lambda_n \in \mathbb{R} : n \in \mathbb{Z}\}$ so that $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ forms a Riesz basis for $L^2(-\pi,\pi)$. The stability of exponential Riesz bases is related to *sampling theorem* as follows. (In this Chapter we prove that the stability of exponential bases is also related to stability of another, important system: the *Sinc basis* $\{\mathrm{sinc}(t-n)\}_{n\in\mathbb{Z}}$.)

If f represents the signal, assuming that $f \in L^2(\mathbb{R})$ (the energy of the signal is finite), then f is said band-limited to $[-\pi,\pi]$ if $\hat{f}$ vanishes outside the set $[-\pi,\pi]$, where $\hat{f}$ denotes the Fourier transform. The space of band-limited to $[-\pi,\pi]$ functions is the Paley-Wiener space, usually denoted by PW_π. The space PW_π play a significant role in signal processing applications [93]. As well known, any function $f \in PW_\pi$, can be expanded in terms of the orthonormal basis $\{e^{int}\}_{n\in\mathbb{Z}}$ (for $\hat{f}$) and $\{\mathrm{sinc}(t-n)\}_{n\in\mathbb{Z}}$ (for f). This is the Shannon's sampling theorem [107], [167]. For the sake of precision, the sampling theorem is associated with Claude Shannon in the West, and with Vladimir Kotelnikov in Russia.

Today, Shannon's work is fundamental in engineering and digital signal processing because it gives a framework for converting analog signals into sequences of numbers [166]. See also [178] and [136]. The sampling theorem establishes an important result: the continuous signal can then be perfectly reconstructed from its samples by means of a discrete-time interpolation operation; the value of the reconstructed signal at the instant t in any continuous is the sum of all samples, which we denote $f(n)$, each weighted with the sinc normalized function centered on the n-th sample and multiplied by the sample $f(n)$. Without loss of generality, the sampling reconstruction formula recovers a function with a frequency bandwidth of $[-\pi,\pi]$ given the function's values at the integers. But the theorem has drawbacks. Foremost, the recovery formula does not converge given certain types of error in the sampled data, as Daubechies and De Vore mention in [60]. They use oversampling to derive an alternative recovery formula which does not have this defect. Furthermore, as already said, for the theorem, the data nodes have to be equally spaced, and nonuniform sampling nodes are not allowed but, from many practical points of view it is necessary to develop sampling theorems for a sequence of samples taken with a nonuniform distribution along the real line.

The functions $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ are also called the *non-harmonic Fourier functions*, where $\{\lambda_n\}_{n\in\mathbb{Z}}$ is a given sequence of complex numbers. The study of these functions was initiated by Paley and Wiener [147]; they obtained the result that if the λ_n are real and such that $|\lambda_n - n| < 1/\pi^2$ then $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ forms a complete sequence in $L^2(-\pi,\pi)$. Later, an advanced result was presented by Levinson [119]. The case of complex λ_n was investigated by Duffin and Eachus [63], obtaining the stability bound $\frac{\log 2}{\pi}$. Their sampling formulae recover a function from nodes $\{\lambda_n\}_n$, where $\{e^{i\lambda_n t}\}_n$ forms a Riesz basis for $L^2(-\pi,\pi)$. As already mentioned, the maximum perturbation of the system $\{e^{int}\}_n$ is found by Kadec [103]. The results on the nonharmonic Fourier bases $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ can be translated into results about nonuniform sampling and reconstruction of band-limited functions: [22], [94], [164], [200]. This plays a very important role in signal theory; it suffices to think for example, that the sampling reconstruction formula expresses the fact that the $\hat{f}$ can be seen as infinite sum of elementary contributions of exponential type complex. Modern digital data processing of functions (or signals or images) always uses a discretized version of the original signal f that is obtained by sampling f on a discrete set. The question then arises whether and how f can be recovered from its samples. Therefore, the objective of research on the sampling problem is twofold. The first goal is to quantify the conditions under which it is possible to recover particular classes of functions from different sets of discrete samples. The second goal is to use these analytical results to develop explicit reconstruction schemes for the analysis and processing of digital data.

This Chapter concentrates also on perturbation of regular sampling: $\lambda_n \in \mathbb{C}$ for $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ and $\lambda_n \in \mathbb{R}$ for $\{\operatorname{sinc}(t-\lambda_n)\}_{n\in\mathbb{Z}}$. The interest on the function $\operatorname{sinc} t$ dates back to the works of Borel [29], Whittaker [193]. The Cardinal function [176]

$$C(f,1)(t) = \sum_{n\in\mathbb{Z}} f(n)\,\operatorname{sinc}(t-n), \quad x \in \mathbb{R}, \tag{3.0.1}$$

occupies an important place in the theory of analytic functions. Whittaker was the first to find a connection with analytic functions. $C(f,1)(t)$ is replete with identities within a Wiener class of functions, $W(\pi)$, of all entire functions of order 1 and type π that are also square integrable over the real line $\mathbb{R}$ [176]. Formula (3.0.1) also appears in the statement of sampling theorem. For this reason sampling theorem is often known as the Whittaker-Kotelnikov-Shannon theorem. Hardy who was referring to (3.0.1) wrote: “It is odd that, although these functions occur repeatedly in analysis, especially in the theory of interpolation, it does not seem to have been remarked explicitly that they form an orthogonal system” [87]. See also: [23], [37], [193].

We now outline the content of the Chapter. Section 3.1 contains a simple and different viewpoint from the literature, to the best of our knowledge, to generalize well known results by Kadec (in $\mathbb{R}$) and Duffin and Eachus (in $\mathbb{C}$), concerning Riesz bases, [188]. The main goal of the section is to overcome, at least partially, the limitations exhibited in the paper of Duffin and Eachus and in the book of Young for the Riesz bases. A consequence of the main theorem and its corollary is that the constant $\frac{\log 2}{\pi}$ can be replaced by 1/4 (for complex λ_n). In Section 3.2 we prove that if $\{\lambda_n\}_{n\in\mathbb{Z}}$ be a sequence of real numbers for which $|\lambda_n - n| \leqq L < \frac{1}{4}$, for all $z \in \mathbb{Z}$, then the sequence $\{\operatorname{sinc}(\lambda_n - z)\}$ satisfies the Paley-Wiener criterion and so forms a Riesz basis for the Paley-Wiener PW_π. This result is stated in [11] with an incorrect bound for L. The constant 1/4 is optimal also for the system $\{\operatorname{sinc}(\lambda_n - z)\}$. We have worked to reobtain the optimal constant 1/4 without going to the exponential basis, i.e., working directly on cardinal series. The goal has not been fully achieved but we have obtained some results in this direction.

3.1 A simple viewpoint for Kadec-1/4 theorem in the complex case.

It is known that exponential Riesz bases $\{e^{i\lambda_n t}\}$ (with $\lambda_n \in \mathbb{R}$) are stable in the sense that a small perturbation of a Riesz basis produces a Riesz basis; it is proved by Paley and Wiener ([199] and [147]). The proof of the Paley-Wiener theorem does not provide an explicit stability bound. The celebrated theorem by M. I. Kadec shows that 1/4 is the stability bound for the exponential basis on $L^2[-\pi, \pi]$.

The proof of theorem, as reported in the Young's textbook[199], applies for sequences of real numbers. Even earlier, however, Duffin and Eachus [63] shows that the Paley - Wiener criterion is satisfied whenever the sequences are complex and $\frac{\log 2}{\pi}$ is a stability bound. For Young (page 38): "Whether the constant $\frac{\log 2}{\pi}$ can be replaced by 1/4 (for complex λ_n) remains an unsolved problem". With Theorem C and Theorem D on [63] they consider sets which are on the borderline of being near a given orthonormal set, while the last part of their paper gives a simple formula for constructing sets near a given orthonormal set. Afterward, Duffin and Eachus apply this result (Theorem D) to the sequence of functions $\{e^{i\lambda_n x}\}$, where $\{\lambda_n\}$, $n = 0, \pm1, \pm2, ...$ is a sequence of complex constants satisfying $|\lambda_n - n| \leq L$ for some constant L. The Duffin and Eachus's approach is deeper and more general than one of Young; in fact their work speaks of orthonormal sets and not of basis. In their paper can be read the following: "The above results on the non-harmonic Fourier series are an extension of previous knowledge in two respects. In the first place, Paley and Wiener were forced to assume that $\{\lambda_n\}$ was a real sequence. Secondly, they obtained the value $1/\pi^2$ where we have $\ln 2/\pi$. The best value for L is not known; however a theorem of Levinson gives an upper limit of 1/4".

Theorem 3.1.3 seeks to overcome the limitations exhibited in the paper of Duffin and Eachus and in the book of Young for the Riesz basis, introducing a limitation on the imaginary part of λ_n. A consequence of theorem 3.1.3 and its corollary, is that the constant $\frac{\log 2}{\pi}$ can be replaced by 1/4 (for complex λ_n).

Lastly, an example that shows 1/4 cannot be replaced by a larger constant for complex case, are given in the appendix. For the latest results on generalizations and extensions of Kadec's theorem see: [178], [136], [48].

3.1.1 $L = 1/4$ as best possible choice.

The two lemmas below follows by Young's book just adapting to complex case in this chapter. For the theory of entire function and the proof of lemma 3.1.1, see chapter 2 and pages 103-105 of Young's book.

Lemma 3.1.1. *If $\lambda_n = n + \varepsilon + i\tau(\varepsilon)$ $(n = 1, 2, 3...)$, where $\varepsilon > -1$, and*

$$H(z) = \prod_n \left(1 - \frac{z^2}{\lambda_n^2}\right)$$

, then

$$H'(\lambda_n) = (-1)^n \Gamma^2(1 + \varepsilon + i\tau(\varepsilon)) \frac{\Gamma(n)}{\Gamma(n + 1 + 2\varepsilon + 2i\tau(\varepsilon))}$$

Now, using the thesis of this lemma, is shown the next result.

Lemma 3.1.2. *If*

$$\lambda_n = \begin{cases} n+\varepsilon+i\tau(\varepsilon), & n>0 \\ 0, & n=0 \\ n-\varepsilon-i\tau(\varepsilon), & n<0 \end{cases} \tag{3.1.1}$$

then, for $\varepsilon \geq 1/4$, the system $\{e^{i\lambda_n t}\}$ is not a Riesz basis for $L^2[-\pi,\pi]$.

Proof. Suppose it were. Then the system of reproducing functions $\{K_n(z)\}$, $K_n(z) = \sin\pi(z-\lambda_n)/\pi(z-\lambda_n)$, would be a Riesz basis for Paley-Wiener space P, since the Fourier transform is an isometry. Put

$$F_n(z) = \frac{F(z)}{F'(\lambda_n)(z-\lambda_n)}$$

where $F(z) = \prod_n (1 - z^2/\lambda_n^2)$. Then $F_n(\lambda_k) = \delta_{nk}$, and F_n belongs to P. Accordingly, $\{F_n(z)\}$ is biorthogonal to $\{K_n(z)\}$ in P and so must also be a Riesz basis for P. In particular, the series

$$\sum_n c_n \frac{F(z)}{F'(\lambda_n)(z-\lambda_n)}$$

must converge in the topology of P, and hence pointwise, whenever $\{c_n(z)\} \in L^2$. By the converse to Hölder's inequality, this can happen only if

$$\sum_{n\neq 0} \left| \frac{1}{\lambda_n F'(\lambda_n)} \right|^2 < \infty.$$

But by Lemma (3.1.1),

$$F'(\lambda_n) = (-1)^n \Gamma^2(1+\varepsilon+i\tau(\varepsilon)) \frac{\Gamma(n)}{\Gamma(n+1+2\varepsilon+2i\tau(\varepsilon))}$$

and Stirling's formula,

$$\frac{\Gamma(n)}{\Gamma(n+a)} \sim e^{-a\ln n}$$

, shows that

$$\sum_{n\neq 0} \left| \frac{1}{\lambda_n F'(\lambda_n)} \right|^2 = \infty$$

for $\varepsilon \geq 1/4$ and the contradiction proves the lemma. □

3.1.2 A class of sequences that improves the estimation of Duffin and Eachus

Theorem 3.1.3. *If $\{\bar{\lambda}_n\} = \{\lambda_n + i\mu_n\}$ is a sequence of complex numbers for which*

$$|\lambda_n - n| \leqq L < \frac{1}{4}, \quad n = 0, \pm 1, \pm 2, ... \tag{3.1.2}$$

and

$$|\mu_n| \leqq \tau(L) < \frac{1}{\pi} \ln\left(\frac{2}{2-\cos\pi L + \sin\pi L} \right), \quad n = 0, \pm 1, \pm 2, ... \tag{3.1.3}$$

then $\{e^{i\bar{\lambda}_n t}\}$ satisfies the Paley-Wiener criterion and so forms a Riesz basis for $L^2[-\pi,\pi]$.

Proof. It is to be shown that $\left\|\sum_n^{+\infty} c_n \left(e^{int} - e^{i\bar{\lambda}_n t}\right)\right\| < 1$ whenever $\sum_n |c_n|^2 \leqq 1$. Write

$$\begin{aligned} e^{int} - e^{i\lambda_n t} &= e^{int}\left(1 - e^{i\delta_n t}\ e^{-\mu_n t}\right) \\ &= e^{int}\left[1 - e^{-\mu_n t} + e^{-\mu_n t}\left(1 - e^{i\delta_n t}\right)\right] \end{aligned} \tag{3.1.4}$$

where $\delta_n = \lambda_n - n$. This time again, the trick is to expand the function $1 - e^{i\delta t}$ $(-\pi \leq t \leq \pi)$ in a Fourier series relative to the complete orthonormal system $\{1, \cos nt, \sin(n - \frac{1}{2})t\}_{n=1}^{\infty}$ and then exploit the fact that $|\lambda_n - n|$ is not too large. Then the expansion of $1 - e^{i\delta t}$ is the same as the previous theorem. Let $\{c_n\}$ be an arbitrary finite sequence of scalars such that $\sum |c_n|^2 \leq 1$. By interchanging the order of summation, using triangle inequality and the notation introduced in the Kadec's theorem on [199], it shows

$$\left\|\sum_n^{+\infty} c_n e^{int}\left[1 - e^{-\mu_n t} + e^{-\mu_n t}\left(1 - e^{i\delta_n t}\right)\right]\right\| \leq \tag{3.1.5}$$

$$\leq \sup_n \left|1 - e^{-\mu_n t}\right| \left\|\sum_n^{+\infty} c_n e^{int}\right\| + \sup_n \left(e^{-\mu_n t}\right)(A + B + C) \tag{3.1.6}$$

From the assumptions of the theorem it is easily seen that $\sup_n \left(e^{-\mu_n t}\right) \leq e^{\tau\pi}$ and $\sup_n \left|1 - e^{-\mu_n t}\right| \leq e^{\tau\pi} - 1$ where $\tau = \tau(L)$. Now, by [103], it has that

$$\left\|\sum_n^{+\infty} c_n \left(e^{int} - e^{i\bar{\lambda}_n t}\right)\right\| \leq e^{|M|} - 1 + e^{|M|}\left(1 - \cos\pi L + \sin\pi L\right) =: \lambda \tag{3.1.7}$$

It is observed that with arbitrary $L < 1/4$ and

$$\tau(L) < \frac{1}{\pi}\ln\left(\frac{2}{2 - \cos\pi L + \sin\pi L}\right) \tag{3.1.8}$$

is obtained $\lambda < 1$. □

The following result shows that, in the hypotheses of the theorem 3.1.3, it has $\{e^{i\bar{\lambda}_n t}\}$ satisfies the Paley-Wiener criterion for $|\bar{\lambda}_n - n| < 1/4$ even when $\{\bar{\lambda}_n\}$ is a complex sequence.

Corollary 3.1.4. *For each $L < \frac{1}{4}$, one has*

$$(i)\quad |\mu_n| \leq \frac{\ln 2}{\pi}; \qquad (ii)\quad |\bar{\lambda}_n - n| \leq \frac{1}{4} \tag{3.1.9}$$

Proof. The proof of first relation (i) is trivial and is left to the reader. Noting that

$$|\bar{\lambda}_n - n| \leq |\lambda_n - n| + |\mu_n| \leq L + \frac{1}{\pi}\ln\left(\frac{2}{2 - \cos\pi L + \sin\pi L}\right) \tag{3.1.10}$$

relation (ii) is verified if $\bar{x} - \ln\left(1 + \frac{\sin\bar{x} - \cos\bar{x}}{2}\right) \leq \frac{\pi}{4}$ with $\bar{x} = \pi L$. Let us consider the function $f(\bar{x})$, defined as follow:

$$f(\bar{x}) = \bar{x} - \ln\left(1 + \frac{\sin\bar{x} - \cos\bar{x}}{2}\right)$$

It comes to prove that the function $f(\bar{x}) - \bar{x} := g(\bar{x})$ is convex. Rewrite the function $g(\bar{x})$ using the relationship $(\sin\bar{x} - \cos\bar{x})/2 = \frac{\sqrt{2}}{2}\sin\left(\bar{x} - \frac{\pi}{4}\right)$ and so $g(x) = -\ln\left(1 + \frac{\sqrt{2}}{2}\sin x\right)$ for

$x = \bar{x} - \pi/4$. Bearing in mind that a function is convex if and only if it is midpoint convex, it must be demonstrated that $2g\left(\frac{x+y}{2}\right) \leq g(x) + g(y)$, and hence

$$-2\ln\left(1 + \frac{\sqrt{2}}{2}\sin\frac{x+y}{2}\right) \leq -\ln\left(1 + \frac{\sqrt{2}}{2}\sin x\right) - \ln\left(1 + \frac{\sqrt{2}}{2}\sin y\right)$$

where $y = \bar{y} - \pi/4$. From properties of logarithms and by applying Prosthaphaeresis formulas, Werner formulas, and half-angle formulae, it has

$$\sqrt{2}\sin\left(-\frac{x+y}{2}\right) \leq \cos^2\frac{x-y}{4}$$

Rewriting $-\frac{x+y}{2} = \frac{\pi}{4} - \frac{\bar{x}-\bar{y}}{2} - \bar{y} \leq \frac{\pi}{4} - t$ with $t = \frac{\bar{x}-\bar{y}}{2} \in [0, \pi/4]$, it becomes $\sqrt{2}\sin\left(\frac{\pi}{4} - t\right) \leq \cos^2\frac{t}{2}$, that is verified over $[0, \pi/4]$. Then $f(x)$ is convex. Denoting with $P_1(0, \ln 2)$, $P_2(\pi/4, \pi/4)$ two points belonging to graphic of $f(x)$ and from an obvious properties of convex functions: $f(x) \leq \frac{\pi - \ln 16}{\pi}x + \ln 2$ (the straight line for P_1, P_2), by the right side term that is less than $\frac{\pi}{4}$ if $x \leq \frac{\pi}{4}$, it is concluded the claim. $\square$

3.2 An explicit bound for stability of sinc bases.

Let f be a function which can be expanded as

$$f(t) = \sum_{n\in\mathbb{Z}} c_n \operatorname{sinc}(t-n) \tag{3.2.1}$$

where

$$\operatorname{sinc}(\alpha) = \begin{cases} \frac{\sin(\pi\alpha)}{\pi\alpha} & \alpha \neq 0, \\ 1 & \alpha = 0, \end{cases} \tag{3.2.2}$$

is the *normalized sinc function*. The RHS of (3.2.1) is called "cardinal series" or *Whittaker cardinal series*. A major factor affecting current interest in the cardinal series is its importance for certain applications as, for example, interpolation based on (3.2.1) which is usually called *ideal bandlimited interpolation* (or sinc interpolation), because it provides a perfect reconstruction for all t, if $f(t)$ is bandlimited in $[-\pi, \pi]$ and if the sampling frequency is greater that the so-called *Nyquist rate*. The system used to implement (3.2.1) is also known in in engineering applications as *ideal DAC* (i.e. digital-to-analog converter, see [129]). The presence of the perturbation could lose the correct reconstruction of the function (signal), so it is important to study the conditions for which the system is still able to reconstruct the function (signal) belonging to a given space. Other applications are sampling theory of band-limited signals in communications engineering [93] or sinc-quadrature method for differential equations [126]. The so-called *sinc numerical methods of computation*, provide procedures for function approximation over bounded or unbounded regions, encompassing interpolation, approximation of derivatives, approximate definite and indefinite integration, and so on [176]. These problems motivated our investigation on sinc systems.

For these reasons, the cardinal series have been widely discussed in the literature; see also [193] and [192]. They are linked to a classical basis, the exponentials $\{e^{int}\}_{n\in\mathbb{Z}}$ in $L^2(-\pi, \pi)$, through the *Fourier transform*, indeed formally

$$\mathcal{F}\left(e^{it\mu}\chi_{[-\pi,\pi]}(t)\right)(\xi) = \int_{-\pi}^{\pi} e^{i(\mu-\xi)t}dt = 2\pi\operatorname{sinc}(\mu - \xi).$$

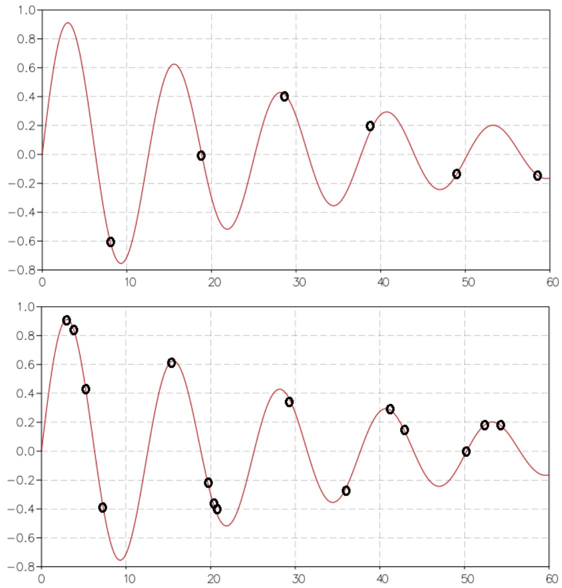


Figure 3.1: Left: A function f defined on $\mathbb{R}$ has been sampled on a uniformly spaced set. Right: The same function f has been sampled on a non-uniformly spaced set.

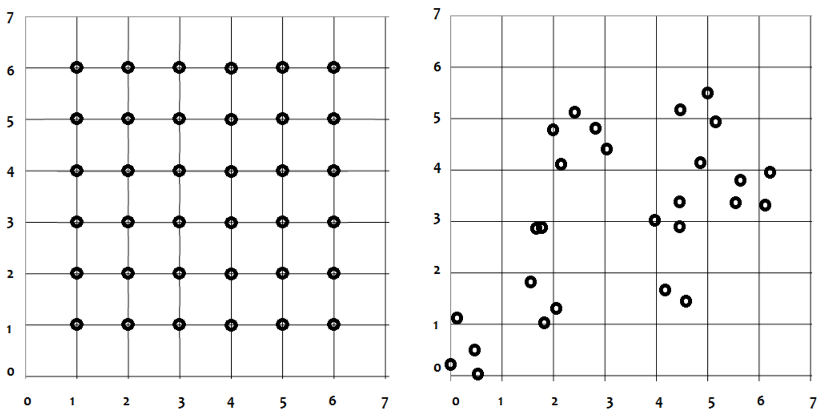


Figure 3.2: Sampling grids. Left: uniform cartesian sampling. Right: A typical nonuniform sampling set as encountered in various signal and image processing applications.

Studies on more general exponential systems $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ find their origin in the celebrated 1934's work of Paley and Wiener [147] in $L^2(0,T)$, where $T > 0$. They proved that if $\lambda_n \in \mathbb{R}, n \in \mathbb{Z}$ and

$$|\lambda_n - n| \leq L < \pi^{-2} \qquad n \in \mathbb{Z}$$

then the system $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ forms a Riesz basis in $L^2[-\pi,\pi]$. A well-known theorem by Kadec [103], [199] shows that $1/4$ is a stability bound for the exponential basis on $L^2(-\pi,\pi)$, in the sense that for $L < 1/4$, $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ is still a Riesz basis in $L^2(-\pi,\pi)$. More than 60 years after Paley and Wiener initiated the study of nonharmonic Fourier series in $L^2[-\pi,\pi]$, many other approaches to exponential Riesz basis problem have emerged in the literature. For other contributions to exponential Riesz basis problem and Kadec's theorem see survey papers, as: [163], [184].

For the system of cardinal sines $\{\mathrm{sinc}(t-n)\}_{n\in\mathbb{Z}}$ we tried to follow the same approach. For simplicity, we refer to $\{\mathrm{sinc}(t-n)\}_{n\in\mathbb{Z}}$ with terms of "sinc system" or *sinc basis*.

The results of this paper are the following theorems.

First of all, we recall and prove the classical result (see also [11]). Below, we denote with PW_π the *Paley-Wiener space.*

Proposition 3.2.1. *Let $\{\lambda_n\}_{n\in\mathbb{Z}}$ be a sequence of real numbers for which*

$$|\lambda_n - n| \leqq L < \infty, \quad n = 0, \pm 1, \pm 2, ... \tag{3.2.3}$$

If $L < \frac{1}{4}$, the sequence $\{\mathrm{sinc}(\lambda_n - t)\}_{n\in\mathbb{Z}}$ satisfies the Paley-Wiener criterion and so forms a Riesz basis for PW_π. Moreover, constant $1/4$ is optimal.

The subsequent results have been achieved in an attempt to reobtain the optimal constant 1/4 without going to the exponential basis, i.e. working directly on cardinal series. Let us consider the following two results.

Theorem 3.2.2. *Let $\lambda_n - n = \frac{A}{|n|^\alpha}$ for $n = \pm 1, \pm 2, \ldots$ and $\lambda_n - n = 0$ for $n = 0$. If $\alpha > 1/2$ and $|A| < \frac{1}{2\pi \sqrt[4]{2}\sqrt{\zeta(2\alpha)}}$ then the system $\{\mathrm{sinc}(\lambda_n - t)\}_{n\in\mathbb{Z}}$ satisfies the Paley-Wiener criterion and so forms a Riesz basis for PW_π.*

Numerical evaluation in the case $\lambda_n - n = \frac{A}{|n|^\alpha}$ is given in Section 3.2.6; in the Tables are showed that, when $\alpha = 2$, for increasing value of A until $A \simeq 0.3868$, the system $\{\mathrm{sinc}(\lambda_n - t)\}_{n\in\mathbb{Z}}$ is a Riesz basis in PW_π. If $n = \pm 1, \pm 2, \ldots$, we have that $\lambda_n - n \leq L$ where L is greater ($\simeq 0.3868\ldots$) of Kadec's bound. This is due to the assumption $\lambda_n - n = \frac{A}{|n|^\alpha}$, that is, to have considered a non-uniform stability bound.

In Section 3.2.5 we study the stability of $\{\mathrm{sinc}(\lambda_n - t)\}_{n\in\mathbb{Z}}$ for $\lambda_n \in \mathbb{C}$, reobtaining a stability bound which depends from Lamb-Oseen constant [142]. This constant was also appeared in previous work [11] although the stability bound was not correct.

In a previous work one of the author studied the extension to complex numbers of Kadec type estimate for exponential bases [186]. The method used there is inspired to work by Duffin and Eachus [63]. In [11] we performed a preliminary study by adapting a previous result on sinc. Here we give a complete result by the following theorem.

Theorem 3.2.3. *If $\{\lambda_n\}$ is a sequence of complex numbers for which*

$$|\lambda_n - n| \leqq L < \frac{1}{\pi}\sqrt{\frac{3\alpha}{8}}, \quad n = 0, \pm 1, \pm 2, ... \tag{3.2.4}$$

then $\{\mathrm{sinc}(\lambda_n - t)\}_{n\in\mathbb{Z}}$ satisfies the Paley-Wiener criterion and so forms a Riesz basis for PW_π.

Observe that the optimality of the bound for the complex case is not studied in our result.

3.2.1 Preliminaries

In this section we will introduce some useful notations and results about cardinal series, with reference to applications in sampling and numerical analysis. Some of the results in the final part of this section are given for the convenience of the reader.

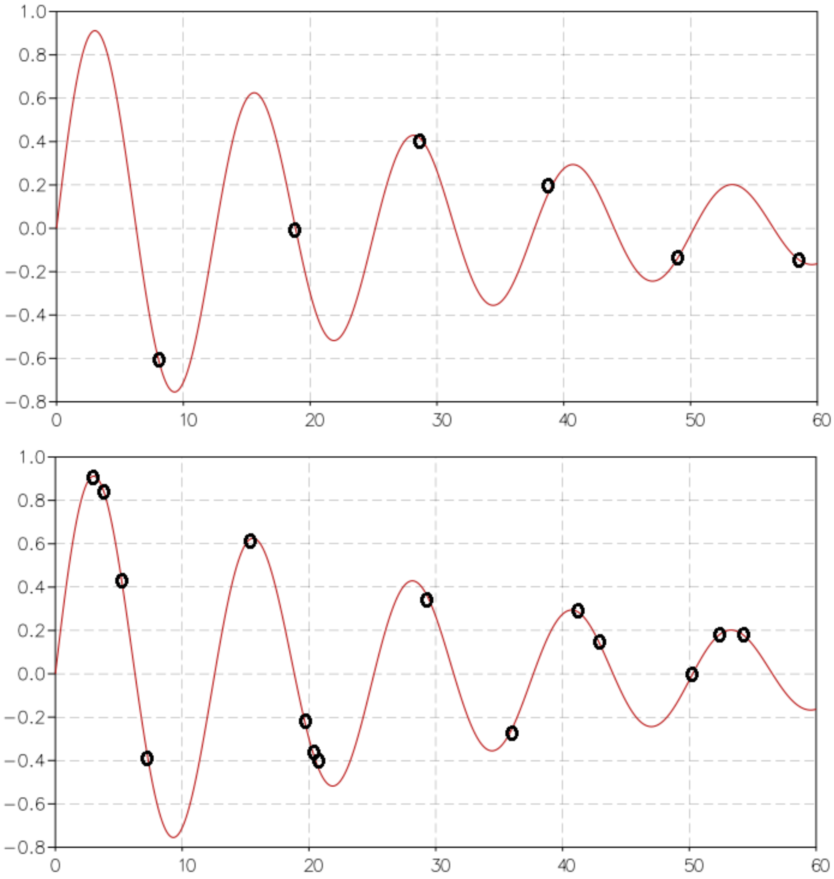


Figure 3.3: Left: A function f defined on $\mathbb{R}$ has been sampled on a uniformly spaced set. Right: The same function f has been sampled on a non-uniformly spaced set.

Sampling Theorem and Stability

By $L^2(-\infty,+\infty)$ we denote the Hilbert space of real functions that are square integrable in Lebesgue's sense:

$$L^2(\mathbb{R}) = \left\{ f : \int_{-\infty}^{+\infty} |f(t)|^2 dt < +\infty \right\}$$

with respect to the inner product and L^2-norm that, on $\mathbb{R}$, are

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\overline{g(t)}dt \qquad ||f|| = \sqrt{\langle f, f\rangle}$$

Given $f \in L^2(\mathbb{R})$ we denote by $\hat{f}$ the Fourier transform of f,

$$\hat{f}(\xi) = \mathcal{F}(f)(\xi) = \int_{-\infty}^{+\infty} f(t)e^{-i\xi t}dt.$$

Let e_n be an orthonormal basis of an Hilbert space H. Then Parseval's identity asserts that for every $x \in H$,

$$\sum_n |\langle x, e_n\rangle|^2 = \|x\|^2.$$

Plancherel identity is expressed, in its common form:

$$\int_{-\infty}^{\infty} f(t)\,\overline{g(t)}\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)\,\overline{\hat{g}(\xi)}\,d\xi.$$

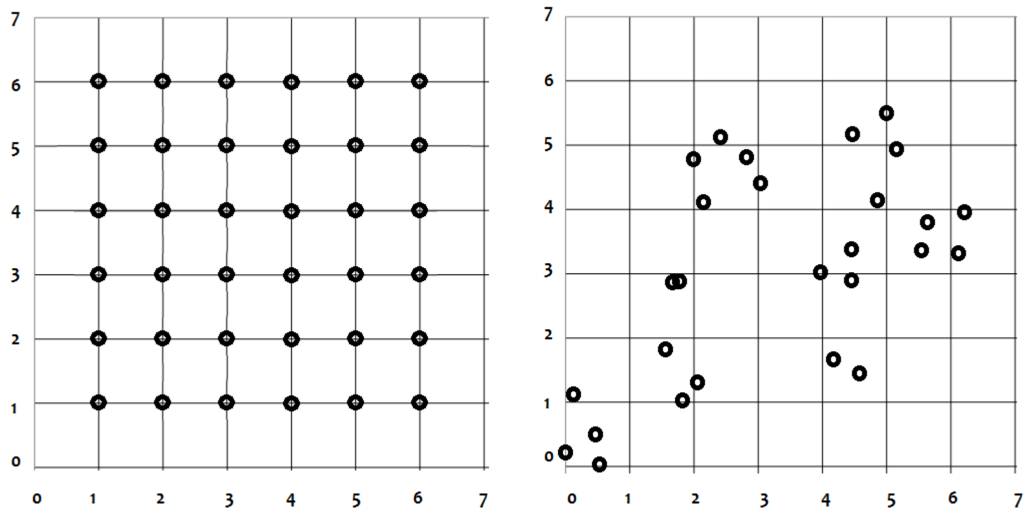


Figure 3.4: Sampling grids. Left: uniform cartesian sampling. Right: A typical nonuniform sampling set as encountered in various signal and image processing applications.

A function $f \in L^2(\mathbb{R})$ is band-limited if the Fourier transform $\hat{f}$ has compact support. The Paley-Wiener space PW_π is the subspace of $L^2(\mathbb{R})$ defined by

$$PW_\pi := \left\{ f \in L^2(\mathbb{R}) \middle| \operatorname{supp} \hat{f} \subseteq [-\pi, \pi] \right\}.$$

We will now recall that the Paley-Wiener space has an orthonormal basis consisting of translates of sinc-function.

Theorem 3.2.4. *(Shannon's sampling theorem) [47] The functions* $\{\operatorname{sinc}(\cdot - n)\}_{n\in\mathbb{Z}}$ *form an orthonormal basis for* PW_π. *If* $f \in PW_\pi$ *is continuous, then*

$$f(t) = \sum_{n\in\mathbb{Z}} f(n) \operatorname{sinc}(t-n). \tag{3.2.5}$$

Taking the Fourier transform in equation (3.2.5) we obtain

$$\hat{f}(\xi) = \sum_{n\in\mathbb{Z}} \left\langle \hat{f}, e^{in\xi} \right\rangle_{L^2(-\pi,\pi)} e^{in\xi}, \tag{3.2.6}$$

where $\langle g, h\rangle_{L^2(-\pi,\pi)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\xi)\overline{h(\xi)}d\xi$.

Sampling theorem expresses the possibility of recovering a certain kind of signals from a sequence of regularly spaced samples. However from many practical points of view it is necessary to develop sampling theorems for a sequence of samples taken with a nonuniform distribution along the real line. Nonuniform sampling of band-limited functions has its roots in the work of Paley, Wiener, and Levinson. In fact, the first answer for this direction was given by Paley and Wiener [147], and later an advanced result was presented by Levinson [119]. Their sampling formulae recover a function from nodes $\{\lambda_n\}_n$, where $\{e^{i\lambda_n\xi}\}_n$ forms a Riesz basis for $L^2[-\pi,\pi]$. The result is related with the perturbation of orthonormal basis $\{e^{in\xi}\}_{n\in\mathbb{Z}}$ for the function space $L^2[-\pi,\pi]$ in such a way that the perturbed sequence $\{e^{i\lambda_n\xi}\}_n$ is also a Riesz basis for the same space. The maximum perturbation of the system $\{e^{inx}\}_n$ is found by Kadec, whose result is the already cited *Kadec-1/4 theorem* [103]. This is a stability result, in the sense that we will explain below.

Usually, it is said that bases in Banach spaces form a *stable* class in the sense that sequences sufficiently close to bases are themselves bases. The fundamental stability criterion, and historically the first, is due to Paley and Wiener [147], [199].

Theorem 3.2.5. *Let $\{x_n\}$ be a basis for a Banach space X, and suppose that $\{y_n\}$ is a sequence of elements of X such that*

$$\left\| \sum_{i=1}^{n} a_i(x_i - y_i) \right\| \leq \lambda \left\| \sum_{i=1}^{n} a_i x_i \right\|$$

for some constant $0 \leqq \lambda < 1$, and all choices of the scalars $a_1, \ldots, a_n$ $(n = 1, 2, 3, \ldots)$. Then $\{y_n\}$ is a basis for X equivalent to $\{x_n\}$.

We reformulate Theorem 3.2.5 to be applied to orthonormal bases.

Theorem 3.2.6. *Let $\{e_n\}$ be an orthonormal basis for a Hilbert space H, and let $\{f_n\}$ be "close" to $\{e_n\}$ in the sense that*

$$\left\| \sum_{i=1}^{n} a_i(e_i - f_i) \right\| \leq \lambda \sqrt{\sum |c_i|^2}$$

for some constant $0 \leqq \lambda < 1$, and all choices of the scalars $a_1, \ldots, a_n$ $(n = 1, 2, 3, \ldots)$. Then $\{f_n\}$ is a Riesz basis for H.

Let $\chi(t)$ be the characteristic function defined as

$$\chi(t) = \begin{cases} 0 & \text{if } |t| > \frac{1}{2} \\ \frac{1}{2} & \text{if } |t| = \frac{1}{2} \\ 1 & \text{if } |t| < \frac{1}{2}. \end{cases} \tag{3.2.7}$$

Theorem 3.2.7 (Kadec $\frac{1}{4}$-Theorem)**.** *Let $\{\lambda_n\}_{n\in\mathbb{Z}}$ be a sequence in $\mathbb{R}$ satisfying*

$$|\lambda_n - n| < \frac{1}{4}, \ n = 0, \pm 1, \pm 2, \ldots$$

then the set $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2[-\pi, \pi]$.

Lambert function W, Lamb-Oseen constant.

The Lambert function W [53], [88], [177] is defined by the equation

$$W(x)e^{W(x)} = x \tag{3.2.8}$$

The function $f(\xi) = \xi e^{\xi}$ for $\xi \in \mathbb{R}$ has a strict minimum point in $\xi = -1$. We draw the picture of $\xi e^{\xi} = f(\xi)$ in figure (3.7). In [11] it is proved the following proposition.

Proposition 3.2.8. *The function $f(\xi) = \xi e^{\xi}$ has an increasing inverse in $(-1, +\infty)$, and a decreasing inverse in $(-\infty, -1)$.*

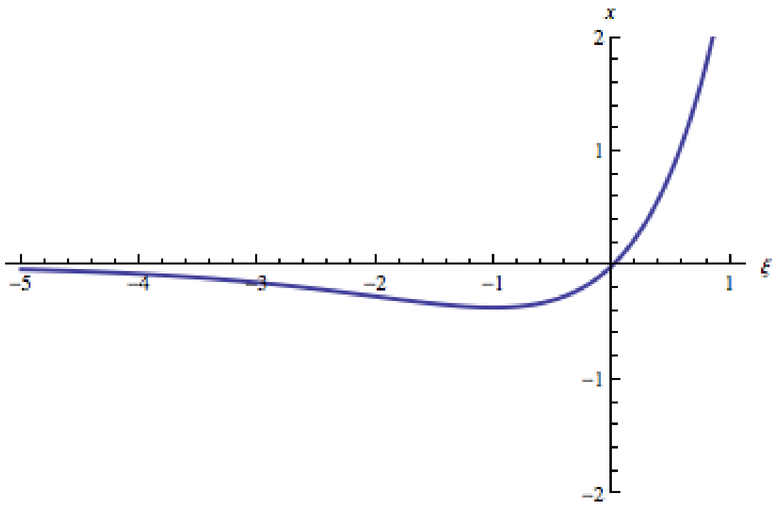


Figure 3.5: Diagram of $\xi e^{\xi} = f(\xi)$.

We consider $f(\xi) = \xi e^{\xi}$ restricted to the interval $(-\infty, -1]$ and we denote by W_{-1} its inverse. W_{-1} is defined in the interval $[-1/e, 0)$. We have two identities arising from the definition of W_{-1}:

$$W_{-1}(\xi e^{\xi}) = \xi, \left[\Leftrightarrow W_{-1}\left[f(\xi)\right] = W_{-1}\left[\xi e^{\xi}\right] = \xi\right] \forall \xi \in (-\infty, -1] \tag{3.2.9}$$

and

$$W_{-1}(\bar{x})e^{W_{-1}(\bar{x})} = \bar{x} \quad [\Rightarrow f\left(W_{-1}(\bar{x})\right) = \bar{x}]\, \forall \bar{x} \in [-\frac{1}{e}, 0) \tag{3.2.10}$$

Also we denote by W_0 the restriction to the interval $[-1/e, 0)$ of the increasing inverse of $f(\xi) = \xi e^{\xi}$. The two identities hold true:

$$W_0(\xi e^{\xi}) = \xi, \quad \left[\Leftrightarrow W_0\left[f(\xi)\right] = W_0\left[\xi e^{\xi}\right] = \xi\right], \quad \forall \xi \in [-1, 0) \tag{3.2.11}$$

and

$$W_0(\bar{x})e^{W_0(\bar{x})} = \bar{x} \quad [\Rightarrow f\left(W_0(\bar{x})\right) = \bar{x}] \quad \forall \bar{x} \in [-\frac{1}{e}, 0) \tag{3.2.12}$$

Let us assume that $\bar{x}$ is a solution of our equation.

$$e^{\bar{x}} - 2\bar{x} = 1 \tag{3.2.13}$$

In order to use the Lambert function W, we observe that from (3.2.13) we get the equivalences

$$e^{\bar{x}} - 2\bar{x} = 1 \Leftrightarrow e^{e^{\bar{x}} - 2\bar{x}} = e$$

whence $-\frac{1}{2}e^{\bar{x}}e^{-\frac{1}{2}e^{\bar{x}}} = -\frac{1}{2}e^{-\frac{1}{2}}$. Therefore we can identifies $-\frac{1}{2}e^{\bar{x}}$ with $W\left(-\frac{1}{2}e^{-\frac{1}{2}}\right)$. Since $-\frac{1}{e} < -\frac{1}{2}e^{-\frac{1}{2}} < 0$, the equation which defines the function W of Lambert, has two branches which verifies the same equation $W(x)e^{W(x)} = -\frac{1}{2}e^{-\frac{1}{2}}$ and we will have

$$-\tfrac{1}{2}e^{\bar{x}} = W_0\left(-\tfrac{1}{2}e^{-\frac{1}{2}}\right) \tag{3.2.14}$$

and

$$-\tfrac{1}{2}e^{\bar{x}} = W_{-1}\left(-\tfrac{1}{2}e^{-\frac{1}{2}}\right). \tag{3.2.15}$$

We call $\bar{x}_1$ the $\bar{x}$ solution of (3.2.14), and $\bar{x}_2$ the solution of (3.2.15).

We state easy that $\bar{x}_1 = 0$. In fact from (3.2.14) we have

$$-\tfrac{1}{2}e^{\bar{x}_1} = -\tfrac{1}{2}\left[= W_0\left(-\tfrac{1}{2}e^{-\frac{1}{2}}\right)\right]$$

and from $e^{\bar{x}_1} = 1$, easily follows $\bar{x}_1 = 0$.

From (3.2.15), and the relation (3.2.10) we get $e^{\bar{x}_2} = -2W_{-1}\left(-\frac{1}{2}e^{-\frac{1}{2}}\right)$, and so $\bar{x}_2$:

$$\ln\left(-2W_{-1}\left(-\frac{1}{2}e^{-\frac{1}{2}}\right)\right) = -\ln\frac{1}{-2W_{-1}\left(-\frac{1}{2}e^{-\frac{1}{2}}\right)}$$

Now we multiply numerator and denominator by $e^{\frac{1}{2}}$

$$= -\ln\left[\frac{-\frac{1}{2}e^{-\frac{1}{2}}}{W_{-1}\left(-\frac{1}{2}e^{-\frac{1}{2}}\right)}e^{\frac{1}{2}}\right] = -\frac{1}{2} - \ln\frac{-\frac{1}{2}e^{-\frac{1}{2}}}{W_{-1}\left(-\frac{1}{2}e^{-\frac{1}{2}}\right)}$$

By (3.2.10) we have

$$\bar{x}_2 = -\frac{1}{2} - W_{-1}\left(-\frac{1}{2}e^{-\frac{1}{2}}\right).$$

The value $-\frac{1}{2} - W_{-1}\left(-\frac{1}{2}e^{-\frac{1}{2}}\right)$ is called the parameter of Oseen, or Lamb-Oseen constant, denoted by α. Numerical estimates give

$$\alpha = 1.25643...$$

We have introduced the Lambert function W in order to give an useful expression to the root of equation

$$e^{\alpha} = 2\alpha + 1. \tag{3.2.16}$$

In [11] we have proved that the real number α is transcendental, through application of the Lindemann - Weierstrass Theorem [16].

Sinc numerical methods

The sinc function is defined on the whole real line by (3.2.2). For the *step size* $h > 0$, the translated sinc functions with evenly spaced nodes are given as

$$S(n,h)(t) = \text{sinc}\left(\frac{t-nh}{h}\right), \quad n = 0, \pm 1, \pm 2, \dots. \tag{3.2.17}$$

If f is defined on the real line, then for $h > 0$ the series

$$C(f,h)(t) = \sum_{n\in\mathbb{Z}} f(hn)\ \text{sinc}\left(\frac{t-nh}{h}\right) \tag{3.2.18}$$

is called the *Whittaker cardinal expansion* of f whenever this series converges. The properties of (3.2.18) have been extensively studied not only in the engineering literature but also in the field of numerical analysis. In fact, we recall that the term "sinc" was introduced to the world of communication; see [107], [167] as widely discussed in [100] and [185]. The employment of sinc function to numerical methods is showed in several textbooks and seminal papers; see [108], [126], [174] as well as [67], [173], [175] and [176]. Sinc methods are based on the use of the *Cardinal function* $C(f,h)(t)$, defined in (3.2.18) and in the literature of sinc computation is empathized the notation

$$S(n,h)\circ(u) := \text{sinc}\left(\frac{u-nh}{h}\right)$$

introduced in [127]. This type of substitution enables use of $C(f,h)$ to approximate functions over intervals other than the real line $\mathbb{R}$.

In this section we illustrate the process of one-dimensional sinc approximation. For this purpose, consider the case of a finite interval (a,b). Define φ by $w = \varphi(z) = \log[(z-a)/(b-z)]$; this function φ provides a conformal transformation of the "eye-shaped" region $\mathfrak{D} = \{z \in \mathbb{C} : |\arg[(z-a)/(b-z)]| < d\}$ onto the strip $\{z \in \mathbb{C} : |\Im z| < d\}$, $d > 0$.

There are two important spaces of functions, $\mathbb{L}_{\alpha,\beta}(\mathfrak{D})$ and $\mathbb{M}_{\alpha,\beta}(\mathfrak{D})$ associated with sinc approximation on the finite interval (a,b) [176]. For the case of $\mathbb{L}_{\alpha,\beta}(\mathfrak{D})$, we assume that α, β and d are arbitrary fixed positive numbers. The space $\mathbb{L}_{\alpha,\beta}(\mathfrak{D})$ consists of the family of all functions f that are analytic and uniformly bounded in the domain $\mathfrak{D}$ defined above, such that,

$$f(z) = \begin{cases} O(|z-a|^{\alpha}), & \text{uniformly as } z \to a \text{ from within } \mathfrak{D}, \\ O(|z-b|^{\beta}), & \text{uniformly as } z \to b \text{ from within } \mathfrak{D}. \end{cases} \tag{3.2.19}$$

In order to define the second space, $\mathbb{M}_{\alpha,\beta}(\mathfrak{D})$, it is convenient to assume that α, β and d are restricted such that α, $\beta \in (0,1]$, and $d \in (0,\pi)$. Then, $\mathbb{M}_{\alpha,\beta}(\mathfrak{D})$ denotes the family of all functions g that are analytic and uniformly bounded in $\mathfrak{D}$, such that $f \in \mathbb{L}_{\alpha,\beta}(\mathfrak{D})$, where f is defined by

$$f = g - \mathfrak{L}g, \tag{3.2.20}$$

and where

$$\mathfrak{L}g(z) = \frac{(b-z)g(a) + (z-a)g(b)}{b-a}. \tag{3.2.21}$$

Sinc approximation in $\mathbb{M}_{\alpha,\beta}(\mathfrak{D})$ is defined as follows [176]. Let N denote a positive integer, and let integers M, and m, a diagonal matrix $D(u)$ and an operator V_m be defined as follows.

$$M = \left[\frac{\beta N}{\alpha}\right], \quad m = M+N+1, \quad D(u) = \operatorname{diag}[u(z_{-M}), \ldots, u(z_N)], \quad V_m(u) = (u(z_{-M}), \ldots, u(z_N))^T, \tag{3.2.22}$$

where $[\cdot]$ denotes the greatest integer function, where u is an arbitrary function defined on Γ, and where T denotes the transpose. Set [176]

$$\begin{gathered} \left(\frac{\pi d}{\beta N}\right)^{1/2}; \quad z_j = \varphi^{-1}(jh) \;\; j \in \mathbb{Z}; \quad \gamma_j = \operatorname{sinc}\left(\frac{\varphi - jh}{h}\right), \quad j = -M, \ldots, N; \\ \omega_j = \gamma_j, \;\; j = -M+1, \ldots, N-1; \quad \omega_{-M} = \frac{1}{1+\rho} - \sum_{j=-M+1}^{N} \frac{1}{1+e^{jh}}\gamma_j; \\ \omega_N = \frac{\rho}{1+\rho} - \sum_{j=-M}^{N-1} \frac{e^{jh}}{1+e^{jh}}\gamma_j; \quad \varepsilon_N = N^{1/2} e^{-(\pi d \beta N)^{1/2}}; \quad \boldsymbol{\omega}_m = (\omega_{-M}, \ldots, \omega_N). \end{gathered} \tag{3.2.23}$$

For given vector $\mathbf{c} = (c_{-M}, \ldots, c_N)^T$, set

$$\boldsymbol{\omega}_m \mathbf{c} = \sum_{j=-M}^{N} c_j\, \omega_j. \tag{3.2.24}$$

This operation $\boldsymbol{\omega}_m \mathbf{c}$ can thus be interpreted as vector dot product multiplication. We shall also define a norm by

$$\|f\| = \sup_{x \in \Gamma} |f(x)|, \tag{3.2.25}$$

and throughout this section C will denote a generic constant, independent of N.

A proof of the following result may be found in [176].

Theorem 3.2.9. *If* $f \in \mathbb{M}_{\alpha,\beta}(\mathfrak{D})$, *then*

$$\|f - \boldsymbol{\omega}_m V_m f\| \le C\varepsilon_N.$$

The constants in the exponent in the definition of ε_N are the best constants for approximation in $\mathbb{M}_{\alpha,\beta}(\mathfrak{D})$ (for details see [176]).

3.2.2 Proof of the main results

In the following we give the results of the paper.

3.2.3 Proof of Proposition 3.2.1

Proof of Proposition 3.2.1. Write

$$\lambda := \left\| \sum_n c_n \left(\operatorname{sinc}(n-\xi) - \operatorname{sinc}(\lambda_n - \xi)\right) \right\|^2_{L^2(\mathbb{R})}.$$

The Fourier transform of the function $t \to e^{it\mu}\chi_{[-\pi,\pi]}(t)$ is $\xi \to 2\pi \operatorname{sinc}(\mu - \xi)$. In fact:

$$\mathcal{F}\left(e^{it\mu}\chi_{[-\pi,\pi]}(t)\right)(\xi) = \int_{-\pi}^{\pi} e^{i(\mu-\xi)t} dt = 2\pi \operatorname{sinc}(\mu - \xi).$$

By Plancherel's theorem

$$\begin{aligned} \left\| \sum_n c_n \left(\operatorname{sinc}(n-\xi) - \operatorname{sinc}(\lambda_n - \xi)\right) \right\|^2_{L^2(\mathbb{R})} &= \left\| \sum_n c_n \chi_{[-\pi,\pi]}(t) \left(e^{int} - e^{i\lambda_n t}\right) \right\|^2_{L^2(\mathbb{R})} \\ &= \left\| \sum_n c_n \left(e^{int} - e^{i\lambda_n t}\right) \right\|^2_{L^2(-\pi,\pi)} \end{aligned}$$

and so, following the proof of Kadec's theorem (see e.g. [199]), when $L < \frac{1}{4}$ then $\lambda \le 1 - \cos(\pi L) - \sin(\pi L) < 1$. Since $\{\operatorname{sinc}(n-\xi)\}$ is a Riesz basis of PW_π, the Paley-Wiener criterion shows that also $\{\operatorname{sinc}(\lambda_n - \xi)\}$ is a Riesz basis of PW_π.

Constant 1/4 is optimal also for $\{\operatorname{sinc}(\lambda_n - \xi)\}$. A counterexample due to Ingham [99] prove that the set $\{e^{i\lambda_n t}\}$ is not a Riesz basis of $L^2(-\pi,\pi)$ when

$$\lambda_n = \begin{cases} n + \frac{1}{4}, & n > 0 \\ 0, & n = 0 \\ n - \frac{1}{4}, & n < 0 \end{cases} \tag{3.2.26}$$

Since PW_π is isometrically equivalent to $L^2(-\pi,\pi)$ via Fourier transform, the set $\{\operatorname{sinc}(\lambda_n - \xi)\}$ is not a Riesz basis. $\square$

Corollary 3.2.10. *Let $\{x'_n\}_{n\in\mathbb{Z}}$ be a system biorthogonal to $\{\mathrm{sinc}(\cdot - \lambda_n)\}_{n\in\mathbb{Z}}$. Let $\{\lambda_n\}_{n\in\mathbb{Z}}$ be a sequence of real numbers for which*

$$|\lambda_n - n| \leqq L < \infty, \quad n = 0, \pm 1, \pm 2, ... \tag{3.2.27}$$

If $L < \frac{1}{4}$, and if $f \in PW_\pi$ is continuous, then

$$f(t) = \sum_{n\in\mathbb{Z}} \langle f, x'_n \rangle_{PW_\pi} \, \mathrm{sinc}(t - \lambda_n). \tag{3.2.28}$$

Proof. Let $\{\lambda_n\}_{n\in\mathbb{Z}}$ be a sequence of real numbers for which

$$|\lambda_n - n| \leqq L < \frac{1}{4}, \quad n = 0, \pm 1, \pm 2, ... \tag{3.2.29}$$

We denote with the sequence $\{x_n\}_{n\in\mathbb{Z}}$ the system $\mathrm{sinc}(t - \lambda_n)$. From Theorem 3.2.1, the sequence $\{x_n\}_{n\in\mathbb{Z}}$ forms a Riesz basis.

Recall that a sequence $\{x_n\}_{n\in\mathbb{Z}}$ in an Hilbert space H is a Riesz basis if and only if any element $x \in H$ has a unique expansion $x = \sum_{n\in\mathbb{Z}} c_n x_n$ with $\{c_n\}_{n\in\mathbb{Z}} \in \ell^2$. If $\{x_n\}_{n\in\mathbb{Z}}$ is a Riesz basis, then in the above expansion the Fourier coefficients c_n are given by $c_n = \langle x, x'_n \rangle$, where $\{x'_n\}_{n\in\mathbb{Z}}$ is a system biorthogonal to $\{x_n\}_{n\in\mathbb{Z}}$, i.e., a system which satisfies the condition $\langle x_k, x'_n \rangle = \delta_{k,n}$ for all k, $n \in \mathbb{Z}$. □

3.2.4 Proof of Theorem 3.2.2

In order to prove Theorem 3.2.2, we prove the following Lemma.

Lemma 3.2.11. *Define*

$$\mathcal{I} = \left\| \sum_n c_n \left[\mathrm{sinc}(\lambda_n - t) - \mathrm{sinc}(n - t) \right] \right\|^2_{L^2(\mathbb{R})}.$$

Then

$$\mathcal{I} \leq 2 \sum_n \left[1 - \mathrm{sinc}(\lambda_n - n) \right], \quad n = 0, \pm 1, \pm 2, \dots. \tag{3.2.30}$$

Proof. Write,

$$\mathcal{I} = \left\| \sum_n c_n \left[\mathrm{sinc}(\lambda_n - t) - \mathrm{sinc}(n - t) \right] \right\|^2_{L^2(\mathbb{R})}. \tag{3.2.31}$$

First, we develop function $\mathrm{sinc}(\lambda_n - t)$ respect to basis $\{\mathrm{sinc}(\lambda_n - t)\}_{n\in\mathbb{Z}}$. We find:

$$\mathrm{sinc}(\lambda_n - t) = \sum_{k\in\mathbb{Z}} \mathrm{sinc}(\lambda_n - k) \, \mathrm{sinc}(k - t) \tag{3.2.32}$$

The convergence in $L^2(\mathbb{R})$ is insured by

$$\sum_{k\in\mathbb{Z}} \mathrm{sinc}^2(\lambda_n - k) = \int_{\mathbb{R}} \mathrm{sinc}^2(\lambda_n - t) dt = 1 \tag{3.2.33}$$

Thanks to equation (3.2.32) we obtain:

$$\sum_n c_n \left[\operatorname{sinc}(\lambda_n - t) - \operatorname{sinc}(n-t)\right] = \sum_n c_n \sum_{k\in\mathbb{Z}} \left[\operatorname{sinc}(\lambda_n - k) - \operatorname{sinc}(n-k)\right] \operatorname{sinc}(k-t). \quad (3.2.34)$$

This transformation is obvious because

$$\operatorname{sinc}(n-k) = \begin{cases} 0 & \text{for } n,k \in \mathbb{Z} \text{ and } n \neq k \\ 1 & \text{for } n = k. \end{cases}$$

We obtain, substituting in (3.2.31):

$$\begin{aligned} \mathcal{I} &= \left\| \sum_n c_n \sum_{k\in\mathbb{Z}} \left[\operatorname{sinc}(\lambda_n - k) - \operatorname{sinc}(n-k)\right] \operatorname{sinc}(k-t) \right\|^2_{L^2(\mathbb{R})} \\ &= \left\| \sum_{k\in\mathbb{Z}} \left\{ \sum_n c_n \left[\operatorname{sinc}(\lambda_n - k) - \operatorname{sinc}(n-k)\right] \right\} \operatorname{sinc}(k-t) \right\|^2_{L^2(\mathbb{R})} \end{aligned} \quad (3.2.35)$$

Applying the Parseval equality,

$$\mathcal{I} = \sum_{k\in\mathbb{Z}} \left| \sum_n c_n \left[\operatorname{sinc}(\lambda_n - k) - \operatorname{sinc}(n-k)\right] \right|^2$$

Using Hölder-Schwarz to the sum of products contained in the absolute value, and the condition on $\sum_n |c_n|^2 \leq 1$ we have:

$$\begin{aligned} \mathcal{I} &\leq \sum_{k\in\mathbb{Z}} \sum_n \left[\operatorname{sinc}(\lambda_n - k) - \operatorname{sinc}(n-k)\right]^2 = \\ &= \sum_n \sum_{k\in\mathbb{Z}\setminus\{n\}} \left[\operatorname{sinc}^2(\lambda_n - k) + (\operatorname{sinc}(\lambda_n - n) - 1)^2\right] \end{aligned} \quad (3.2.36)$$

From (3.2.33),

$$\sum_{k\in\mathbb{Z}\setminus\{n\}} \operatorname{sinc}^2(\lambda_n - k) = 1 - \operatorname{sinc}^2(\lambda_n - n) \quad (3.2.37)$$

Finally, from (3.2.36) and (3.2.37), we obtain

$$\mathcal{I} \leq 2 \sum_n \left[1 - \operatorname{sinc}(\lambda_n - n)\right]. \quad (3.2.38)$$

□

Proof of Theorem 3.2.2. Let us consider

$$\mathcal{I} \leq 2 \sum_n \left[1 - \operatorname{sinc}(\lambda_n - n)\right]$$

for $\lambda_n - n = \frac{A}{|n|^\alpha}$:

$$\begin{aligned} \mathcal{I} &\leq 2 \sum_{n\neq 0} \left[1 - \operatorname{sinc}\left(\frac{A}{|n|^\alpha}\right)\right] = \\ &= 2 \sum_{n<0} \left[1 - \operatorname{sinc}\left(\frac{A}{|n|^\alpha}\right)\right] + 2 \sum_{n>0} \left[1 - \operatorname{sinc}\left(\frac{A}{|n|^\alpha}\right)\right] \end{aligned}$$

$$= 4 \sum_{n \in \mathbb{N}} \left[1 - \operatorname{sinc}\left(\frac{A}{|n|^\alpha} \right) \right].$$

Since, for $x \in (-\pi/2, \pi/2)$

$$\left| 1 - \frac{\sin x}{x} \right| = \left| \frac{\sin x}{x} \right| \left| \frac{x}{\sin x} - 1 \right| \leq \frac{1}{\cos x} - 1 = \frac{\sin^2 x}{\cos x(1 + \cos x)} \leq \frac{\sin^2 x}{\cos x},$$

we have:

$$\mathcal{I} \leq 4 \sum_{n \in \mathbb{N}} \left[1 - \operatorname{sinc}\left(\frac{A}{|n|^\alpha} \right) \right] \leq 4 \sum_{n \in \mathbb{N}} \frac{\sin^2\left(\frac{\pi A}{|n|^\alpha}\right)}{\cos\left(\frac{\pi A}{|n|^\alpha}\right)}.$$

It is verified if $\frac{\pi A}{|n|^\alpha} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Let, for example, $\frac{\pi A}{|n|^\alpha} \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$. Hence,

$$\mathcal{I} \leq 4 \sum_{n \in \mathbb{N}} \frac{\sin^2\left(\frac{\pi A}{|n|^\alpha}\right)}{\cos\left(\frac{\pi A}{|n|^\alpha}\right)} \leq 4\sqrt{2}(\pi A)^2 \sum_{n \in \mathbb{N}} \frac{1}{n^{2\alpha}} = 4\sqrt{2}(\pi A)^2 \zeta(2\alpha).$$

Then

$$\mathcal{I} \leq 4\sqrt{2}(\pi A)^2 \zeta(2\alpha) < 1$$

if

$$|A| < \frac{1}{2\pi \sqrt[4]{2} \sqrt{\zeta(2\alpha)}}.$$

Moreover, $\zeta(2\alpha) < 1$ and, under condition on A and for $\alpha > 1/2$, it is confirmed that $\frac{\pi A}{|n|^\alpha} \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$. □

3.2.5 Proof of Theorem 3.2.3.

In this Section we study the system $\{\operatorname{sinc}(\lambda_n - t)\}_{n \in \mathbb{Z}}$ for $\lambda_n \in \mathbb{C}$ and $|\lambda_n - n| \leq L < \infty$. First, we state the following Lemma whose proof is left to the reader.

Lemma 3.2.12. *Let n, $k \in \mathbb{N}$. The Fourier transform of the function $t \to \frac{d^k}{dt^k} \operatorname{sinc}(t - n)$ is $\xi \to (i\xi)^k \chi_{[-\pi,\pi]}(t)\, e^{-in\xi}$.*

The following is a stability result for $\{\operatorname{sinc}(\lambda_n - t)\}_{n \in \mathbb{Z}}$ when $\lambda_n \in \mathbb{C}$. The result involves the Lamb-Oseen constant, as already announced in [11]. However the stability bound for the case $\lambda_n \in \mathbb{C}$ is $\frac{1}{\pi}\sqrt{\frac{3\alpha}{8}}$ (and not $\frac{\alpha}{\pi}$, as written in [11]).

Proof of Theorem 3.2.3. Let us consider,

$$\mathcal{I} = \left\| \sum_n c_n \left[\operatorname{sinc}(\lambda_n - t) - \operatorname{sinc}(n - t) \right] \right\|^2_{L^2(\mathbb{R})}. \tag{3.2.39}$$

whenever $\sum_n |c_n|^2 \leqq 1$. We use the Taylor series of $\operatorname{sinc}(\lambda_n - t)$:

$$\operatorname{sinc}(n - t) + \sum_{k=1}^{+\infty} \frac{(\lambda_n - n)^k}{k!} \frac{d^k}{dx^k} \operatorname{sinc}(x - t)\Big|_{x=n}$$

Then

$$
\begin{aligned}
\mathcal{I} &= \left\| \sum_n c_n \sum_{k=1}^{+\infty} \frac{(\lambda_n - n)^k}{k!} \frac{d^k}{dx^k} \operatorname{sinc}(x-t)\Big|_{x=n} \right\|^2_{L^2(\mathbb{R})} \\
&= \left\| \sum_{k=1}^{+\infty} \frac{1}{k!} \sum_n c_n \, (\lambda_n - n)^k \, \frac{d^k}{dx^k} \operatorname{sinc}(x-t)\Big|_{x=n} \right\|^2_{L^2(\mathbb{R})} \\
&\leq \sum_{k=1}^{+\infty} \frac{1}{k!} \left\| \sum_n c_n \, (\lambda_n - n)^k \, \frac{d^k}{dx^k} \operatorname{sinc}(x-t)\Big|_{x=n} \right\|^2_{L^2(\mathbb{R})}
\end{aligned} \tag{3.2.40}
$$

The term $\|\cdot\|$ is reducible to

$$
\left\| \sum_n c_n \, (\lambda_n - n)^k \, \frac{d^k}{dx^k} \operatorname{sinc}(x-t)\Big|_{x=n} \right\|^2_{L^2(\mathbb{R})} =
$$

$$
= \sum_{n,m} a_n \overline{a_m} \, \left\langle \frac{d^k}{dx^k} \operatorname{sinc}(x-t)\Big|_{x=n}, \frac{d^k}{dx^k} \operatorname{sinc}(x-t)\Big|_{x=m} \right\rangle_{L^2(\mathbb{R})}
$$

where $a_n := c_n \, (\lambda_n - n)^k$. Observing that

$$
\frac{d^k}{dx^k} \operatorname{sinc}(x-t)\Big|_{x=n} = \begin{cases} -\frac{d^k}{dt^k} \operatorname{sinc}(t-n), & k \text{ odd} \\ \frac{d^k}{dt^k} \operatorname{sinc}(t-n), & k \text{ even} \end{cases}
$$

i.e., $\frac{d^k}{dx^k} \operatorname{sinc}(x-t)\Big|_{x=n} = (-1)^k \frac{d^k}{dt^k} \operatorname{sinc}(t-n)$. Then

$$
\begin{aligned}
&\left\langle \frac{d^k}{dx^k} \operatorname{sinc}(x-t)\Big|_{x=n}, \frac{d^k}{dx^k} \operatorname{sinc}(x-t)\Big|_{x=m} \right\rangle_{L^2(\mathbb{R})} \\
&= \int_{\mathbb{R}} (-1)^k \frac{d^k}{dt^k} \operatorname{sinc}(t-n) \, \overline{(-1)^k \frac{d^k}{dt^k} \operatorname{sinc}(t-m)} \, dt
\end{aligned} \tag{3.2.41}
$$

From Plancherel's equality and Lemma 3.2.12, we have

$$
\int_{\mathbb{R}} (-1)^k \frac{d^k}{dt^k} \operatorname{sinc}(t-n) \, \overline{(-1)^k \frac{d^k}{dt^k} \operatorname{sinc}(t-m)} \, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \xi^{2k} \, e^{i(m-n)\xi} \, d\xi.
$$

Hence,

$$
\left\langle \frac{d^k}{dx^k} \operatorname{sinc}(x-t)\Big|_{x=n}, \frac{d^k}{dx^k} \operatorname{sinc}(x-t)\Big|_{x=m} \right\rangle_{L^2(\mathbb{R})} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \xi^{2k} \, e^{i(m-n)\xi} \, d\xi. \tag{3.2.42}
$$

Taking equation (3.2.42) in (3.2.40), we obtain:

$$
\mathcal{I} \leq \frac{1}{2\pi} \sum_{k=1}^{+\infty} \frac{1}{k!} \sum_{n,m} a_n \overline{a_m} \, \int_{-\pi}^{\pi} \xi^{2k} \, e^{i(m-n)\xi} \, d\xi := \omega_1 + \omega_2
$$

where ω_1, ω_2 are the cases, respectively, when $n = m$ and $n \neq m$. Thereby,

$$
\omega_1 = \frac{1}{\pi} \sum_{k=1}^{+\infty} \frac{1}{k!} \sum_n |a_n|^2 \, \int_0^{\pi} \xi^{2k} \, d\xi \leq \sum_{k=1}^{+\infty} \frac{(\pi L)^{2k}}{k!(2k+1)}. \tag{3.2.43}
$$

and using integration by parts for ω_2, we see that

$$\begin{aligned}
\omega_2 &= \frac{1}{2\pi}\sum_{k=1}^{+\infty}\frac{1}{k!}\sum_{\substack{n,m\\ n\neq m}}\frac{a_n\,\overline{a_m}}{i(m-n)}\int_{-\pi}^{\pi}\xi^{2k}\,\left[e^{i(m-n)\xi}\right]'\,d\xi\\
&= \frac{1}{2\pi}\sum_{k=1}^{+\infty}\frac{1}{k!}\sum_{\substack{n,m\\ n\neq m}}\frac{a_n\,\overline{a_m}}{i(m-n)}\left[\left(\xi^{2k}\,e^{i(m-n)\xi}\right)_{-\pi}^{\pi}-2k\int_{-\pi}^{\pi}\xi^{2k-1}\,e^{i(m-n)\xi}d\xi\right]\\
&= \frac{i}{\pi}\sum_{k=1}^{+\infty}\frac{1}{(k-1)!}\sum_{\substack{n,m\\ n\neq m}}\frac{a_n\,\overline{a_m}}{m-n}\int_{-\pi}^{\pi}\xi^{2k-1}\,e^{i(m-n)\xi}d\xi. \qquad (3.2.44)
\end{aligned}$$

Putting double series into integral,

$$\omega_2 = \frac{i}{\pi}\sum_{k=1}^{+\infty}\frac{1}{(k-1)!}\int_{-\pi}^{\pi}\xi^{2k-1}\sum_{\substack{n,m\\ n\neq m}}\frac{a_n e^{-in\xi}\,\overline{a_m e^{-im\xi}}}{m-n}d\xi$$

and denoting $b_n := a_n e^{-in\xi}$, ω_2 is estimable from above as:

$$|\omega_2| \leq \frac{1}{\pi}\sum_{k=1}^{+\infty}\frac{1}{(k-1)!}\int_{-\pi}^{\pi}|\xi|^{2k-1}\left|\sum_{\substack{n,m\\ n\neq m}}\frac{b_n\overline{b_m}}{m-n}\right|d\xi.$$

From Hilbert's inequality for the double series into the integral, we obtain

$$|\omega_2| \leq \sum_{k=1}^{+\infty}\frac{1}{(k-1)!}\int_{-\pi}^{\pi}|\xi|^{2k-1}\sum_{n}|b_n|^2\,d\xi \leq \sum_{k=1}^{+\infty}\frac{(\pi L)^{2k}}{k!}. \qquad (3.2.45)$$

Then,

$$\mathcal{I} \leq |\omega_1| + |\omega_2| \leq \sum_{k=1}^{+\infty}\frac{(\pi L)^{2k}}{k!(2k+1)} + \sum_{k=1}^{+\infty}\frac{(\pi L)^{2k}}{k!} = \sum_{k=1}^{+\infty}\frac{(\pi L)^{2k}}{k!}\left[1+\frac{1}{2k+1}\right].$$

We notice that

$$\frac{(\pi L)^{2k}}{k!}\left[1+\frac{1}{2k+1}\right] \leq \frac{(\pi L)^{2k}}{(k+1)!}\left(\frac{8}{3}\right)^k$$

for all $k \in \mathbb{N}$ and, in fact,

$$2\,\frac{k+1}{2k+1} \leq \frac{1}{k+1}\left(\frac{8}{3}\right)^k$$

is verified for all $k \in \mathbb{N}$. Only for $k = 1$ the above inequality becomes an equality. Accordingly,

$$\mathcal{I} \leq \sum_{k=1}^{+\infty}\frac{x^k}{(k+1)!} = \frac{1}{x}\left(e^x - x - 1\right), \quad \text{where } x = \frac{8}{3}\,\pi^2 L^2.$$

Set $\lambda = \frac{1}{x}\left(e^x - x - 1\right)$ where $x = \frac{8}{3}\,\pi^2 L^2$. In order to get $\lambda < 1$, we solve, in a first moment, the equation $\lambda = 1$, that is

$$e^x = 2x + 1\,. \qquad (3.2.46)$$

From the considerations done in Section 3.2.1 for equation (3.2.16) we obtain the thesis. $\square$

Numerical estimates give

$$\frac{1}{\pi}\sqrt{\frac{3\alpha}{8}} = 0.218492\ldots.$$

3.2.6 Tables

From proof of Theorem 3.2.2 one has

$$\mathcal{I} \leq 4 \sum_n \sum_{l=1}^{\infty} (-1)^{l+1} \frac{(\pi A)^{2l}}{(2l+1)!} \frac{1}{n^{2l\alpha}} = 4 \sum_{l=1}^{\infty} (-1)^{l+1} \frac{(\pi A)^{2l}}{(2l+1)!} \zeta(2\alpha l)$$

where $\zeta(2l\alpha)$ is the Riemann zeta function. We have the estimate

$$\mathcal{I} \leq 4 \sum_{l=1}^{\infty} (-1)^{l+1} \frac{(\pi A)^{2l}}{(2l+1)!} + 4 \sum_{l=1}^{\infty} (-1)^{l+1} \frac{(\pi A)^{2l}}{(2l+1)!} \left[\zeta(2\alpha l) - 1\right]$$

$$= 4\,(1 - \operatorname{sinc} A) + 4 \sum_{l=1}^{\infty} (-1)^{l+1} \frac{(\pi A)^{2l}}{(2l+1)!} \left[\zeta(2\alpha l) - 1\right]$$

In the following we evaluate numerically the expression

$$\lambda := 4 \left(1 - \frac{\sin \pi A}{\pi A}\right) + 4 \sum_{l=1}^{\infty} (-1)^{l+1} \frac{(\pi A)^{2l}}{(2l+1)!} \left[\zeta(2l\alpha) - 1\right].$$

Parameters α and A derive from position $\lambda_n - n = \frac{A}{|n|^\alpha}$, for $\alpha > \frac{1}{2}$. From Paley-Wiener λ must be less than 1. For a better comprehension we fix:

$$\lambda_1 = 4 \left(1 - \frac{\sin \pi A}{\pi A}\right), \quad \lambda_2 = 4 \sum_{l=1}^{\infty} (-1)^{l+1} \frac{(\pi A)^{2l}}{(2l+1)!} \left[\zeta(2l\alpha) - 1\right]$$

Below we try with $A = 0.25$ and varying α.

α	A	λ_1	λ_2	λ
0.9	0.25	0.398735	0.361336	0.760071
0.85	0.25	0.398735	0.431806	0.830541
0.8	0.25	0.398735	0.526643	0.925377
0.78	0.25	0.398735	0.574332	0.973067
0.77	0.25	0.398735	0.600889	0.999623

Notice that for $\lambda_n - n = \frac{0.25}{|n|^\alpha}$ (0.25 is just the Kadec's bound for exponential bases), when we have, for example, $\alpha = 0.9$ (first row of previous table) the parameter λ is still far from 1, which is the maximum value for λ in the Paley-Wiener criterion. For decreasing value of α, when $\alpha = 0.77$, λ is very close to 1. If $n = \pm 1, \pm 2, \dots,$ $\lambda_n - n \leq 0.25$.

We now fix $\alpha = 2$ while A is variable.

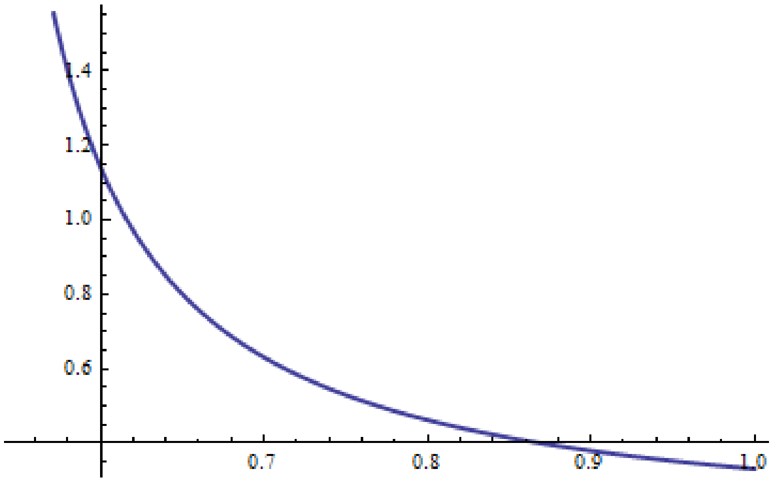


Figure 3.6: Plot of $4\left(1-\frac{\sin\pi A}{\pi A}\right)+4\sum_{l=1}^{\infty}(-1)^{l+1}\frac{(\pi A)^{2l}}{(2l+1)!}[\zeta(2l\alpha)-1]$. Here $A=0.25$, horizontal axis is referred to α and the graphics is obtained in the range $\alpha\in[0.7,1]$.

α	A	λ_1	λ_2	λ
2	0.25	0.398735	0.0338024	0.432537
2	0.3	0.566425	0.0486428	0.615068
2	0.35	0.758672	0.0661557	0.824828
2	0.36	0.799829	0.0699778	0.869807
2	0.38	0.884663	0.077941	0.962604
2	0.3868	0.914244	0.0807451	0.994989

At this point, we have $\lambda_n - n = \frac{A}{|n|^2}$. When $A = 0.25$ (first row of previous table) the parameter λ is still far from value 1 of λ in the Paley-Wiener criterion. For increasing value of A, when $A \simeq 0.3868$, λ is very close to 1. If $n = \pm1, \pm2, \ldots,$ $\lambda_n - n \leq L$ where L seems to be approximately $0.3868\ldots$, which is greater of Kadec's bound. We have completed here the study announced in [11], giving a whole proof for stability of sinc bases.

3.3 An application of Sinc Bases: the Ideal DAC.

Let $f(t)$ be a signal; we refer to the following definition of *energy*.

Definition 3.3.1. The energy in the signal $f(t)$ is

$$E_f := \int_{-\infty}^{\infty} |f(t)|^2 dt.$$

The results obtained here concern a generalization of the Parseval's identity for the sequence of functions $\{\text{sinc}(t-\lambda_n)\}_{n\in\mathbb{Z}}$, where $\lambda_n \in \mathbb{R}$. In fact, it is well-known that, for a function such that

$$f(t) = \sum_{n\in\mathbb{Z}} a_n \,\text{sinc}(t-n),$$

its energy is:

$$E_f = \sum_{n\in\mathbb{Z}} |a_n|^2.$$

This is a Parseval identity for the sequence of functions $\{\mathrm{sinc}(t-n)\}_{n\in\mathbb{Z}}$, and it is based on the identity

$$\int_{\mathbb{R}} \mathrm{sinc}\big(\tau-\lambda\big)\mathrm{sinc}\big(\tau-\nu\big)d\tau = \mathrm{sinc}(\lambda-\nu). \tag{3.3.1}$$

occurred for any real numbers λ and ν. But Parseval identity ceases to be true if n is substitutes with $\lambda_n \in \mathbb{R}$. This motivates the result of the Section, which is described in the next Theorem and also reported in [122].

Theorem 3.3.2. *Let $I = \{n\,|1 \le n \le R,\ R \in \mathbb{N}\}$ be a finite set of integers, and let*

$$f(t) = \sum_{n\in I} a_n\,\mathrm{sinc}(t-\lambda_n), \tag{3.3.2}$$

where the λ_n are real and satisfy

$$|\lambda_n - \lambda_m| \ge \gamma > \sqrt{\frac{1}{3}+\frac{\pi^2}{12}}, \quad \forall n, m \in I.$$

Then

$$E_f \asymp \sum_{n\in I} |a_n|^2. \tag{3.3.3}$$

Remark 3.3.3. Write $E_f \asymp \sum_{n\in I} |a_n|^2$ means that

$$c_1 \sum_{n\in I} |a_n|^2 \le E_f \le c_2 \sum_{n\in I} |a_n|^2$$

with two constants $c_1, c_2 > 0$, independent of the particular form of $f(t)$, except for the assumption $|\lambda_n - \lambda_m| \ge \gamma > \sqrt{\frac{1}{3}+\frac{\pi^2}{12}}$, $\forall n, m \in I$.

This result applies to the so-called *ideal bandlimited interpolation*

$$f(t) = \sum_{n\in\mathbb{Z}} a_n\,\mathrm{sinc}(t-n). \tag{3.3.4}$$

It provides a perfect reconstruction for all t, if $f(t)$ is bandlimited in f_m and if the sampling frequency f_s is such that $f_s \ge 2f_m$. The system used to implement (3.3.4), which is known as an *ideal DAC* (i.e. digital-to-analog converter, see [129]), is depicted in block diagram form in figure 3.7.

DACs are essential components for measuring instruments (such as arbitrary waveform signal generators) and communication systems (such as transceivers). *Sampling clock jitter* is the deviation of a signal's timing event from its intended (ideal) occurrence in time, often in relation to a reference clock source. Thus, time jitter is an important parameter for determining the performance of digital systems. For a review how time jitter impacts the performance of digital systems, see [157]. For digital sampling in analog-to-digital and digital-to-analog converters, it is shown that noise power or multiplicative decorrelation noise generated by sampling clock jitter is a major limitation on the bit resolution (effective number of bits) of these devices, [157].

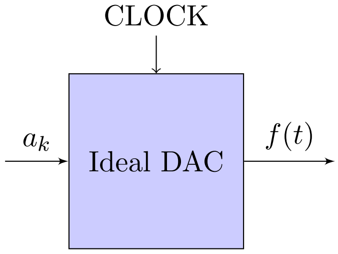


Figure 3.7: Representation of the ideal digital-to-analog converter (DAC) or ideal bandlimited interpolator. According to 3.3.4.

As it has been well argued in previous works ([9], [110]), theory dealing with major aspects concerning DAC time base jitter, quantization noise, and nonlinearity is still incomplete; unexpected changes and distortions of waveforms generated via DAC are occasionally supported by simulations and barely investigated by means of experimental activities, [9] and references therein. See also: [54], where stochastic analysis is presented in order to predict the average switching rate; [168], where time jittering is modeled as a random variable uniformly distributed; [6], [183], where jitter effect is assumed as a random variable normally distributed.

In [110] authors analyze the clock jitter effects on DACs, (Fig. 1 therein), considering a DAC where a digital input is applied with a sampling clock CLK. Ideally the sampling clock CLK operates with a sampling period of T_s for every cycle, however in reality its timing can fluctuate (see Fig. 2 in [110]). Phase and frequency fluctuations have therefore been the subject of numerous studies; well-known references include: [1], [61], [85], [156]. In [9], authors focus on zero-order-hold DACs and, in particular, on how the presence of jitter that can affect their time base modifies the desired features of the analog output waveform. They study more deterministic jitter and develop an analytical model which is capable of describing the spectral content of the analog signal at the output of a DAC, the time base of which suffers from (or is modulated by) sinusoidal jitter. See also: [56], where is introduced a model capable of describing the functioning of a real DAC affected by horizontal quantization, clock modulation, vertical quantization and integral nonlinearity.

Theorem 3.3.2 gives one-sided energy inequality for the output signal of an ideal DAC, in presence of *sampling clock jitter*. Although the energy inequality can be derived for the Fourier transform by the system of complex exponentials [99], here we present a direct proof, based on sinc functions and on the result showed in [132]. We denote jitter as ϵ_n, then the n-th sampling timing of CLK is $nT_s + \epsilon_n$ instead of nT_s. Since we have assumed that $T_s = 1$, in the Section sampling timing of CLK is $n + \epsilon_n$ but the results for $T_s \neq 1$ one can obtain in an obvious way. Hence, equation (3.3.4) becomes

$$f(t) = \sum_{n\in\mathbb{Z}} a_n \operatorname{sinc}(t - \lambda_n)\,, \tag{3.3.5}$$

where $\lambda_n = n + \epsilon_n$.

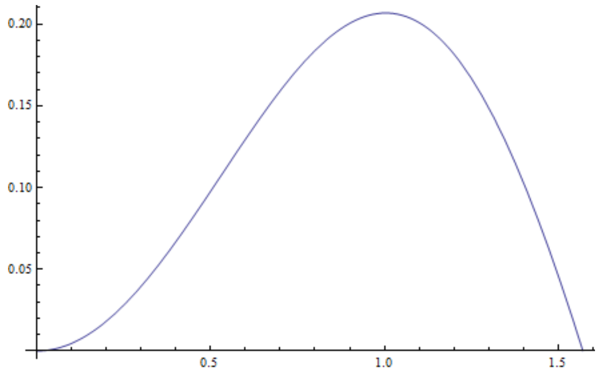


Figure 3.8: Graphic of $g(\theta) = \frac{\pi^2}{4}\sin^2\theta - \theta^2(1+\cos\theta)$ for $\theta \in [0, \pi/2]$.

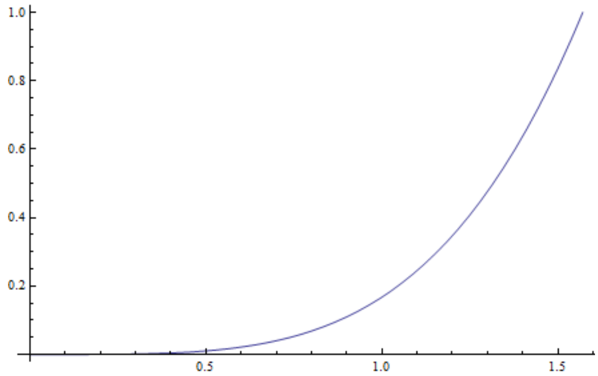


Figure 3.9: Graphic of $\sin^2\theta - \theta^2\cos\theta$ for $\theta \in [0, \pi/2]$.

3.3.1 Proof of the Result

For the our purposes, we will use a well-known inequality. *Hilbert's inequality* states that

$$\left|\sum_{n\neq m} \frac{a_n \bar{a}_m}{n-m}\right| \leq \pi \sum_n |a_n|^2$$

for any set of complex a_n, where the best possible constant π was found by Schur [162]. In [132] authors obtained a precise bound for the more general bilinear forms:

$$\sum_{n\neq m} a_n \overline{a}_m \csc \pi(x_r - x_s), \qquad \sum_{n\neq m} \frac{a_n \overline{a}_m}{\lambda_r - \lambda_s}.$$

In the following, $\|\theta\|$ denotes the distance from θ to the nearest integer, that is, $\|\theta\| = \min_n |\theta - n|$. Moreover, $\min_+ f$ will denotes the least positive value when f ranges over a finite set of non-negative values. We now give an useful Lemma.

Lemma 3.3.4. *The inequalities*

$$\csc^2 \pi x + |\cot \pi x \csc \pi x| \leq \frac{1}{4}\|x\|^{-2} \tag{3.3.6}$$

and

$$|\cot \pi x \csc \pi x| \leq \pi^{-2}\|x\|^{-2} \tag{3.3.7}$$

hold for all real x.

Proof. Let $\theta = \pi x$. We notice that, for an integer n, $0 \le \|x\| = \min_n |x - n| \le \frac{1}{2}$ and so $0 \le \theta \le \pi/2$. For inequality (3.3.6), it is sufficient to show that $g(\theta) \ge 0$ in $[0, \pi/2]$, where

$$g(\theta) = \frac{\pi^2}{4}\sin^2\theta - \theta^2(1 + \cos\theta).$$

For inequality (3.3.7) one shows that:

$$\sin^2\theta - \theta^2\cos\theta \ge 0$$

for $\theta \in [0, \pi/2]$. See Figures 3.8 and 3.9. □

Now we readapt and prove a part of Theorem 1, taken from [132].

Lemma 3.3.5. *Let $x_1, x_2, ..., x_R$ and $y_1, y_2, ..., y_R$ denote real numbers which are distinct modulo 1, and suppose that*

$$\delta = \min_{n,m}{}^{+}\|x_n - y_m\|, \quad x_n \neq y_m \ \forall n, m = 1, \ldots, R.$$

Then

$$\left|\sum_{n,m} a_n \overline{a}_m \csc\pi(x_n - y_m)\right| \le \delta^{-1}\sqrt{\frac{1}{3} + \frac{\pi^2}{12}}\sum_{n=1}^{R}|a_n|^2. \tag{3.3.8}$$

where n and m are distinct.

Proof. Our proof is modelled on Montgomery and Vaughan's proof [132] of Hilbert's inequality. In [132] authors proven that the bilinear form

$$\sum_{n,m} a_n \overline{a}_m \csc\pi(x_n - x_m),$$

where $n \neq m$, is skew-Hermitian. For this proof we consider the bilinear form:

$$\sum_{n,m} a_n \overline{a}_m \csc\pi(y_n - y_m)$$

for $n \neq m$. Let us consider

$$\sum_{n} a_n \ \csc\pi(y_n - y_m) = \sum_{n} a_n \, c_{n,m}$$

where $c_{n,m} = \csc\pi(y_n - y_m)$. The RHS is the product of eigenvector $\mathbf{a} = (a_1, \ldots, a_R)^t$ for the mth column of matrix $C := (c_{n,m})$. Since the bilinear form under consideration is skew-Hermitian, eigenvalues of matrix C are all purely imaginary or zero, namely there exists a real number μ such that: $\mathbf{a}^t C \mathbf{a} = i\mu$. Hence,

$$\sum_{n} a_n \ \csc\pi(y_n - y_m) = i\mu a_m \tag{3.3.9}$$

for $m \neq n$ and $1 \le n, m \le R$. Also, we may normalize so that $\sum_n |a_n|^2 = 1$. By Cauchy's inequality,

$$\left|\sum_{n,m} a_n \overline{a}_m \csc\pi(x_n - y_m)\right|^2 \le \sum_{n}\left|\sum_{m}{}' \overline{a}_m \csc\pi(x_n - y_m)\right|^2$$

where $\sum_m{}'$ means that all indexes are different. Also,

$$\sum_n \left| \sum_m{}' \bar{a}_m \csc \pi(x_n - y_m) \right|^2 =$$

$$= \sum_{m,p} \bar{a}_m a_p \sum_n{}' \csc \pi(x_n - y_m) \csc \pi(x_n - y_p)$$

$$= S_1 + S_2, \tag{3.3.10}$$

where

$$S_1 = \sum_m |a_m|^2 \sum_n{}' \csc^2 \pi(x_n - y_m) \tag{3.3.11}$$

and

$$S_2 = \sum_{m \neq p} \bar{a}_m a_p \sum_n{}' \csc \pi(x_n - y_m) \csc \pi(x_n - y_p). \tag{3.3.12}$$

In S_2 we may write

$$\csc \pi(x_n - y_m) \csc \pi(x_n - y_p) =$$

$$= \csc \pi(x_m - y_p) \left[\cot \pi(x_n - y_m) - \cot \pi(x_n - y_p) \right].$$

According to [132] (Proof of Theorem 1, p. 79) we use this to split S_2 in the following way: $S_2 = S_3 - S_4 + 2 \operatorname{Re} S_5$, where

$$S_3 = \sum_{n,m,p}{}' \bar{a}_m a_p \csc \pi(y_m - y_p) \cot \pi(x_n - y_m), \tag{3.3.13}$$

$$S_4 = \sum_{n,m,p}{}' \bar{a}_m a_p \csc \pi(y_m - y_p) \cot \pi(x_n - y_p), \tag{3.3.14}$$

and

$$S_5 = \sum_{n,m}{}' \bar{a}_m a_n \csc \pi(x_n - y_m) \cot \pi(x_n - y_m). \tag{3.3.15}$$

We show now that $S_3 = S_4$. We see from (3.3.9) and (3.3.13) that

$$S_3 = \sum_{n,m}{}' \bar{a}_m \cot \pi(x_n - y_m) \sum_p{}' a_p \csc \pi(y_m - y_p)$$

$$= \sum_{n,m}{}' \bar{a}_m \cot \pi(x_n - y_m) (-i\mu a_m)$$

$$= -i\mu \sum_{n,m}{}' |a_m|^2 \cot \pi(x_n - y_m). \tag{3.3.16}$$

Similarly, from (3.3.9) and (3.3.14),

$$S_4 = \sum_{n,p}{}' a_p \cot \pi(x_n - y_p) \sum_m{}' \bar{a}_m \csc \pi(y_m - y_p)$$

$$= \sum_{n,p}{}' a_p \cot \pi(x_n - y_p) (-i\mu \bar{a}_p)$$

$$= -i\mu \sum_{n,p}{}' |a_p|^2 \cot \pi(x_n - y_p). \tag{3.3.17}$$

Therefore, $S_3 = S_4$, so that $S_1 + S_2 = S_1 + 2 \operatorname{Re} S_5 \leq S_1 + 2|S_5|$. We use the inequality $2|a_n a_m| \leq |a_n|^2 + |a_m|^2$ in (3.3.15), so that (3.3.11) and (3.3.15) give

$$S_1 + S_2 \leq \sum_{m,n}{}' |a_m|^2 \csc^2 \pi(x_n - y_m) +$$

$$+\sum_{n,m}{}' \left(|a_n|^2+|a_m|^2\right) |\csc\pi(x_n-y_m)\, \cot\pi(x_n-y_m)|$$

$$=\sum_{m,n}{}'|a_m|^2\Big(\csc^2\pi(x_n-y_m)+$$

$$+|\csc\pi(x_n-y_m)\, \cot\pi(x_n-y_m)|\Big)+$$

$$+\sum_{m,n}{}'|a_n|^2\ |\csc\pi(x_n-y_m)\, \cot\pi(x_n-y_m)|\,.$$

By Lemma 3.3.4 this is

$$\leq \frac{1}{4}\sum_m |a_m|^2\sum_n{}'\|x_n-y_m\|^{-2}+$$

$$+\frac{1}{\pi^2}\sum_n |a_n|^2\sum_m{}'\|x_n-y_m\|^{-2}.$$

A remark similar to that conducted in [132], leads to be conclude that the x_n and the y_m are spaced from each other by at least δ, so that

$$\sum_m{}'\|x_n-y_m\|^{-2}\leq 2\sum_{k=1}^{\infty}(k\delta)^{-2}=\frac{\pi^2}{3}\delta^{-2}.$$

Hence,

$$S_1+S_2\leq\frac{\pi^2}{3}\delta^{-2}\left(\frac{1}{\pi^2}+\frac{1}{4}\right)$$

where we have considered $\sum_n|a_n|^2=1$. □

We now able to prove the result of the Section.

Proof of Theorem 3.3.2. Put, by hypothesis,

$$\gamma=\min_{n,m}{}_+|\lambda_n-\lambda_m|>\sqrt{\frac{1}{3}+\frac{\pi^2}{12}}.$$

Write $\int_{-\infty}^{\infty}|f(t)|^2dt$:

$$\sum_{m,n}a_n\bar{a}_m\int_{-\infty}^{+\infty}\operatorname{sinc}(\lambda_n-t)\operatorname{sinc}(\lambda_m-t)dt$$

which is equal to

$$\sum_n|a_n|^2+\sum_{m,n}{}'a_n\bar{a}_m\operatorname{sinc}(\lambda_n-\lambda_m). \tag{3.3.18}$$

Furthermore,

$$\frac{\sin\pi(\lambda_n-\lambda_m)}{\pi(\lambda_n-\lambda_m)}=\frac{1}{\frac{\pi\lambda_n}{\sin\pi(\lambda_n-\lambda_m)}-\frac{\pi\lambda_m}{\sin\pi(\lambda_n-\lambda_m)}}$$

$$=\frac{1}{x_n+x_m}$$

where $x_n := \frac{\pi\lambda_n}{\sin\pi(\lambda_n-\lambda_m)}$. Putting $y_m = -x_m$ above equality is rewritten as $\frac{1}{x_n-y_m}$, and

$$\sum_{m,n}{}' a_n\bar{a}_m \operatorname{sinc}(\lambda_n - \lambda_m) = \sum_{m,n}{}' \frac{a_n\bar{a}_m}{x_n - y_m}.$$

To prove the Theorem, we note that if x is any member of a bounded interval, then $\|\varepsilon x\| = \varepsilon|x|$ whenever ε is sufficiently small. Moreover,

$$\frac{1}{x_n - y_m} = \lim_{\varepsilon\to 0} \pi\varepsilon \csc \pi\varepsilon(x_n - y_m)$$

so that we can appeal to Lemma 3.3.5:

$$\left|\sum_{m,n}{}' \frac{a_n\bar{a}_m}{x_n - y_m}\right| = \pi\varepsilon \left|\sum_{m,n}{}' a_n\bar{a}_m \csc \pi\varepsilon(x_n - y_m)\right|$$

$$\leq \pi\varepsilon\, \delta^{-1}\sqrt{\frac{1}{3} + \frac{\pi^2}{12}} \sum_{n\in I} |a_n|^2,$$

where, for $\varepsilon \to 0$,

$$\delta = \min_{n,m}{}_{+}\|\varepsilon x_n - \varepsilon y_m\| = \varepsilon \min_{n,m}{}_{+}|x_n - y_m|,$$

$$x_n \neq y_m \quad \forall n, m = 1, \ldots, R.$$

Since $x_n := \frac{\pi\lambda_n}{\sin\pi(\lambda_n-\lambda_m)}$, $y_m = -x_m$,

$$\delta = \varepsilon \min_{n,m}{}_{+} \left|\frac{\pi\lambda_n - \pi\lambda_m}{\sin\pi(\lambda_n - \lambda_m)}\right|$$

and since $|\sin\pi(\lambda_n - \lambda_m)| \leq 1$, we have

$$\delta \geq \varepsilon\pi \min_{n,m}{}_{+}|\lambda_n - \lambda_m| = \varepsilon\pi\gamma$$

Accordingly,

$$\left|\sum_{m,n}{}' \frac{a_n\bar{a}_m}{x_n - y_m}\right| = \pi\varepsilon \left|\sum_{m,n}{}' a_n\bar{a}_m \csc \pi\varepsilon(x_n - y_m)\right|$$

$$\leq \gamma^{-1}\sqrt{\frac{1}{3} + \frac{\pi^2}{12}} \sum_{n\in I} |a_n|^2.$$

Thus, an appeal to (3.3.18) completes the proof of the Theorem:

$$E_f \geq \left(1 - \gamma^{-1}\sqrt{\frac{1}{3} + \frac{\pi^2}{12}}\right) \sum_{n\in I} |a_n|^2.$$

□

As one reads on [132], it follows from a paper of Hellinger and Toeplitz ([91] and [132]) that Theorem 3.3.2 and Lemma 3.3.5 hold also for infinite sums, provided that $\min_{+} f$ is replaced by $\inf_{+} f$. It is also possible to consider bilateral series if we put $\lambda_{-n} = -\lambda_n$ for $n = 1, 2, \ldots$.

An estimate from above is immediate employing same steps involved used in the proof of theorem 3.3.2. Indeed, from equation (3.3.18) and by triangle inequality:

$$\sum_n |a_n|^2 + \sum_{m,n}{}' a_n \bar{a}_m \operatorname{sinc}(\lambda_n - \lambda_m) \leq$$

$$\leq \left(1 + \gamma^{-1}\sqrt{\frac{1}{3} + \frac{\pi^2}{12}}\right) \sum_n |a_n|^2$$

where γ is defined as in theorem 3.3.2.

3.4 Conclusions and perspectives.

Theorem 3.1.3 and corollary 3.1.4 responding to the outstanding questions of Duffin, Eachus and Young, essentially because this chapter shows that the constant $\frac{\log 2}{\pi}$ can be replaced by 1/4, also for the complex case. Moreover, from corollary 3.1.4, it has $\{e^{i\bar{\lambda}_n t}\}$ satisfies the Paley-Wiener criterion for $|\bar{\lambda}_n - n| < 1/4$ even when $\{\bar{\lambda}_n\}$ is a complex sequence. Two lemmas present in appendix (an extension to complex case of result present on [199]) prove that Kadec's 1/4-theorem is "best possible": the system $\{e^{i\bar{\lambda}_n t}\}$ constitutes a basis for $L^2[-\pi, \pi]$ whenever every $\bar{\lambda}_n$ is complex and $|\lambda_n - n| \leqq L$, $|\mu_n| \leqq \tau(L)$ but not constitute a basis when $L = 1/4$. Equally interesting is the fact that $\tau(L)$ is not specified in the proofs of lemmas 3.1.1 and 3.1.2 and, into this proofs, it is not necessary that it assumes the logarithmic expression (3.1.3).

In Duffin and Eachus [63] one reads: "It is a curious parallelism that $\log 2/\pi$ and $1/4$ are in the same ratio as the limits of Takenaka and Schoenberg in a somewhat similar unsolved problem". See: [192], [182]. In [192] is reported a particular case of one of Takenaka's theorems [182]: "If every derivative of an integral function $f(z)$ has a zero inside or on the unit circle and if $\limsup_{r\to\infty} \frac{\log M(r)}{r} < \log 2$ then $f(z)$ is a costant". ($M(r)$ is the maximum modulus in $|z| \leq r$ of function). The author write that this condition is probably not "best possible": $\sin \frac{\pi}{4} z - \cos \frac{\pi}{4} z$ shows that $\log 2$ cannot be replaced by any number larger than $\pi/4$, and this may well be the true value. A possible development of the topics covered in this chapter would be compare proof of Kadec's-1/4 theorem (complex case) with question in [192].

Let us consider Section 3.2. As mentioned, Kadec's theorem states that if $\{\lambda_n\}$ is a sequence of real numbers for which $|\lambda_n - n| \leq L < \frac{1}{4}$ for $n \in \mathbb{Z}$, then $\{e^{i\lambda_n x}\}$ forms a Riesz basis for $L^2(-\pi, \pi)$. For the multivariate case, many authors have wondered about similar question. Let $\mathbf{n} = (n_1, ..., n_d) \in \mathbb{Z}^d$, $\lambda_n = (\lambda_1, ..., \lambda_{n_d}) \in \mathbb{R}^d$. The aim is to find a constant θ_d such that $\{e^{i\langle\lambda_n,\omega\rangle} : \mathbf{n} \in \mathbb{Z}^d\}$ is a Riesz basis for $L^2(-\pi, \pi)^d$ whenever $\sup_{\mathbf{n}\in\mathbb{Z}^d} ||\lambda_n - \mathbf{n}||_\infty$ computed as

$$\mathrm{sup}_{\mathbf{n}\in\mathbb{Z}^d}\mathrm{sup}_{1\leq k\leq d}|\lambda_{n_k} - n_k| < \theta_d$$

We call θ_d a stability bound. Many author have approached this problem. Favier and Zalik [71], C. Chui, and X. Shi [50] presented a multivariate version of Kadec's theorem. But their result contains an additional condition $B_d(L) < 1$ and lead to very small stability bounds. Later works, as [179] and [180], show that additional conditions may be deleted, giving an optimal stability bound for the multivariate trigonometric systems.

A possible development of our work could transport these arguments on sinc bases, studying a multivariate version of Theorem 3.2.1.

Another area of research may be the one of the frame. Especially Gabor frame. Let $\{f_k\}_k$ a sequence that generates a Hilbert space H. It is known that this sequence is a Riesz basis for H if and only if there exist positive constants A and B such that for any finite sequence of numbers $c_1, ..., c_n$ it has

$$A\sum_{k=1}^{n}|c_k|^2 \leq ||\sum_{k=1}^{n}c_kf_k|| \leq B\sum_{k=1}^{n}|c_k|^2$$

The notion of frame is a generalization of Riesz basis in a Hilbert space. Let H be a Hilbert space, and let $\{f_k\}_k$ a sequence of H.

We say that this family is a frame for H if there are constants $A > 0$, $B > 0$ such that for each $f \in H$ it has

$$A||f||^2 \leq \sum_{k=1}^{n}|\langle f, f_k\rangle|^2 \leq B||f||^2$$

A frame that ceases to be a frame when any one of its elements is removed is said to be an *exact frame*. We now recall, from [182], a fundamental theorem involving Riesz bases.

Theorem 3.4.1. *Each sequence of vectors belonging to a separable Hilbert space, is a basis of Riesz if and only if it is a exact frame.*

Furthermore, see: [5], for investigate further aspects of the frame; [22], for understand the crucial role of frames in nonuniform sampling, and [150] for understand the process of signal reconstruction via frames.

Let us consider $a, b \in \mathbb{R}$, we indicate with T_a the time-shift operator and with M_b the frequency-shift operator, such that

$$T_af(x) = f(x-a), \quad M_bf(x) = e^{2\pi ibx}f(x)$$

Given a function $g \in L^2(\mathbb{R})$ and real parameters $\alpha, \beta > 0$, the collection

$$G(g, \alpha, \beta) = \{T_{\alpha,k}M_{\beta,n}g : k, n \in \mathbb{Z}\}$$

is called a Gabor system. That is, (g, α, β) generates a Gabor frame for $L^2(\mathbb{R})$ if

$$\{T_{\alpha,k}M_{\beta,n}g : k, n \in \mathbb{Z}\}$$

is a frame for $L^2(\mathbb{R})$. This means that there are constants $A > 0$, $B > 0$ such that

$$A||f||^2 \leq \sum_n\sum_k|\langle f, T_{\alpha,k}M_{\beta,n}g\rangle|^2 \leq B||f||^2$$

For the related conditions between Kadec's theorem and the stability of Gabor frames see, for example, [180]. Accordingly, a possible progress in our work, it might be to look for a link between theorem 3.2.1 and some properties of Gabor frames, as the stability property.

A family of complex exponentials $\{e^{inx}\}_{n\in\mathbb{Z}}$ is called a *Fourier frame* [141], if it constitutes a frame for $L^2(-\pi, \pi)$ or, explicitly, if there exist $0 < A \leq B < \infty$ such that $\forall f \in L^2(-\pi, \pi)$:

$$A\int_{-\pi}^{\pi}|f(x)|^2dx \leq \sum_{n\in\mathbb{Z}}\left|\int_{-\pi}^{\pi}f(x)e^{-i\lambda_nx}dx\right|^2$$

and

$$\sum_{n\in\mathbb{Z}}\left|\int_{-\pi}^{\pi} f(x)e^{-i\lambda_n x}dx\right|^2 \leq B\int_{-\pi}^{\pi}|f(x)|^2dx$$

[15] and [49] gave the stability bounds of Fourier frames. After defining in a suitable way the concept of "sinc-frame" - a transposition of the definition of Fourier frame to sinc basis - another aspect of future work, would be to get the results of Balan and Christensen for these *sinc-frames.*

Chapter 4

The class of Lucas-Lehmer polynomials and its applications.

In this Chapter we study a particular class of orthogonal polynomials, the class of Lucas-Lehmer polynomials. The subject of orthogonal polynomials finds its origins in the XVIIIth century, thanks to the works of Legendre, Laplace and Lagrange. The history of orthogonal polynomials probably originated from the Legendre polynomials, firstly employed in the determination of the force of attraction exerted by solids of revolution [104]; their orthogonal properties were established by A. M. Legendre during last years of XVIIIth century. They can be viewed as solutions of Legendre's differential equation,

$$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P_n(x)\right]+n(n+1)P_n(x)=0.$$

These polynomials, usually denoted P_0, P_1, $\dots$, may be defined by Rodrigues formula,

$$P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^n\right].$$

C. Hermite, in the XIXth century, introduced the class of Hermite polynomials, which answered to the problem of obtaining expansions of unknown functions in order to solve ordinary differential equations [104]. There are two different kinds of Hermite polynomials:

$$He_n(x)=(-1)^n e^{\frac{x^2}{2}}\frac{d^n}{dx^n}e^{-\frac{x^2}{2}}=\left(x-\frac{d}{dx}\right)^n\cdot 1,$$

used more commonly in probability, and

$$H_n(x)=(-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}=\left(2x-\frac{d}{dx}\right)^n\cdot 1,$$

used more commonly in physics. Each of these two definitions is a rescaling of the other.

Almost in the same period E. Laguerre, working on the relations between polynomials and continued fraction [21], discovered polynomials known today as Laguerre polynomials. They are solutions of a second-order linear differential equation, the Laguerre equation:

$$xy''+(1-x)y'+ny=0.$$

Each Legendre polynomial $L_n(x)$ is an nth-degree polynomial. It may be expressed using Rodrigues formula:

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}\left(e^{-x}x^n\right) = \frac{1}{n!}\left(\frac{d}{dx} - 1\right)^n x^n.$$

Starting from 1807 J. Fourier introduced some series of exponentially weighted sine functions for the purpose of solving the heat equation in a metal plate [73], extending subsequently the initial idea to represent any arbitrary function as an infinite sum of sine and cosine functions [74]. He refers to them also with the term of trigonometric polynomials. The *Chebyshev polynomials* [18, 79, 161] are related to Fourier cosine series through a change of variables. The class of Chebyshev polynomials assumes here a fundamental role. In Section 4.1, from [186], we introduce a new sequence of polynomials, which follows the same recursive rule of the well-known Lucas-Lehmer integer sequence. We show the most important properties of this sequence, relating them to the Chebyshev polynomials of the first and second kind.

Section 4.2 is devoted to reinvestigate the structure of the solution of a well-known Love's problem, related to the electrostatic field generated by two circular co-axial conducting disks, in terms of orthogonal polynomial expansions, enlightening the role of the recently introduced class of the Lucas-Lehmer polynomials [189]. Love's integral equation is a Fredholm equation of the second kind. An equation of this kind has the form

$$f(x) = \phi(x) + \lambda \int_a^b K(x,s) f(s)\,\mathrm{d}s.$$

Given the kernel $K(x,s)$, and the function $\phi(x)$, the problem consists typically in finding the function $f(x)$. We can find a Chebyshev-series solution writing

$$f(x) = \sum_{n=0}^{\infty} a_n T_n(x), \qquad \text{where } T_n(x) \text{ are the Chebyshev polynomials.}$$

Moreover we show that the solution can be expanded more conveniently with respect to a Riesz basis obtained starting from Chebyshev polynomials.

In Section 4.3 we discuss some relations between zeros of Lucas-Lehmer polynomials and Gray code. Gray codes have a very long history; see, for instance, [89] on the origin of binary codes and [77] on some entertaining aspects of Gray codes. A Gray code represents each number in the sequence of integers $\{0...2^N - 1\}$ as a binary string of length N in an order such that adjacent integers have Gray code representations that differ in only one bit position. Marching through the integer sequence therefore requires flipping just one bit at a time. For example, the usual binary coding of $\{0, \dots, 7\}$ is $\{000, 001, 010, 011, 100, 101, 110, 111\}$, while its Gray coding is $\{000, 001, 011, 010, 110, 111, 101, 100\}$.

We apply this binary law to the study of nested square roots of 2 expressed by (4.1.2), associating bits 0 and 1 to $\oplus$ and $\ominus$ signs in the nested form. This gives the possibility to obtain an ordering for the zeros of Lucas-Lehmer polynomials, which assume the form of nested square roots of 2 expressed by (4.1.2). This is the cornerstone of the results shown in Section 4.4, where we obtain π as the limit of a sequence related to the zeros of the class of polynomials $L_n(x)$. The results obtained here are based on the placement of the zeros of the polynomials $L_n(x)$. Since zeros have a structure of nested radicals, in this way we can build infinite sequences of nested radicals converging to π.

4.1 The class of Lucas-Lehmer polynomials

In this Section we study a class of polynomials $L_n(x) = L_{n-1}^2(x) - 2$, which, at the best of our knowledge, were introduced for the first time in [186], created by means of the same iterative formula used to build the well-known Lucas-Lehmer sequence, employed in primality tests [125, 118, 160, 35, 106]. It is clearly crucial to choose the first term of the polynomial sequence. In this chapter we consider $L_0 = x$.

We show some properties of these polynomials, in particular discussing the link among the Lucas-Lehmer polynomials and the Chebyshev polynomials of the first and second kind [18, 79, 161]. The Chebyshev polynomials are well-known and, although they have been known and studied for a long time, continue to play an important role in recent advances in many areas of mathematics such as Algebra, Numerical Analysis, Differential Equations and Number Theory (see, for instance: [12, 20, 28, 40, 59, 84, 131, 181, 198, 24]) and new other properties of theirs continue to be discovered ([20, 40, 57, 62]).

In particular, in the spirit of some existing results on the Chebyshev polynomials and the nested square roots (see, for example, [134, 196]), we show that the zeros of the Lucas-Lehmer polynomials can be written in terms of nested radicals.

There are many classes of polynomials which are related to the Chebyshev polynomials, such as [28, 58, 95, 96, 195]. In the spirit of some of these works - if $L_n(x)$, $T_n(x)$, $U_n(x)$ denote (respectively) the nth Lucas-Lehmer polynomials, the Chebyshev polynomials of the first and second kind - we can consider the polynomials L_n as a generalization of the so-called *modified or shifted Chebyshev polynomials*, by introducing an appropriate change of variable $t = f(x)$.

We will now outline the content of this chapter. In Sections 4.1.1 and 4.1.2 we introduce the Lucas-Lehmer polynomials and show their main properties. Furthermore, we give a recursive formula for the sequence of the first nonnegative zeros of $L_n(x)$, in terms of nested radicals. In Section 4.1.3 we show some relations among the Lucas - Lehmer polynomials $L_n(x)$ and the Chebyshev polynomials of the first and second kind, determining several new properties for the former.

In Section 4.1.4 we show some generalizations of the Lucas-Lehmer map, having the same properties of L_n.

4.1.1 First iterations of the Lucas-Lehmer map.

Let us consider the iterative map

$$L_n(x) = L_{n-1}(x)^2 - 2 \qquad ; \qquad L_0(x) = x \ . \tag{4.1.1}$$

Assuming $L_0 = x$ as the initial value, let us construct the first terms of the sequence. The function $L_1(x) = x^2 - 2$ represents a parabola with two zeros $z_{1,2} = \pm\sqrt{2}$ and one minimum point in $(0, -2)$; $L_2(x) = (x^2-2)^2-2 = 2\left(1 - 2x^2 + \frac{x^4}{2}\right)$, shown in Fig. 4.1, contains four zeros:

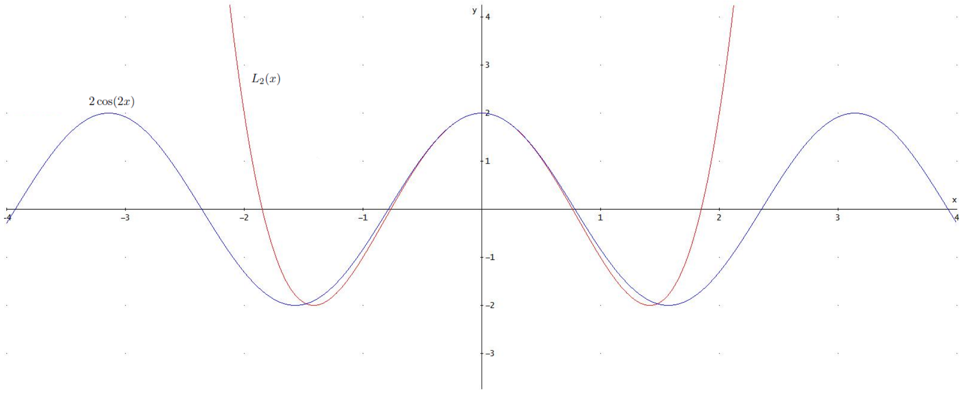


Figure 4.1: comparison between $L_2(x)$ and $2\cos(2x)$.

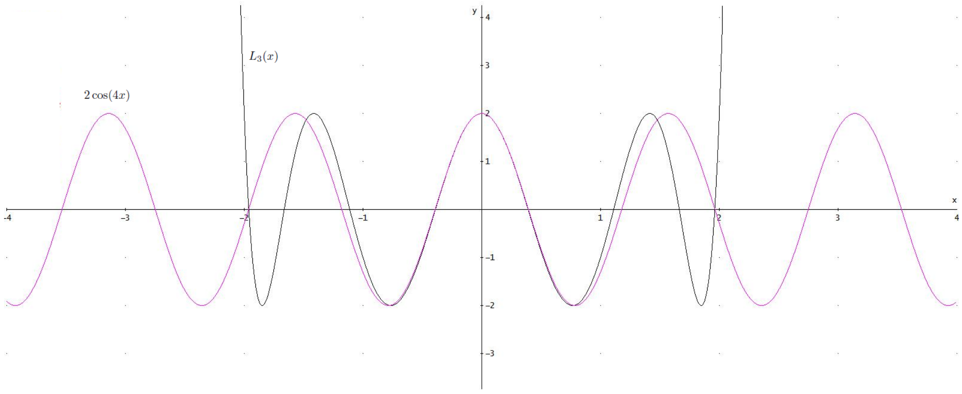


Figure 4.2: comparison between $L_3(x)$ and $2\cos(4x)$.

$z_{1\div 4} = \pm\sqrt{2 \pm \sqrt{2}}$. From the derivative of $L_2(x)$, $L_2'(x) = 4x \cdot (x^2 - 2) = 4x \cdot L_1(x)$ it is possible to determine the critical points of the function: $x_1 = 0$ (maximum), $x_{2,3} = \pm\sqrt{2}$ (minimum). Since $L_2(x) = 2\left(1 - 2x^2 + \frac{x^4}{2}\right) = 2\cos(2x) + o(x^3)$, for $x \to 0$ we have $L_2(x) \sim 2\cos(2x)$.

The zeros of the function $L_3(x) = ((x^2-2)^2-2)^2 - 2 = 2\,(1 - 8x^2 + o(x^3))$, whose graph is shown in Fig. 4.2, are eight: $z_{1\div 8} = \pm\sqrt{2 \pm \sqrt{2 \pm \sqrt{2}}}$. The critical points are: $x_1 = 0$, $x_{2,3} = \pm\sqrt{2}$, $x_{4,5,6,7} = \pm\sqrt{2 \pm \sqrt{2}}$. Besides $L_3(x) \sim 2\cos(4x)$ for $x \to 0$. The zeros of the function $L_4(x) = (((x^2-2)^2-2)^2-2)^2 - 2$ (shown in Fig. 4.3) are sixteen: $z_{1\div 16} =$

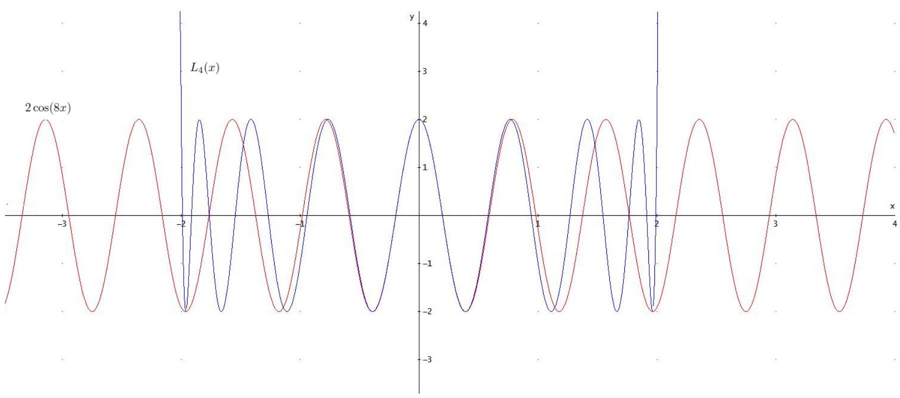


Figure 4.3: comparison between $L_4(x)$ and $2\cos(8x)$.

$\pm\sqrt{2\pm\sqrt{2\pm\sqrt{2\pm\sqrt{2}}}}$. The critical points follow the same general rule which is possible to guess observing the previous iterations; moreover it results again $L_4(x) \sim 2\cos(8x)$ for $x \to 0$. It must be noted that: $L_1(\pm\sqrt{2}) = 0$, $L_2(\pm\sqrt{2}) = -2$, $L_n(\pm\sqrt{2}) = 2 \quad \forall n \geq 3$; $L_0(0) = 0$, $L_1(0) = -2$, $L_n(0) = 2 \quad \forall n \geq 2$; $L_0(-2) = -2$, $L_n(-2) = 2 \quad \forall n \geq 1$; $L_n(2) = 2 \quad \forall n \geq 0$.

Let us observe that the numerical sequence $L_n(\sqrt{6})$ corresponds to the sequence of Lucas-Lehmer numbers (OEIS, On-Line Encyclopedia of Integer Sequences, `http://oeis.org/A003010`) used, as we said before, in the Lucas-Lehmer primality test [106, 160, 35].

4.1.2 Zeros and critical points.

Taking into account the considerations of the previous section, we can in general state the following proposition (whose proof is quite simple and is omitted for brevity)

Proposition 4.1.1. *At each iteration the zeros of the map $L_n (n \geq 1)$ have the form*

$$\pm\sqrt{2\pm\sqrt{2\pm\sqrt{2\pm\sqrt{2\pm\ldots\pm\sqrt{2}}}}} \tag{4.1.2}$$

How can we order these zeros? Considering only positive zeros (being every L_n a symmetric function), the first sign on the left inside the root must be negative. Let us set $\sqrt{2\pm\sqrt{x_2}} =: \sqrt{x_1}$ and $\sqrt{2\pm\sqrt{y_2}} =: \sqrt{y_1}$ be two generic roots chosen from those expressed in (4.1.2). We wonder when $\sqrt{x_1} > \sqrt{y_1}$. Then we have several options.

opt1 If $x_1 = 2+\sqrt{x_2}, y_1 = 2+\sqrt{y_2}$ we have that: $\sqrt{x_1} > \sqrt{y_1} \leftrightarrow 2+\sqrt{x_2} > 2+\sqrt{y_2}$ whence $x_2 > y_2$, that is the examination moves to the next step (and we apply again opt1,2,3,4).

opt2 If $x_1 = 2-\sqrt{x_2}, y_1 = 2+\sqrt{y_2}$ we have that: $\sqrt{x_1} > \sqrt{y_1} \leftrightarrow 2-\sqrt{x_2} > 2+\sqrt{y_2}$, that is impossible, so we have $\sqrt{x_1} < \sqrt{y_1}$.

opt3 If $x_1 = 2+\sqrt{x_2}, y_1 = 2-\sqrt{y_2}$ we have that: $\sqrt{x_1} > \sqrt{y_1} \leftrightarrow 2+\sqrt{x_2} > 2-\sqrt{y_2}$, always satisfied and we have $\sqrt{x_1} > \sqrt{y_1}$.

opt4 If $x_1 = 2-\sqrt{x_2}, y_1 = 2-\sqrt{y_2}$ we have that: $\sqrt{x_1} > \sqrt{y_1} \leftrightarrow 2-\sqrt{x_2} > 2-\sqrt{y_2}$, whence $x_2 < y_2$ then we have to check the following step (we apply again opt1,2,3,4).

We show now what we argued in the previous steps.

Theorem 4.1.2. *For $n \geq 2$ we have*

$$L_n(x) = 2\cos(2^{n-1}x) + o(x^3) \tag{4.1.3}$$

Proof. Taking into consideration the McLaurin expansion of the cosine, to prove formula (4.1.3) is equivalent to show that

$$L_n(x) = 2 - 2^{2n-2}x^2 + o(x^3) = 2 - 4^{n-1}x^2 + o(x^3)\ . \tag{4.1.4}$$

Let us proceed by means of induction principle. For $n = 2$ we have $L_2(x) = (x^2 - 2)^2 - 2 = x^4 - 4x^2 + 2 = 2 - 4x^2 + o(x^3)$. Consider then the McLaurin polynomial of the second order of $2\cos(2x)$: it is $\frac{4}{3}x^4 - 4x^2 + 2$, which proves the relation for $n = 2$. Suppose now as true formula (4.1.3) for a generic index n and proceed to check the case $n + 1$:

$$\begin{aligned} L_{n+1} &= L_n^2 - 2 = [2 - 4^{n-1}x^2 + o(x^3)]^2 - 2 = \\ &= 2 - 4^n x^2 + o(x^2) \end{aligned} \tag{4.1.5}$$

It is also known that the McLaurin polynomial of $2\cos(2^n x)$ is $2 - 2^{2n} \cdot x^2 + R_3$. We can therefore conclude that $2\cos(2^{n-1}x)$ and $L_n(x)$ have the same coefficients up to the second order, which concludes the proof. □

We are interested in determining the distribution of minima and maxima for each L_n. To this aim, now we are going to show an important property of the polynomials L_n.

Lemma 4.1.3. *For each $n \geq 2$ we have*

$$\frac{d}{dx} L_n(x) = 2^n \; x \prod_{i=1}^{n-1} L_i(x) \tag{4.1.6}$$

Proof. Let us proceed by induction. If $n = 2$

$$\frac{d}{dx} L_2(x) = \frac{d}{dx}[(x^2 - 2)^2 - 2] = 4x(x^2 - 2) = 4x \; L_1(x) \tag{4.1.7}$$

Now we are going to check it for $n + 1$. For the function (4.1.1)

$$\frac{d}{dx} L_{n+1}(x) = \frac{d}{dx}[L_n^2(x)] = 2L_n(x) \; \frac{d}{dx} L_n(x) \tag{4.1.8}$$

Replacing it with (4.1.6) we will have at the end:

$$\frac{d}{dx} L_{n+1}(x) = 2L_n(x) \cdot \left[2^n \; x \prod_{i=1}^{n-1} L_i(x) \right] = 2^{n+1} \; x \prod_{i=1}^{n} L_i(x) \tag{4.1.9}$$

□

Let M_n be the set of the critical points and be Z_n the set of the zeros of $L_n(x)$; we obtain the following results.

Proposition 4.1.4. *For each $n \geq 2$ we have*

$$M_n = M_{n-1} \cup Z_{n-1} = M_1 \cup \bigcup_{i=1}^{n-1} Z_i \tag{4.1.10}$$

with $card(Z_n) = 2^n$.

Proof. Let us first find the critical points of $L_n(x)$, imposing $\frac{d}{dx}L_n(x) = 0$, from which it results

$$2L_{n-1}(x)\,\frac{d}{dx}L_{n-1}(x) = 0 \tag{4.1.11}$$

which vanishes either if $L_{n-1}(x) = 0$ (finding the points of Z_{n-1}) or if $\frac{d}{dx}L_{n-1}(x) = 0$ (determining the points of M_{n-1}). Therefore it is proved that $M_n = M_{n-1} \cup Z_{n-1}$. To prove the second equality, it is sufficient to observe that the right hand side of (4.1.6) vanishes either if $x = 0$ or if $L_i(x) = 0$ for some $i = 1, ..., n-1$; thus we obtain the set $\bigcup_{i=1}^{n-1} Z_i$, which proves the statement. □

Proposition 4.1.5. *For each $n \in N$ we have $card(M_n) = 2^n - 1$. Furthermore, let M_n^+ be the set of the positive critical points of $L_n(x)$; we have that $card(M_n^+) = 2^{n-1} - 1$.*

Proof. We must show that $card(M_n) = 2^n - 1$; proceeding by induction: if $n = 1$, then $L_1(x) = x^2 - 2$ is a parabola having only a minimum, at the point $(0, -2)$. Now we are going to check it for $n + 1$, having assumed it true for a generic $n \geq 2$. From proposition 4.1.4, we have, for $n > 1$:

$$M_{n+1} = M_n \cup Z_n \;\Rightarrow\; card(M_{n+1}) = card(M_n) + card(Z_n) \tag{4.1.12}$$

(the intersection between M_n and Z_n being empty). From proposition 4.1.4, we have that $card(Z_n) = 2^n$; besides, by hypothesis, we know that $card(M_n) = 2^n - 1$. Then $card(M_{n+1}) = 2^n - 1 + 2^n = 2^{n+1} - 1$. Furthermore, if we don't consider the maximum in the origin, we will have $2^n - 2$ critical points, half of which are positive. Therefore $card(M_n^+) = 2^{n-1} - 1$, □

Proposition 4.1.3 is very useful because allows us to obtain some interesting properties for the critical points of $L_n(x)$. We already observed that, for every $n \geq 2$, if $x = 0$, then $((0-2)^2 ...)^2 - 2 = 2$ and the point is a maximum. Moreover, for every natural number j such that $1 < j < n-1$ we have that the points x_0 such that $L_j(x_0) = 0$ are maximum points for L_n. Indeed $\left(...(\underbrace{L_j}_{=0} - 2)^2 ... \right)^2 - 2 = 2$. Instead, the points x such that $L_{n-1}(x) = 0$, being $L_n(x) = \underbrace{L_{n-1}^2(x)}_{=0} - 2 = -2$, are minimum points for L_n. Now, from proposition (4.1.3) there aren't other critical points; thus we have shown that the set of maximum points of $L_n(x)$ is: $\bigcup_{i=1}^{n-2} Z_i \cup \{x = 0\}$, while the set of minimum points of $L_n(x)$ is Z_{n-1}.

Remark 4.1.6. Minimum points for $L_n(x)$ become maximum points for $L_{n+1}(x)$, maximum points for $L_n(x)$ remain maximum points for $L_{n+1}(x)$. This implies that all the local maxima of every L_n are equal to 2.

Corollary 4.1.7. *All zeros and critical points of L_n belong to the interval $(-2, 2)$* [1].

[1] Because of the symmetry of Lucas-Lehmer polynomials, we will study only positive zeros.

4.1.3 Relationships between Lucas-Lehmer polynomials and Chebyshev polynomials of the first and second kind, and additional properties.

As we know [161, 18, 79], the *Chebyshev polynomials of the first kind* satisfy the recurrence relation

$$\begin{cases} T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) & n \geq 2 \\ T_0(x) = 1, \quad T_1(x) = x \end{cases}$$

from which it easily follows that for the n-th term:

$$T_n(x) = \frac{\left(x - \sqrt{x^2-1}\right)^n + \left(x + \sqrt{x^2-1}\right)^n}{2} \tag{4.1.13}$$

This formula is valid in $\mathbb{R}$ for $|x| \geq 1$; here we assume instead that T_n, defined in $\mathbb{R}$, can take complex values, too.

Proposition 4.1.8. *For each $n \geq 1$ we have*

$$L_n(x) = 2\ T_{2^{n-1}}\left(\frac{x^2}{2} - 1\right) \tag{4.1.14}$$

Proof. We must show that

$$\begin{aligned} L_n(x) = &\left(\frac{x^2}{2} - 1 - \sqrt{\left(\frac{x^2}{2}-1\right)^2 - 1}\right)^{2^{n-1}} + \\ &+ \left(\frac{x^2}{2} - 1 + \sqrt{\left(\frac{x^2}{2}-1\right)^2 - 1}\right)^{2^{n-1}} \end{aligned} \tag{4.1.15}$$

This formula is real for $|x| \geq 2$ and complex for $|x| < 2$ and is true for $n = 1$:

$$L_1(x) = x^2 - 2 = \left[\frac{x^2}{2} - 1 - \sqrt{\left(\frac{x^2}{2}-1\right)^2 - 1}\right] + \left[\frac{x^2}{2} - 1 + \sqrt{\left(\frac{x^2}{2}-1\right)^2 - 1}\right] . \tag{4.1.16}$$

We assume true (4.1.14) for a natural n and write:

$$\begin{aligned} &L_{n+1}(t(x)) = L_n^2(t(x)) - 2 = \\ &= \left(t - \sqrt{t^2-1}\right)^{2^n} + \left(t + \sqrt{t^2-1}\right)^{2^n} + \\ &+ 2\left[\left(t - \sqrt{t^2-1}\right)\ \left(t + \sqrt{t^2-1}\right)\right]^{2^{n-1}} - 2 \end{aligned} \tag{4.1.17}$$

where $t(x) = \dfrac{x^2}{2} - 1$. Observing that $\left(t - \sqrt{t^2-1}\right)\ \left(t + \sqrt{t^2-1}\right) = 1$, we lastly obtain

$$L_{n+1}(t(x)) = \left(t - \sqrt{t^2-1}\right)^{2^n} + \left(t + \sqrt{t^2-1}\right)^{2^n} \tag{4.1.18}$$

which concludes the proof.

It is observed that the (4.1.18) is true for a generic function $t(x)$. If $n = 1$, instead, the only function that satisfies the (4.1.18) is $t(x) = \dfrac{x^2}{2} - 1$. $\square$

Proposition 4.1.9. *The polynomials $L_n(x)$ are orthogonal with respect to the weight function $\frac{1}{4\sqrt{4-x^2}}$ defined on $x \in (-2, 2)$.*

Proof. Let us consider Chebyshev polynomials of the first kind; then:

$$\int_{-1}^{1} (1 - x^2)^{-1/2} T_n(x) T_m(x) dx = 0$$

if $m \neq n$ and $m, n \in \mathbb{N}$. Using this relationship, we must prove that:

$$\frac{1}{4} \int_{-2}^{2} \frac{1}{\sqrt{4 - x^2}} L_n(x) L_m(x) dx = 0 \quad m \neq n \tag{4.1.19}$$

or, by (4.1.14):

$$2 \int_{-2}^{2} \frac{1}{\sqrt{4 - x^2}} T_{2^{n-1}} \left(\frac{x^2}{2} - 1 \right) T_{2^{m-1}} \left(\frac{x^2}{2} - 1 \right) dx \tag{4.1.20}$$

for $m \neq n$ and $m, n \in \mathbb{N}$. From symmetry of the integrand function, putting $t = \frac{x^2}{2} - 1$ and solving the integral we obtain the thesis. $\square$

Corollary 4.1.10. *Let $x = 2\cos\theta$; then the polynomials $L_n(x)$ admit the representation*

$$L_n(2\cos\theta) = 2\cos(2^n\theta)\ . \tag{4.1.21}$$

Proof. Note that, in this case, $|x| \leq 2$. Therefore we need to work with radicals of negative numbers. Substituting $x = 2\cos\theta$ in (4.1.15) we have:

$$L_n(2\cos\theta) = (\cos 2\theta - \imath \sin 2\theta)^{2^{n-1}} + (\cos 2\theta + \imath \sin 2\theta)^{2^{n-1}}$$

which can be rewritten by applying *Euler's identity*:

$$\left(e^{-\imath 2\theta}\right)^{2^{n-1}} + \left(e^{+\imath 2\theta}\right)^{2^{n-1}} = 2\cos(2^n\theta)$$

$\square$

We resume approximation (4.1.3) of $L_n(x)$ to prove that locally and for $|x_0| \leq 2$ the function $L_n(x)$ behaves like a cosine, while globally, in $[-2, 2]$, it oscillates with shorter and shorter periods in the neighborhoods of the endpoints, by means of the following theorem.

Theorem 4.1.11. *Let x_0 a generic maximum point of $L_n(x)$. For $n \geq 2$ we have*

$$L_n(x) = 2\cos(2^{n-1}k(x - x_0)) + o((x - x_0)^2) \tag{4.1.22}$$

where k is such that $|k| \geq 1$ and is increasing with x_0, for fixed n.

Proof. For $n = 2$ it is sufficient recall Theorem 4.1.2. In this case $k = 1$. Let us now suppose the claim to be true for some natural n and proceed by induction for $n + 1$:

$$\begin{aligned} L_{n+1}(x) = & L_n^2(x) - 2 = [2\cos(2^{n-1}k(x - x_0)) + o((x - x_0)^2)]^2 - 2 = \\ = & \ 4\cos^2[2^{n-1}k(x - x_0)] + o[(x - x_0)^4] + \\ + & 4 \ \cos[2^{n-1}k(x - x_0)] \ o[(x - x_0)^2] - 2 \end{aligned} \tag{4.1.23}$$

from which, by means of well known trigonometric formulas, we arrive to $L_{n+1}(x) = L_n^2 - 2 = 2\cos(2^n k(x - x_0)) + o((x - x_0)^2)$ if $x \to x_0$. From Remark (4.1.6), the point x_0 is a maximum point for $L_n(x)$ and $L_{n+1}(x)$. Now we aim to prove that $|k| \geq 1$. The second order Taylor expansion of the right hand side of (4.1.22), centered in x_0, is

$$2 - 2^{2(n-1)}k^2(x - x_0)^2 + o((x - x_0)^2) \tag{4.1.24}$$

For what concerns the left hand side of (4.1.22), we observe that $L_n(x_0) = 2$, being x_0 a maximum point. Let us observe that equation (4.1.15)

$$\begin{aligned} L_n(x) = & \left(\frac{x^2}{2} - 1 + \sqrt{\left(\frac{x^2}{2} - 1\right)^2 - 1}\ \right)^{2^{n-1}} + \\ + & \left(\frac{x^2}{2} - 1 - \sqrt{\left(\frac{x^2}{2} - 1\right)^2 - 1}\ \right)^{2^{n-1}} = L_n^+(x) + L_n^-(x) \end{aligned} \tag{4.1.25}$$

must be understood with values in the complex field, because, due to

$$\underbrace{\sqrt{2 + \sqrt{2 + \sqrt{2 + ... + \sqrt{2}}}}}_{n} = 2 \ \cos\left(\frac{\pi}{2^{n+1}}\right) < 2\,,$$

all the critical points have absolute value less or equal to 2. Then the derivative of $L_n(x)$ is

$$L_n'(x) = \frac{d}{dx}\left(L_n^+(x) + L_n^-(x)\right) \tag{4.1.26}$$

with

$$\frac{d}{dx}L_n^+(x) = 2^{n-1} \ \frac{x \ L_n^+(x)}{\sqrt{\left(\frac{x^2}{2} - 1\right)^2 - 1}}$$

and

$$\frac{d}{dx}L_n^-(x) = -2^{n-1} \ \frac{x \ L_n^-(x)}{\sqrt{\left(\frac{x^2}{2} - 1\right)^2 - 1}}$$

whence

$$L_n'(x) = 2^{n-1} \ \frac{x}{\sqrt{\left(\frac{x^2}{2} - 1\right)^2 - 1}} \left[L_n^+(x) - L_n^-(x)\right] = \frac{2^n}{\sqrt{x^2 - 4}}\left[L_n^+(x) - L_n^-(x)\right] \tag{4.1.27}$$

which must vanish when calculated in $x = x_0$, maximum point. For the sake of simplicity, let us consider only $x > 0$. The second order derivative is

$$L_n''(x) = \frac{2^n \left\{(x^2 - 4)\left[\frac{d}{dx}L_n^+(x) - \frac{d}{dx}L_n^-(x)\right] - x\left(L_n^+(x) - L_n^-(x)\right)\right\}}{(x^2 - 4)\sqrt{x^2 - 4}} \tag{4.1.28}$$

which can be rewritten as

$$L_n''(x) = 2^n \left[\frac{\frac{d}{dx}\left(L_n^+(x) - L_n^-(x)\right)}{\sqrt{x^2-4}} - \frac{xL_n'(x)}{2^n(x^2-4)} \right] =$$
$$= 2^n \left[\frac{2^n\left(L_n^+(x) + L_n^-(x)\right)}{(\sqrt{x^2-4})^2} - \frac{xL_n'(x)}{2^n(x^2-4)} \right] \ . \tag{4.1.29}$$

We calculate it in $x = x_0$:

$$L_n''(x_0) = 2^n \left[\frac{2^n L_n(x_0)}{x_0^2-4} - \frac{x_0 L_n'(x_0)}{2^n(x_0^2-4)} \right] = \frac{2^{2n}L_n(x_0)}{x_0^2-4} = \frac{2^{2n+1}}{x_0^2-4} \tag{4.1.30}$$

since $L_n'(x_0) = 0$ and $L_n(x_0) = 2$. Thus we have the Taylor expansion

$$L_n(x) = 2 + \frac{2^{2n}}{x_0^2-4}(x-x_0)^2 + o((x-x_0)^2) \ . \tag{4.1.31}$$

Equating it to (4.1.24) gives

$$\frac{2^{2n}}{x_0^2-4} = -2^{2(n-1)}k^2 \ \Rightarrow \ \frac{4}{4-x_0^2} = k^2 \ \Rightarrow \ k = \pm\frac{1}{\sqrt{1-x_0^2/4}} \tag{4.1.32}$$

It is easy to verify that k is such that $|k| \geq 1$ and increasing with $x_0 > 0$. □

As those of the first kind, the *Chebyshev polynomial of the second kind* are defined by a recurrence relation [161, 18, 79]:

$$\begin{cases} U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x) & n \geq 2 \\ U_0(x) = 1, \quad U_1(x) = 2x \end{cases}$$

which is satisfied by

$$U_n(x) = \sum_{k=0}^{n} (x+\sqrt{x^2-1})^k (x-\sqrt{x^2-1})^{n-k} \quad \forall x \in [-1,1] \ . \tag{4.1.33}$$

This relation is equivalent to

$$U_n(x) = \frac{\left(x+\sqrt{x^2-1}\right)^{n+1} - \left(x-\sqrt{x^2-1}\right)^{n+1}}{2\sqrt{x^2-1}} \tag{4.1.34}$$

where the radicals assume real values for each $x \in (-1,1)$. From continuity of function (4.1.33), we observe that (4.1.34) can be extended by continuity in $x = \pm 1$, too. It can therefore be put $U_n(\pm 1) = (\pm 1)^n(n+1)$ in (4.1.34).

Proposition 4.1.12. *For each $n \geq 1$ we have*

$$\prod_{i=1}^{n} L_i(x) = U_{2^n-1}\left(\frac{x^2}{2} - 1\right) \tag{4.1.35}$$

Proof. Also in this case, the formulas are defined on complex numbers. By (4.1.34) we must demonstrate that:

$$\prod_{i=1}^{n} L_i(x) = \frac{\left(\frac{x^2}{2} - 1 + \sqrt{\left(\frac{x^2}{2}-1\right)^2 - 1}\right)^{2^n}}{2\sqrt{\left(\frac{x^2}{2}-1\right)^2 - 1}} - $$
$$+ \frac{\left(\frac{x^2}{2} - 1 - \sqrt{\left(\frac{x^2}{2}-1\right)^2 - 1}\right)^{2^n}}{2\sqrt{\left(\frac{x^2}{2}-1\right)^2 - 1}}. \tag{4.1.36}$$

We proceed by induction on n. First of all, let us observe that when $n = 1$ we have $L_1(x) = x^2 - 2 = U_1\left(\frac{x^2}{2} - 1\right)$.

For the inductive step, let $n > 1$ be an integer, and assume that the proposition holds for n; by multiplying both sides of (4.1.35) by $L_{n+1}(x)$ we obtain:

$$\prod_{i=1}^{n+1} L_i(x) = U_{2^n-1}\left(\frac{x^2}{2} - 1\right)\ L_{n+1}(x) \tag{4.1.37}$$

Thus, the proposition holds for $n + 1$ if

$$U_{2^{n+1}-1}\left(\frac{x^2}{2} - 1\right) = U_{2^n-1}\left(\frac{x^2}{2} - 1\right)\ L_{n+1}(x) \tag{4.1.38}$$

Let's focus on the right hand side, setting $t = \dfrac{x^2}{2} - 1$:

$$= \underbrace{\sum_{k=0}^{2^n-1} (t + \sqrt{t^2-1})^{k+2^n}(t - \sqrt{t^2-1})^{2^n-1-k}}_{B} +$$
$$+ \underbrace{\sum_{k=0}^{2^n-1} (t + \sqrt{t^2-1})^{k}(t - \sqrt{t^2-1})^{2^{n+1}-1-k}}_{A} \tag{4.1.39}$$

where

$$\begin{aligned}
A &= \sum_{k=0}^{2^{n+1}-1} (t + \sqrt{t^2-1})^{k}(t - \sqrt{t^2-1})^{2^{n+1}-1-k} + \\
&- \sum_{k=2^n}^{2^{n+1}-1} (t + \sqrt{t^2-1})^{k}(t - \sqrt{t^2-1})^{2^{n+1}-1-k} \\
B &= \sum_{k=0}^{2^n-1} (t + \sqrt{t^2-1})^{k+2^n}(t - \sqrt{t^2-1})^{2^n-1-k} \\
&= \sum_{j=2^n}^{2^{n+1}-1} (t + \sqrt{t^2-1})^{j}(t - \sqrt{t^2-1})^{2^{n+1}-1-j}
\end{aligned} \tag{4.1.40}$$

therefore $A + B$ is just equal to $U_{2^{n+1}-1}(t)$, and this completes the proof. □

After having calculated $L_n(2\cos\theta)$ in (4.1.21), now let us calculate $U_{2^n-1}(x^2/2-1)$ for $x = 2\cos\theta$ by (4.1.36).

$$\prod_{i=1}^{n} L_i(2\cos\theta) = \frac{(\cos 2\theta + \imath \sin 2\theta)^{2^n} - (\cos 2\theta - \imath \sin 2\theta)^{2^n}}{2\imath \sin 2\theta}$$

from which and *Euler identity*, we get

$$\prod_{i=1}^{n} L_i(2\cos\theta) = \frac{\left(e^{+\imath 2\theta}\right)^{2^n} - \left(e^{-\imath 2\theta}\right)^{2^n}}{2\imath \sin 2\theta} = \frac{\sin\left(2^{n+1}\theta\right)}{\sin 2\theta} \ . \tag{4.1.41}$$

For $|x| \leq 2$ we can show another formula for L_n. Let us come back to (4.1.15):

$$L_n(x) = \left(\frac{x^2}{2} - 1 - \sqrt{\left(\frac{x^2}{2} - 1\right)^2 - 1}\right)^{2^{n-1}} + \\ + \left(\frac{x^2}{2} - 1 + \sqrt{\left(\frac{x^2}{2} - 1\right)^2 - 1}\right)^{2^{n-1}} \tag{4.1.42}$$

In this case $|x| \leq 2$; we change the sign inside the radical, factorizing out the imaginary unit:

$$L_n(x) = \left(\frac{x^2}{2} - 1 - \imath\sqrt{1 - \left(\frac{x^2}{2} - 1\right)^2}\right)^{2^{n-1}} + \\ + \left(\frac{x^2}{2} - 1 + \imath\sqrt{1 - \left(\frac{x^2}{2} - 1\right)^2}\right)^{2^{n-1}} \tag{4.1.43}$$

We then calculate the powers of two complex conjugate numbers L_n^+ and L_n^-, depending on the variable x. With the notation introduced in (4.1.25), the absolute value of both complex numbers is unitary, since

$$|L_n^+| = |L_n^-| = \sqrt{\left(\frac{x^2}{2} - 1\right)^2 + 1 - \left(\frac{x^2}{2} - 1\right)^2} = 1 \ . \tag{4.1.44}$$

Moreover, since $L_1(\pm\sqrt{2}) = 0$; $L_2(\pm\sqrt{2}) = -2$; $L_n(\pm\sqrt{2}) = 2 \quad \forall n \geq 3$, then the argument of $L_n(\pm\sqrt{2})$ is 0 for every $n \geq 3$. In the other cases, since, when $|x| \leq 2$, we can write $x = 2\cos(\vartheta)$, thus $\dfrac{x^2}{2} - 1 = \cos(2\vartheta)$; thus for $|x| \neq \sqrt{2}$ we can also put

$$\vartheta(x) = \frac{1}{2}\arctan\left[\frac{\sqrt{1 - \left(\frac{x^2}{2} - 1\right)^2}}{\frac{x^2}{2} - 1}\right] + b\pi \tag{4.1.45}$$

where b is a binary digit; thus, using (4.1.21), we obtain $L_n(x) = 2\cos\left(2^n \vartheta(x)\right)$.

By setting further

$$\theta(x) = \frac{1}{2}\arctan\left[\frac{\sqrt{1-\left(\frac{x^2}{2}-1\right)^2}}{\frac{x^2}{2}-1}\right] \tag{4.1.46}$$

we can write:

$$L_n(x) = 2\cos\left(2^n\theta(x) + 2^n b\pi\right) = 2\cos\left(2^n\theta(x)\right) \ . \tag{4.1.47}$$

On the other hand, for very large $|x|$, considering the iterative structure of the map L_n, we deduce immediately the asymptotic formula $L_n(x) \sim (x^2-2)^{2^{n-1}}$.

4.1.4 $M_n^a = 2a\left(M_{n-1}^a\right)^2 - \frac{1}{a}$ map.

The considerations made in the previous sections on the map L_n can be extended to an entire class of maps, obtained through the iterated formula $M_n^a = 2a\left(M_{n-1}^a\right)^2 - \frac{1}{a}$, $a > 0$, with $M_0^a(x) = x$. It follows that

$$M_0^a(x) = x \quad ; \quad M_1^a(x) = 2ax^2 - \frac{1}{a} \quad ; \quad M_2^a(x) = 8a^3x^4 - 8ax^2 + \frac{1}{a} \quad ... \tag{4.1.48}$$

Note that the map L_n is a particular case of M_n^a, obtained by setting $a = 1/2$. We briefly show that the map M_n^a satisfies similar properties as those proven for L_n.

Proposition 4.1.13. *For $n \geq 2$ we have*

$$M_n^a(x) = \frac{1}{a} \cdot \cos(a\ 2^n x) + o(x^2) \tag{4.1.49}$$

Proof. We must show that:

$$M_n^a(x) = \frac{1}{a} - a2^{2n-1}x^2 + o(x^2) \tag{4.1.50}$$

where we take into account the McLaurin polynomial of cosine. We proceed by induction. For $n = 2$:

$$M_2^a(x) = 2a\left(2ax^2 - \frac{1}{a}\right)^2 - \frac{1}{a} = \frac{1}{a} - 8ax^2 + o(x^2) \tag{4.1.51}$$

Let us consider the second order McLaurin polynomial of $\frac{1}{a}\cdot\cos(4ax)$: it is just $\frac{1}{a} - 8ax^2 + o(x^2)$, thus verifying the relation for $n = 2$. Let us now assume (4.4.33) is true for a generic n, and deduce that it is also true for $n+1$:

$$\begin{aligned} M_{n+1}^a &= 2a\left(M_n^a\right)^2 - \frac{1}{a} = 2a\left[\frac{1}{a} - a2^{2n-1}x^2 + o(x^2)\right]^2 - \frac{1}{a} = \\ &= \frac{1}{a} - a2^{2n+1}x^2 + o(x^2) \end{aligned} \tag{4.1.52}$$

which is in fact the McLaurin polynomial of $\frac{1}{a}\cdot\cos(a\ 2^{n+1}x)$. ∎

Proposition 4.1.14. *At each iteration the zeros of the map $M_n^a (n \geq 1)$ have the form*

$$\pm\frac{1}{2a} \cdot \sqrt{2 \pm \sqrt{2 \pm \sqrt{2 \pm \sqrt{2 \pm ... \pm \sqrt{2}}}}} \tag{4.1.53}$$

Proof. It is obvious that at $n = 1$ this statement is valid. Now assume that the (4.4.37) is valid for n. We have to prove that it is valid for $n + 1$:

$$x^2 = \frac{1}{2a^2} \pm \frac{1}{4a^2} \cdot \sqrt{2 \pm \sqrt{2 \pm \sqrt{2 \pm \sqrt{2 \pm ... \pm \sqrt{2}}}}} \tag{4.1.54}$$

and placing under the radical sign

$$x = \pm\sqrt{\frac{1}{2a^2} \pm \frac{1}{4a^2} \cdot \sqrt{2 \pm \sqrt{2 \pm \sqrt{2 \pm \sqrt{2 \pm ... \pm \sqrt{2}}}}}} \tag{4.1.55}$$

the thesis is obtained. □

Proposition 4.1.15. *For each $n \geq 1$ we have*

$$M_n^a(x) = \frac{1}{a}\, T_{2^{n-1}}\left(2a^2x^2 - 1\right) \tag{4.1.56}$$

Proof. We must show that

$$M_n^a(x) = \frac{\left(2a^2x^2 - 1 - \sqrt{\left(2a^2x^2 - 1\right)^2 - 1}\right)^{2^{n-1}}}{2a} + \\ + \frac{\left(2a^2x^2 - 1 + \sqrt{\left(2a^2x^2 - 1\right)^2 - 1}\right)^{2^{n-1}}}{2a} \tag{4.1.57}$$

This is verified for $n = 1$:

$$M_1^a(t(x)) = \frac{t + \sqrt{t^2 - 1} + t - \sqrt{t^2 - 1}}{2a} = \frac{2t}{2a} = 2ax^2 - \frac{1}{a} \tag{4.1.58}$$

where $t = 2a^2x^2 - 1$. By assumption, we suppose (4.1.56) true for n and by $M_{n+1}^a(t(x)) = 2a(M_n^a)^2(t(x)) - \frac{1}{a}$; we get finally the thesis for $n + 1$:

$$M_{n+1}^a(t(x)) = \frac{\left(t - \sqrt{t^2 - 1}\right)^{2^n}}{2a} + \frac{\left(t + \sqrt{t^2 - 1}\right)^{2^n}}{2a} \tag{4.1.59}$$

□

Remark 4.1.16. For $|x| \leq \dfrac{1}{a}$, substituting $x = \frac{1}{a}\cos\theta$ in (4.1.57) we obtain:

$$M_n^a\left(\frac{1}{a}\cos\theta\right) = \frac{1}{a}\cos\left(2^n\theta\right)\ . \tag{4.1.60}$$

Proposition 4.1.17. *For each $n \geq 1$ we have*

$$\prod_{i=1}^{n} M_i^a(x) = \left(\frac{1}{2a}\right)^n U_{2^n-1}\left(2a^2x^2 - 1\right) \tag{4.1.61}$$

Proof. We must first show that the formula is true for $n = 1$:

$$\frac{1}{2a}\,U_1(t) = \frac{\left(t + \sqrt{t^2-1}\right)^2 - \left(t - \sqrt{t^2-1}\right)^2}{4a\sqrt{t^2-1}} = \frac{t}{a} \tag{4.1.62}$$

which is true because $\frac{t}{a} = 2ax^2 - \frac{1}{a} = M_1^a(x)$. We assume, then, that formula (4.1.61) is true for n. We must now show that it is also true for $n+1$. To this aim, let us multiply both sides of (4.1.61) by $M_{n+1}(t)$, expressed in (4.1.59); the right-hand side becomes

$$\begin{aligned}&\left(\frac{1}{2a}\right)^{n+1} \frac{\left(t + \sqrt{t^2-1}\right)^{2^{n+1}} - \left(t - \sqrt{t^2-1}\right)^{2^{n+1}}}{2\sqrt{t^2-1}} = \\ &= \left(\frac{1}{2a}\right)^{n+1} U_{2^{n+1}-1}\left(2a^2x^2 - 1\right)\end{aligned} \tag{4.1.63}$$

□

Proposition 4.1.18. *For each $n \geq 2$ we have*

$$\frac{d}{dx}M_n^a(x) = (4a)^n\ x \prod_{i=1}^{n-1} M_i^a(x) \tag{4.1.64}$$

Proof. We have first to prove it is true for $n = 2$:

$$\frac{d}{dx}M_2^a(x) = \frac{d}{dx}\left[2a\left(2ax^2 - \frac{1}{a}\right)^2 - \frac{1}{a}\right] = 4axM_1^a(x) \tag{4.1.65}$$

Assume it is true for n and deduce that (4.1.64) is true for $n+1$, too. In fact $M_{n+1}^a = 2a(M_n^a)^2 - \frac{1}{a}$. Write

$$\frac{d}{dx}M_{n+1}^a(x) = 2a\frac{d}{dx}(M_n^a)^2 = 4aM_n^a\ \frac{d}{dx}M_n^a \tag{4.1.66}$$

and using (4.1.64) we arrive to:

$$\frac{d}{dx}M_{n+1}^a(x) = 4aM_n^a \cdot \left[(4a)^n\ x \prod_{i=1}^{n-1} M_i^a(x)\right] = (4a)^{n+1}\ x \prod_{i=1}^{n} M_i^a(x) \tag{4.1.67}$$

□

Remark 4.1.19. It can be easily shown that, when $|x| \leq \dfrac{1}{a}$, replacing $x = \frac{1}{a}\cos\theta$ in the expression of $U_{2^n-1}\left(2a^2x^2-1\right)$ and taking into account that $2a^2x^2 - 1 = \cos(2\theta)$:

$$\frac{\left(2a^2x^2 - 1 + \sqrt{(2a^2x^2-1)^2 - 1}\right)^{2^n} - \left(2a^2x^2 - 1 - \sqrt{(2a^2x^2-1)^2-1}\right)^{2^n}}{2\sqrt{(2a^2x^2-1)^2-1}} \tag{4.1.68}$$

we again get the trigonometric expression (4.1.41).

Factorizing out the minus sign in (4.1.57) and carrying out the imaginary unit from radical, we obtain:

$$M_n^a(x) = \frac{1}{2a}\left[(M_n^{a,+})^{2^{n-1}} + (M_n^{a,-})^{2^{n-1}}\right]$$

The module of both complex numbers $M_n^{a,+}$ and $M_n^{a,-}$ is unitary; in fact:

$$|M_n^{a,+}(x)| = |M_n^{a,-}(x)| = \sqrt{(2a^2x^2-1)^2 + 1 - (2a^2x^2-1)^2} = 1 \tag{4.1.69}$$

Then

$$\begin{aligned} M_n^a(x) &= \frac{e^{i2^n\vartheta(x)} + e^{-i2^n\vartheta(x)}}{2a} = \frac{1}{a}\cos\left(2^n\vartheta(x)\right) \\ \vartheta(x) &= \frac{1}{2}\arctan\left[\frac{\sqrt{1-(2a^2x^2-1)^2}}{2a^2x^2-1}\right] + b\pi = \theta(x) + b\pi \end{aligned} \tag{4.1.70}$$

with b a binary digit, and

$$M_n^a(x) = \frac{1}{a}\cos\left(2^n\theta(x) + 2^n b\pi\right) = \frac{1}{a}\cos\left(2^n\theta(x)\right) \tag{4.1.71}$$

If $x = \pm\frac{\sqrt{2}}{2a}$: $M_1^a(\pm\frac{\sqrt{2}}{2a}) = \frac{1}{a}\cos\left(\frac{\pi}{2}\right) = 0$; $M_2^a(\pm\frac{\sqrt{2}}{2a}) = \frac{1}{a}\cos(\pi) = -\frac{1}{a}$; $M_n^a(\pm\frac{\sqrt{2}}{2a}) = \frac{1}{a}\cos(2^{n-2}\pi) = \frac{1}{a}$; $n \geq 3$. Then the argument of $M_n^a\left(\pm\frac{\sqrt{2}}{2a}\right)$ is 0 for every $n \geq 3$. For very large $|x|$, considering the iterative structure of the map M_n^a, we deduce immediately the asymptotic formula: $M_n^a \sim (2a)^{2^{n-1}-1}\left(2ax^2 - \frac{1}{a}\right)^{2^{n-1}}$, $\forall n \geq 1$.

4.2 Orthogonal polynomials and Riesz bases applied to the solution of Love's equation.

In 1949, E. R. Love [123] considered the electrostatic field generated by two identical circular co-axial conducting disks either at equal, or at equal and opposite, potentials, the potential at infinity being taken equal to zero. He established a celebrated expression for the potential, involving the solution of an integral equation of well-known type, much simpler than that considered by other authors in previous works.

Love's integral equation is a Fredholm equation of the second kind. It has found many applications in several applied physics fields such as polymer structures, aerodynamics, fracture mechanics, hydrodynamics and elasticity engineering. Recently, a polynomial expansion scheme has been proposed by M. Agida and A. S. Kumar [4], as an analytical method for solving Love's integral equation in the case of a rational kernel. Their study is concerned with the calculation of the normalized field created conjointly by two similar plates of radius R, separated by a distance kR, where k is a positive real parameter, and at equal or opposite potential, with zero potential at infinity; the solution of this problem solves a Love's second kind integral equation (see also [124], [158]).

We propose two different approaches to this problem. In section 4.2.1, starting from a classical technique, based on the expansion of the solution in orthogonal polynomials, we employ a class of polynomials introduced in [186], in order to solve a modified version of the original Love's equation. In section 4.2.2 we recall a work by M. Norgren and B. L. G. Jonsson [138], and we show that their results are still valid expanding the solution of the Love's integral equation with respect to a non-harmonic Fourier cosine series, which is a particular case of Riesz basis [179].

For literature related to the numerical solutions of singular integral equations of the deterministic type, we refer to the fundamental book by L. Fox and I. B. Parker [75], where different analytical methods for the solution of random integral equations have been investigated.

4.2.1 Chebyshev polynomials approach to Love's problem

Two leading cases of the problem are here considered. They are: to specify the field generated by two identical circular co-axial conducting disks a) at equal, and b) at equal and opposite, potentials, the potential at infinity being taken as zero. The results established by Love are as follows, the upper sign referring to the case of equally charged disks and the lower to that of oppositely charged disks. For theorem 4.2.1 we refer to figs. 1 and 2 in [123].

Theorem 4.2.1. *[123] In the two leading cases described above, the potential at any point (p, ζ, ζ'), specified by its distance $r = \rho a$ from the axis of the disks and its axial distances $z = \zeta a$ and $z' = \zeta' a$ from their planes, is*

$$\frac{V_0}{\pi}\int_{-1}^{1}\left[\frac{1}{\sqrt{\rho^2+(\zeta+it)^2}}\pm\frac{1}{\sqrt{\rho^2+(\zeta'+it)^2}}\right]f(t)dt, \tag{4.2.1}$$

where V_0 is potential of the disks, a is the radius of the disks, each square root has positive real part, and $f(t)$ is the solution of the integral equation

$$f(x)\pm\frac{1}{\pi}\int_{-1}^{1}\frac{k}{k^2+(x-t)^2}f(t)dt=1,\quad (|x|\leq 1) \tag{4.2.2}$$

where k is the spacing parameter.

Theorem 4.2.2. *[123] For every positive k, equation (4.2.2) has a continuous solution, and no other solution: it is real and even, and is specifiable by the Neumann series*

$$f(x)=1+\sum_{n=1}^{\infty}(\mp 1)^n\int_{-1}^{1}K_n(x,t)dt, \tag{4.2.3}$$

where the iterated kernels $K_n(x,t)$ are given by

$$K_1(x,t)=\frac{1}{\pi}\frac{k}{k^2+(x-t)^2},\quad K_n(x,t)=\int_{-1}^{1}K_{n-1}(x,s)K_1(s,t)ds.$$

for $n\in\mathbb{N}$, $n>1$.

Theorem 4.2.3. *[123] The capacity of each disk in the two cases is*

$$\frac{a}{\pi}\int_{-1}^{1}f(t)dt,$$

and the components of the field at all points not on the disks are given by the appropriate formal differentiations of (4.2.1).

For the solution of the problems we will refer to [75]. When the upper and lower disks are at potentials V_0 and $\pm V_0$, the potential V at any point, whose spheroidal coordinates are (μ, η) with respect to the upper disk and (μ', η') with respect to the lower one, is expressed in terms of Legendre functions. The upper disk, specified in cylindrical polar coordinates (r, θ, z) by $r \leq a$ and $z = 0$, is taken as "focal disk" $\eta = 0$ of spheroidal coordinates (μ, η); in actual study these are such that $-2 \leq \mu \leq 2$, $\eta \geq 0$.

Then equation (4.2.1) can be rewritten in the form

$$\frac{V_0}{2\pi}\int_{-2}^{2}\left[\frac{1}{\sqrt{\rho^2+(\zeta+it/2)^2}} \pm \frac{1}{\sqrt{\rho^2+(\zeta'+it/2)^2}}\right] f(t/2)dt, \tag{4.2.4}$$

where each square root has positive real part, and $f(t)$ is the solution of the integral equation

$$f(x) \pm \frac{1}{2\pi}\int_{-2}^{2}\frac{k}{k^2+(x-t/2)^2} f(t/2)dt = 1, \quad (|x| \leq 2) \tag{4.2.5}$$

By linear transformation $t = 2y$, both equations can be reduced to Love's original form. In (4.2.5) we put $k = 1$ and consider positive sign, so

$$f(x) + \frac{1}{2\pi}\int_{-2}^{2}\frac{1}{1+\left(x-\frac{t}{2}\right)^2} f\left(\frac{t}{2}\right) dt = 1, \quad (|x| \leq 2) \tag{4.2.6}$$

We replace $x \mapsto \frac{x^2}{2} - 1$ in equation (4.2.6), thus

$$f\left(\frac{x^2}{2}-1\right) + \frac{1}{2\pi}\int_{-2}^{2}\frac{1}{1+\left(\frac{x^2-t}{2}-1\right)^2} f\left(\frac{t}{2}\right) dt = 1$$

We can find a Chebyshev-series solution if we write

$$f(x) = \sum_{r=0}^{\infty} a_r T_r(x)$$

substitute it in (4.2.6), interchange the order of integration and summation in the first term, arriving at the equation

$$\sum_{r=0}^{\infty} a_r T_r\left(\frac{x^2}{2}-1\right) + \frac{1}{2\pi}\sum_{s=0}^{\infty} a_s \int_{-2}^{2}\frac{T_s\left(\frac{t}{2}\right)}{1+\left(\frac{x^2-t}{2}-1\right)^2}dt = 1 \tag{4.2.7}$$

for $|x| \leq 2$. If we can now determine the expansion

$$\frac{1}{2}\int_{-2}^{2}\frac{T_s\left(\frac{t}{2}\right)}{1+\left(\frac{x^2-t}{2}-1\right)^2}dt = \sum_{r=0}^{\infty} b_{sr} T_r\left(\frac{x^2}{2}-1\right),$$

we can equate the corresponding coefficients of each $T_r(x)$ on both sides of equation (4.2.6), which is legitimate since the Chebyshev polynomials form a complete set of independent functions, to produce an infinite set of algebraic equations for the required coefficients a_r, given by

$$a_r + \sum_{s=0}^{\infty} a_s b_{sr} = 0, \quad r = 1, 2, \ldots \tag{4.2.8}$$

and, for $r = 0$:

$$a_0 + \sum_{s=0}^{\infty} a_s b_{s,0} = 1.$$

The a_r will decrease rapidly for sufficiently large r, so that in a convenient method of solving (4.2.8) we select the first $n+1$ rows and columns, perform Gauss elimination and back-substitution for the last few coefficients a_n, a_{n-1}, a_{n-2}, say, decide by inspection whether convergence is sufficiently rapid for the required precision with this selected value of n, and if necessary add some extra rows and columns with only a small additional amount of work.

Let's go back to equation (4.2.7). Let $J \subset \mathbb{N}$ be the subset of natural numbers so defined:

$$J = \{1, 2, 4, \dots\} = \{2^{r-1} | r \in \mathbb{N}\}$$

and rewrite (4.2.7) in this way:

$$\sum_{r=0}^{\infty} a_r T_r \left(\frac{x^2}{2} - 1 \right) + \sum_{s=0}^{\infty} a_s \sum_{r=0}^{\infty} c_{sr} T_r \left(\frac{x^2}{2} - 1 \right) = 1$$

where $c_{sr} = \frac{b_{sr}}{\pi}$. Then:

$$\sum_{r \in J} a_r T_r \left(\frac{x^2}{2} - 1 \right) + \sum_{r \notin J} a_r T_r \left(\frac{x^2}{2} - 1 \right)$$

$$+ \sum_{s=0}^{\infty} a_s \left[\sum_{r \in J} c_{sr} T_r \left(\frac{x^2}{2} - 1 \right) + \sum_{r \notin J} c_{sr} T_r \left(\frac{x^2}{2} - 1 \right) \right] = 1$$

By Proposition 4.1.8 in Section 4.1, we have:

$$\frac{1}{2} \sum_{r=1}^{\infty} a_r L_r(x) + \sum_{r \notin J} a_r T_r \left(\frac{x^2}{2} - 1 \right) +$$

$$+ \frac{1}{2} \sum_{s=0}^{\infty} a_s \sum_{r=1}^{\infty} c_{sr} L_r(x) + \sum_{s=0}^{\infty} a_s \sum_{r \notin J} c_{sr} T_r \left(\frac{x^2}{2} - 1 \right) = 1$$

We also note that solving (4.2.8), a subset of first $n+1$ rows and columns selected to perform Gauss elimination, is due to Lucas-Lehmer polynomials. They not only cannot by themselves guarantee the convergence to the solution, but also their contributes can be neglected. In fact, by above reasoning, since

$$f(x) = \sum_{r \notin J} a_r T_r \left(\frac{x^2}{2} - 1 \right) + \frac{1}{2} \sum_{r \in J} a_r L_r(x)$$

hence

$$\left| f(x) - \sum_{r \notin J} a_r T_r \left(\frac{x^2}{2} - 1 \right) \right| \le \frac{1}{2} \sum_{r \in J} |a_r| = \frac{1}{2} \sum_{r=1}^{\infty} |a_{2^{r-1}}|$$

Accordingly, when the term in the right-hand side can be considered "small" with respect to other contributions, a convenient method of solving (4.2.8) should be to select the first $n+1$ rows and columns, perform Gauss elimination and back-substitution for the last few coefficients a_n, a_{n-1}, a_{n-2}, say, and delete terms due to Lucas-Lehmer polynomials.

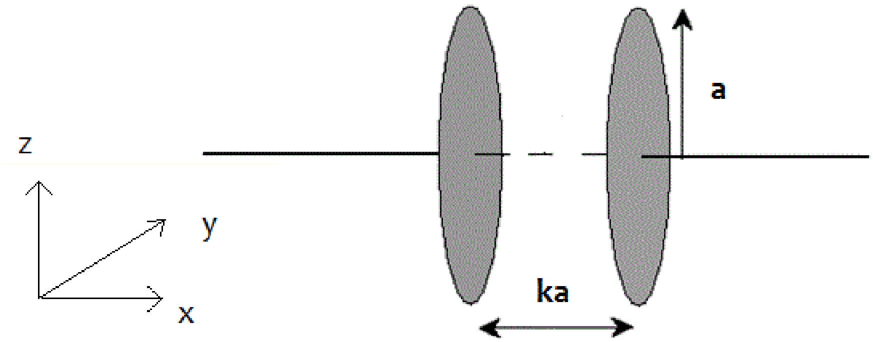


Figure 4.4: The circular parallel plate capacitor, viewed as cylindrical volume, where the bases of the cylinder are the circular armors of the capacitor.

4.2.2 An alternative approach to Love's problem: Non-harmonic Fourier series

The capacitance of the circular parallel plate capacitor can be calculated by expanding the solution of the Love's integral equation in terms of a Fourier cosine series. In previous literature, this kind of expansion has been carried out numerically, leading to accuracy problems at small plate separations. M. Norgren and B. L. G. Jonsson [138] calculate analytically all expansion integrals in terms of the Sine and Cosine integrals. Hence, they approximate the kernel, using considerably large matrices, resulting in improved numerical accuracy for the capacitance. Previously, G. T. Carlson and B. L. Illman [38], solve the Love's equation through an expansion of the kernel into a Fourier-cosine series. To calculate the expansion coefficients of the kernel, Carlson and Illman use numerical integration. Hence, as noted in [138], their method is limited by a combination of the accuracy of the integration and the large number of terms needed. The accumulated errors effectively limit the expansion to about 100 terms, which is insufficient for the convergence at very small separations. Let us observe that both the methods here recalled make use of orthogonal expansions.

From Chapter 3 we know that the family of exponentials $\{e^{int}\}_{n\in\mathbb{Z}}$ forms an orthonormal basis in $L^2(-\pi,\pi)$, and that $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ is still a Riesz basis under assumption of Kadec's result (Theorem 3.2.7). Using this result we now approach the problem described in [138]. The circular parallel plate capacitor is depicted in Figure 4.4. The distance between the circular plates is here put equal to their common radius. Accordingly, the normalized separation between the plates, k constant, is set for the sake of simplicity equal to 1. The model is idealized in the sense that the plates have zero thickness.

The capacitance of the parallel plate capacitor is [38]

$$C = 4\varepsilon_0 a \int_0^1 f(s)ds, \tag{4.2.9}$$

where a is the radius of the circular plate and the function $f(s)$ is the solution of the modified Love's integral equation

$$f(s) - \int_0^1 K(s,t)f(t)dt = 1, \quad 0 \le s \le 1, \tag{4.2.10}$$

with kernel

$$K(s,t) = \frac{1}{\pi}\left[\frac{1}{1+(s-t)^2} + \frac{1}{1+(s+t)^2}\right]. \tag{4.2.11}$$

To solve equation (4.2.10) numerically, we follow the approach in [38] and expand the kernel and the unknown function into the (non-harmonic) Fourier-cosine expansion in terms of the functions

$$\widetilde{\psi_n(s)} = \sqrt{2-\delta_{n0}}\cos(\lambda_n s), \quad n = 0, 1, \dots$$

which in our study have been orthonormalized to fulfil the orthogonality relation

$$\int_0^\pi \psi_n(s)\psi_m(s)ds = \delta_{mn}$$

and satisfy Kadec's assumption on $L = \sup_n |\lambda_n - n| < 1/4$. Here δ_{mn} denotes the Kronecker delta function.

This orthonormalization process is shown in the following

Theorem 4.2.4 (Orthonormalization process)**.** *Let us consider $L^2(-\pi,\pi)$ and a sequence of real numbers $\{\lambda_n\}_{n\in\mathbb{Z}}$ which satisfies Kadec's assumption. Let $P = (I-S)^{-1} = \sum_{m=0}^{\infty} S^m$, where $S(f)(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n)(e^{inx} - e^{i\lambda_n x})$ and $\{\hat{f}(n)\}$ are the Fourier coefficients of f. Then $P(e^{i\lambda_n x}) = e^{inx}$ for each $n \in \mathbb{Z}$.*

Proof. By Kadec's theorem, we have that $\|S\| < 1$. Hence, $P = (I-S)^{-1} = \sum_{m=0}^{\infty} S^m$. To show that $P(e^{i\lambda_n x}) = e^{inx}$, we write

$$e^{i\lambda_n x} = (I-S)e^{inx} = e^{inx} - \sum_k c_k(e^{ikx} - e^{i\lambda_k x})$$

where $c_k = \langle e^{inx}, e^{ikx}\rangle$, thus

$$e^{inx} - e^{i\lambda_n x} = \sum_k \delta_{n,k}(e^{ikx} - e^{i\lambda_k x})$$

which proves the theorem. □

In this way we have orthonormalized the Riesz basis $\{e^{i\lambda_n x}\}$, in an easy way. Further results on the orthonormalization of more complex Riesz bases $\{\phi(t-n)\}_{n\in\mathbb{Z}}$, applied for example to the study of a "digital filter", can be found in [130]. For our purposes, it is sufficient theorem 4.2.4.

Carrying out the expansions of $f(s)$ and $K(s,t)$ in terms of $\{\psi_n\}$, we obtain

$$f(s) = \sum_{m=0}^{\infty} f_m\psi_m(s), \quad f_m = \int_0^\pi f(s)\psi_m(s)ds \tag{4.2.12}$$

$$K(s,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} K_{mn}\psi_m(s)\psi_n(t), \tag{4.2.13}$$

where

$$K_{mn} = \int_0^\pi \int_0^\pi K(s,t)\psi_n(t)\psi_m(s)dsdt,$$

These equations yield the following infinite linear system of equations for the coefficients $\{f_n\}_{n=0}^\infty$:

$$\sum_{n=0}^\infty (\delta_{mn} - K_{mn})f_n = \delta_{m0}, \quad m = 0, 1, \ldots \tag{4.2.14}$$

From (4.2.9), (4.2.12), from the orthonormalization process described in theorem 4.2.4 and guaranteed by Kadec's assumption, which allows us to expand the kernel and the unknown function into the (non-harmonic) Fourier-cosine expansion in terms of the functions $\{\cos(\lambda_n s)\}$, the capacitance reduces to $C = 4\varepsilon_0 a f_0$ where f_0 is simply the $(0,0)$-element in the inverse of the matrix with elements $\delta_{mn} - K_{mn}$, as obtained in [138].

Furthermore, Norgren and Jonsson derive the analytical expressions for the expansion of the kernel $K(s,t)$. Proceeding as in [138], it is easy to prove that, in the general case when $m \neq n$ and $m, n > 0$:

$$K_{mn} = \frac{2}{\pi}\tilde{I}_3(n\pi, m\pi) \tag{4.2.15}$$

where $\tilde{I}_3(n\pi, m\pi) = P\ I_3(\lambda_n\pi, \lambda_m\pi)$, with P as in theorem 4.2.4 and I_3 is defined as in [138]. The application of the operator P denotes here the orthonormalization process performed on the set of functions $\{\cos(\lambda_n s)\}_{n\in\mathbb{Z}}$.

We have extended the results of [38] and [138] to (non-harmonic) Fourier-cosine expansion in terms of the set of functions $\{\cos(\lambda_n s)\}_{n\in\mathbb{Z}}$, employing a simple procedure, due to theorem 4.2.4, to orthonormalize the Riesz basis $\{e^{i\lambda_n x}\}$ under Kadec's assumption. Therefore, we have found a further expansion of the solution that it is not in terms of orthogonal polynomials, but in terms of non-harmonic functions $\cos(\lambda_n s)$, $s \in \mathbb{R}$.

4.3 Ordering of nested square roots of 2 according to Gray code

Starting from the seminal papers by Ramanujan ([155], [26] pp. 108-112), there is a vast literature studying the properties of the so-called continued radicals as, for example: [92, 31, 171, 101, 66, 128]. Other authors investigated the properties of more general continued operations and their convergence. For a nice review of these results, see for example [102], which focuses mainly on continued reciprocal roots.

Nested square roots of 2 have been also studied in two works of Servi [165] and Nyblom [140], while Efthimiou [65] proved that the radicals given by

$$a_0\sqrt{2 + a_1\sqrt{2 + a_2\sqrt{2 + a_3\sqrt{2 + \ldots}}}}, \quad a_i \in \{-1, 1\}$$

are related to the Chebyshev polynomials $T_{2^n}(x)$. See also [134, 135], for other relations between the nested square roots and the Chebyshev polynomials of degree 2^n in a complex variable.

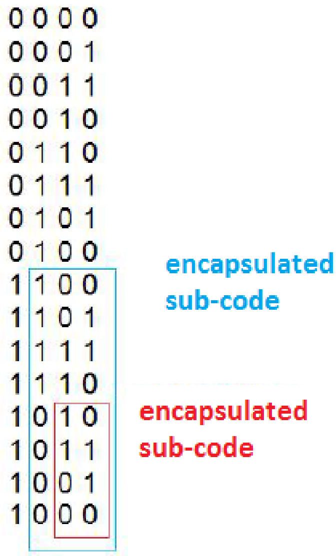


Figure 4.5: Sub-codes for $m = 2$, $m = 3$.

In this Section we give an ordering for zeros of Lucas-Lehmer polynomials (which assume the form of nested square roots of 2 expressed by (4.1.2)) using the Gray code which, at the best of our knowledge, is used to this aim for the first time in [187]. Lucas-Lehmer polynomials were introduced in Section 4.1.

Although our results are similar to (4.4.4), this approach is different because we study square roots of 2 expressed by (4.1.2) applying a "binary code" that associates bits 0 and 1 to $\oplus$ and $\ominus$ signs in the nested form that expresses generic zeros of L_n.

In Section 4.3.1 we recall some important properties of the Gray code [78, 137], useful for Section 4.3.2, where we show that the zeros of every $L_n(x)$ follow the same ordering rule of this code, where the signs $\oplus$ and $\ominus$ in the nested radicals are respectively substituted by the digits 0 e 1.

4.3.1 Gray code.

In this section, we will introduce some useful definitions about Gray code, a particular binary code which is widely used in Informatics. Given a binary code, its **order** is the number of bits with which the code is built, while its **length** is the number of strings that compose it. The celebrated Gray code [78, 137] is a binary code of order n and length 2^n.

We briefly recall below how a Gray Code is generated; if the code for $n-1$ bits is formed by binary strings

$$\begin{array}{l} g_{n-1,1} \\ \dots \\ g_{n-1,2^{n-1}-1} \\ g_{n-1,2^{n-1}} \end{array} \tag{4.3.1}$$

the code for n bits is built from the previous one in the following way:

$$\begin{array}{l} 0g_{n-1,1} \\ ... \\ 0g_{n-1,2^{n-1}-1} \\ 0g_{n-1,2^{n-1}} \\ 1g_{n-1,2^{n-1}} \\ 1g_{n-1,2^{n-1}-1} \\ ... \\ 1g_{n-1,1} \end{array} \tag{4.3.2}$$

Just as an example, we have

$$\begin{array}{llll} g_{3,1} = 000 & g_{3,2} = 001 & g_{3,3} = 011 & g_{3,4} = 010 \\ g_{3,5} = 110 & g_{3,6} = 111 & g_{3,7} = 101 & g_{3,8} = 100 \ , \end{array} \tag{4.3.3}$$

and so on.

Definition 4.3.1. Let us consider a Gray code of order n and length 2^n. A *sub-code* is a Gray code of order $m < n$ and length 2^m.

Definition 4.3.2. Let us consider a Gray code of order n and length 2^n. An *encapsulated sub-code* is a sub-code built starting from the last string of the Gray code of order n that contains it.

Figure (4.5) contains some examples of encapsulated sub-codes inside a Gray code (with order 4 and length 16).

4.3.2 Gray code and nested square roots.

It is known (Propositions 4.1.8, 4.1.4 and Corollary 4.1.7) that L_n has 2^n zeros, symmetric with respect to the origin. Let us consider the signs $\oplus, \ominus$ in the nested form that expresses generic zeros of L_n, as follows:

$$\sqrt{2 \pm \underbrace{\sqrt{2 \pm \sqrt{2 \pm \sqrt{2 \pm ... \pm \sqrt{2 \pm \sqrt{2}}}}}}} \tag{4.3.4}$$

Obviously the underbrace encloses $n-1$ signs $\oplus$ or $\ominus$, each one placed before each nested radical. Starting from the first nested radical we apply a code (i.e., a system of rules) that associates bits 0 and 1 to $\oplus$ and $\ominus$ signs, respectively.

Let us define with $\{\omega(g_{n-1}, j)\}_{j=1,...,2^{n-1}}$ the set of all the 2^{n-1} nested radicals of the form

$$\underbrace{2 \pm \sqrt{2 \pm \sqrt{2 \pm \sqrt{2 \pm ... \pm \sqrt{2 \pm \sqrt{2}}}}}}_{n-1 \quad signs} = \omega(g_{n-1,1 \div 2^{n-1}}) \ , \tag{4.3.5}$$

where each element of the set differs from the others for the sequence of $\oplus$ and $\ominus$ signs. Then:

$$\sqrt{2 \pm \underbrace{\sqrt{2 \pm \sqrt{2 \pm \sqrt{2 \pm \ldots \pm \sqrt{2 \pm \sqrt{2}}}}}}_{n-1 \quad signs}} = \sqrt{\omega(g_{n-1,1 \div 2^{n-1}})} \tag{4.3.6}$$

where the notation $n-1, 1 \div 2^{n-1}$ means that it is possible obtain 2^{n-1} strings formed by $n-1$ bit.

Theorem 4.3.3. *The strings with which we code the 2^{n-1} positive zeros of L_n (sorted in decreasing order) follow the sorting of Gray code. That is, if*

$$\begin{array}{l} g_{n-1,1} \\ \ldots \\ g_{n-1,2^{n-1}-1} \\ g_{n-1,2^{n-1}} \end{array} \tag{4.3.7}$$

is the Gray Code, then

$$\sqrt{\omega(g_{n-1,1})} > \ldots > \sqrt{\omega(g_{n-1,2^{n-1}-1})} > \sqrt{\omega(g_{n-1,2^{n-1}})} \tag{4.3.8}$$

Proof. We first prove (4.3.8) for $n = 2$; here the Gray Code is reduced to bits $0, 1$ (with this order). Indeed we have $\sqrt{\omega(0)} > \sqrt{\omega(1)}$ because:

$$\sqrt{2+\sqrt{2}} > \sqrt{2-\sqrt{2}} \Leftrightarrow 2+\sqrt{2} > 2-\sqrt{2} \Leftrightarrow 2\sqrt{2} > 0 \tag{4.3.9}$$

Let us now suppose that (4.3.8) is true for the Gray Code of order $n-1$. We know that

$$z_{(n)} = \pm\sqrt{2 \pm z_{(n-1)}} \tag{4.3.10}$$

where $z_{(n)}$ and $z_{(n-1)}$ are the generic zeros of L_n and L_{n-1}. For the symmetry of the zeros we can consider only positive zeros. Therefore

$$z_{(n)} = \sqrt{2 \pm z_{(n-1)}} \tag{4.3.11}$$

But the generic zero of L_{n-1}, according to the hypothesis, is precisely one among $\sqrt{\omega(g_{n-1,1})}$,... ,$\sqrt{\omega(g_{n-1,2^{n-1}-1})}$, $\sqrt{\omega(g_{n-1,2^{n-1}})}$; then the generic zero can be indicated with $\sqrt{\omega(g_{n-1,1 \div 2^{n-1}})}$, in a more compact form. Then, from (4.3.11) we can separate the cases $\oplus$ and $\ominus$, obtaining either

$$z_{(n)} = \underbrace{\sqrt{2 + \sqrt{\omega(g_{n-1,1 \div 2^{n-1}})}}}_{\sqrt{\omega(0g_{n-1,1 \div 2^{n-1}})}} \tag{4.3.12}$$

because $\oplus$ corresponds to 0, or

$$z_{(n)} = \underbrace{\sqrt{2 - \sqrt{\omega(g_{n-1,1 \div 2^{n-1}})}}}_{\sqrt{\omega(1g_{n-1,1 \div 2^{n-1}})}} \tag{4.3.13}$$

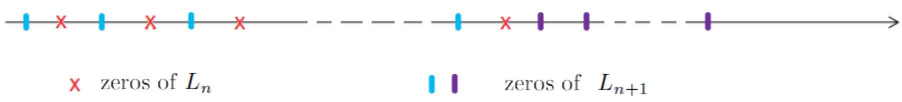


Figure 4.6: Disposition of the zeros of L_n and L_{n+1} on the real axis.

because $\ominus$ corresponds to 1. Thesis follows if we show the following:

$$\begin{aligned}&\sqrt{\omega(0, g_{n-1,1})} > \sqrt{\omega(0, g_{n-1,2})} > \ldots > \sqrt{\omega(0, g_{n-1,2^{n-1}})} > \\ &> \sqrt{\omega(1, g_{n-1,2^{n-1}})} > \ldots > \sqrt{\omega(1, g_{n-1,1})}\end{aligned} \tag{4.3.14}$$

or equivalently

$$\begin{aligned}&\omega(0, g_{n-1,1}) > \omega(0, g_{n-1,2}) > \ldots > \omega(0, g_{n-1,2^{n-1}}) > \\ &> \omega(1, g_{n-1,2^{n-1}}) > \ldots > \omega(1, g_{n-1,1})\end{aligned} \tag{4.3.15}$$

We start proving inequality

$$\omega(0, g_{n-1,i}) > \omega(0, g_{n-1,i+1}) \qquad \forall i = 1, 2, \ldots, 2^{n-1} - 1 \tag{4.3.16}$$

or

$$\begin{aligned}&2 + \sqrt{\omega(g_{n-1,i})} > 2 + \sqrt{\omega(g_{n-1,i+1})} \Leftrightarrow \\ &\Leftrightarrow \omega(g_{n-1,i}) > \omega(g_{n-1,i+1})\end{aligned} \tag{4.3.17}$$

true by virtue of hypothesis (4.3.8). We prove now the inequality

$$\omega(1, g_{n-1,i+1}) > \omega(1, g_{n-1,i}) \qquad \forall i = 1, 2, \ldots, 2^{n-1} - 1 \tag{4.3.18}$$

It can be rewritten

$$2 - \sqrt{\omega(g_{n-1,i+1})} > 2 - \sqrt{\omega(g_{n-1,i})} \Leftrightarrow \omega(g_{n-1,i}) > \omega(g_{n-1,i+1}) \tag{4.3.19}$$

true for assumption. Finally, the relation

$$\omega(0, g_{n-1,2^{n-1}}) > \omega(1, g_{n-1,2^{n-1}}) \tag{4.3.20}$$

follows naturally from

$$\begin{aligned}&2 + \sqrt{\omega(g_{n-1,2^{n-1}})} > 2 - \sqrt{\omega(g_{n-1,2^{n-1}})} \Rightarrow \\ &\Rightarrow 2\sqrt{\omega(g_{n-1,2^{n-1}})} > 0\end{aligned} \tag{4.3.21}$$

always true. $\square$

In Table 4.1 we give an example of ordering of the zeros of $L_n(x)$ for $n = 4$.

$g_{3,j}$	Binary string	Radicals	Approx.
$g_{3,1}$	000	$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$	$1.99\ldots$
$g_{3,2}$	001	$\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2}}}}$	$1.91\ldots$
$g_{3,3}$	011	$\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2}}}}$	$1.76\ldots$
$g_{3,4}$	010	$\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2}}}}$	$1.54\ldots$
$g_{3,5}$	110	$\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2}}}}$	$1.26\ldots$
$g_{3,6}$	111	$\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2}}}}$	$0.94\ldots$
$g_{3,7}$	101	$\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2}}}}$	$0.58\ldots$
$g_{3,8}$	100	$\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}$	$0.19\ldots$

Table 4.1: In the Table we consider the 2^3 positive zeros of $L_4(x)$ and their ordering due to Gray code. It is according to Theorem 4.3.3.

Since, as shown in Proposition 4.1.9, $\{L_n\}$ is a set of orthogonal polynomials, it follows that between two zeros of $L_n(x)$ there exists one and only one zero of $L_{n+1}(x)$ (see [79]).

Theorem 4.3.4. *Let us consider the 2^{n-1} zeros of $L_n(x)$*

$$\sqrt{\omega(g_{n-1,2^{n-1}})} < \sqrt{\omega(g_{n-1,2^{n-1}-1})} < \ldots < \sqrt{\omega(g_{n-1,1})} \ . \tag{4.3.22}$$

Then the zeros of $L_{n+1}(x)$ are arranged on the real axis in this way:

- **i)** *The first zero of $L_{n+1}(x)$ (i.e. $\sqrt{\omega(1, g_{n-1,1})}$) is on the left of the first zero of $L_n(x)$: $\sqrt{\omega(1, g_{n-1,1})} < \sqrt{\omega(g_{n-1,2^{n-1}})}$.*
- **ii)** *The $2^{n-1}-1$ zeros of $L_{n+1}(x)$, which can be represented in the form*

$$\sqrt{\omega(1, g_{n-1,2\div 2^{n-1}})},$$

  are arranged one by one in the $2^{n-1}-1$ intervals which have consecutive zeros of $L_n(x)$; i.e.: $(\sqrt{\omega(g_{n-1,k})},\ \sqrt{\omega(g_{n-1,k-1})})$.
- **iii)** *The remaining zeros, expressed as $\sqrt{\omega(0, g_{n-1,1\div 2^{n-1}})}$, are on the right of the last zero of $L_n(x)$:$\sqrt{\omega(g_{n-1,1})}$.*

The above is schematically shown in Figure (4.6).

Proof. For the proof we first need to dispose on the real axis the 2^{n-1} zeros of L_n; in the interval $J = (a, b)$, where a and b are the first and the last zeros:

$$J = \left(\sqrt{\omega(g_{n-1,2^{n-1}})},\ \sqrt{\omega(g_{n-1,1})}\right) \tag{4.3.23}$$

we can identify $2^{n-1}-1$ subintervals

$$J_{k,k-1} = \left(\sqrt{\omega(g_{n-1,k})},\ \sqrt{\omega(g_{n-1,k-1})}\right) \quad k = 2, 4, 8, ..., 2^{n-1} \tag{4.3.24}$$

whose endpoints are consecutive zeros of L_n. Since L_n is a Chebyshev polynomial, and therefore it is an orthogonal polynomial, it follows that between two consecutive zeros of $L_n(x)$ there exists one and only one zero of $L_{n+1}(x)$. Therefore in each interval $J_{k,k-1}$ we find only one zero of $L_{n+1}(x)$. Let us understand how they are distributed. Let us start with the first 2^{n-1} zeros of $L_{n+1}(x)$, i.e.:

$$\sqrt{\omega(1, g_{n-1,1})} < \ ... \ < \sqrt{\omega(1, g_{n-1,2^{n-1}})} \tag{4.3.25}$$

The statements **i)** and **ii)** are true if we show that

$$\sqrt{\omega(1, g_{n-1,1})} < \sqrt{\omega(g_{n-1,2^{n-1}})} \tag{4.3.26}$$

and

$$\sqrt{\omega(1, g_{n-1,2})} > \sqrt{\omega(g_{n-1,2^{n-1}})} \tag{4.3.27}$$

In fact, let us recall that

$$\begin{aligned}
&\sqrt{\omega(1, g_{n-1,1})} \ \text{ is related to the sequence } \ 1\underbrace{0\ ...\ 0}_{n-1}\\
&\sqrt{\omega(g_{n-1,2^{n-1}})} \ \text{ is related to the sequence } \ 1\underbrace{0\ ...\ 0}_{n-2}\\
&\sqrt{\omega(1, g_{n-1,2})} \ \text{ is related to the sequence } \ 1\underbrace{0\ ...\ 0}_{n-2}1
\end{aligned} \tag{4.3.28}$$

From the first two relations, we can show (4.3.26):

$$\begin{aligned}
&\sqrt{\omega(1, g_{n-1,1})} < \sqrt{\omega(g_{n-1,2^{n-1}})} \quad \Leftrightarrow \sqrt{2 - \sqrt{\underbrace{2 + \sqrt{2 + \sqrt{2 + ... + \sqrt{2}}}}_{n-1}}} < \\
&< \sqrt{2 - \sqrt{\underbrace{2 + \sqrt{2 + \sqrt{2 + ... + \sqrt{2}}}}_{n-2}}} \Leftrightarrow \sqrt{\underbrace{2 + \sqrt{2 + \sqrt{2 + ... + \sqrt{2}}}}_{n-1}} > \\
&> \sqrt{\underbrace{2 + \sqrt{2 + \sqrt{2 + ... + \sqrt{2}}}}_{n-2}}
\end{aligned} \tag{4.3.29}$$

noting that

$$\sqrt{\underbrace{2 + \sqrt{2 + \sqrt{2 + ... + \sqrt{2}}}}_{n-1}} = \sqrt{2 + \sqrt{\underbrace{2 + \sqrt{2 + ... + \sqrt{2}}}_{n-2}}} \tag{4.3.30}$$

is greater than

$$\sqrt{\underbrace{2 + \sqrt{2 + \sqrt{2 + ... + \sqrt{2}}}}_{n-2}}\ . \tag{4.3.31}$$

Let us prove (4.3.27) by the same reasoning used previously.

$$\sqrt{\omega(1,g_{n-1,2})} > \sqrt{\omega(g_{n-1,2^{n-1}})} \quad \Leftrightarrow \sqrt{2-\sqrt{2+\underbrace{\sqrt{2+\sqrt{2+...-\sqrt{2}}}}_{n-1}}} >$$

$$> \sqrt{2-\sqrt{2+\underbrace{\sqrt{2+\sqrt{2+...+\sqrt{2}}}}_{n-2}}} \quad \Leftrightarrow \sqrt{2+\underbrace{\sqrt{2+\sqrt{2+...-\sqrt{2}}}}_{n-1}} <$$

$$< \sqrt{2+\underbrace{\sqrt{2+\sqrt{2+...+\sqrt{2}}}}_{n-2}} \tag{4.3.32}$$

By squaring iteratively the $n-2$ radicals and simplifying, we obtain $-\sqrt{2} < 0$, which is true.

iii) follows immediately noting that

$$\omega(g_{n-1,1}) < 2 \quad ; \quad \omega(0, g_{n-1,2^{n-1}}) = 2 + \sqrt{\omega(g_{n-1,2^{n-1}})} > 2\ . \tag{4.3.33}$$

The three points of the thesis are proven. □

Remark 4.3.5. In the previous work [186] and in Section 4.1.4, the considerations made on the map L_n were extended to an entire class of maps, obtained through the iterated formula $M_n^a = 2a\left(M_{n-1}^a\right)^2 - \frac{1}{a}$, $a > 0$, with $M_0^a(x) = x$. At each iteration the zeros of the map $M_n^a(n \geq 1)$ have the form

$$\pm\frac{1}{2a}\cdot\sqrt{2\pm\sqrt{2\pm\sqrt{2\pm\sqrt{2\pm...\pm\sqrt{2}}}}}$$

and it is clear that the results obtained in this Section are also valid for polynomials obtained through the iterated formula on M_n^a.

4.4 New formulas for π involving infinite nested square roots and Gray code.

Viète's formula [170], developed in 1593 and subsequently generalized (see, for example [135]), is probably the oldest exact result derived for π and is based on an infinite product of nested radicals. Another formula involving nested radicals is given by [55, 72]

$$\pi = \lim_{n\to\infty} 2^{n+1}\cdot\sqrt{2-\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+...+\sqrt{2}}}}}_{n}}\ . \tag{4.4.1}$$

which, following J. Munkhammar [2], [197], can be rewritten in the form

$$\pi = \lim_{n\to\infty} 2^{n+1}\ \pi_n \tag{4.4.2}$$

where π_n is defined in an iterative way:

$$\pi_{n+1} = \sqrt{\left(\frac{1}{2}\pi_n\right)^2 + \left[1 - \sqrt{1 - \left(\frac{1}{2}\pi_n\right)^2}\right]^2} \tag{4.4.3}$$

with $\pi_0 = \sqrt{2}$ [197].

The brilliant English mathematician John Wallis, who lived in the XVIIth Century, highly able in detecting formal schemes and regularity in mathematical structures [34] and celebrated for his formula for π [72], defended the legitimacy of any method that could help the discovery of the truth, even if not corroborated by a rigorous proof. He even stated that Archimedes should have been more blamed because he did not explain the logical processes used for his discoveries than admired for his very elegant proofs [34].

Wallis even got to assert that the contemplation of a finite number of particular cases is all that can be defined as a proof. This kind of contemplation allows us the understanding of the general rule leading to the expected formula. Let us however recall that for many other mathematicians of his time (first of all Fermat) there was not yet the attitude to build a proof, as we can understand today.

Modern Mathematics follows other, more rigorous, ways. However, all the results here shown and proved are introduced as unavoidable consequences of this kind of "contemplation", as suggested by Wallis; thus, several mathematical proofs in this section are conducted by mathematical induction, which can be viewed as a useful way to prove that some statements are true not only for "a finite number of particular cases" but for every value of $n = 1, 2, \ldots$.

The section is organized in the following way. We give a recursive formula for the sequence of the first nonnegative zeros of $L_n(x)$, in terms of nested radicals. We apply this formula in order to prove again (4.4.1). We will start from this formula, in order to introduce the techniques we will use to study the generalized sequences converging to π, in Section 4.4.2. Moreover, in this Section, we show that the generalizations of the Lucas-Lehmer map, M_n^a for $a > 0$ introduced in 4.1.4, have the same properties of L_n, for what concerns the distribution of the zeros and the approximations of π. We also obtain π not as the limit of a sequence, but equal to an expression involving the zeros of the polynomials L_n and M_n^a for $a > 0$. Finally, still in Section 4.4.2, we introduce two relationships between π and the golden ration φ.

4.4.1 A comparison with other results.

The cornerstone of the results shown in this section is the ordering of a class of continued radicals, the nested square roots of 2, introduced in [187] and recalled just above.

The nested square roots of 2 are a special case of the wide class of continued radicals. They have been studied by several authors. In particular, Cipolla [51] obtained a very elegant formula for

$$\sqrt{2 + i_n\sqrt{2 + i_{n-1}\sqrt{2 + \cdots + i_1\sqrt{2}}}}\ ,$$

in terms of $2\cos\left(k_n\frac{\pi}{2^{n+2}}\right)$, where $i_k \in \{-1,1\}$ and k_n is a constant depending on $i_1, \ldots, i_n$. A rigorous treatment of continued radicals of arbitrary, nonnegative terms was found, probably for the first time, in a problem proposed by Pólya [152], solved by Szegö at a later time and included in their famous problem book [153].

Servi [165] rediscovered and extended Cipolla's formula, tying the evaluation of nested square roots of the form

$$R(b_k,...,b_1) = \frac{b_k}{2}\sqrt{2+b_{k-1}\sqrt{2+b_{k-2}\sqrt{2+...+b_2\sqrt{2+2\sin\left(\frac{b_1\pi}{4}\right)}}}} \tag{4.4.4}$$

where $b_i \in \{-1,0,1\}$ for $i \neq 1$, to expression

$$\left(\frac{1}{2}-\frac{b_k}{4}-\frac{b_k b_{k-1}}{8}-...-\frac{b_k b_{k-1}...b_1}{2^{k+1}}\right)\pi \tag{4.4.5}$$

to obtain, amongst other results, some nested square roots representations of π:

$$\pi = \lim_{k\to\infty}\left[\frac{2^{k+1}}{2-b_1}R\left(\underbrace{1,-1,1,1,\ldots,1,1,b_1}_{k\ \text{terms}}\right)\right] \tag{4.4.6}$$

where $b_1 \neq 2$. Nyblom [140], citing Servi's work, derived a closed-form expression for (4.4.4) with a generic $x \geq 2$ that replaces $2\sin\left(\frac{b_1\pi}{4}\right)$ in (4.4.4). Efthimiou [65] proved that the radicals given by

$$a_0\sqrt{2+a_1\sqrt{2+a_2\sqrt{2+a_3\sqrt{2+\cdots}}}},\quad a_i \in \{-1,1\}$$

have limits two times the fixed points of the Chebyshev polynomials $T_{2^n}(x)$, unveiling an interesting relation between these topics. Previous formula is equivalent to (4.1.2).

In [134, 135], the authors report a relation between the nested square roots of depth n as

$$\pm\sqrt{2\pm\sqrt{2\pm\sqrt{2\pm\cdots\pm\sqrt{2+2z}}}},\quad z \in \mathbb{C}\ ,$$

and the Chebyshev polynomials of degree 2^n in a complex variable, generalizing and unifying Servi and Nyblom's formulas. In [134], the authors propose an ordering of the continued roots

$$b_k\sqrt{2+b_{k-1}\sqrt{2+\cdots+b_1\sqrt{2+2\xi}}}\ , \tag{4.4.7}$$

where $\xi = 1$ and each b_i is either 1 or -1, according to formula

$$j(b_k,\ldots,b_1) = \frac{1}{2}\left(2^k - \left(\sum_{j=1}^{k}\left(2^{k-j}\prod_{i=0}^{j-1}b_{k-i}\right)\right)+1\right), \tag{4.4.8}$$

for each positive integer k. Formula (4.4.7) expresses the nested square roots of 2 in (4.1.2), and in [187] we gave an alternative ordering for them involving the so-called Gray code which, to the best of our knowledge, is applied for the first time to these topics.

Actually, there is a strong connection between [134] and [187]. From (4.4.8), we have, for example,

$$
\begin{aligned}
j(1,1,1) &= 1 & j(1,1,-1) &= 2\\
j(1,-1,-1) &= 3 & j(1,-1,1) &= 4\\
j(-1,-1,1) &= 5 & j(-1,-1,-1) &= 6\\
j(-1,1,-1) &= 7 & j(-1,1,1) &= 8\ .
\end{aligned}
$$

If we associate bit 0 to number $b_i = 1$ and bit 1 to number $b_i = -1$, in the above expression of index j, we obtain

$$
\begin{array}{rcl}
(1,1,1) & \mapsto & (0,0,0)\\
(1,1,-1) & \mapsto & (0,0,1)\\
(1,-1,-1) & \mapsto & (0,1,1)\\
(1,-1,1) & \mapsto & (0,1,0)\\
(-1,-1,1) & \mapsto & (1,1,0)\\
(-1,-1,-1) & \mapsto & (1,1,1)\\
(-1,1,-1) & \mapsto & (1,0,1)\\
(-1,1,1) & \mapsto & (1,0,0)\ ,
\end{array}
$$

which are just the strings $\{g_{3,i}\}_{i=1}^{8}$ shown in (4.3.3).

4.4.2 Main results: π formulas involving nested radicals.

Formulas for π which are based on nested radicals have forever received much attention from mathematicians because of their inherent elegance. We give below a short discussion of the π formulas and nested radicals. For this purpose we refer to some comprehensive reviews: [25], [72] and [83], as well as [197].

Let S_n denote the length of a side of a regular polygon of 2^{n+1} sides inscribed in a unit circle, with $S_1 = \sqrt{2}$. More generally, $S_n = 2\sin\left(\frac{\pi}{2^{n+1}}\right)$. Hence, by the half-angle formula,

$$
S_n = \sqrt{2 - \sqrt{4 - S_{n-1}^2}}
$$

The length of this polygon of 2^{n+1} sides is $2^{n+1}S_n$ and tends to 2π as $n \to \infty$ [72]. Therefore (4.4.1) is reobtained. In [52] we find a geometric viewpoint of these recursions.

Viète [190] obtained an elegant expansion with infinitely many nested square roots:

$$
\frac{2}{\pi} = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2+\sqrt{2}}}{2}\cdot\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots\ . \tag{4.4.9}
$$

It is included in a more general formula due to Osler [143]:

$$
\frac{2}{\pi} = \prod_{n=1}^{p}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}}}}}\cdot\prod_{n=1}^{\infty}\frac{2^{p+1}n-1}{2^{p+1}n}\cdot\frac{2^{p+1}n+1}{2^{p+1}n}, \tag{4.4.10}
$$

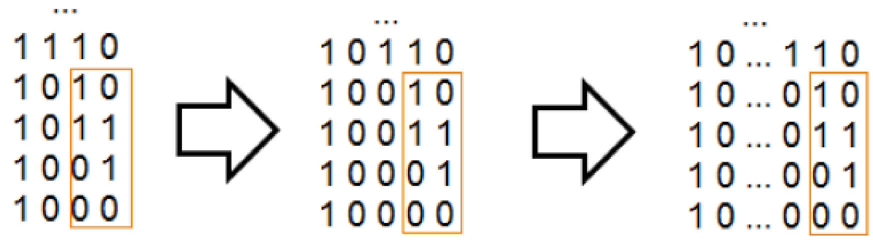


Figure 4.7: A possible subcode (orange), where the meaning of the limit (4.4.26) is highlighted: in this way the number of symbols 0, on the left of the sub-code, increases.

where he reobtains Viète's formula for $p = \infty$ and Wallis' product for $p = 0$. In [144] and [145] new generalizations of this formula for π are obtained.

Another expression with products and radicals was discovered by Sondow [172]:

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \left[1^{(-1)^1 \binom{n}{0}} \cdot 2^{(-1)^2 \binom{n}{1}} \ldots (n+1)^{(-1)^{n+1} \binom{n}{n}} \right]^{1/2^n} . \tag{4.4.11}$$

Besides Osler, in [146], derives an infinite product representation for the AGM (arithmetic-geometric mean) of two positive numbers. The factors of this product are nested radicals recalling Viète's product for π.

The main result contained in the next subsection is in the spirit of formula (4.4.6). Some of the results described in (4.4.6) were already known in the VIth century, thanks to Aryabhata, the famous Indian mathematician and astronomer; see, for example [201].

Infinite sequences tending to π

Let us consider the writing:

$$\omega(\underbrace{*\ldots*}_{n-m} g_{m,h}) = \omega_{n-m}(*\ldots * g_{m,h})$$

where the asterisks represent $n-m$ bits 0 and 1. Then we give the following results.

Lemma 4.4.1. *For all $m \in \mathbb{N}$ one has:*

$$\sqrt{\omega_2(10g_{m,h+1})} = 2\sin\left(\frac{2h+1}{2^{m+4}}\pi\right) \qquad h \in [0, 2^m - 1] \tag{4.4.12}$$

Proof. We proceed with induction principle for m to prove (4.4.12). If $m = 1$:

$$\sqrt{\omega_2(10g_{1,h+1})} = 2\sin\left(\frac{2h+1}{2^5}\pi\right) \qquad h \in [0, 1] \tag{4.4.13}$$

i.e.

$$\sqrt{\omega_2(10g_{1,1})} = 2\sin\left(\frac{\pi}{2^5}\right)$$

for $h = 0$, and

$$\sqrt{\omega_2(10g_{1,2})} = 2\sin\left(\frac{3\pi}{2^5}\right)$$

for $h = 1$, where $g_{1,1} = 0$ and $g_{1,2} = 1$. These formulas are easy to check. Now we are going to check (4.4.12) for $m+1$:

$$\sqrt{\omega_2(10g_{m+1,h+1})} = 2\sin\left(\frac{2h+1}{2^{m+5}}\pi\right) \qquad h \in [0, 2^{m+1}-1] \tag{4.4.14}$$

having assumed it true for $m \geq 1$. From Gray Code's definition we have that either a) $g_{m+1,h+1} = (0, g_{m,h+1})$ or b) $g_{m+1,h+1} = (1, g_{m,2^m-h})$. In the former case:

$$\sqrt{\omega_2(10g_{m+1,h+1})} = \sqrt{\omega_2(100g_{m,h+1})}$$

where

$$\begin{aligned}\sqrt{\omega_2(100g_{m,h+1})} &= \sqrt{2 - \sqrt{\omega_2(00g_{m,h+1})}}\\ &= \sqrt{2 - \sqrt{2 + \sqrt{\omega_2(0g_{m,h+1})}}}\end{aligned} \tag{4.4.15}$$

But in fact: $\omega_2(10g_{m,h+1}) = 2 - \sqrt{\omega_2(0g_{m,h+1})}$, so (4.4.15) becomes

$$\begin{aligned}\sqrt{\omega_2(100g_{m,h+1})} &= \sqrt{2 - \sqrt{2 + \sqrt{\omega_2(0g_{m,h+1})}}}\\ &= \sqrt{2 - \sqrt{4 - \omega_2(10g_{m,h+1})}}\\ &= \sqrt{2 - \sqrt{4 - 4\sin^2\left(\frac{2h+1}{2^{m+4}}\pi\right)}}\\ &= \sqrt{2 - 2\cos\left(\frac{2h+1}{2^{m+4}}\pi\right)}\\ &= 2\sin\left(\frac{2h+1}{2^{m+5}}\pi\right)\end{aligned} \tag{4.4.16}$$

Therefore (4.4.14) is proved for the case a).

Now we assume that $g_{m+1,h+1} = (1, g_{m,2^m-h})$:

$$\sqrt{\omega_2(10g_{m+1,h+1})} = \sqrt{\omega_2(101g_{m,2^m-h})}$$

thus

$$\begin{aligned}\sqrt{\omega_2(101g_{m,2^m-h})} &= \sqrt{2 - \sqrt{\omega_2(01g_{m,2^m-h})}}\\ &= \sqrt{2 - \sqrt{2 + \sqrt{\omega_2(1g_{m,2^m-h})}}}\\ &= \sqrt{2 - \sqrt{2 + \sqrt{2 - \sqrt{\omega_2(g_{m,2^m-h})}}}}\end{aligned} \tag{4.4.17}$$

Noting that

$$\omega_2(0g_{m,2^m-h}) = 2 + \sqrt{\omega_2(g_{m,2^m-h})}$$

it follows that

$$\sqrt{\omega_2(101g_{m,2^m-h})} = \sqrt{2 - \sqrt{2 + \sqrt{4 - \omega_2(0g_{m,2^m-h})}}} \tag{4.4.18}$$

From $\omega_2(10g_{m,2^m-h}) = 2 - \sqrt{\omega_2(0g_{m,2^m-h})}$, equation (4.4.18) becomes

$$\sqrt{\omega_2(101g_{m,2^m-h})} = \sqrt{2 - \sqrt{2 + \sqrt{4 - [2 - \omega_2(10g_{m,2^m-h})]^2}}} \tag{4.4.19}$$

From (4.4.12), we have

$$\sqrt{\omega_2(10g_{m,2^m-h})} = 2\sin\left(\frac{2^{m+1} - (2h+1)}{2^{m+4}}\pi\right)$$

and equation (4.4.19) can be rewritten

$$\begin{aligned}
\sqrt{\omega_2(101g_{m,2^m-h})} &= \sqrt{2 - \sqrt{2 + \sqrt{4 - [2 - \omega_2(10g_{m,2^m-h})]^2}}} \\
&= \sqrt{2 - \sqrt{2 + \sqrt{4 - \left[2 - 4\sin^2\left(\frac{2^{m+1} - (2h+1)}{2^{m+4}}\pi\right)\right]^2}}} \\
&= \sqrt{2 - \sqrt{2 + \sqrt{4 - 4\cos^2\left(\frac{2^{m+1} - (2h+1)}{2^{m+3}}\pi\right)}}} \\
&= \sqrt{2 - \sqrt{2 + 2\sin\left(\frac{2^{m+1} - (2h+1)}{2^{m+3}}\pi\right)}} \\
&= \sqrt{2 - \sqrt{2 + 2\cos\left(\frac{\pi}{2} - \frac{2^{m+1} - (2h+1)}{2^{m+3}}\pi\right)}} \\
&= \sqrt{2 - 2\cos\left(\frac{\pi}{4} - \frac{2^{m+1} - (2h+1)}{2^{m+4}}\pi\right)}
\end{aligned} \tag{4.4.20}$$

Accordingly:

$$\sqrt{\omega_2(101g_{m,2^m-h})} = 2\sin\left(\frac{2(h+2^m)+1}{2^{m+5}}\pi\right)$$

Since the term $h + 2^m \in [2^m, 2^{m+1} - 1]$ for $h \in [0, 2^m - 1]$, then (4.4.14) is fully shown and, with it, the whole proposition. □

Proposition 4.4.2. *For each $n \geq m+2$, $h \in \mathbb{N}$ such that $h \in [0, 2^m - 1]$:*

$$\sqrt{\omega_{n-m}(10...0g_{m,h+1})} = 2\sin\left(\frac{2h+1}{2^{n+2}}\pi\right) \tag{4.4.21}$$

Proof. Put $n-m=\sharp$, $n-m-1=\sharp'$, $n-m-2=\sharp''$, ..., $n-m-k=\sharp^{(k)}$ for $0\leq k\leq n-m-2$. Let us proceed by means of induction principle on n. Fixing m, suppose formula (4.4.21) to be true for a generic index $\sharp'$,

$$\sqrt{\omega_{\sharp'}(10...0g_{m,h+1})} = 2\sin\left(\frac{2h+1}{2^{n+1}}\pi\right) \tag{4.4.22}$$

and proceed to check the case $\sharp$. We work on both sides of (4.4.22):

$$\begin{aligned}
\omega_{\sharp'}(10...0g_{m,h+1}) &= 4\sin^2\left(\frac{2h+1}{2^{n+1}}\pi\right)\\
2-\sqrt{\omega_{\sharp''}(0...0g_{m,h+1})} &= 4-4\cos^2\left(\frac{2h+1}{2^{n+1}}\pi\right)\\
-\sqrt{\omega_{\sharp''}(0...0g_{m,h+1})} &= 2-4\cos^2\left(\frac{2h+1}{2^{n+1}}\pi\right)\\
2+\sqrt{\omega_{\sharp''}(0...0g_{m,h+1})} &= 4\cos^2\left(\frac{2h+1}{2^{n+1}}\pi\right)\\
\omega_{\sharp'}(0...0g_{m,h+1}) &= 4\cos^2\left(\frac{2h+1}{2^{n+1}}\pi\right)
\end{aligned} \tag{4.4.23}$$

whence

$$\sqrt{\omega_{\sharp'}(0...0g_{m,h+1})} = 2\left|\cos\left(\frac{2h+1}{2^{n+1}}\pi\right)\right| = 2\cos\left(\frac{2h+1}{2^{n+1}}\pi\right)$$

Thus:

$$\begin{aligned}
\sqrt{\omega_{\sharp'}(0...0g_{m,h+1})} &= 2\left(1-2\sin^2\left(\frac{2h+1}{2^{n+2}}\pi\right)\right)\\
&\Downarrow\\
2-\sqrt{\omega_{\sharp'}(0...0g_{m,h+1})} &= 4\sin^2\left(\frac{2h+1}{2^{n+2}}\pi\right)
\end{aligned} \tag{4.4.24}$$

and

$$\omega_{\sharp}(10...0g_{m,h+1}) = 4\sin^2\left(\frac{2h+1}{2^{n+2}}\pi\right)$$

hence,

$$\sqrt{\omega_{\sharp}(10...0g_{m,h+1})} = 2\left|\sin\left(\frac{2h+1}{2^{n+2}}\pi\right)\right| = 2\sin\left(\frac{2h+1}{2^{n+2}}\pi\right).$$

The absolute value can be removed by the proposition's assumptions. Therefore, the inductive step is proved. Let us consider the base step: $\sharp=2$. Indeed:

$$\sqrt{\omega_2(10g_{m,h+1})} = 2\sin\left(\frac{2h+1}{2^{n-m+2}2^m}\pi\right)$$

or,

$$\sqrt{\omega(10g_{m,h+1})} = 2\sin\left(\frac{2h+1}{2^{m+4}}\pi\right) \quad h\in[0,2^m-1] \tag{4.4.25}$$

which is proved, for all $m\in\mathbb{N}$, in Lemma 4.4.1. $\square$

Theorem 4.4.3.

$$\lim_{n\to\infty}\frac{2^{n+1}}{2h+1}\sqrt{\omega_{n-m}(10...0g_{m,h+1})} = \pi \tag{4.4.26}$$

for every $h\in\mathbb{N}$ such that $h\in[0,2^m-1]$ and $n>m+1$.

Proof. From (4.4.21), we have

$$\frac{2^{n+1}}{2h+1}\ \sqrt{\omega_{n-m}(10...0g_{m,h+1})} = \frac{2^{n+2}}{2h+1}\ \sin\left(\frac{2h+1}{2^{n+2}}\pi\right) \tag{4.4.27}$$

that, for a well-know limit, tends to π for $n \to \infty$. □

Example 4.4.4. With the help of computational tools we show below some iterations of a sequence described by

$$\frac{2^{n+1}}{2h+1}\ \sqrt{\omega_{n-m}(10...0g_{m,h+1})}\ .$$

Let us consider $m = 3$; then

$$g_{3,1} = 000\ ;\ g_{3,2} = 001\ ;\ g_{3,3} = 011\ ;\ g_{3,4} = 010\ ;$$

$$g_{3,5} = 110\ ;\ g_{3,6} = 111\ ;\ g_{3,7} = 101\ ;\ g_{3,8} = 100.$$

We choose the binary string $g_{3,6} = 111$; in this case, if $m = 3$, one has $h + 1 = 6$ and so $h = 5$. This means that we are iterating

$$\frac{2^{n+1}}{11}\ \sqrt{\omega_{n-3}(\underbrace{10...0}_{n-3}111)} = \frac{2^{n+1}}{11}\ \sqrt{\omega(10...0111)}\ .$$

Hence, for $n = 8$:

$$\frac{2^9}{11}\ \sqrt{\omega(10000111)} =$$

$$\frac{2^9}{11}\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2}}}}}}}}} \simeq 3.\mathbf{140}996\ldots$$

For $n = 12$:

$$\frac{2^{13}}{11}\ \sqrt{\omega(100000000111)} =$$

$$\frac{2^{13}}{11}\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2}}}}}}}}}}}}}$$

$$\simeq 3.\mathbf{14159}0324\ldots$$

and so on.

An asymptotic relationship between the golden ratio and π A simple application of Theorem 4.4.3 allows us to obtain an asymptotic relationship between the golden ratio φ and π. There are not many known relations between π and φ. From [72], we recall a geometric application of the golden mean, which arises when inscribing a regular pentagon within a given circle by ruler and compass. This is related to the fact that

$$2\cos\left(\frac{\pi}{5}\right) = \varphi, \quad 2\sin\left(\frac{\pi}{5}\right) = \sqrt{3-\varphi}.$$

Perhaps it is the simplest connection that one can find between π and φ. But there are not many others. We also mention the four Rogers-Ramanujan continued fractions shown in [72] at pages 7-8, and the harmonious and even unexpected links between the constants, in nature and in architecture, illustrated in [201]. Despite also φ, like π, may be expressed in terms of nested radicals, we are not aware of expressions that bind φ and π with infinite nested radicals.

Let $k = 2$. Therefore, from (4.4.26) we have

$$\pi \sim \frac{2^n}{5} \cdot \sqrt{\omega(11)} \tag{4.4.28}$$

whence

$$5 \sim \frac{2^n}{\pi} \cdot \sqrt{\omega(11)} \tag{4.4.29}$$

for which

$$\sqrt{5} \sim \frac{2^{n/2}}{\sqrt{\pi}} \cdot \sqrt[4]{\omega(11)} \tag{4.4.30}$$

dividing by 2 and adding 1/2, then

$$\varphi \sim \frac{2^{n/2-1}}{\sqrt{\pi}} \cdot \sqrt[4]{\omega(11)} + \frac{1}{2} \tag{4.4.31}$$

where $\varphi = \dfrac{\sqrt{5}+1}{2}$ is the golden ratio.

$M_n^a = 2a\left(M_{n-1}^a\right)^2 - \frac{1}{a}$ **map.**

In [186] we introduced an extension of the map L_n, obtained through the iterated formula $M_n^a = 2a\left(M_{n-1}^a\right)^2 - \frac{1}{a}$, $a > 0$, with $M_0^a(x) = x$. It follows that

$$M_0^a(x) = x \quad ; \quad M_1^a(x) = 2ax^2 - \frac{1}{a} \quad ; \quad M_2^a(x) = 8a^3x^4 - 8ax^2 + \frac{1}{a} \quad ... \tag{4.4.32}$$

Note that the map L_n is a particular case of M_n^a, obtained by setting $a = 1/2$. We briefly show that the map M_n^a leads to the same π formulas stated in the previous sections.

Proposition 4.4.5. *For $n \geq 2$ we have*

$$M_n^a(x) = \frac{1}{a} \cdot \cos(a\ 2^n x) + o(x^2) \tag{4.4.33}$$

Proof. We must show that:

$$M_n^a(x) = \frac{1}{a} - a2^{2n-1}x^2 + o(x^2) \tag{4.4.34}$$

where we take into account the McLaurin polynomial of cosine. We proceed by induction. For $n = 2$:

$$M_2^a(x) = 2a\left(2ax^2 - \frac{1}{a}\right)^2 - \frac{1}{a} = \frac{1}{a} - 8ax^2 + o(x^2) \tag{4.4.35}$$

Let us consider the second order McLaurin polynomial of $\frac{1}{a}\cdot\cos(4ax)$: it is just $\frac{1}{a}-8ax^2+o(x^2)$, thus verifying the relation for $n=2$. Let us now assume (4.4.33) is true for a generic n, and deduce that it is also true for $n+1$:

$$\begin{aligned} M^a_{n+1} &= 2a\,(M^a_n)^2-\frac{1}{a}=2a\left[\frac{1}{a}-a2^{2n-1}x^2+o(x^2)\right]^2-\frac{1}{a}= \\ &=\frac{1}{a}-a2^{2n+1}x^2+o(x^2) \end{aligned} \tag{4.4.36}$$

which is in fact the McLaurin polynomial of $\frac{1}{a}\cdot\cos(a\;2^{n+1}x)$. □

Proposition 4.4.6. *At each iteration the zeros of the map $M^a_n(n\geq 1)$ have the form*

$$\pm\frac{1}{2a}\cdot\sqrt{2\pm\sqrt{2\pm\sqrt{2\pm\sqrt{2\pm...\pm\sqrt{2}}}}} \tag{4.4.37}$$

Proof. It is obvious that at $n=1$ this statement is valid.

Now assume that the (4.4.37) is valid for n. We have to prove that it is valid for $n+1$.

$$2ax^2-\frac{1}{a}=\pm\frac{1}{2a}\cdot\sqrt{2\pm\sqrt{2\pm\sqrt{2\pm\sqrt{2\pm...\pm\sqrt{2}}}}} \tag{4.4.38}$$

or

$$x^2=\frac{1}{2a^2}\pm\frac{1}{4a^2}\cdot\sqrt{2\pm\sqrt{2\pm\sqrt{2\pm\sqrt{2\pm...\pm\sqrt{2}}}}} \tag{4.4.39}$$

and placing under the radical sign

$$x=\pm\sqrt{\frac{1}{2a^2}\pm\frac{1}{4a^2}\cdot\sqrt{2\pm\sqrt{2\pm\sqrt{2\pm\sqrt{2\pm...\pm\sqrt{2}}}}}} \tag{4.4.40}$$

the thesis is obtained. □

It is possible to prove that zeros of the map M^a_{n+1} are related to those of M^a_n, $n\geq 1$.

π-formulas: not only approximations.

From (4.1.46) and (4.1.47) we obtained [186] the following formula:

$$L_n(x)=2\cos\left[2^{n-1}\arctan\left(\frac{\sqrt{1-\left(\frac{x^2}{2}-1\right)^2}}{\frac{x^2}{2}-1}\right)\right] \tag{4.4.41}$$

valid for $x \in [-2,2]$ and $x \neq \pm\sqrt{2}$. This expression is equivalent to

$$L_n(x) = \left(\left(\left(x^2-2\right)^2-2\right)^2 \ \dots \ -2\right)^2 - 2 \tag{4.4.42}$$

Moreover, we already observed that, for $|x| = \sqrt{2}$, we have

$$L_0(\sqrt{2}) = \sqrt{2} \quad ; \quad L_1(\sqrt{2}) = 0 \quad ; \quad L_2(\sqrt{2}) = -2 \quad ; \quad L_n(\sqrt{2}) = 2 \ \ \forall n \geq 3 \ . \tag{4.4.43}$$

The right hand side of (4.4.41) vanishes when

$$2^{n-1} \arctan\left[\frac{\sqrt{1-\left(\frac{x^2}{2}-1\right)^2}}{\frac{x^2}{2}-1}\right] = \pm\frac{\pi}{2}\ (2h+1) \ ; \ h \in N \ ; \ x \neq \pm\sqrt{2} \tag{4.4.44}$$

i.e.,

$$-\frac{\pi}{2} < \arctan\left[\frac{\sqrt{1-\left(\frac{x^2}{2}-1\right)^2}}{\frac{x^2}{2}-1}\right] = \pm\frac{\pi}{2^n}\ (2h+1) < \frac{\pi}{2} \quad , \quad x \neq \pm\sqrt{2} \tag{4.4.45}$$

whence

$$\sqrt{1-\left(\frac{x^2}{2}-1\right)^2} = \left(\frac{x^2}{2}-1\right) T_{n,h} \tag{4.4.46}$$

where $T_{n,h} = \tan\left[\pm\frac{\pi}{2^n}\ (2h+1)\right]$, for $h = 0, 1, \dots, h_{max}$, and h_{max} defined in this way: from (4.4.44) and boundedness of inverse tangent function we have

$$\frac{\pi}{2^n}(2h+1) < \frac{\pi}{2}$$

from which

$$h < 2^{n-2} - \frac{1}{2}$$

therefore $h_{max} = 2^{n-2} - 1$, for $n \geq 2$.

If the factor $T_{n,h}$ is negative, the solutions of (4.4.46) belong to the interval $(-\sqrt{2}, \sqrt{2})$; otherwise $x \in [-2, -\sqrt{2}) \ \cup \ (\sqrt{2}, 2]$, if $T_{n,h} > 0$. We have:

$$1 - \left(\frac{x^2}{2}-1\right)^2 = \left(\frac{x^2}{2}-1\right)^2 T_{n,h}^2 \ \Rightarrow \ \frac{x^2}{2} - 1 = \pm\frac{1}{\sqrt{1+T_{n,h}^2}} \tag{4.4.47}$$

Therefore we can write the zeros of L_n in the form

$$x_h^n = \pm\sqrt{2 \pm \frac{2}{\sqrt{1+\tan^2\left[\frac{\pi}{2^n}\ (2h+1)\right]}}} \quad , \quad n \geq 2 \ ; \ 0 \leq h \leq 2^{n-2}-1 \tag{4.4.48}$$

Moreover, we know that, for every $n \geq 2$, the h-th positive zero of $L_n(x)$ has the form:

$$\sqrt{\omega(g_{n-1,2^{n-1}-h})} \tag{4.4.49}$$

where $0 \le h \le 2^{n-2}-1$. Equating the two expressions, one finds:

$$\frac{1}{1+\tan^2\left[\frac{\pi}{2^n}\ (2h+1)\right]} = \left[\frac{1}{2}\omega(g_{n-1,2^{n-1}-h}) - 1\right]^2 \tag{4.4.50}$$

whence

$$\pi = \frac{2^n}{2h+1} \arctan\sqrt{\frac{1}{\left[\frac{1}{2}\omega(g_{n-1,2^{n-1}-h}) - 1\right]^2} - 1}\ . \tag{4.4.51}$$

In this way we obtain infinite formulas giving π not as the limit of a sequence, but through an equality involving the zeros of the polynomials L_n which is true for every choice of n and h as in (4.4.48).

Similar considerations can be made for the polynomials M_n^a. Since, for $|x| \neq \dfrac{\sqrt{2}}{2a}$,

$$M_n^a(x) = \frac{1}{a}\cos\left(2^{n-1}\arctan\left[\frac{\sqrt{1-(2a^2x^2-1)^2}}{2a^2x^2-1}\right]\right) \tag{4.4.52}$$

vanishes if

$$2^{n-1}\arctan\left[\frac{\sqrt{1-(2a^2x^2-1)^2}}{2a^2x^2-1}\right] = \pm\frac{\pi}{2}\ (2h+1) \tag{4.4.53}$$

i.e.,

$$\arctan\left[\frac{\sqrt{1-(2a^2x^2-1)^2}}{2a^2x^2-1}\right] = \pm\frac{\pi}{2^n}\ (2h+1)\ , \tag{4.4.54}$$

then

$$\sqrt{1-(2a^2x^2-1)^2} = \left(2a^2x^2-1\right)T_{n,h} \tag{4.4.55}$$

where $T_{n,h} = \tan\left[\pm\frac{\pi}{2^n}\ (2h+1)\right]$, with $h = 0, 1, 2..., 2^{n-2}-1$.

Furthermore:

$$\left(2a^2x^2-1\right)\ T_{n,h} > 0 \tag{4.4.56}$$

Inequality $2a^2x^2-1>0$ is verified for $x < -\frac{\sqrt{2}}{2a}\ \vee\ x > \frac{\sqrt{2}}{2a}$. If $T_{n,h}$ is negative, the solutions of (4.4.55) belong to the interval $\left(-\frac{\sqrt{2}}{2a}, \frac{\sqrt{2}}{2a}\right)$, otherwise $x \in \left[-\frac{1}{a}, -\frac{\sqrt{2}}{2a}\right) \cup \left(\frac{\sqrt{2}}{2a}, \frac{1}{a}\right]$, if $T_{n,h} > 0$. On the other hand:

$$1-\left(2a^2x^2-1\right)^2 = T_{n,h}^2\ \left(2a^2x^2-1\right)^2 \Rightarrow 2a^2x^2-1 = \pm\frac{1}{\sqrt{1+T_{n,h}^2}} \tag{4.4.57}$$

from which:

$$x_h^n = \pm\frac{1}{2a}\sqrt{2 \pm \frac{2}{\sqrt{1+\tan^2\left[\frac{\pi}{2^n}\ (2h+1)\right]}}} \tag{4.4.58}$$

Since, from (4.4.37), the zeros of $M_n^a(x)$ are proportional to the zeros of $L_n(x)$, we can say that also *the* 2^{n-1} *positive zeros of* M_n^a*, in decreasing order, follow the order given by the Gray code*:

$$\frac{1}{2a}\sqrt{\omega(g_{n-1,2^{n-1}-h})} \tag{4.4.59}$$

Equating the two expressions we find again the identity:

$$\pi = \frac{2^n}{2h+1}\arctan\sqrt{\frac{1}{\left[\frac{1}{2}\omega(g_{n-1,2^{n-1}-h})-1\right]^2}-1} \tag{4.4.60}$$

An exact relationship between the golden ratio and π From the previous section we have

$$2h+1 = \frac{2^n}{\pi}\arctan\sqrt{\frac{1}{\left[\frac{1}{2}\omega(g_{n-1,2^{n-1}-h})-1\right]^2}-1} \tag{4.4.61}$$

Applying the root to both members of this equality, for $h=2$, it becomes

$$\sqrt{5} = \frac{2^{n/2}}{\sqrt{\pi}}\sqrt{\arctan\sqrt{\frac{1}{\left[\frac{1}{2}\omega(g_{n-1,2^{n-1}-2})-1\right]^2}-1}} \tag{4.4.62}$$

dividing by 2 and adding 1/2, one has:

$$\varphi = \frac{2^{n/2-1}}{\sqrt{\pi}}\sqrt{\arctan\sqrt{\frac{1}{\left[\frac{1}{2}\omega(g_{n-1,2^{n-1}-2})-1\right]^2}-1}}+\frac{1}{2} \tag{4.4.63}$$

As already seen for formula (4.4.51), let us remark that this is an exact formula, without involving any limiting process.

4.5 Conclusions and perspectives.

In this chapter we introduced a class of polynomials which follow the same recursive formula as the Lucas-Lehmer numbers. We showed several properties of the polynomials, including important links with the Chebyshev polynomials, proving their orthogonality with respect to a suitable weight.

This chapter intended just to introduce this new class of polynomials. Much more aspects need to be deepened, concerning the properties of the polynomials and their applications.

Thanks to their strict link with the Chebyshev polynomials, we could determine other properties of the Lucas-Lehmer polynomials, mainly of integral and asymptotic type. These topics will be subject of future studies. Moreover, it would be interesting to determine and study different classes of Lucas-Lehmer polynomials, for example modifying suitably the first term of the sequence.

Orthogonal functions, other classes of polynomials and Riesz bases have shown to be very powerful for the search of solutions of several problems in disparate fields, from Physics to Engineering, from Economics to Biology and so on. In Section 4.2 we applied Lucas-Lehmer polynomials [186] and the tool of Riesz bases [147] in order to reinvestigate a classical problem, due to Love [123], obtaining a further expansion of the solution that it is not in terms of orthogonal polynomials, but in terms of non-harmonic functions $\cos(\lambda_n s)$, $s \in \mathbb{R}$, suitably orthonormalized thanks to Theorem 4.2.4, which uses the celebrated result due to Kadec [103], as stated in Chapter 3.

In Section 4.3 we have studied the distribution of the zeros of L_n, that can be expressed in terms of nested radicals of 2; it allows us to give an ordering for nested square roots of 2 expressed by (4.1.2) thanks to a binary code employed in Informatics (the Gray code). In Section 4.4, these zeros are used to obtain two new formulas for π: the first (i.e., formula (4.4.26)) can be seen as a generalization of the known formula (4.4.1), because the latter can be seen as the case related to the smallest positive zero of L_n; the second (i.e., formula (4.4.51)) gives infinite formulas reproducing π not as the limit of a sequence, but through an equality involving the zeros of the polynomials L_n. We also introduce two relationships between π and the golden ratio φ: (4.4.31) and (4.4.63).

In Section 4.4 we used Proposition 4.4.2 to prove new formulas for π. Actually, Proposition 4.4.2 can be fundamental for further studies. In fact, it not only allows to get the main results of this section, but also allows the evaluation of nested square roots of 2 as:

$$\sqrt{\omega_{n-m}(10...0g_{m,h+1})} = \sqrt{2 - \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2 \pm \sqrt{2 \pm \cdots \pm \sqrt{2}}}}}}$$

for each $n \geq m+2$, $h \in \mathbb{N}$ such that $h \in [0, 2^m - 1]$. This is a result to put in evidence and to generalize in future researches, for example following interesting insights suggested by paper [202], where the authors defined the set S_2 of all continued radicals of the form

$$a_0\sqrt{2 + a_1\sqrt{2 + a_2\sqrt{2 + a_3\sqrt{2 + \ldots}}}}$$

(with $a_0 = 1$, $a_k \in \{-1, 1\}$ for $k = 0, 1, \ldots, n-1$) and investigated some of its properties by assuming that the limit of the sequence of radicals exists.

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