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Symmetry reductions and conservation laws of the short pulse equation

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ABSTRACT

In this letter, we study invariance properties of the nonlinear short pulse equation through Lie symmetry analysis. We show that this leads to several reductions yielding solutions of the short pulse equation. Furthermore, we obtain two conservation laws of the equation through the direct method. We show that two resulting nonlocally related systems yield no nonlocal symmetries of the short pulse equation. Some remarks and appropriate conclusions are drawn at the end.

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1. Introduction

Although the nonlinear Schrödinger equation (NLSE) [1] forms the basis for optimizing existing fiber links, it is inadequate for describing ultra-short pulses. This leads to the derivation of the short pulse equation, as a model equation, to approximate the evolution of very short optical pulses in nonlinear media [2,3]. The short pulse equation (SPE) serves as a basic governing equation for many physical/mathematical models describing different processes in many scientific areas [4–7]. Several authors have analyzed the SPE in different contexts by utilizing various methods and obtained diverse classes of solutions [8–19]. References [11,15] have obtained traveling wave solutions. In [15], authors have used Lie symmetries: i.e., translation symmetries and did detailed analysis, however, they only listed series type solutions through scaling symmetries and no analysis regarding combination of different symmetries, etc. Although extensive literature is available on the SPE, the quest to explore the nonlinearity of the equation is still far from being complete. In the last few decades, active research efforts have been made on the derivation of conservation laws for PDEs. Consequently, several methods such as Noether's theorem [20] for variational problems, multiplier approach (direct method) [21–24], symmetry action on a known conservation law [25], and partial Noether approach [26], etc., have been employed to construct these. To the best of our knowledge, so far there has been just one attempt [27], to obtain the conservation laws of SPE, which however is incomplete. Therefore, the purpose of the letter is to complete and extend the study of SPE through Lie symmetry analysis and conservation laws. Here, we study the SPE through the classical Lie symmetry approach [21,28–30], finding its conservation laws through the direct method as well as seeking nonlocal symmetries from nonlocally related systems arising from conservation laws. We consider the SPE given by [2]

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$$u_{xt} = \alpha u + \frac{1}{3}\beta(u^3)_{xx}, \tag{1}$$

which represents a model equation describing the propagation of ultra-short light pulses in silica optical fibers with α and β as parameters. For the case $\beta = 0$, (1) reduces to the Klein–Gordon equation that possesses an infinite-dimensional Lie symmetry algebra.

2. Reduction and exact solutions of SPE via Lie point symmetries

The Lie point symmetries generators of SPE, obtained in [15], are

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = x\frac{\partial}{\partial x} - t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}.$$

The non-zero commutators for the infinitesimal generators of the SPE are

$$[X_1, X_3] = X_1, \quad [X_2, X_3] = -X_2, \tag{2}$$

where $[X_i, X_j] = X_iX_j - X_jX_i$.

Remark 1

Using the relation

$$\mathrm{Ad}(\exp(\varepsilon X_i))X_j = X_j - \varepsilon[X_i, X_j] + \frac{1}{2}\varepsilon^2[X_i, [X_i, X_j]] - \cdots,$$

where ϵ is a real number and [,] is the Lie bracket, we obtain the adjoint transformations and the following adjoint table from the infinitesimal generators of the Lie algebra of (1).

Ad	X_1	X_2	X_3
X_1	X_1	X_2	$-\epsilon X_1 + X_3$
X_2	X_1	X_2	$\epsilon X_2 + X_3$
X_3	$e^{\epsilon}X_1$	$e^{-\epsilon}X_2$	X_3

Furthermore, from the commutators of the Lie point symmetries of (1) given through (2) and the adjoint representations of the symmetry group on its Lie algebra given in the above table, we can obtain an optimal system of one-dimensional subalgebras [28] given by $\{X_1, X_2 + \delta X_1, X_3\}$, where $\delta = 0, \pm 1$.

Now, we use the Lie symmetries of the SPE (1) as well as their linear combinations to obtain some reductions leading to invariant solutions of (1). We first use the combination of generators X_1 and X_2 that is, the obvious invariance of the SPE (1) under translations in x and t. The change of variables $y = x - \mathrm{M}t$, $u = v(y)$, where M is a constant, reduces (1) to the ODE

$$-Mv'' = \alpha v + \beta(2vv'^2 + v^2v''), \tag{3}$$

where a prime ($'$) denotes d/dy. Now, (3) may be reduced further from its invariance under translation in y represented by the symmetry generator $Y_1 = (\partial/\partial y)$. Consequently, ODE (3) is reduced to the first order ODE

$$-\mathrm{MK}\frac{\mathrm{dK}}{\mathrm{dl}} = \alpha l + 2\beta l K^2 + \beta l^2 K\frac{\mathrm{dK}}{\mathrm{dl}},$$

which after some simplifications reduces to

$$\frac{\mathrm{dv}}{\mathrm{dy}} = K = \left[\frac{1}{2\beta}\left(\frac{c_1}{(M + \beta v^2)^2} - \alpha\right)\right]^{1/2},$$

which can be put in terms of the original variable as

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$$x - \mathrm{Mt} = y = \int \left[\frac{1}{2\beta} \left(\frac{c_1}{(M + \beta u^2)^2} - \alpha \right) \right]^{-1/2} \mathrm{du} + c_2, \tag{4}$$

where c_1, c_2 are constant of integrations and parameters α, β are the same as given in (1). The solution of (4) yields u as a function of x and t. The traveling wave solutions obtained in [14] can be recovered from solution (4) by scaling both u and t, appropriately. Also, the arbitrariness of α and β in solution (4), offers the freedom of choosing any appropriate values of these parameters. This fact will enable us to explicitly observe the effect of the nonlinear term given through β in (1). Fig. 1 presents a numerical simulation for (4) of SPE (1), using Mathematica, where the parameter values are chosen as $M = 1$, $\alpha = 1$, $\beta = (1/2)$ with initial conditions $v(0) = 0, v'(0) = 1$.

Now, we consider the reduction of the SPE (1) through its invariance under scalings corresponding to the infinitesimal generator X_3. In particular (1), in terms of new variables $y = xt$, $u = vx$, reduces to the ODE

$$2\mathrm{xv}' + \mathrm{tx}^2 v'' = \alpha \mathrm{xv} + \beta \left(2\mathrm{xv}[v + \mathrm{xtv}']^2 + \mathrm{tx}^2 v^2 [2v' + \mathrm{xtv}''] \right),$$

which after multiplication by t and then rearrangement can be written as

$$D(y^2 v') = \alpha \mathrm{yv} + \beta D(y^2 v^3 + y^3 v^2 v'), \tag{5}$$

where $D = (d/dy)$. Now we consider $\alpha = 0$, for which we get the following first ODE for v corresponding to an extreme nonlinear situation:

$$v' = \beta v^3 + \beta \mathrm{yv}^2 v' + \frac{c_3}{y^2}, \tag{6}$$

where c_3 is a constant of integration. By using the special transformation

$$v = \frac{1}{y} g(y), \tag{7}$$

(6) reduces to

$$(y - \beta g^2) \frac{\mathrm{dg}}{\mathrm{dy}} - g = c_3,$$

which further can be re-written as

$$\frac{\mathrm{dy}}{\mathrm{dg}} - \frac{1}{g + c_3} y = -\frac{\beta g^2}{g + c_3},$$

and has the following solution

$$y = (g + c_3)[c_4 - \beta g + 2c_3 \ln[c_3 + g]] + \beta c_3^2, \tag{8}$$

where c_4 is a constant of integration. From (7) and (8), one gets the solution v for (6) which ultimately leads to the solution of the SPE (1) for $\alpha = 0$, which holds for any non-zero β.

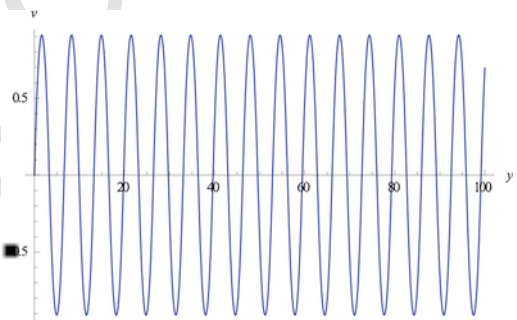

Fig. 1. Plot of the solution (3).

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Moreover, in the special case when $c_3 = 0$ in (8), one obtains

$$g^2 - \frac{c_4}{\beta} g + \frac{1}{\beta} y = 0,$$

which after simplification in terms of v, y has the solution

$$v = \frac{c_4 \pm \sqrt{c_4^2 - 4\beta y}}{2\beta}. \tag{9}$$

In term of the original variables of the SPE (1), Eq. (9) yields

$$u = \frac{c_4 \pm \sqrt{c_4^2 - 4\beta \mathrm{xt}}}{2\beta t}. \tag{10}$$

The bounded solution (10) of SPE (1) holds for any non-zero β. Furthermore, it has an obvious singularity as $t \to 0$. Note that if necessary, one can generalize the solution (10) by translating the variables t, x, i.e., $t \to t + A, x \to x + B$, to avoid a singularity in a region of interest. Hence, the most general boundary conditions for (10) in a generic case will be

$$u(x_\sigma, t) = A \quad \text{as} \quad t \to 0$$
$$\lim_{t \to \infty} u(x, t) = 0.$$

Now, consider reduction of the SPE (1) through a combination of generators X_1, X_2, X_3, in particular, $X_1 + X_2 + X_3$, though this algebra is not in the optimal list. Under this algebra (1), in terms of invariance variables $y = (1 + x)(1 - t), \mathrm{u} = (1 + x)v$, reduces to the ODE

$$2v' + \mathrm{y}\mathrm{v}'' + \alpha v + \beta\left(2v^3 + 6\mathrm{y}\mathrm{v}^2 v' + 2y^2 v (v')^2 + y^2 v^2 v''\right) = 0,$$

which after rearrangement can be written as

$$\alpha v + v' + D(\mathrm{y}\mathrm{v}') + \beta D(2\mathrm{y}\mathrm{v}^3) + \beta y^2 (2v (v')^2 + v^2 v'') = 0, \tag{11}$$

where $D = (d/dy)$. Now, we consider $\alpha = 0$, for which (11) reduces to

$$2v' + \mathrm{y}\mathrm{v}'' + \beta\left(2v^3 + 6\mathrm{y}\mathrm{v}^2 v' + 2y^2 v (v')^2 + y^2 v^2 v''\right) = 0, \tag{12}$$

which has the particular solution $v = (1/y)$. Thus in terms of its variables, the corresponding solution of the SPE (1) turns out to be

$$u = \frac{1}{1 - t}. \tag{13}$$

The time dependent solution (13) is independent of β. In order to find a β-dependent solution of the SPE (1), we first note its invariance under scalings, corresponding to its infinitesimal generator

$$X = -2y\frac{\partial}{\partial y} + v\frac{\partial}{\partial v}.$$

Correspondingly, we obtain the invariant solution of ODE (12) given by

$$v = \pm \frac{1}{\sqrt{\beta}} y^{-1/2}. \tag{14}$$

In terms of the original variables of SPE (1), Eq. (14) yields the solution of (1) given by

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$$u = \pm \frac{1}{\sqrt{\beta}} \sqrt{\frac{1+x}{1-t}}. \quad (15)$$

The bounded solutions (13) and (15) become singular when $t \to 1$. Again, this singularity, if necessary can be shifted by translating the variable t.

Now consider the differential invariants of the scaling symmetry given by

$$V = \mathrm{yv}^2, \quad W = y^2\mathrm{vv}'.$$

In terms of these variables, Eq. (12) transforms to the first ODE

$$\frac{\mathrm{d}W}{\mathrm{d}V} = \frac{W^2 - V(4\mathrm{WV} + W^2 + 2V^2)}{V(V+2W)(1+\beta V)}. \quad (16)$$

Approaches such as homotopy analysis and numerical techniques can be used to find solutions of (16) and hence by the process of back substitutions to corresponding solutions to SPE (1).

3. Conservation laws of SPE

Since the SPE (1) does not have a variational principle, it follows that the classical Noether's theorem cannot be used to obtain conservation laws. We seek conservation laws using the direct method [22–24]. The multipliers and the conserved fluxes were obtained through use of the GeM package [31]. Omitting details of the calculation, we directly obtain two conservation laws with the flows given by

(i)
$$T_1^t = \tfrac{1}{2}u_x u_t + \tfrac{1}{2}\mathrm{uu}_{\mathrm{tx}} - \tfrac{1}{2}\alpha u^2 - \tfrac{1}{4}\beta u^3 u_{\mathrm{xx}} - \tfrac{3}{4}\beta (u_x)^2 u^2,$$
$$T_1^x = \tfrac{1}{4}u\left(-\beta \mathrm{uu}_x u_t + \beta u^2 u_{\mathrm{tx}} + 2\beta^2 u^3 (u_x)^2 + \alpha\beta u^3 - 2u_{\mathrm{tt}}\right),$$

(ii)
$$T_2^t = -\frac{\sqrt{2\beta(u_x)^2+\alpha}}{2\beta},$$
$$T_2^x = -\tfrac{1}{2}u^2\sqrt{2\beta(u_x)^2 + \alpha},$$

arising from multipliers $Q_1 = u_t - \beta u^2 u_x$ and $Q_2 = (u_x/(\sqrt{2\beta(u_x)^2+\alpha})$, respectively. The first conservation law is equivalent to the conservation law obtained in [27], where as the second conservation is not obtained previously.

4. Lie symmetries of SPE via nonlocally related systems

The potential symmetry approach [22,32,33] provides a mechanism for seeking nonlocal symmetries of a PDE system from conservation laws of the system. Each conservation law allows the introduction of auxiliary potential variables which are nonlocally defined with respect to the original dependent variables. A Lie symmetry of the resulting potential system yields a nonlocal symmetry of a given PDE system provided its infinitesimals for the variables of the given PDE system have an essential dependence on potential variables. We start with the first conservation law. The procedure outlined in [22,32,33] yields the nonlocally related system

$$u_x u_t + uu_{\mathrm{xt}} - \alpha u^2 - \tfrac{\beta}{2}u^3 u_{\mathrm{xx}} - \tfrac{3\beta}{2}u_x^2 u^2 + 2F_x = 0,$$
$$-\tfrac{\beta}{2}u^2 u_x u_t + \tfrac{\beta}{2}u^3 u_{\mathrm{xt}} + \beta^2 u^4 u_x^2 + \frac{\alpha\beta}{2}u^4 - uu_{\mathrm{tt}} - 2F_t = 0,$$

which has Lie symmetries

$$X_1 = \tfrac{\partial}{\partial F}, \quad \mathrm{X}_2 = \tfrac{\partial}{\partial x}, \quad \mathrm{X}_3 = \tfrac{\partial}{\partial t},$$
$$X_4 = 3F\tfrac{\partial}{\partial F} + u\tfrac{\partial}{\partial u} - t\tfrac{\partial}{\partial t} + x\tfrac{\partial}{\partial x}.$$

The second conservation law yields the nonlocally related system

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$$F_x - \frac{1}{2}\left(\frac{-\sqrt{\alpha}+\sqrt{2\beta(u_x)^2+\alpha}}{\beta}\right) = 0,$$
$$F_t + \frac{1}{2}u^2\sqrt{2\beta(u_x)^2+\alpha} = 0,$$

with Lie symmetries

$$X_1 = \frac{\partial}{\partial F}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial t},$$
$$X_4 = F\frac{\partial}{\partial F} + u\frac{\partial}{\partial u} - t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}.$$

Clearly, these symmetries correspond to Lie point symmetries of the SPE (1).

Remark 2

One can also use the symmetry-based method [34] to obtain a nonlocally related PDE system, which unlike the conservation law-based method (as above) does not require the existence of a nontrivial local conservation law of a given PDE system.

5. SPE via double reduction

The recently developed notion of the association between Lie point symmetry generators and conservation laws lead to double reductions of the underlying equation and ultimately to exact/invariant solutions for higher-order nonlinear partial differential equations. The procedure is straightforward, and therefore we omit the details and directly write the results. For the details of the results and the procedure, readers are strongly refereed to [35,36].

The possible associations between Lie point symmetries and conservation laws are presented in the form of table given below.

	X_1	X_2	X_3
T_1	0	0	≠0
T_2	0	0	≠0

It is evident from the table, that we can reduce SPE by combination of X_1 and X_2 on either of the conservation laws. The reduction of SPE by combination of X_1 and X_2 on the first conservation law generates the following solution [36]

$$x - t = r = \pm\int \frac{3(u^2\beta + 2)u}{\left[9c_5 - 36u^2 - 9(\beta c_6 + 2\alpha)u^4 - 6\alpha\beta u^6\right]^{1/2}}\mathrm{du} + c_7, \tag{17}$$

where c_5, c_6, c_7 are constant of integrations and parameters α, β are the same as given in (1). However, for the case of second conservation laws, the reductions yields the trivial solution [35].

In summary, from the invariance of the short pulse equation under translations and scalings, we have obtained several reductions leading to a class of exact solutions of the SPE. The solution (4) is a traveling wave solution, for arbitrary values of its parameters α and β. Due to the structure of the solution (4), previously known solutions can be recovered, by a suitable scaling of u and t. The solutions (8) and (10), obtained through scaling symmetries, have not been obtained previously. Through a combination of translating and scaling symmetries, we have obtained previously unknown solutions (13) and (15). The solution (13) is independent of β, whereas solution (15) holds for $\beta > 0$. Moreover, the reduced ODE in (16) can be further pursued by suitable numerical techniques. We have also calculated conservation laws of the SPE, which could be useful for appropriate boundary conditions for the SPE. Furthermore, we used the conservation law based method to construct nonlocally related systems (potential systems) of the SPE but the Lie symmetries of the constructed systems yielded no nonlocal symmetries of the SPE. Also, we analyzed the SPE through double reduction procedure and able to obtain a solution (17), which could not be obtainable through Lie symmetry method.

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References

[1] G.P. Agrawal, Nonlinear Fiber Optics, Academic Press, Boston, 1989.

UNCORRECTED PROOF

[2] T. Schafer, C.E. Wayne, Propagation of ultra-short optical pulses in cubic nonlinear media, Physica D 196 (2004) 90–105.
[3] Y. Chung, C. Jones, T. Schafer, C.E. Wayne, Ultra-short pulses in linear and nonlinear media, Nonlinearity 18 (2005) 1351–1374.
[4] E.B. Brown, E. Wu, W. Zipfel, W.W. Webb, Measurement of molecular diffusion in solution by multiphoton fluorescence photobleaching recovery, Biophys. J. 77 (1999) 2837–2849.
[5] L. Xu, M.W. Kimmel, P. O'Shea, R. Trebino, R.S. Windeler, J.K. Ranka, A.J. Stentz, Measuring the intensity and phase of ultrabroadband continuum, Springer Ser. Chem. Phys. 66 (2001) 129–131.
[6] J.C. Knight, J. Broeng, T.A. Birks, P.St.J. Russell, Photonic band gap guidance in optical fibers, Science 282 (1998) 1476–1478.
[7] N. Karasawa, S. Nakamura, N. Nakagawa, M. Shibata, R. Morita, H. Shigekawa, M. Yamashita, Comparison between theory and experiment of nonlinear propagation for a-few-cycle and ultrabroadband optical pulses in a fused- silica fiber, IEEE J. Quant. Electron. 37 (2001) 398–404.
[8] A. Sakovich, S. Sakovich, The short pulse equation is integrable, J. Phys. Soc. Jpn. 74 (2005) 239–241.
[9] J.C. Brunelli, The short pulse hierarchy, J. Math. Phys. 46 (2005) 123–507.
[10] J.C. Brunelli, The bi-Hamiltonian structure of the short pulse equation, Phys. Lett. A 353 (2006) 475–478.
[11] A. Sakovich, S. Sakovich, Solitary wave solutions of the short pulse equation, J. Phys. A 39 (2006) L361–L367.
[12] K.K. Victor, B.B. Thomas, T.C. Kofane, On exact solutions of the Schafer–Wayne short pulse equation: WKI eigenvalue problem, J. Phys. A 39 (2007) 5585–5596.
[13] Y. Chung, T. Schafer, Stabilization of ultra-short pulses in cubic nonlinear media, Phys. Lett. A 361 (2007) 63–69.
[14] E.J. Parkes, Some periodic and solitary travelling-wave solutions of the short-pulse equation, Chaos Solitons Fractals 38 (2008) 154–159.
[15] H. Liu, J. Li, Lie symmetry analysis and exact solutions for the short pulse equation, Nonlinear Anal. Theor. 71 (2009) 2126–2133.
[16] A. Huber, The short pulse equation – a symmetry study, J. Comput. Methods Sci. Eng. 10 (2010) 79–87.
[17] Z. Fu, Z. Chen, L. Zhang, J. Mao, S. Liu, Novel exact solutions to the short pulse equation, Appl. Math. Comput. 215 (2010) 3899–3905.
[18] M.L. Yu, L.Q. Bang, Some new Jacobi elliptic function solutions for the short-pulse equation via a direct symbolic computation method, J. Appl. Math. Comput. 40 (2012) 683–690.
[19] U. Saleem, M.U. Hassan, Darboux transformation and multisoliton solutions of the short pulse equation, J. Phys. Soc. Jpn. 81 (2012) 094008–094016.
[20] E. Noether, Invariante Variationsprobleme, Nachr. K önig. Gesell. Wiss. G öttingen., Math.-phys. Kl. Heft 2 (1918) 235 (English translation: Invariant variation problems, Transport Theory Stat. Phys. 1 (1971) 183–207).
[21] G.W. Bluman, S. Anco, Symmetry and Integration Methods for Differential Equations, Springer, New York, 2002.
[22] G.W. Bluman, A. Cheviakov, S. Anco, Applications of Symmetry Methods to Partial Differential Equations, Springer, New York, 2010.
[23] S.C. Anco, G.W. Bluman, Direct construction of conservation laws from field equations, Phys. Rev. Lett. 78 (1997) 2869–2873.
[24] S.C. Anco, G.W. Bluman, Direct construction method for conservation laws of partial differential equations. Part I. Examples of conservation law classifications, Eur. J. Appl. Math. 13 (2002) 545–566.
[25] G.W. Bluman, Temuerchaolu, S.C. Anco, New conservation laws obtained directly from symmetry action on a known conservation law, J. Math. Anal. Appl. 322 (2006) 233–250.
[26] A.H. Kara, F.M. Mahomed, Noether-type symmetries and conservation laws via partial Lagrangians, Nonlinear Dynam. 45 (2006) 367–383.
[27] N.H. Ibragimov, Nonlinear Self-Adjointness in Constructing Conservation Laws, 201163–67. arXiv:1109.1728.
[28] P.J. Olver, Application of Lie Groups to Differential Equations, Springer, Berlin, 1986.
[29] G.W. Bluman, S. Kumei, Symmetries and Differential Equations, Springer, New York, 19891989.
[30] N.H. Ibragimov (Ed.), CRC Handbook of Lie Group Analysis of Differential Equations, vols. 1–3, CRC Press, Boca Raton, FL, 1994–1996.
[31] A.F. Cheviakov, GeM software package for computation of symmetries and conservation laws of differential equations, Comput. Phys. Commun. 176 (2007) 48–61.
[32] G.W. Bluman, A.F. Cheviakov, Framework for potential systems and nonlocal symmetries: algorithmic approach, J. Math. Phys. 46 (2005) 123506–123519.
[33] G.W. Bluman, A.F. Cheviakov, N.M. Ivanova, Framework for nonlocally related partial differential equations systems and nonlocal symmetries: extension, simplification, and examples, J. Math. Phys. 47 (2006) 113505–113523.
[34] G.W. Bluman, Z. Yang, A symmetry-based method for constructing nonlocally related PDE systems, J. Math. Phys. 54 (2013) 093504–093522.
[35] K. Fakhar, A.H. Kara, The reduction of Chazy classes and other third-order differential equations related to boundary layer flow models, Chin. Phys. Lett. 29 (2012). 060202:1-4.
[36] J.B. Zik Hong, K. Fakhar, S. Ahmad, A.H. Kara, On double reduction of short pulse equation, In: AIP Conf. Proc, 1682, 2015, p. 020029.

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