12-2016

PLANAR GRAPHS, BIPLANAR GRAPHS AND GRAPH THICKNESS

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Planar Graphs, Biplanar Graphs and Graph Thickness

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

by

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Abstract

A graph is planar if it can be drawn on a piece of paper such that no two edges cross. The smallest complete and complete bipartite graphs that are not planar are K_5 and $K_{3,3}$. A biplanar graph is a graph whose edges can be colored using red and blue such that the red edges induce a planar subgraph and the blue edges induce a planar subgraph. In this thesis, we determine the smallest complete and complete bipartite graphs that are not biplanar.

Acknowledgements

I would like to thank a few people for making this thesis possible. First, my loving wife Jessica. Without your support and motivation I don't think I would of completed this task. Second, to my parents for supporting me and allowing me a place to live during my entire masters and thesis process. Finally, Dr. Aikin for all his time, effort and dedication to helping me through this entire process. Without your encouragement, patience and guidance I would not have nearly as polished of a thesis. It was your support that has granted me this opportunity to grow as a Mathematician.

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Chapter 1

Introduction

Within the branch of combinatorics exists a field of mathematics called graph theory. Graphs can be used to model many everyday circumstances, such as transportation networks and electrical circuits. Consider a map of a given region of the world. There exist man made cities with transportation pathways such as roads, freeways, or railways between the cities. Through graph theory, such a map can be modeled in a very simple way. A city could be represented by a circle, called a vertex, with lines, called edges, drawn between the circles to represent transportation pathways between different cities. Consider Figure 1.1.

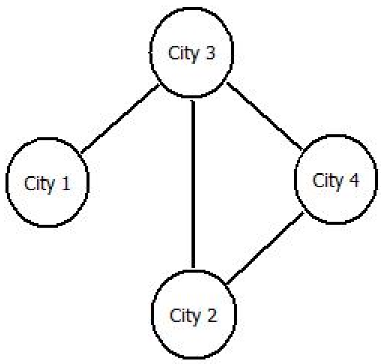

Figure 1.1: An example of a transportation network.

Note that not every city is directly connected by a road to every other city, such as Cities 1 and 2. Thus to travel from one city to another, a person may need to take a path through multiple cities to get to their destination. In order to reach the destination of City 2 from City 1, a person would need to travel to City 3 first then to City 2, or travel from City 1 to City 3 to City 4 to City 2. Another interesting observation is if transportation through City 3 was blocked, then City 1 would be, in a sense, isolated from Cities 2 and 4. In this way, graphs can be used to encode important information from a transportation network, giving us a relatively simplified perspective of an otherwise complicated system.

Graphs can also be used to model electrical circuits in a computer or even the electrical circuit within a single computer chip. In these cases we would model separate electrical components as circles with lines drawn between them to represent electrical pathways. This is a very similar model to that of the model of transportation networks. Consider Figure 1.2.

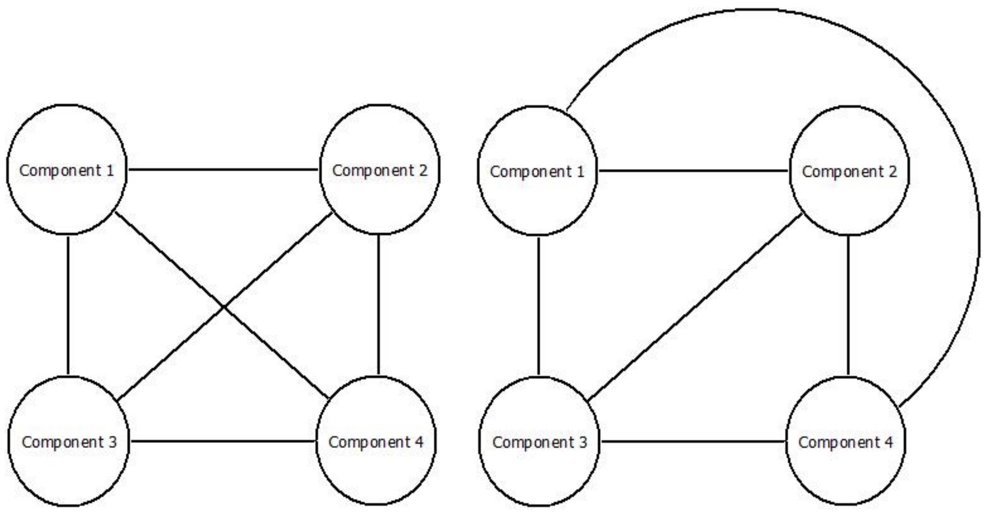

Figure 1.2: Two models of the same circuit.

Although both models are equally valid and encode the same information about components and electrical pathways, there exists a significant difference in structure. In the model on the left in Figure 1.2, the electrical pathway connecting Components 1 and 4 crosses the electrical pathway connecting Components 2 and 3. In contrast, the model on the right in Figure 1.2 does not contain any pathways that cross. This is a key structural difference that was not important within the transportation network. On a circuit board,if electrical pathways cross the communication between the components will not function properly. With this in mind, the graph on the right in Figure 1.2

would provide a better model for an electrical circuit. A graph that can be drawn on a piece of paper without having any two lines cross is called a planar graph. Modeling electrical circuits as planar graphs provides our motivation for studying planar graphs. In particular, we are interested in being able to determine precisely when a given graph is planar. For, if it is, such a graph could potentially serve as a model for an electrical circuit embedded on a circuit board.

Intuitively, an example may exist in which the number of electrical pathways is to great in comparison to the number of electrical components preventing us from placing the circuit on a single circuit board. We might view such a circuit as one that requires the use of the front and back sides of a circuit board to place the electrical pathways in a way that no two pathways cross. Modeling these more complex circuits would require graphs that can be decomposed in some way to two planar graphs. We call a graph biplanar when such a decomposition is possible. Hence, biplanar graphs model electric circuits which require the use of at most two sides of a circuit board. Figure 1 provides an example of a biplanar model of the planar model in Figure 1.2.

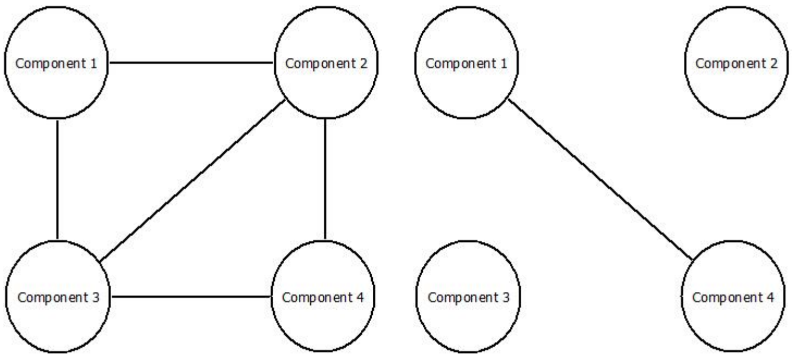

Figure 1.3: Biplanar model of the electrical circuit in Figure 1.2.

As electrical circuits become more complex there would become a point at which the graph model of the circuit would require even more than two sides of a circuit board. An electrical circuit of significant complexity may require more than two layers in order to connect all components in a prescribed way. The graph modeling such a circuit would then require a decomposition consisting of more than two planar graphs. The number of planar graphs required to model such a circuit is a parameter we will investigate in this thesis and is called the thickness of the graph that models the circuit.

Given a graph that serves as a model for an electrical circuit, determining the thickness of a graph will tell us the minimum number of layers needed in a computer chip in order to successfully build the circuit. Assuming more layers implies an increased cost in production, we are in some sense minimizing the cost of microchip production. This application is one factor that motivates the study of biplanar graphs.

Chapter 2

Graphs and Planar Graphs

2.1 Graphs

The concept of a graph grew from a problem Leonhard Euler had developed concerning the city of Königsberg.

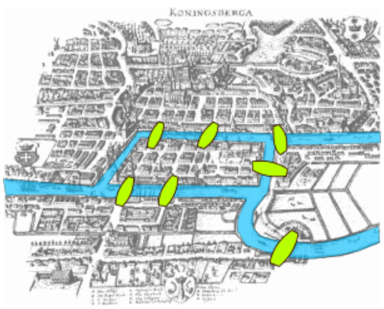

Figure 2.1: The bridges of Königsberg [CLZ10].

The city of Königsberg, at the time when Euler contemplated it, was located in Prussia. The city was divided into four sections with seven bridges as seen in Figure 2.1. Euler's original problem asked the inhabitants of Königsberg whether or not it was possible to walk around the city and cross each bridge exactly once. Euler was not concerned with the starting and ending point to be the same. Although the problem was quite simple, Euler needed to derive a new concept in order to generate a proof that would be rigorous enough to withstand mathematical scrutiny.

Thus Euler developed the graph as a tool to solve his problem. A *graph* G is a finite nonempty set V of objects, called *vertices* together with a possibly empty set E of 2-element subsets of V called *edges*. Graphs are commonly viewed as drawings where the vertices are points or circles and the edges as lines occurring between two vertices. Two distinct vertices u and v are *adjacent* if the edge $\{u, v\}$ is in contained in the edge set E of G. Rather than denoting the edges as a two element subset of the vertex set, we will shorten the notation to the edge uv for any two connected vertices u and v.

Example 2.1. *Let G have the vertex set $V = \{v_1, v_2, v_3, v_4\}$ and the edge set $E = \{\{v_1, v_2\}, \{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_1, v_4\}, \{v_2, v_4\}\}$. The graph of G can be drawn in the following way:*

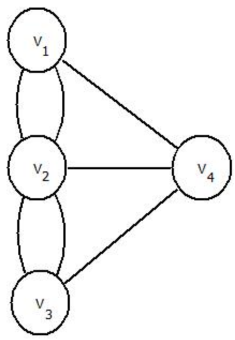

Figure 2.2: A graph representaion of Königsberg.

Euler constructed the graph in Figure 2.2 to model the bridges of Königsberg. The vertices represent the land masses separated by the river Pregel, and the edges represent the bridges. By developing some of the first theorems about graphs, Euler was able to give a rigorous proof that no such walk around Königsberg was possible. In particular, Euler proved that there was no such way to traverse the corresponding graph model unless there were either an even number of edges at all vertices, or there were exactly two vertices at which there were an odd number of edges.

Notation should be clarified at this point. Let G be a graph with edge set E and vertex set V. We will use $|E| = m$ to denote the number of elements in the edge set E. Additionally, $|V| = n$ will denote the number of elements in the vertex set V. Finally we will denote the number of 2-element sets of E containing a vertex v an an element as $deg(v)$. A graph $G' = (V', E')$ is a *subgraph* of a graph $G = (V, E)$ if V' is a subset of V

and E' is a subset of E. Additionally for any vertex v in V but not in V', no 2-element set of E' can contain v. A graph $G' = (V', E')$ is a *proper subgraph* of a graph $G = \{V, E\}$ if V' is a proper subset of V and E' is a proper subset of E such that E'. Throughout this thesis we will denote the subgraph of $G = (V, E)$ obtained by removing a single edge e as $G - e$. Additionally, a proper subgraph of G may be obtained by removing a vertex v from V and removing all elements that include v from E. This is denoted by $G - v$.

A graph is *simple* if its edge set has no repeated elements and every element of the edge set consists of two distinct vertices of the set V. Although there are other types of graphs, it is important to note that for the remainder of the thesis.

Since the graphs we study here are simple there exists a maximum number of edges a graph could have. A graph G is a *complete graph* if for any two distinct vertices $v, u \in V$ then the edge $\{v, u\}$ is contained in the edge set E. A complete graph with n distinct vertices is denoted K_n. Additionally, for a complete graph K_n, it is important to note there are exactly $n(n-1)/2$ edges and $deg(v) = n - 1$ for ever $v \in V$.

A graph G is *complete bipartite* if the vertex set V can be partitioned into two parts V_1 and V_2 such that the edge $\{u, w\} \in E$ if and only if u and w are in different parts. Complete bipartite graphs are denoted by $K_{a,b}$ where $|V_1| = a$ and $|V_2| = b$. Additionally, for a complete bipartite graph $K_{a,b}$ has exactly $a \cdot b$ edges.

It is important to properly define complete and complete bipartite graphs because they play a vital roll throughout graph theory and especially within this thesis.

2.2 Planar Graphs

As mentioned in Chapter 1, it may be possible to draw a graph on a piece of paper in such a way that the edges do not cross, although this is not always possible. A graph G is *planar* if there exists a way to draw the graph without edges crossing. Such a drawing is called a *planar embedding* of G.

A *path* in a graph G is a sequence of vertices $(v_1, v_2, ..., v_n)$ such that v_i and v_{i+1} are adjacent for $1 \leq i \leq n$ and each vertex may only be only appear once in the sequence. Such a path starting at a vertex v_i and terminating at a vertex v_j is called a v_i, v_j-path. A *cycle* in a graph G is a path that begins and ends at the same vertex.

A graph G is *connected* if between any two vertices $u, v \in V$, there exists a u, v-path. A graph G has *components* if G is disconnected. That is, G is disconnected if there exist at least two proper subgraphs H_1 and H_2 of G such that for any $v \in H_1$ and $u \in H_2$ then there does not exist a v, u-path.

Let $G = (V, E)$ be a graph with κ components. The graph G contains a *cut vertex* if there exists a vertex v such that the number of components in the proper subgraph $G - v$ increases. A graph G is *n-connected* if n vertices must be removed in order to increase the number of components of G.

In order to better understand the introduced definitions, consider the following example:

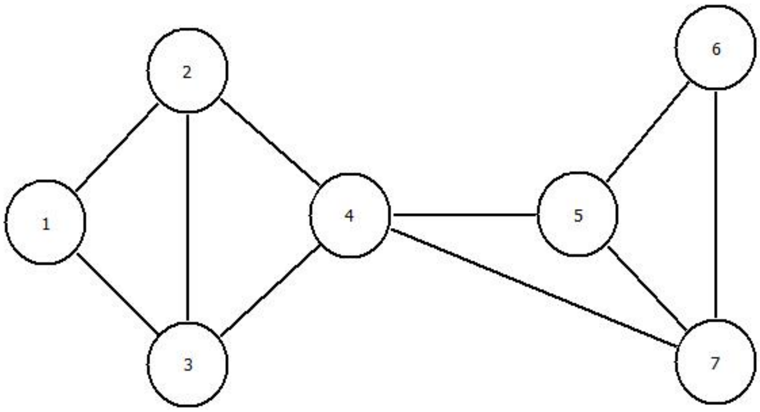

Figure 2.3: A simple graph. .

Between any two vertices i and j in Figure 2.3 there exists an i, j-path. For example there exists a $1, 7$-path by traveling along the edges $\{1, 3\}$,$\{3, 4\}$,$\{4, 7\}$. Thus, this graph is connected and has exactly one component. Additionally Vertex 4 is a cut-vertex in G since $G - 4$ results in components G_1 with the vertex set $V_1 = \{1, 2, 3\}$ and G_2 with the vertex set $V_2 = \{5, 6, 7\}$ Since G contains a cut vertex, G is only 1-connected.

The simplest form of a planar graph is a tree or a forest. A graph G is a *tree* if it does not contain any cycles. A graph G is a *forest* if each component of G is a tree.'A *region* of a graph is a section of a graph that is enclosed by a cycle. Additionally there exists an unbounded region on the exterior of the graph. The vertices and edges that create the cycle enclosing the region are said to be *incident* with the region.

Theorem 2.2 (Euler's Polyhedral Formula)**.** *Let G be a connected graph with n vertices, m edges, and r regions. The graph G obeys the equality $n - m + r = 2$.*

Proof. We proceed by induction on m. Let $m = 0$. Since G is a connected graph, $n = 1$ and since the only region of G that exists is the exterior region, then $r = 1$. We obtain the equality $1 - 0 + 1 = 2$ and so the formula holds for $m = 0$.

Let $m = 1$. The only edge must be between two vertices. Thus $n = 2$. Since G can not contain any cycles, then the only region is the exterior region. Thus $r = 1$. We obtain the equality $2 - 1 + 1 = 2$ and so the formula holds for $m = 1$.

Now consider $m \geq 2$. The graph G is either a tree or contains a cycle. If G is a tree, then $m = n - 1$ and since the only region is the exterior region then $r = 1$. The formula holds true since $n - (n - 1) + 1 = 2$. If G contains a cycle, then there exists an edge e such that e is incident with two regions. Consider the subgraph $G - e$ having n' vertices, m' edges and r' regions. The new graph of $G - e$, $n' = n$, $m' = m - 1$ and $r' = r - 1$. By the induction hypothesis, $m' \leq m$ so the equality $n' - m' + r' = 2$ holds true. We obtain $n - (m - 1) + (r - 1) = 2$ implies $n - m + r = 2$ is true. That is, Euler's polyhedral formula holds for G, and so holds in general.

□

Euler's Polyhedral Formula is the first important tool in which to study planar graphs. Through this identity, an upper bound on the number of edges in a planar graph G with n vertices can be found.

Theorem 2.3. *If G is a planar graph with $n \geq 3$ vertices and m edges then $m \leq 3n - 6$.*

Proof. Let G be a planar graph. Since a simple graph with three vertices cannot have more than three edges the result holds when $n = 3$. Suppose $n \geq 4$. Since G is a planar graph than it obeys Euler's Identity, having n vertices, m edges and f faces, $n - m + f = 2$. Since every region of G is composed of at least 3 edges, and every edge is incident with exactly two regions we see that inequality $3r \leq 2m$ holds true. By multiplying both sides of Euler's Polyhedral Identity we see that $6 = 3n - 3m + 3r$. Since we know that $3r \leq 2m$, then $3n - 3m + 3r \leq 3n - m$. By combining the equations we obtain $6 \leq 3n - m$ or $m \leq 3n - 6$.

□

A graph G is *maximal planar* if G is planar and the addition of an edge e to the edge set E of G produces a nonplanar graph.

Corollary 2.4. *If G is a maximal planar graph with $n \geq 3$ vertices and m edges, then $m = 3n - 6$*

We will now establish some lemmas for maximal planar graphs that we will utilize later on in chapter 3.

Lemma 2.5. *Let G be a maximal planar graph. Every region of G is of length three.*

Proof. Let G be a maximal planar graph. Suppose the G had a region R of length greater than three. Since R is bounded by a cycle, then there exists a path P of length three. Let v_1, v_2 and v_3 be the vertices of P. At least two among v_1, v_2 and v_3 are nor adjacent or else P is part of a cycle of length three. Without lost of generality let v_1 and v_3 not be adjacent. Adding the edge $\{v_1, v_3\}$ to the edge set of G, G would still be planar. But this is a contradiction to the definition of G being maximal planar.

□

Lemma 2.6. *Let G be a maximal planar graph of order $n \geq 4$*

Proof. Let $G = (V, E)$ be a maximal planar graph. Suppose there exists a vertex v such that $\deg(v) \leq 2$. Let $\deg(v) = 0$ than clearly the edge $\{v, u\}$ can be added to G for any vertex u distinct from v and the resulting graph will still be planar. Hence G is not maximal planar which is a contradiction. Suppose $\deg(v) = 1$. Thus v is adjacent to exactly one vertex, say $u \in V$. Then u must be adjacent to some other vertex say w. Then the edge $\{v, w\}$ can be added to G. since u and w are adjacent, adding the edge $\{v, w\}$ would create a u, w-path through v which is homeomorphic to the edge $\{u, w\}$ [GM12]. Therefore G is not maximal planar which is a contradiction.

Suppose $\deg(v) = 2$. Let v_i and v_j be the vertices adjacent to v. At most v_i and v_j are adjacent. There must exists a face containing v, v_i and v_j plus at least another vertex. Then there exists some vertex v_k adjacent to v_i or v_j. Then the edge v_i, v_k can be added to E. Thus G is not maximal planar. Thus for a maximal planar graph, every vertex must be of at least degree three.

□

Lemma 2.7. *Let G be a maximal planar graph with a vertex v of degree three. Let v_1, v_2 and v_3 be the three vertices adjacent to v. Then v_1, v_2 and v_3 are pairwise adjacent.*

Proof. Let G be a graph with a vertex v of degree three. Suppose at least two of the three vertices are not adjacent say v_1 and v_2. Then the vertices v, v_1 and v_2 are all part of the same region. Since G is maximal planar, by Lemma 2.5 every region is a cycle of length three. Then v, v_1, and v_2 are a cycle of length three. Thus v_1 and v_2 are adjacent which is a contradiction. Therefore all three vertices adjacent to v are pairwise adjacent. □

Theorem 2.8. *K_5 is not planar*

Proof. Suppose that K_5 is planar. Then K_5 has exactly five vertices and ten edges. Then K_5 satisfies the inequality $m \leq 3n - 6$. But $10 \leq 9$, which is false. Thus K_5 is not planar. □

We will now prove that $K_{3,3}$ is not planar. However, $K_{3,3}$ has six vertices and nine edges which does not contradict Theorem 2.3. Hence we will need to employ a different technique.

Theorem 2.9. *$K_{3,3}$ is not planar.*

Proof. Suppose $K_{3,3}$ is planar. We obtain the equality, by Euler's Polyhedral Formula, $n - m + r = 2$ where $n = 6$ and $m = 9$. It follows that $r = 5$ regions. Additionally, since $K_{3,3}$ contains no cycles of length three, then each region must be incident with at least four edges. By summing the edges over each face, we count each edge twice and each face at least four times. Hence, $4r \leq 2m$. Since $r = 5$ and $m = 9$, we obtain $20 \leq 18$, a contradiction. Thus $K_{3,3}$ is cannot be planar. □

Since neither K_5 nor $K_{3,3}$ are planar, it is clear that any graph G that contains either K_5 or $K_{3,3}$ as a subgraph, cannot be planar. Thus, we have a necessary but not sufficient condition for a graph to be planar.

A graph H is a *subdivision* of a graph G if H is obtained from G by replacing any of the edges of G by arbitrarily long paths. Two graphs H and H' are *homeomorphic* if there is some graph G from which H and H' are each obtained from a series of subdivisions.

Theorem 2.10. *Let G be a plane graph. Then for any region R having boundary B, there exists an embedding such that B is on the boundary of the exterior region.*

Proof. Let G be a given plane graph with regions $R_1, R_2, ...R_n$. Then the boundary of a region R_i is the collection of edges incident with the region. Let $B_1, B_2, ..., B_n$ be the boundaries of $R_1, R_2, ...R_n$, respectively. We construct a planar embedding of G having B_1 on the boundary of the exterior region. We may assume R_1 is not already the exterior region otherwise we are done. If G is not connected, we construct the desired embedding working only in the component of G containing R_1. Thus we lose no generality in assuming G is connected. We begin by embedding the boundary B_1 in the plane. Next we identify a region R_i with boundary B_i that is incident with at least one vertex of B_1. We then embed the boundary B_i on the interior of boundary B_1. Next we identify a region R_j with boundary B_j which is incident to at least one vertex of either R_1 or R_i and embed the boundary B_j on the interior of B_1.

If at some point in this process we are unable to embed any additional boundaries on the interior of B_1, then either no remaining boundaries have a vertex in common with any previously embedded boundaries or no matter how we try to embed another boundary B_k on the interior of B_1, planarity is violated. In the first case, we have a collection of boundaries that are disjoint from embedded boundaries. Then G must be disconnected, which is a contradiction to our assumption that G is connected. In the second case, since every edge of G that is not a bridge is incident with exactly two regions, all remaining boundaries can only share edges with boundaries that are not already in common to two embedded boundaries. Thus the set of edges that are contained in exactly one embedded boundary bound a region R in the current partial embedding of G. It follows that all remaining boundaries must be on the interior of R. Hence, any of the remaining boundaries may be embedded without violating planarity.

□

From Theorem 2.11, it is easy to see that for any vertex v or edge e of a planar graph G, there exists a planar embedding of G such that v or e, respectively, is may also be positioned incident to the exterior region. This fact allows us to construct new planar graphs from existing planar graphs. Consider two planar graphs G_1 and G_2 with given planar embeddings. We construct a *planar gluing* by identifying either a vertex or edge in each of G_1 and G_2, making the edge or vertex incident with the exterior region in each graph and taking the union of the planar embeddings of G_1 and G_2 by overlapping the planar embeddings precisely at the vertex or edge. Through planar gluing we are able to construct larger planar graphs by gluing smaller planar graphs together.

Theorem 2.11 (Kuratowski's Theorem)**.** *A graph G is planar if and only if G contains no subgraph that is homeomorphic to K_5 or $K_{3,3}$*

Proof. By Theorem 2.5 and Theorem 2.6, if G is a planar graph, then G cannot contain a subgraph homeomorphic to K_5 or $K_{3,3}$. Thus we will verify that if a graph contains no subgraph homeomorphic to K_5 or $K_{3,3}$, then G is planar.

To the contrary, suppose there exists a nonplanar graph that contains no subgraph homeomorphic to K_5 or $K_{3,3}$. We may assume that G is an example of such a graph that is minimal with respect to the cardinality of its edge set. We lose no generality in assuming G is connected, for if G is disconnected, our argument applies to each connected component of G. If G is not 2-connected, then there exists a vertex v in G whose deletion results in a disconnected graph. Let C_1 be a component of $G - v$ and let C_2 be the union of the remaining components of $G - v$. By the minimality of G, we see that the subgraph of G obtained by deleting all the vertices of C_1 is planar. Similarly, the subgraph of G obtained by deleting all the vertices of C_2 is planar. That is, the induced subgraph G_1 on $V(C_1) \cup \{v\}$ is planar, as is the induced subgraph G_2 on $V(C_2) \cup \{v\}$. Thus, we can construct a planar glueing of G_1 and G_2 at the vertex v to produce G. As G is nonplanar, this contradiction implies G must be 2-connected. We now show G is 3-connected.

Suppose G is not 3-connected. Then there exist two vertices v_1 and v_2 such that $G - \{v_1, v_2\}$ is a disconnected graph. Consider the partition $\{H_1, H_2\}$ of the edge set of G such that H_1 is one of the components of $G - \{v_1, v_2\}$ and H_2 the union of the remaining components of $G - \{v_1, v_2\}$.

Suppose that v_1 and v_2 are adjacent. Consider the partitions $F_1 = H_1 \cup \{v_1, v_2\}$ and $F_2 = H_2 \cup \{v_1, v_2\}$. By the minimality of G, each of F_1 and F_2 induce planar subgraphs of G. Thus we can construct a planar gluing of F_1 and F_2 at v_1v_2 producing the graph G. Thus G is planar, a contradiction.

Now suppose v_1 and v_2 are not adjacent. Since G is nonplanar, the graph $G' = G \cup v_1v_2$ is also nonplanar. Moreover, since $G - \{v_1, v_2\}$ is disconnected, so is $G' - \{v_1, v_2\}$. Let H_1 be a connected component of $G - \{v_1, v_2\}$ and let H_2 be the collection of the remaining components of $G - \{v_1, v_2\}$. Let $F_1 = (G - H_2) + v_1v_2$ and $F_2 = (G - H_1) + v_1v_2$ be subgraphs of G'. Since G' is not planar, either F_1 or F_2 is not planar. Indeed, suppose both F_1 and F_2 are planar, then we may construct an embedding of F_1 with v_1v_2 on the exterior region by Theorem 2.11 and also construct an embedding of F_2 with v_1v_2 as an exterior region. Hence we may assume F_1 is not planar.

Since $F_1 - v_1v_2$ is a proper subgraph of G then, by minimality, $F_1 - v_1v_2$ is planar. Since G is 2-connected, then there exists some v_1, v_2-path contained F_2 other than the edge v_1v_2. Thus, in G there exists a subgraph homeomorphic to F_1. Therefore G has a subgraph that is not planar, contradicting the minimality of G. We conclude that G is 3-connected.

The minimality of G guarantees that there exists an edge $e = uv$ such that the subgraph $H = G - e$ is planar and 2-connected. For the remainder of the proof, we assume H is embedded in the plane. Since H is 2-connected, there exists a cycle in H containing u and v. Among all such cycles, let C be one of maximum length. We let $C = v_0v_1v_2...v_kv_0$, and we may assume $u = v_0$ and $v = v_l$ for some l with $0 \leq l \leq k$.

We will call an x, y-path in G that only has vertices x and y in common with C an *x,y-chordal path of* C. If for all s and t with $0 < s < l$ and $l < t \leq k$, there does not exist a v_s, v_t-chordal path of C in H, then e can be added to H while preserving planarity, a contradiction. Therefore, there must exist such a v_s, v_t-path in H. We illustrate this structure in Figure 2.2.

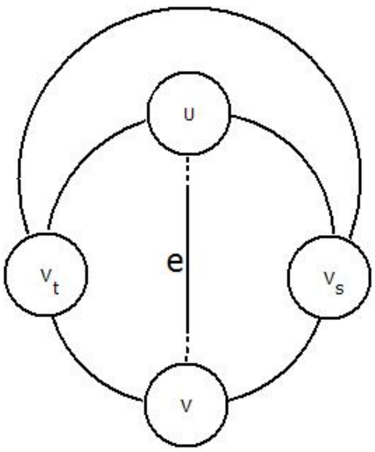

Figure 2.4: A picture of the subgraph H.

We will now develop a case analysis with the cases being determined by where cordial paths terminate on C.

Case 1: Let the v_a,v_b-chordal path of C exist such that neither v_a nor v_b are any of v_0, v_s, v_l or v_t. Suppose v_a and v_b were between v_0 and v_l. Then the v_a,v_b-chordal path could be embedded parallel to both the v_0,v_l path and the edge e. Thus such a path would not cross the edge e and so e could be added to G preserving planarity. Similarly, if v_a and v_b are between the vertices from v_i for $l < i \leq k$, then we again are able to generate a planar embedding of G.

Suppose v_a and v_b are between the vertices from v_i for $s < i < t$. Such a v_a, v_b-chordal path would be able to be drawn on the exterior of of our cycle in such a way that it was parallel to to the cycle. Thus such an embedding of the chordal path would not affect the planarity of G. Similarly, if v_a and v_b are between the vertices from v_i for $t < i \leq k$ or $0 < i < s$ then the v_a,v_b-chordal path could be placed parallel to the cycle and the v_s,v_t-path in such a way to not affect the planarity of G when e is embedded.

Thus finally, without lost in generality consider if v_a is one of the vertices v_i such that $0 < i < s$ and v_b is one of the vertices v_j for $l < j < t$. Such a v_a,v_b chordal path must intersect the embedding of the edge e as seen in Figure 2.5. Thus if such a chordal path existed G is not planar. But such a graph G contains a subgraph homeomorphic to $K_{3,3}$. This is a contradiction to our assumption G has no subgraph homeomorphic to neither $K_{3,3}$ nor K_5. Thus such a chordal path can not exist. A similar argument holds for if v_a was one of the vertices v_i such that $s < i < l$ and v_b was one of the vertices v_j for $t < j \leq k$.

Case 2: Let the v_a,v_b-chordal path of C exist such that one of v_a or v_b is either v_0, v_s, v_l or v_t. Without lost of generality let v_a be v_0, and let v_b be one one the cycle vertices but not v_s, v_l, or v_t. Regardless of our choice of v_b, we are able to embed the $v_a v_b$ chordal path in such a way that it is parallel to e and thus does not intersect it. Such a chordal path results in G being planar which is a contradiction to our assumption that G is not planar. So consider if v_b was one of the vertices v_i such that $l < i < t$ and there existed a vertex v_c such that $s < c < l$. Then we know from Case 1 a v_b,v_c-chordal path is planar. Then consider if the v_a,v_b-chordal path and the $v_b v_c$ chordal path shared at least a vertex w. We are then no longer able to embed the chordal paths in such a way that G is planar as seen in Figure 2.6. But such chordal paths result in G having a subgraph that is homeomorphic to $K_{3,3}$ which is a contradiction to our assumption G has no such subgraph. Regardless of our choices for v_a and v_b, we are always able to identify a v_c such that this argument holds.

Case 3: Let the v_a,v_b-chordal path of C exist such that v_a is either v_0, v_s, v_l or v_t and v_b is either v_0, v_s, v_l or v_t but not the same as v_a. Without lost of generality let v_a be v_s and v_b be v_t. If this was the only $v_a v_b$-chordal path in G then we would be able to embed the chordal path parallel to the $v_s v_t$-path in such a way that the chordal path does not intersect the edge e and thus G would be planar, which is a contradiction to our assumption that G is not planar. A similar argument holds true for if v_a is v_0 and v_b is v_l. Thus consider if two such chordal paths existed the first being a v_0,v_l-chordal path and the other being a v_s,v_t-chordal path. Since we know if the chordal paths do not intersect we are able to embed them in such a way that G is planar, let the two chordal paths intersect.

Suppose the v_0,v_l-chordal path and the v_s,v_t-chordal path intersect and share at least two common vertices. As demonstrated in Figure 2.7, such a structure must in fact intersect some part of G, mainly it must intersect with some part of the edge e. Thus G is not planar. But, G would have a subgraph that is homeomorphic to $K_{3,3}$ which is a contradiction to our assumption that G has no such subgraph. Thus suppose that v_0,v_l-chordal path and the v_s,v_t-chordal path intersect at exactly one vertex, namely w. The structure would again intersect e and thus G is not planar as seen in Figure 2.8. But G would have a subgraph that is homeomorphic to K_5 which is a contradiction to our assumption that G has no such subgraph.

From the case analysis. it becomes clear that with any such chordal path, either we are able to embed the chordal path in such a way that G is planar, or anytime G is not planar, G has a subgraph that is homeomorphic to either $K_{3,3}$ or K_5. Thus since no such chordal path exists then there is no such graph G.

□

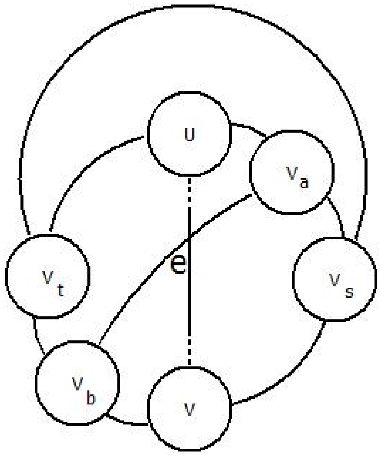

Figure 2.5: Structure of Case 1 of Kuratowski's Theorem.

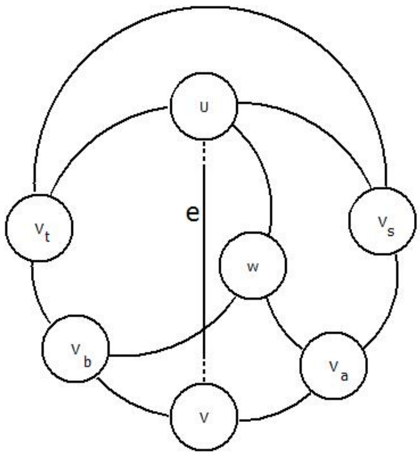

Figure 2.6: Structure of Case 2 of Kuratowski's Theorem.

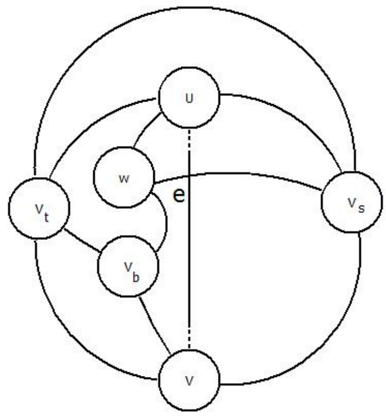

Figure 2.7: Structure of Case 3 of Kuratowski's Theorem.

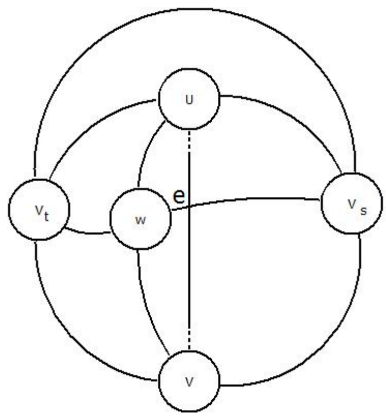

Figure 2.8: Structure of Case 4 of Kuratowski's Theorem.

Chapter 3

Biplanar Graphs

Our study of biplanar graphs is inspired by the model of an electrical circuit that was given in Chapter 1, in which it was necessary to utilize two sides of a chip. A graph $G = (V, E)$ is *biplanar* if there exists a partition of E into two parts E_1,E_2 such that subgraphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ are planar graphs. If a graph G is biplanar then we will state that G has a *biplanar decomposition* into the subgraphs G_1 and G_2. The ultimate goal in our study of biplanar graphs is to provide a characterization of forbidden subgraph analogous to Kuratowski's Theorem. Unfortunately, there exist example of nonbiplanar graphs that are homeomorphic to biplanar graphs (See Corollary 3.10). Hence, a completely analogous characterization is impossible. With this in mind, we dedicate this chapter to an investigation of the smallest complete and complete bipartite graphs that are not biplanar.

The goal of this chapter is to prove the following Theorems.

Theorem 3.1. *The graph K_9 is the smallest complete graph that is not biplanar.*

Theorem 3.2. *The graphs $K_{7,7}$, $K_{6,9}$ and $K_{5,12}$ are the smallest complete bipartite graphs that are not biplanar.*

3.1 Complete Biplanar Graphs

We will first show that the complete graph K_9 is the smallest complete graph that is not biplanar.

Lemma 3.3. *The complete graph K_8 is biplanar.*

Proof. As seen in Figure 3.1 K_8 has a biplanar decomposition.

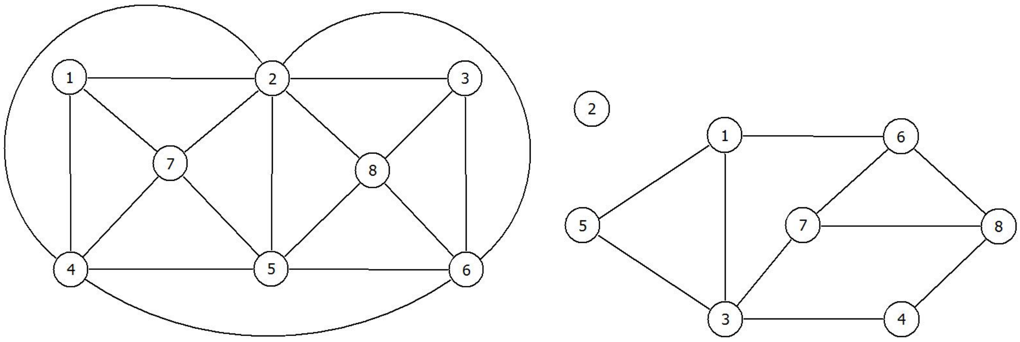

Figure 3.1: A biplanar decomposition of K_8.

□

In order to show that K_9 is not biplanar, then we will use Lemmas, 2.5, 2.6 and 2.7 from chapter two. The original inspiration for the proof can be found here [BHK62].

Lemma 3.4. *The complete graph K_9 is not biplanar.*

Proof. Suppose that K_9 is biplanar. Then K_9 has a biplanar decomposition into $G = (V, E_1)$ and $\overline{G} = (V, E_2)$. We first show that both G and $\overline{G}$ must be connected. Suppose, to the contrary, that either G or $\overline{G}$ is not connected. With no loss in generality, we may assume $\overline{G}$ is not connected. Consider the following cases determined by the number of components in $\overline{G}$:

Case 1: Suppose $\overline{G}$ has at least four components. Then there exists a partition of the components into two parts P_1 and P_2 such that each part has at least three vertices. Then, in G, every vertex in P_1 is adjacent to every vertex in P_2. Since both parts are of size at least three, G has a $K_{3,3}$ subgraph. By Kuratowski's Theorem, G is not planar, which is a contradiction to our assumption that K_9 is biplanar.

Case 2: Suppose $\overline{G}$ has exactly three components C_1, C_2 and C_3. If each component has more than one vertex, then we can partition the components into two parts P_1 and P_2 such that each part has at least three vertices. Since each of P_1 and P_2 have at least three vertices, G has a $K_{3,3}$ subgraph. By Kuratowski's Theorem, G is not planar, which is a contradiction to our assumption that K_9 is biplanar.

Now suppose that exactly one component, say C_1, of $\overline{G}$ contains exactly one vertex. Further assume that C_3 has at least as many vertices as C_2. Since there are exactly nine vertices, it follows that C_3 must have at least three vertices. Consider a partition of vertices of $\overline{G}$ into two parts P_1 and P_2, such that P_1 contains the vertex of C_1 and vertices of C_2, and P_2 contains the vertices of C_3. Since both P_1 and P_2 contain at least three vertices, G has a $K_{3,3}$ subgraph. By Kuratowski's Theorem, G is not planar which again is a contradiction to our assumption that K_9 is biplanar.

Finally, suppose that two components, say C_1 and C_2, have exactly one isolated vertex. Let C_3 contain the remaining vertices. This is actually a proper subgraph of Case 3 in which the vertices in C_1 and C_2 would be adjacent. So instead consider Case 3.

Case 3: Suppose $\overline{G}$ has exactly two components. Let C_1 and C_2 be the components of $\overline{G}$ and assume C_2 has at least as many vertices as C_1. Suppose C_1 has at least three vertices in it, then it follows C_2 must also have at least three vertices. Then, G contains a $K_{3,3}$. By Kuratowski's Theorem, G is not planar, which is a contradiction to our assumption that K_9 is biplanar.

Now suppose C_1 has exactly two vertices, say v_1 and v_2. Then, in G, v_1 and v_2 are not adjacent but are both adjacent to all the vertices $v_3, ..., v_9$. In order to minimize the edges in $\overline{G}$, we will let G be maximal. It follows that $v_3, ..., v_9$ are degree at most four. If $v_i, 3 \leq i \leq 9$ is degree at least 5 then G contains a $K_{3,3}$ as a subgraph making G not planar by Kuratowski's Theorem which is a contradiction to our assumption K_9 is biplanar. Thus the vertices $v_3, ..., v_9$ form a cycle. Then $\overline{G}$ contains the graph as in Figure 3.2. Thus $\overline{G}$ contains a subdivision of K_5 and is not planar by Kuratowski's Theorem, which is a contradiction to our assumption that K_9 is biplanar.

Suppose C_1 has exactly one vertex, say v_1. In order to reduce the number of edges in $\overline{G}$, suppose that G is maximal planar. Then, by Corollary 2.4, G has exactly $3n - 6 = 21$ edges. Since we know eight of the edges are incident with v_1 there exist thirteen edges between $v_2, ..., v_9$. In order for $\overline{G}$ to be planar,by Theorem 2.3, it must have $m \leq 3n-6$ edges. Since we know v_1 is an isolated vertex, all edges must be between $v_2, ..., v_9$ except for the thirteen edges in G. Thus $\overline{G}$ must have less than $m \leq 3(8)-6 = 18$ edges. But $\overline{G}$ must have all the remaining edges except the thirteen in G and thus has $8\dot{7}/2 - 13 = 19$ edges. Thus $\overline{G}$ is not planar by Theorem 2.3 which is a contradiction to our assumption that K_9 is biplanar.

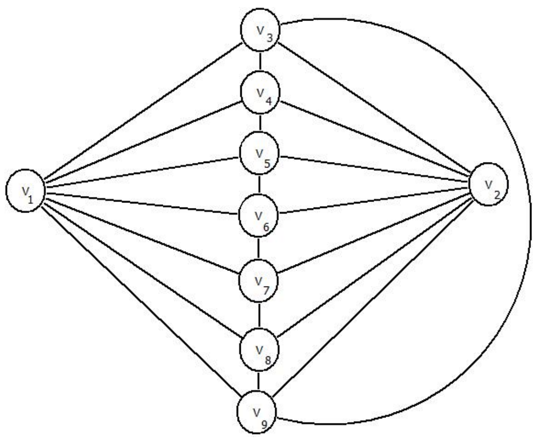

Figure 3.2: Structure of G in Case 3.1.

From Cases 1, 2, and 3, it is clear that G and $\overline{G}$ must both be connected if K_9 biplanar. Let $d_1, d_2, ..., d_9$ be the degrees of the vertices of G such that $d_1 \leq d_2 \leq ... \leq d_9$. With no loss in generality, we may assume G is maximal planar so that the cardinality of E_2 is minimized.

If any vertex has degree eight in G, then that vertex must be an isolated vertex in $\overline{G}$. Since $\overline{G}$ is connected, this is impossible. Additionally since G is maximal planar, by Lemma 2.6, every vertex has degree at least three in G. Thus $3 \leq d_i \leq 7$ for $i = 1, 2, ..., 9$. In the case analysis that follows, we narrow down the possibilities of the degree sequence for G. In Case 1, we will show no two vertices may have degree seven. In case two we show that there must exist at most one vertex of degree three. In case three we show there must exist at most one vertex of degree four. Then finally, we show the only degree sequence possible does not have a corresponding planar graph.

Case 1: Suppose that G has a degree sequence such that $d_1 = d_2 = 7$. Let the two vertices of degree seven be labeled v_1 and v_2. Suppose v_1 and v_2 are not adjacent in G. Then v_1 and v_2 are adjacent in $\overline{G}$. Then v_1 and v_2 together with the edge $\{v_1, v_2\}$ are a component in $\overline{G}$. Thus $\overline{G}$ has two components which is a contradiction to $\overline{G}$ being connected. Thus v_1 and v_2 are adjacent in G.

Case 1.1: Suppose there exists a vertex v_9 in G that is not adjacent to v_1 and is not adjacent to v_2. Then since v_1 and v_2 are adjacent in G and of degree seven in G, then v_1 and v_2 are adjacent to vertices $v_3, v_4, ...v_8$. Suppose that v_9 was adjacent to three or more of $v_3, v_4, ...v_8$. Then the graph G contains a $K_{3,3}$ subgraph since v_1, v_2 and v_9 are adjacent to at least three common vertices. Therefore, v_9 must have degree less than three. But this contradicts Lemma 2.6 and thus there every vertex must be adjacent to either v_1, v_2, or both.

Hence there must exist two vertices v_3 and v_4 such that v_3 is adjacent to v_1 but not v_2 in G and v_4 is adjacent to v_2 but not v_1 in G. Suppose that either v_3 or v_4 has degree greater than four. We may assume v_4 is this vertex. Then v_4 is adjacent to v_2 and at least four other vertices. Thus v_4 is adjacent to at least three vertices among $v_5, v_6...v_9$. We may assume that v_4 is adjacent to v_5, v_6 and v_7. It follows G contains a $K_{3,3}$ subgraph having a partite sets $\{v_1, v_2, v_4\}$ and $\{v_5, v_6, v_7\}$. We may conclude that v_3 and v_4 have degree at most four.

Suppose v_4 has degree four. Since v_4 is adjacent to v_2 and not v_1, then v_4 is adjacent to three other vertices among $v_3, v_5, v_6, ..., v_9$. If v_4 is adjacent to three vertices among $v_5, v_6, ..., v_9$, then G once again contains a $K_{3,3}$ subgraph. This implies that G would not be planar by Kuratowski's Theorem which is a contradiction to our assumption that K_9 is biplanar. We may conclude that v_4 is adjacent to v_3 and two other vertices, say v_5 and v_6.

Suppose that v_3 is either not adjacent to v_5 or not adjacent to v_6. We may assume v_3 is not adjacent to v_5. Then v_3, v_4, v_5 are incident with some region R. But since v_3 and v_5 are not adjacent then there exists some vertex v_i also a incident with R for $i \in 7, 8, 9$. Thus R is a region of length at least four in the maximal planar graph G. But by Lemma 2.5, every region of a maximal planar graph is of length three. With this contradiction, we conclude that v_3 and v_4 are both adjacent to v_5 and v_6. Figure 3.1 illustrates the structure present in G within our subcase.

Suppose that none of v_7, v_8 and v_9 are adjacent to v_5. In G, none of v_7, v_8 and v_9 are not adjacent to v_3 or v_4 since each of v_3 and v_4 are degree four. Then in $\overline{G}$, the vertices v_3, v_4, and v_5 are adjacent to all the vertices v_7, v_8, and v_9 creating a $K_{3,3}$ subgraph in $\overline{G}$. Thus, by Kuratowski's Theorem, $\overline{G}$ is not planar which is a contradiction to our assumption that K_9 is biplanar. A similar argument holds when v_7, v_8 and v_9 are all not adjacent to v_6.

The other possible case is if there exists a path P of length three consisting of two vertices among v_7, v_8 and v_9 with one of the end points being either v_5 or v_6. Without

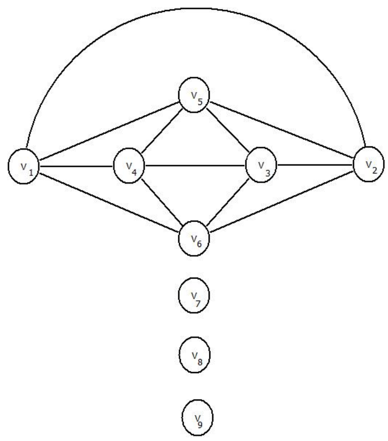

Figure 3.3: Core structure of Case 1.3.

loss of generality, let $P = v_9v_8v_5$. Then v_7 must be adjacent to v_6. Then we obtain the graph in Figure 3.4.

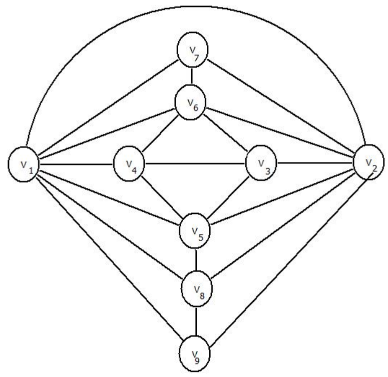

Figure 3.4: Structure of Case 1.3 with path P.

In $\overline{G}$ the vertices v_3 and v_4 are adjacent to v_7, v_8 and v_9. Additionally, v_6 is adjacent to v_8 and v_9. But v_5 must be adjacent to v_6 and v_7. Then through a sub divided edge, v_6 is adjacent to v_7. Thus $\overline{G}$ contains a subgraph homeomorphic to $K_{3,3}$ making $\overline{G}$ and thus, by Kuratowski's Theorem, $\overline{G}$ nonplanar which is a contradiction to our assumption that K_9 is biplanar. Note that regardless of choices for the vertices for P, the same argument can be applied. The degree of vertices v_3 and v_4 must be equal to three.

Since both v_3 and v_4 are of degree three, then by Lemma 2.7, v_3 and v_4 may have exactly zero, one or two neighbors in common. These three possibilities are the following cases.

Case 1.4: Let v_3 and v_4 be of degree three and be adjacent to exactly zero common vertices. Then both v_3 and v_4 are adjacent to four distinct vertices, say v_5, v_6, v_7 and v_8. Let v_3 be adjacent to v_5 and v_6, and let v_4 be adjacent to v_7 and v_8. Then v_9 is not adjacent to v_3 or v_4. Additionally, v_9 is adjacent to either one or two of v_5, v_6, v_7, or v_8.

Suppose that v_9 is adjacent to one of v_5, v_6, v_7, or v_8, say v_5. Then v_9, v_5, v_3 are not adjacent to v_4, v_7 or v_8 in G. Thus in $\overline{G}$, v_9, v_5, v_3 are adjacent to v_4, v_7 and v_8 creating a $K_{3,3}$ subgraph in $\overline{G}$. Then $\overline{G}$ is not planar which is a contradiction. Then v_9 must be adjacent to two of v_5, v_6, v_7, or v_8.

Suppose that v_9 is adjacent v_5 and v_6. Then v_1, v_3 and v_9 are all adjacent to v_2, v_5 and v_6 in G. Thus G has a $K_{3,3}$ subgraph and, by Kuratowski's Theorem, G is not planar which is a contradiction to our assumption that K_9 is biplanar. A similar argument holds if v_9 is adjacent to v_7 and v_8.

Thus v_9 must be adjacent to one of v_5 or v_6 and one of v_7 or v_8. Let v_9 be adjacent to v_5 and v_7. Then v_5 and v_7 are adjacent to exactly two vertices of $v_5, v_6, ...v_9$. If either v_5 or v_7 were adjacent to any additional vertex of the set v_6, v_8, v_9, then G would have a $K_{3,3}$ and, by Kuratowski's Theorem, G is not planar which is a contradiction to our assumption that K_9 is biplanar. So v_5 is not adjacent to v_4, v_7 nor v_8 and v_7 is not adjacent to v_3, v_5 nor v_6. Then, in $\overline{G}$, v_3 and v_5 are adjacent to v_4, v_7 and v_8. Also v_6 is adjacent to v_4 and v_7 in $\overline{G}$. Additionally, v_9 is adjacent to v_6 and v_8 in $\overline{G}$. Thus $\overline{G}$ has a subgraph homeomorphic to $K_{3,3}$ and, by Kuratowski's Theorem, is not planar which is a contradiction to our assumption that K_9 is biplanar.

Case 1.5: Let v_3 and v_4 be of degree three and be adjacent to exactly one common vertex. Let v_5 be that common vertex, let v_6 be adjacent to v_3 and v_7 be adjacent to v_4. Suppose that there existed a path P of length three consisting of v_8 and v_9 with either v_6 or v_7. Let $P = v_6, v_8, v_9$. Then G has the graph seen in Figure 3.5.

Then in $\overline{G}$, vertices v_3 and v_6 are adjacent to vertices v_4, v_7 and v_9. Additionally v_8 is adjacent to v_4 and v_7. Finally, since v_5 is adjacent to v_8 and v_9, then $\overline{G}$ has a subgraph homeomorphic to $K_{3,3}$. Then $\overline{G}$ is not planar which is a contradiction.

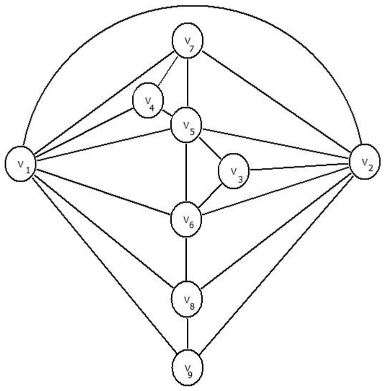

Figure 3.5: Structure of Case 1.5 with path P.

Since there does not exist a path P of length three, then vertices v_8 and v_9 must not be adjacent. Then v_8 must be adjacent to v_6 or v_7, and v_9 must be adjacent to the opposite vertex. Without loss of generality, let v_8 be adjacent to v_7 and v_9 be adjacent to v_6. Then vertices v_3, v_6 and v_9 are not adjacent to v_4, v_7 nor v_8. Then in $\overline{G}$, vertices v_3, v_6 and v_9 are adjacent to v_4, v_7 and v_8. Then $\overline{G}$ has a $K_{3,3}$ subgraph and, by Kuratowski's Theorem, $\overline{G}$ is nonplanar which is a contradiction to our assumption that K_9 is biplanar.

Case 1.6: Let v_3 and v_4 be of degree three and be adjacent to exactly two common vertices. This case follows a similar argument from that in case 1.2 in which there either exists a path P of length three resulting in $\overline{G}$ has a subgraph homeomorphic to $K_3, 3$ and, by Kuratowski's Theorem, $\overline{G}$ is nonplanar which is a contradiction to our assumption that K_9 is biplanar.

From Cases 1.1 through 1.6 we conclude that, there does not exist a maximal planar graph G with an edge set E_1 with two vertices are of degree seven such that $\overline{G}$ is planar.

Case 2: Let G have a degree sequence such that $d_8 = d_9 = 3$. Since G has two vertices of degree three let them be labeled v_8 and v_9. By Lemma 2.7, v_8 and v_9 may have either zero, one, or two common neighbors. We will consider these possibilities in the following subcases.

Subcase 2.1: Suppose v_8 and v_9 have exactly two common vertices. Let the two common vertices be v_6 and v_7. Let v_5 be the other vertex adjacent v_8 and let v_4 to be the vertex adjacent to v_9. Since every vertex adjacent to v_8 is adjacent and every vertex adjacent to v_9 is adjacent, then v_6 and v_7 are of at least degree five. Since we are considering the decomposition of K_9, then there exists three other vertices v_1, v_2 and v_3. A graph of Subcase 2.1 can be seen in Figure 3.6.

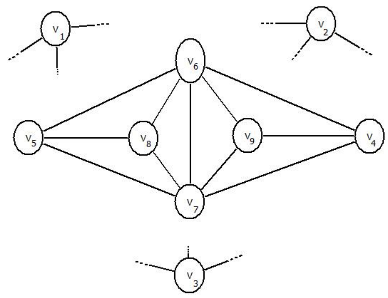

Figure 3.6: Structure of Subcase 2.1.

By Case 1, it is clear that no two vertices may be of degree seven. In particular, v_7 and v_8 may not both be of degree seven. Additionally, since no vertex may be of degree eight, v_1, v_2 and v_3 may not all be adjacent to either v_6 nor v_7.

Suppose that v_1, v_2 and v_3 were not adjacent to neither v_6 nor v_7. Then in $\overline{G}$, v_6, v_8 and v_9 would all be adjacent to v_1, v_2 and v_3. Thus $k_{3,3}$ would be a subgraph of $\overline{G}$ making $\overline{G}$ not planar. But this is a contradiction, so at least one of v_1, v_2 or v_3 is adjacent to v_6.A similar argument can be made for v_7, and so at least one of v_1, v_2 or v_3 is adjacent to v_7. Without lost of generality, let the vertices v_1 and v_6 be adjacent and let the vertices v_3 and v_7 be adjacent. We will now consider two subcases for if the edge v_4, v_5 is in the edge set E_1 or if v_4, v_5 is in the edge set E_2.

Subcase 2.1.1: Suppose the edge v_4, v_5 was in the edge set E_1. Then without lost of generality, we may consider the edge v_4, v_5 to be part of the region with vertices v_7, v_4 and v_5. Since v_7 must be adjacent to at least one of v_1, v_2 or v_3 then without lost of generality let v_3 be adjacent to v_7. Suppose v_3 was the only vertex of v_1, v_2 or v_3 to be adjacent to v_7. Then a graph of subcase 2.1.1 can be seen in Figure 3.7.

The vertices v_8 and v_9 are not adjacent to v_1, v_2 or v_3, and vertices v_3 and v_7 are not adjacent to vertices v_1 nor v_2. Then all of these vertices must be adjacent in $\overline{G}$. Then $\overline{G}$ contains a subgraph homeomorphic to K_5 and, by Kuratowski's Theorem, is not planar. This is a contradiction to our assumption that K_9 is biplanar.

This is the only case to consider with the edge v_4v_5 being in the edge set E_1.

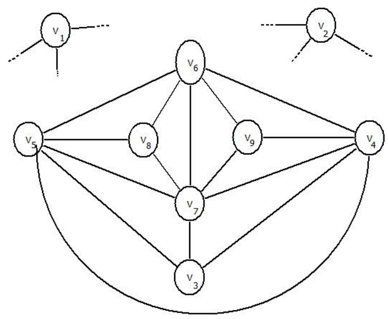

Figure 3.7: Structure of Case 2.1.1.

If two vertices of v_1, v_2 or v_3 were adjacent to v_7 and the last vertex was adjacent to v_6, then the edge going around v_7 may instead be drawn around v_6 creating an isomorphic graph.

Subcase 2.1.2: Suppose the edge v_4, v_5 was not in the edge set E_1. Suppose that v_2 was not adjacent to v_6 nor v_7. Then in a very similar argument to Case 2.1.1, $\overline{G}$ has a subgraph isomorphic to $K_{3,3}$ making $\overline{G}$ not planar which is a contradiction.

Subcase 2.2: Suppose v_8 and v_9 have one common vertex. Let v_7 be that common vertex. Let v_5 and v_6 be the other two vertices adjacent to v_8 and let v_3 and v_4 be the vertices adjacent to v_9. Let v_1 and v_2 be vertices of at least degree three. Then Case 2.2 has a graph like below.

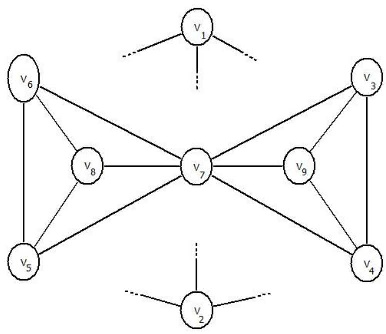

Figure 3.8: Structure of Case 2.2.

Since v_7 is already of degree six, then v_7 can at most be adjacent to either v_1 or v_2. Then consider the following cases

Subcase 2.2.1 Suppose neither v_1 nor v_2 are adjacent to v_7. Suppose further that the edge v_5, v_4 is not in the edge set E_1. Then in the graph G, v_5, v_4 and v_7 are part of some face. But since v_5, v_4 is not in E_1 then there must exist some other vertex v_i part of the same face. Then G has a face of size greater then three which is a contradiction since G is maximal. Then the edge v_5, v_4 is in E_1. Similarly, v_3, v_6 is in E_1.

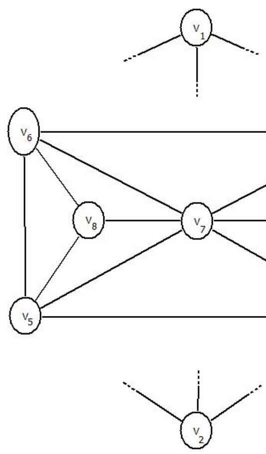

Figure 3.9: Structure of Case 2.2.1.

Subcase 2.2.2 Suppose either v_1 or v_2 is adjacent to v_7. Without lost of generality, let v_1 be adjacent to v_7. Similar to in Case 2.2.1, then either the edge v_6, v_3 or the edge v_4, v_5 is in E_1. Without lost of generality, let the edge v_4, v_5 be in E_1.

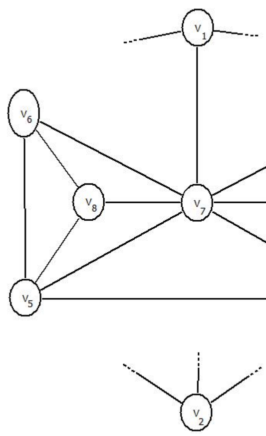

Figure 3.10: Structure of Case 2.2.2.

Suppose that v_1 was not adjacent to v_3. Then v_1, v_7 and v_3 are part of some face with at least one more vertex. Then either v_1 is adjacent to v_2, or one of v_4 or v_5. If v_1 was adjacent to v_2 then v_2 is adjacent to some v_i. In order to make G maximal planar, every face must be of size three. Thus there exists an edge that can be added to E_1. But since the edge v_2, v_7 makes v_7 a degree eight vertex, then v_1 must be adjacent to v_3. A similar argument holds for v_1 and v_6 to be adjacent.

Subcase 2.3 Suppose v_8 and v_9 have zero common vertices. Then v_8 is adjacent to three vertices that are adjacent. Let these three vertices be v_5, v_6, v_7. Then v_9 is adjacent to three vertices that are adjacent. Let these three vertices be v_2, v_3, v_4. Then v_1 is the remaining vertex. Not from Cases 2.1 and 2.2, it is not possible for v_1 to be of degree three or else it would fall into one of the previous cases of degree three adjacency. Thus v_1 must be of degree greater then three. Additionally, v_1 is not adjacent to v_9 nor v_9 and thus could only be of max degree six.

Case 3: Let G have a degree sequence such that $d_i = d_j = 4$. By Case 1 and Case 2, then G must have at most one degree seven vertex, and one degree three vertex. Let v_8 and v_9 be the vertices of degree four. For each construction of the maximal planar graph G with having two vertices of degree four, every case ends up with G having at least two vertices of degree three. Thus such a case is not possible

From the Case 2 and Case 3 it follows that G must have exactly one vertex of degree three and exactly one vertex of degree four. Then the degree sequence of G must take on the form $5, 5, 5, 5, 5, 5, 5, 4, 3$. It is left to show that this degree sequence does not have a planar graph representation. Suppose that the degree sequence did have a planar graph representation. Then there would exist a degree five vertex that was adjacent to five other degree five vertices. Each of these vertices would then be at least adjacent to two other degree vertices in the set of five degree five vertices. This would form a cycle like construct seen in Figure 3.11.

Since there is a degree three vertex, it must be adjacent to exactly two adjacent of the outer degree five vertices. If the degree three vertex was adjacent to three adjacent then the center of the three degree three vertices would be a degree four. A similar argument holds true for the degree four vertex. Then there exist exactly three cases, if the degree three and degree four vertex are adjacent to exactly zero one or two of the same vertices.

Case 1: The degree three and degree four vertex are adjacent to exactly zero of

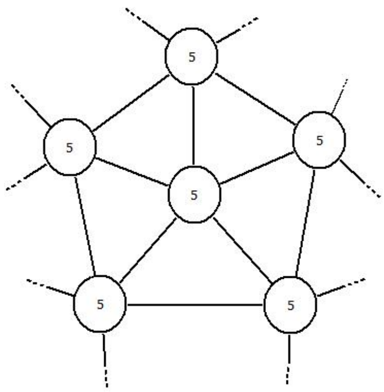

Figure 3.11: Structure of degree five cycle.

the same vertices. Then the graph G has the following structure:

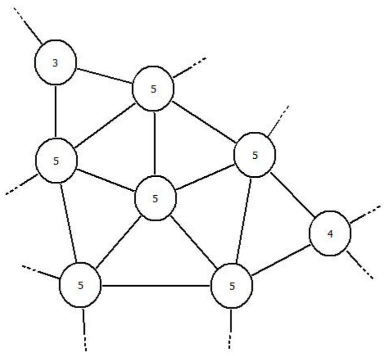

Figure 3.12: Structure of degree sequence Case 1.

The degree four vertex would have to be adjacent to at least one of the other vertices in the structure. If the degree four vertex was adjacent to one of the other vertices, then it would form a cycle of length three. One of the vertex would then be locked into an interior region and would not be able to have any additional edges adjacent to it forcing the degree to be less then five, a contradiction.

Case 2: The degree three and degree four vertex are adjacent to exactly one of the same vertices. Then the graph G has the following structure:

Since the vertex with all edges complete is degree five, and since every region must be of length three, then the degree three and degree four vertices are adjacent. In a similar fashion, the degree four and the degree five vertex that is also adjacent to the

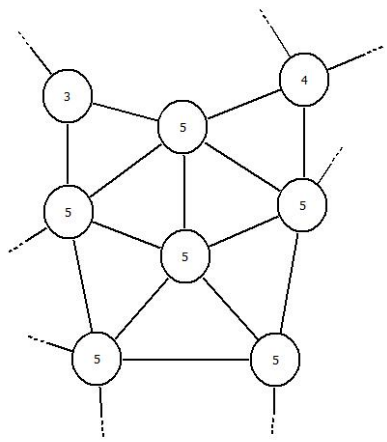

Figure 3.13: Structure of degree sequence Case 2.

degree three vertex are adjacent creating the structure seen in Figure 3.14.

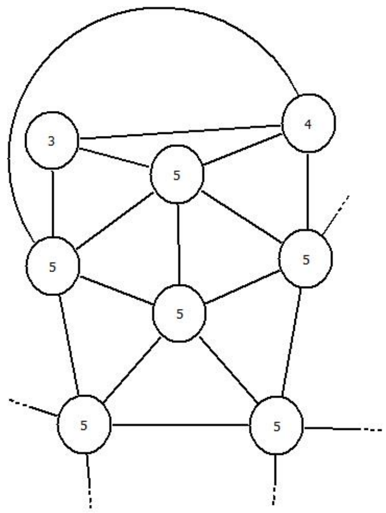

Figure 3.14: End structure of degree sequence Case 2.

From this it is clear that the last vertex could only be of maximum degree three. Then the graph G is not maximum planar which is a contradiction.

Case 3: The degree three and degree four vertex are adjacent to exactly two of the same vertices. Then the graph G has the structure seen in Figure 3.15.

The vertices in this graph labeled with a and b are of degree five. Similar to in Case 2, the two vertices adjacent to the degree four are already degree five, so in order

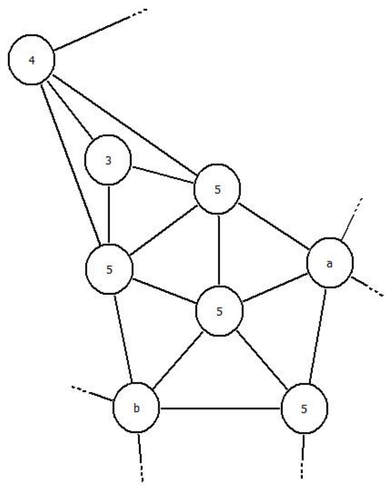

Figure 3.15: Structure of degree sequence Case 3.

to keep each region a cycle of length three, then the degree four must be adjacent to a and b making it a degree five, a contradiction.

From these cases it becomes clear that it is not possible to construct a maximal degree sequence of $5, 5, 5, 5, 5, 5, 5, 5, 4, 3$. So there does not exist a degree sequence correlating to a maximal planar graph G. Thus K_9 is not planar.

□

Lemma 3.5. *The graph K_9 is minimal non-biplanar.*

Proof. As seen in Figure 3.16, it is possible to remove a single edge $e = \{2, 4\}$ and generate a biplanar decomposition. Thus the graph $K_9 - e$ is biplanar.

□

3.2 Complete Bipartite Biplanar Graphs

Although the proof that K_9 is not biplanar is very technical, it follows directly from expanding Euler's Polyhedral Identity to show the complete bipartite graphs $K_{7,7}$ and $K_{6,9}$ are not biplanar. First we will expand Euler's Polyhedral Identity to biplanar bipartite graphs.

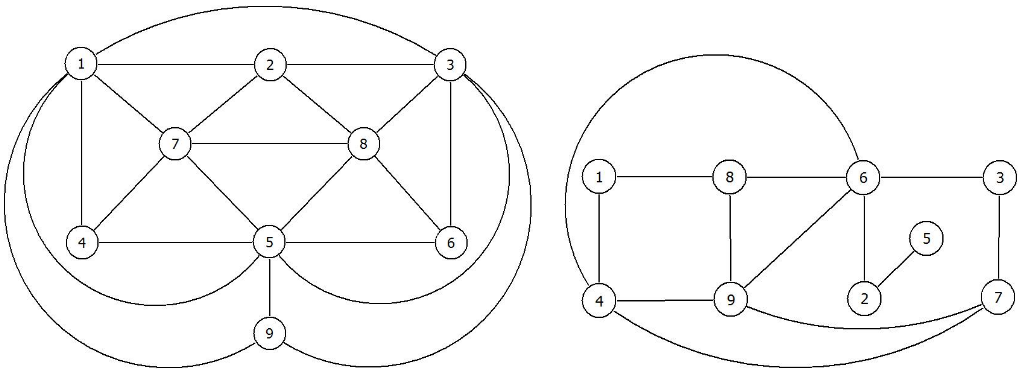

Figure 3.16: A biplanar decomposition of $K_9 - \{2, 4\}$.

Theorem 3.6. *A the size and order of a biplanar bipartite graph must satisfy the inequality $m \leq 4n - 8$.*

Proof. Let G be a planar bipartite graph. By Euler's Polyhedral Identity, in order for G to be planar it must satisfy the equation $n - m + r = 2$ or $r = 2 + m - n$. Since G is bipartite, G contains no cycles of length three. Then every region of G must be incident with at least four edges and since every edge is incident with exactly two regions we obtain the following inequality $2m \geq 4r$. By substitution, we obtain the inequality $2 + m - n \leq m/2$, which simplifies to $m \leq 2n - 4$. Since a biplanar decomposition of a bipartite graph results in two planar graphs, the number of edges and vertices in a biplanar bipartite graph must satisfy the inequality $m \leq 4n - 8$.

□

Lemma 3.7. *The complete bipartite graph $K_{7,7}$ is not biplanar.*

Proof. Suppose the complete bipartite graph $K_{7,7}$ is biplanar. By Theorem 3.2, the size and order of $K_{7,7}$ must satisfy the inequality $m \leq 4n - 8$. By substituting 49 for m and 14 for n, we obtain $49 \leq 4(14) - 8$, or $49 \leq 48$, which is a contradiction. Thus $K_{7,7}$ is not biplanar.

□

Lemma 3.8. *The complete bipartite graph $K_{6,9}$ is not biplanar.*

Proof. Suppose that the complete bipartite graph $K_{6,9}$ is biplanar. By Theorem 3.2, the size and order of $K_{6,9}$ must satisfy the inequality $m \leq 4n - 8$. By substituting 54 for m

and 15 for n we obtain $6(9) \leq 4(15) - 8$ or $54 \leq 52$ which is a contradiction. Thus $K_{6,9}$ is not biplanar.

□

Lemma 3.9. *The complete bipartite graph $K_{5,13}$ is not biplanar.*

Proof. Suppose that the complete bipartite graph $K_{5,13}$ is biplanar. By Theorem 3.2, the size and order of $K_{5,13}$ must satisfy the inequality $m \leq 4n - 8$. By substituting 60 for m and 18 for n we obtain $60 \leq 4(18) - 8$ or $65 \leq 64$ which is a contradiction. Thus $K_{5,13}$ is not biplanar.

□

Lemma 3.10. *The graphs $K_{7,7}$, $K_{6,9}$ and $K_{5,13}$ are the smallest complete bipartite graphs that are not biplanar .*

Proof. As seen in Figure 3.2, the complete bipartite graph $K_{6,8}$ is biplanar.

Since $K_{6,7}$ is a proper subgraph of $K_{6,8}$, it follows that $K_{6,7}$ is biplanar as seen in Figure 3.17.. Additionally, $K_{5,12}$ has a biplanar decomposition as seen in Figure 3.18.

□

Corollary 3.11. *Biplanar graphs are not closed under subdivision of an edge.*

Proof. Let G be the graph $K_9 - \{v_2, v_4\}$ as in Lemma 3.5. Let the edge $e = \{v_2, v_4\}$ be subdivided with a vertex v such that the resulting edges are $\{2, v\}$ and $\{v, 4\}$. By placing the edge $\{2, v\}$ into one edge set and $\{v, 4\}$ into the other edge set, each decomposition of G is still is still planar. However, since K_9 is not biplanar, biplanarity is not closed under subdivisions of edges.

□

Note that Corollary 3.10 shows biplanar graphs are not closed under subdivision of an edge, then biplanar graphs are not closed under graph homeomorphism.

Theorem 3.12. *The graphs K_9, $K_{7,7}$, $K_{6,9}$ and $K_{5,13}$ are the smallest complete and complete bipartite graphs that are not biplanar.*

Theorem 3.12 follows directly from Lemmas 3.5, 3.5, and 3.12. As with Kuratowski's Theorem, we have identified the smallest complete and complete bipartite graphs that are not biplanar. Thus for any graph G to be biplanar, G must not have one of K_9, $K_{7,7}$, $K_{6,9}$ nor $K_{5,13}$ as a subgraph.

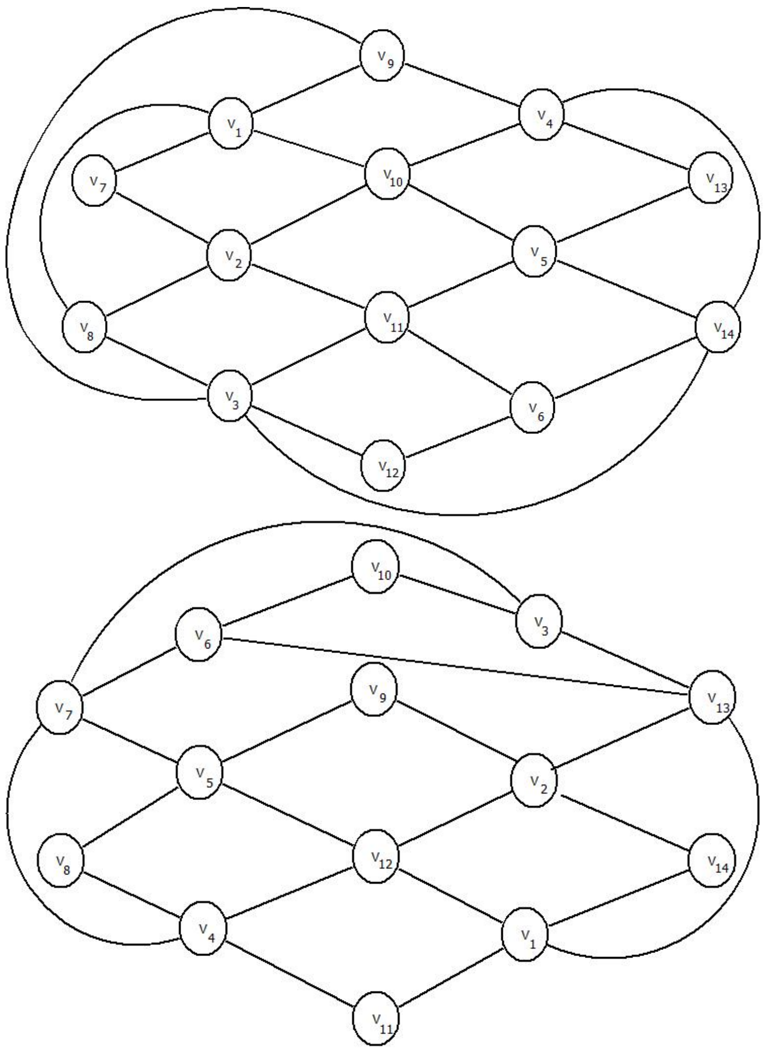

Figure 3.17: A biplanar decomposition of $K_{6,8}$.

Chapter 4

Graph Thickness

As with planar graphs, biplanar graphs can only have a certain number of edges in comparison to the number of vertices before the decomposition becomes impossible. This was shown through the cases of $K_9, K_{7,7}$ and $K_{6,9}$. The *thickness* of a graph G is smallest number partitions of the edge set E of G such that each subgraph $G_1 = (V, E_1), G_2 = (V, E_2)...G_n = (V, E_n)$ is a planar graph. The thickness of a graph will be denoted by $\theta(G)$

From the previous section, it is clear that $\theta(K_8) = 2$ since K_8 is biplanar.

Theorem 4.1. *For a graph G, $\theta(G) \geq \lceil m/(3n-6) \rceil$*

Proof. Proof: Let G be a given graph. A maximal planar graph has $3n - 6$ edges. By $m/(3n - 6)$ would give a minimum number of planes in which it would take to embed G. Since $m/(3n - 6)$ may be rational and not an integer, it is sufficient to consider $\lceil m/(3n-6) \rceil$ as the minimum number of planes needed to planar embed G. Thus $\theta(G) \geq \lceil m/(3n - 6) \rceil$.

□

It directly follows from Theorem 4.2 that a complete graph has a minimum thickness.

Theorem 4.2. *For a complete graph K_n, then $\theta(K_n) \geq \lfloor (n + 1)/6 \rfloor + 1$*

Proof. Let $G = K_n$ for some n. By the Theorem 4.2, $\theta(G) \geq \lceil m/(3n - 6) \rceil$. Since G is complete, then it has exactly $n(n - 1)/2$ edges. Then $\theta(G) \geq \lceil n(n - 1)/2(3n - 6) \rceil$

or $\theta(G) \geq \lceil n(n-1)/6(n-2) \rceil$. Then by dividing with remainder, $\theta(G) \geq \lceil (n+1)/6 + 2/6(n-2) \rceil \geq \lfloor (n+1)/6 \rfloor + 1$. Thus $\theta(K_n) \geq \lfloor (n+1)/6 \rfloor + 1$.

□

In a very similar manor of expanding maximal planar, we are able to expand the result of Eulers Polyhedral Identity for biplanar graphs to give a minimum for graph thickness.

Theorem 4.3. *For all simple bipartite graphs G, $\theta(G) \geq \lceil m/(2n-4) \rceil$*

Proof. Let G be a bipartite graph. As seen before, $m \leq 2n-4$. Then for any given partition of the edges of a bipartite graph, there may exist at most $2n-4$. Then by considering $\lceil m/(2n-4) \rceil$ would give a minimum number of parts needed for the partition of the edges of G. Thus for all bipartite graphs G, $\theta(G) \geq \lceil m/(2n-4) \rceil$

□

Although these are theorems only develop the foundations for studying graph thickness, they create a very powerful tool set. From these theorems, exact formulas for graph thickness have been determined for all complete graphs and most complete bipartite graphs. More information that we did not have time to investigate in this thesis is available in [Bei97].

Chapter 5

Conclusion

Through this thesis, we were able to develop the required tools from introducing a graph to major theorems that play a key role in determining which graphs are planar and biplanar along with graph thickness. By building up to the proof that K_9 is not biplanar, we introduced many structural theorems that create an idea of what maximal planar graphs can look like. Although this thesis is self contained, there is still a lot of work to be done in the topic of biplanar graphs.

Within future research, we would further investigate more classifications of biplanar graphs including multipartite. Additionally, we would add edges to smaller biplanar bipartite graphs to determine if there exist any other cases of graphs that are not complete and are not biplanar. After these cases are studied, we would be able to generate all forbidden subgraphs for biplanar graphs.

In an attempt to still expand Kuratowski's Theorem, we would also further investigate additional structures within biplanar graphs. Through these structures we would hope to find some function on graphs that would assist in determining if a given graph is biplanar or not.

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