Keplerian integrals, elimination theory and identification of very short arcs in a large database of optical observations

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Abstract

Modern asteroid surveys produce an increasingly large number of observations, which are grouped into very short arcs (VSAs) each containing a few observations of the same object in one single night. To decide whether two VSAs collected in different nights correspond to the same observed object we can attempt to compute an orbit with the observations of both arcs: this is called the *linkage* problem. Since the number of linkages to be attempted is very large, we need efficient methods of orbit determination. Using the first integrals of Kepler's motion we can write algebraic equations for the linkage problem, which can be put in polynomial form. In [7] these equations are reduced to a polynomial equation of degree 9: the unknown is the topocentric distance of the observed body at the mean epoch of one VSA. Here we derive the same equations in a more concise way, and show that the degree 9 is optimal in a sense that will be specified in Section 3.3. We also introduce a procedure to join three VSAs: from the conservation of angular momentum we obtain a polynomial equation of degree 8 in the topocentric distance at the mean epoch of the second VSA. For both identification methods, with two and three VSAs, we discuss how to discard solutions. Finally we present some numerical tests showing that the new methods give satisfactory results and can be used also when the time separation between the VSAs is large. The low polynomial degree of the new methods makes them well suited to deal with the very large number of asteroid observations collected by the modern surveys.

1 Introduction

We consider very short arcs (VSAs) of optical observations of a solar system body whose motion is dominated by the gravitational attraction of the Sun. These small sets of observations are called *tracklets*, see [8], and the corresponding arc described in the sky is usually too short to compute a least squares orbit, see [12, Chap.8]. In each observing night we can detect thousands of these data, thus it is difficult to decide whether two such arcs, collected in different nights, correspond to the same body. This gives rise to an identification problem, that can be solved by attempting to compute an orbit with the information contained in two or more tracklets. The efficiency of the existing identification methods needs to be improved, as shown by the large database of unidentified tracklets of asteroid observations, the isolated tracklet file (ITF) currently available at the MPC website, which now (July 2016) contains about 12 millions of observations.

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Using the classical methods of initial orbit determination, e.g., those by [9] or [4], we usually can not compute a preliminary orbit with three observations belonging to the same VSA because they are too close in time to provide us with information on the radial distance. In fact, in this case, the geodesic curvature of the arc is usually not statistically significant, and this quantity appears in the formulas of the classical methods, see [12, Chap.9]. For the same reason, even using observations taken from two different VSAs of the same object it may be difficult to compute an orbit. In most cases the methods of Laplace and Gauss work well if we use observations from three different VSAs at the same apparition. To compute a preliminary orbit with these methods we have to find the roots of a univariate polynomial of degree 8 (see [12]), which correspond to the possible values of the radial distance (geocentric for Laplace, topocentric for Gauss) of the observed body at a given epoch (the mean epoch of the observations $\sum_{h=1}^{3} t_h/3$ for Laplace, the central epoch t_2 for Gauss).

Assume for simplicity that we deal with this identification problem using observations made by a single telescope performing an asteroid survey, like Pan-STARRS[1], or the next generation telescope LSST[2]. The average number of observations per night is $N \approx 10^4$ for Pan-STARRS[3] and this number will be larger for the next generation surveys. To systematically perform the identification by Gauss' method using the data of three observing nights we should test $O(N^3)$ triples of observations. This is clearly a cumbersome task. For this reason the computation of preliminary orbits with only two VSAs has been investigated. The identification of two VSAs is usually called *linkage*. Among the different methods currently in use for the linkage there are the following: statistical ranging [15], systematic ranging [1, 16], sampling of the admissible region [10, 11] and kd-tree algorithms [8]. These methods can be used successfully to link VSAs over relatively short time spans, but they can not be employed when the time separation between the tracklets is large. However, when the time separation is large, we can use the Keplerian integrals methods introduced in [5, 6, 7], based on the conservation laws of the two-body motion. A common feature of these three works is that they use polynomial equations for the linkage, leading to univariate polynomials of degree 48, 20 and 9 respectively. The polynomial equations introduced in [7] are derived in a more concise way in Section 3. Moreover, using elimination theory [2], we show that 9 is the minimum degree for the univariate polynomial equations that are consequence of the conservation laws of Kepler's problem, provided that we drop the dependence between the inverse of the heliocentric distance $1/|\mathbf{r}|$ and the topocentric distance ρ (see Section 3.3 for the details). This approach avoids the squaring operations needed in [5, 6] to bring the selected equations[4] in polynomial form. In Section 3.4 we sketch a method to check the validity of the identification: this is done by checking some compatibility conditions for the solutions, similar to the ones in [5], that use the full two-body dynamics.

In this paper we also deal with the identification of three VSAs: in Section 4 we introduce a univariate polynomial equation of degree 8 to join three VSAs of optical observations by means of the conservation of angular momentum only. Then the other laws of Kepler's motion can be used to set up restrictive compatibility conditions, allowing us to test the identification and select solutions.

Assume we set up an identification procedure with a large database of asteroid observations. For simplicity, we can consider three observing nights, in which we collect $O(N)$ VSAs of observations per night. We can try to identify pairs of VSAs belonging to the first two nights by applying $O(N^2)$ times the linkage algorithm introduced in [7] and reviewed in Section 3. The output is composed by preliminary orbits obtained with pairs of VSAs. If the thresholds in the

[1] Panoramic Survey Telescope & Rapid Response System, `http://pan-starrs.ifa.hawaii.edu/public/`

[2] Large Synoptic Survey Telescope, `http://www.lsst.org/`

[3] see `http://hamilton.dm.unipi.it/astdys/index.php?pc=2.1.1&o=F51`

[4] In these papers not all the algebraic conservation laws are used.

controls for acceptance (see Section 3.4) are well selected, we do not obtain more than $O(N)$ pairs of VSAs, in fact the number of different objects observed in the two nights is $O(N)$. Then we can apply the method to join three VSAs introduced in Section 4 to the $O(N)$ selected pairs and the $O(N)$ VSAs of the third observing night. We conclude that this identification problem can be faced with $O(N^2)$ computations of roots of a polynomial of degree 9 or 8, instead of $O(N^3)$ computations of roots of Gauss' polynomial.

Finally, in Section 5 we present some numerical tests showing that the Keplerian integrals methods give satisfactory results and can be used also when the time separation between the VSAs is large.

2 Keplerian integrals

We consider the Keplerian motion of a celestial body around a center of force, set at the origin of a given reference system, which in the asteroid case corresponds to the center of the Sun. Optical observations of the body are made by a telescope whose heliocentric position is a known function of time. Then the heliocentric position and velocity of the body are given by

$$\mathbf{r} = \rho\mathbf{e}^\rho + \mathbf{q}, \qquad \dot{\mathbf{r}} = \dot{\rho}\mathbf{e}^\rho + \rho\boldsymbol{\eta} + \dot{\mathbf{q}}, \tag{1}$$

where $\mathbf{q}, \dot{\mathbf{q}}$ are the heliocentric position and velocity of the observer, $\rho, \dot{\rho}$ are the topocentric radial distance and velocity, $\mathbf{e}^\rho$ is the *line of sight* unit vector, which can be written in terms of the topocentric right ascension α and declination δ as

$$\mathbf{e}^\rho = (\cos\delta\cos\alpha, \cos\delta\sin\alpha, \sin\delta).$$

Moreover in (1) we use the *proper motion* vector

$$\boldsymbol{\eta} = \dot{\alpha}\cos\delta\mathbf{e}^\alpha + \dot{\delta}\mathbf{e}^\delta,$$

where

$$\mathbf{e}^\alpha = (\cos\delta)^{-1}\frac{\partial\mathbf{e}^\rho}{\partial\alpha}, \qquad \mathbf{e}^\delta = \frac{\partial\mathbf{e}^\rho}{\partial\delta},$$

and $\dot{\alpha}, \dot{\delta}$ are the angular rates. The Keplerian integrals, represented by the angular momentum vector $\mathbf{c}$, the Laplace-Lenz vector $\mathbf{L}$ and the energy $\mathcal{E}$, are defined by

$$\mathbf{c} = \mathbf{r}\times\dot{\mathbf{r}}, \qquad \mu\mathbf{L} = \Big(|\dot{\mathbf{r}}|^2 - \frac{\mu}{|\mathbf{r}|}\Big)\mathbf{r} - (\mathbf{r}\cdot\dot{\mathbf{r}})\dot{\mathbf{r}}, \qquad \mathcal{E} = \frac{1}{2}|\dot{\mathbf{r}}|^2 - \frac{\mu}{|\mathbf{r}|}. \tag{2}$$

Given the values of $\alpha, \delta, \dot{\alpha}, \dot{\delta}$, they can be written as algebraic functions of $\rho, \dot{\rho}$ using relations (1).

3 Linking two VSAs

Given a very short arc of optical observations $(\alpha_i,\ \delta_i)$, $i = 1\ldots m$, obtained at the same station at different epochs t_i, it is often possible to compute the *attributable* vector (see [13])

$$\mathcal{A} = (\alpha, \delta, \dot{\alpha}, \dot{\delta})$$

at the mean epoch $\bar{t} = \frac{1}{m}\sum_{i=1}^m t_i$. The missing quantities to obtain a preliminary orbit are the topocentric distance and velocity ρ, $\dot{\rho}$ at $t = \bar{t}$. When the second derivatives $(\ddot{\alpha}, \ddot{\delta})$ are either not available (if $m = 2$), or not statistically significant due to the errors in the observations, then the attributable summarizes essentially all the information contained in the VSA. In this case a preliminary orbit can be obtained by linking together two different VSAs.

The key idea of the linkage method presented here is to use the conservation of the Keplerian integrals $\mathbf{c}$, $\mathbf{L}$, $\mathcal{E}$ at the two mean epochs $\bar{t}_1, \bar{t}_2$ of two attributables $\mathcal{A}_1, \mathcal{A}_2$:

$$\mathbf{c}_1 = \mathbf{c}_2, \qquad \mathbf{L}_1 = \mathbf{L}_2, \qquad \mathcal{E}_1 = \mathcal{E}_2, \tag{3}$$

where the indexes $1, 2$ refer to the epoch.

Below we derive the polynomial equations for the linkage problem introduced in [7] in a more concise way, and we review the procedure to obtain the univariate polynomial of degree 9 giving the possible values for the topocentric distance ρ_2. Moreover, we show an optimal property of this polynomial.

3.1 Conservation of angular momentum

The angular momentum as function of $\rho, \dot{\rho}$ can be written as

$$\mathbf{c}(\rho, \dot{\rho}) = \mathbf{r} \times \dot{\mathbf{r}} = \mathbf{D}\dot{\rho} + \mathbf{E}\rho^2 + \mathbf{F}\rho + \mathbf{G},$$

where

$$\begin{aligned}
&\mathbf{D} = \mathbf{q} \times \mathbf{e}^\rho, \\
&\mathbf{E} = \dot{\alpha}\cos\delta \mathbf{e}^\rho \times \mathbf{e}^\alpha + \dot{\delta}\mathbf{e}^\rho \times \mathbf{e}^\delta = \dot{\alpha}\cos\delta \mathbf{e}^\delta - \dot{\delta}\mathbf{e}^\alpha, \\
&\mathbf{F} = \dot{\alpha}\cos\delta \mathbf{q} \times \mathbf{e}^\alpha + \dot{\delta}\mathbf{q} \times \mathbf{e}^\delta + \mathbf{e}^\rho \times \dot{\mathbf{q}}, \\
&\mathbf{G} = \mathbf{q} \times \dot{\mathbf{q}}.
\end{aligned}$$

Then the equation

$$\mathbf{c}_1 = \mathbf{c}_2,$$

represents the conservation of the angular momentum and is written

$$\mathbf{D}_1\dot{\rho}_1 - \mathbf{D}_2\dot{\rho}_2 = \mathbf{J}(\rho_1, \rho_2), \tag{4}$$

where

$$\mathbf{J}(\rho_1, \rho_2) = \mathbf{E}_2\rho_2^2 - \mathbf{E}_1\rho_1^2 + \mathbf{F}_2\rho_2 - \mathbf{F}_1\rho_1 + \mathbf{G}_2 - \mathbf{G}_1. \tag{5}$$

We can eliminate the radial velocities $\dot{\rho}_1, \dot{\rho}_2$ from (4) by making the scalar product with $\mathbf{D}_1 \times \mathbf{D}_2$, that gives the quadratic equation

$$q(\rho_1, \rho_2) := \mathbf{D}_1 \times \mathbf{D}_2 \cdot \mathbf{J}(\rho_1, \rho_2) = 0 \tag{6}$$

in the variables ρ_1, ρ_2. The radial velocities are given by

$$\dot{\rho}_1(\rho_1, \rho_2) = \frac{(\mathbf{J} \times \mathbf{D}_2) \cdot (\mathbf{D}_1 \times \mathbf{D}_2)}{|\mathbf{D}_1 \times \mathbf{D}_2|^2}, \quad \dot{\rho}_2(\rho_1, \rho_2) = \frac{(\mathbf{J} \times \mathbf{D}_1) \cdot (\mathbf{D}_1 \times \mathbf{D}_2)}{|\mathbf{D}_1 \times \mathbf{D}_2|^2}. \tag{7}$$

These expressions are obtained by projecting (4) onto the vectors $\mathbf{D}_1 \times (\mathbf{D}_1 \times \mathbf{D}_2)$ and $\mathbf{D}_2 \times (\mathbf{D}_1 \times \mathbf{D}_2)$, generating the plane orthogonal to $\mathbf{D}_1 \times \mathbf{D}_2$. Therefore using such expressions of $\dot{\rho}_1, \dot{\rho}_2$ we have

$$(\mathbf{c}_1 - \mathbf{c}_2) \times (\mathbf{D}_1 \times \mathbf{D}_2) = \mathbf{0}$$

whatever the values of ρ_1, ρ_2.

3.2 The univariate polynomial $\mathfrak{u}$

By relations (7) we can eliminate the dependence on $\dot\rho_1, \dot\rho_2$ in the Laplace-Lenz and energy conservation laws

$$\mathbf{L}_1 = \mathbf{L}_2, \qquad \mathcal{E}_1 = \mathcal{E}_2. \tag{8}$$

These are algebraic equations in ρ_1, ρ_2 that are not polynomial because of the terms $1/|\mathbf{r}_1|, 1/|\mathbf{r}_2|$. However, in the equation

$$\boldsymbol{\xi} := \big[\mu(\mathbf{L}_1 - \mathbf{L}_2) - (\mathcal{E}_1\mathbf{r}_1 - \mathcal{E}_2\mathbf{r}_2)\big] \times (\mathbf{r}_1 - \mathbf{r}_2) = \mathbf{0}, \tag{9}$$

which is a consequence of (8), the terms $1/|\mathbf{r}_1|, 1/|\mathbf{r}_2|$ cancel out. The monomials of $\boldsymbol{\xi}$ with the highest total degree, i.e. 6, are all parallel to $\mathbf{e}_1^\rho \times \mathbf{e}_2^\rho$, so that the bivariate polynomials

$$p_1 = \boldsymbol{\xi} \cdot \mathbf{e}_1^\rho, \qquad p_2 = \boldsymbol{\xi} \cdot \mathbf{e}_2^\rho \tag{10}$$

have total degree 5. In [7] the authors show that the over-determined bivariate polynomial system

$$q = 0, \qquad \boldsymbol{\xi} = \mathbf{0}$$

is consistent, i.e. its set of solutions in $\mathbb{C}^2$ is not empty, and is equivalent to

$$q = p_1 = p_2 = 0.$$

Moreover, if we consider the resultants (see [2])

$$\mathfrak{u}_1 = \mathrm{Res}(p_1, q, \rho_1), \qquad \mathfrak{u}_2 = \mathrm{Res}(p_2, q, \rho_1),$$

which are both univariate polynomials in the variable ρ_2 of degree 10, then their greatest common divisor

$$\mathfrak{u} = \gcd(\mathfrak{u}_1, \mathfrak{u}_2) \tag{11}$$

has degree 9 (see [7, Theorem 1]).

Remark 1. *Since in this problem the role of ρ_1 and ρ_2 is symmetric, for a generic choice of the data $\mathcal{A}_j, \mathbf{q}_j, \dot{\mathbf{q}}_j, j = 1, 2,$ we obtain an analogous result by eliminating the variable ρ_2, instead of ρ_1, from p_1, p_2.*

We also recall the construction used in [7] to compute $\mathfrak{u}_j$, $j = 1, 2$. We can write

$$q(\rho_1, \rho_2) = \sum_{h=0}^{2} b_h(\rho_2)\rho_1^h,$$

where

$$b_0(\rho_2) = q_{0,2}\rho_2^2 + q_{0,1}\rho_2 + q_{0,0}, \qquad b_1 = q_{1,0}, \qquad b_2 = q_{2,0},$$

with the coefficients $q_{h,k}$ depending only on the data $\mathcal{A}_j, \mathbf{q}_j, \dot{\mathbf{q}}_j$, $j = 1, 2$. Moreover, we have

$$p_1(\rho_1, \rho_2) = \sum_{h=0}^{4} a_{1,h}(\rho_2)\rho_1^h, \qquad p_2(\rho_1, \rho_2) = \sum_{h=0}^{5} a_{2,h}(\rho_2)\rho_1^h, \tag{12}$$

for some polynomials $a_{k,h}$ whose degrees are described by the upper small circles used to construct Newton's polygons (see [14]) of p_1, p_2 in Figure 1. Assume $q_{2,0}, q_{0,2} \neq 0$. From $q = 0$ we obtain

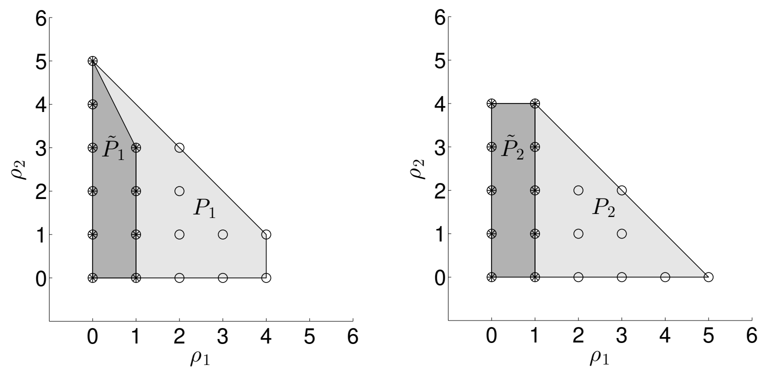

Figure 1: We draw Newton's polygons P_j, $\tilde{P}_j$ for the polynomials $p_j, \tilde{p}_j$, $j = 1, 2$. In this figure the polygons are overlapping: the nodes with circles correspond to the (multi-index) exponents of the monomials in p_j; the nodes with asterisks correspond to the exponents of the monomials in $\tilde{p}_j$.

$$\rho_1^h = \beta_h \rho_1 + \gamma_h, \qquad h = 2, 3, 4, 5, \tag{13}$$

where

$$\beta_2 = -\frac{b_1}{b_2}, \qquad \gamma_2 = -\frac{b_0}{b_2},$$

and

$$\beta_{h+1} = \beta_h \beta_2 + \gamma_h, \qquad \gamma_{h+1} = \beta_h \gamma_2, \qquad h = 2, 3, 4.$$

Inserting (13) into (12) we obtain

$$\tilde{p}_j(\rho_1, \rho_2) = \tilde{a}_{j,1}(\rho_2)\rho_1 + \tilde{a}_{j,0}(\rho_2), \qquad j = 1, 2, \tag{14}$$

where

$$\tilde{a}_{1,1} = a_{1,1} + \sum_{h=2}^{4} a_{1,h}\beta_h, \qquad \tilde{a}_{1,0} = a_{1,0} + \sum_{h=2}^{4} a_{1,h}\gamma_h, \tag{15}$$

$$\tilde{a}_{2,1} = a_{2,1} + \sum_{h=2}^{5} a_{2,h}\beta_h, \qquad \tilde{a}_{2,0} = a_{2,0} + \sum_{h=2}^{5} a_{2,h}\gamma_h. \tag{16}$$

In Figure 1 we also draw Newton's polygons of $\tilde{p}_1$, $\tilde{p}_2$. In this case the nodes with asterisks correspond to the exponents of the monomials in $\tilde{p}_j$ and the upper asterisks describe the degrees of the polynomials $\tilde{a}_{k,h}$. Let us also introduce the resultants

$$\mathfrak{v}_1 = \mathrm{Res}(\tilde{p}_1, q, \rho_1), \qquad \mathfrak{v}_2 = \mathrm{Res}(\tilde{p}_2, q, \rho_1).$$

We can show the following result, mentioned without proof in [7].

Lemma 1. *By the properties of resultants we find that*

$$\mathfrak{u}_1 = q_{2,0}^3 \mathfrak{v}_1, \qquad \mathfrak{u}_2 = q_{2,0}^4 \mathfrak{v}_2. \tag{17}$$

Proof. We prove the first relation, the proof of the second one being similar. We have

$$
\mathfrak{u}_1 = \mathrm{Res}(p_1, q, \rho_1) = \det \begin{bmatrix} a_{1,0} & 0 & b_0 & 0 & 0 & 0 \\ a_{1,1} & a_{1,0} & b_1 & b_0 & 0 & 0 \\ a_{1,2} & a_{1,1} & b_2 & b_1 & b_0 & 0 \\ a_{1,3} & a_{1,2} & 0 & b_2 & b_1 & b_0 \\ a_{1,4} & a_{1,3} & 0 & 0 & b_2 & b_1 \\ 0 & a_{1,4} & 0 & 0 & 0 & b_2 \end{bmatrix}.
$$

By performing raw operations and by the properties of determinants we obtain

$$
\begin{aligned}
\mathrm{Res}(p_1, q, \rho_1) &= \det \begin{bmatrix} a_{1,0} & 0 & b_0 & 0 & 0 & 0 \\ a_{1,1} + \gamma_2 a_{1,3} + \gamma_3 a_{1,4} & \tilde{a}_{1,0} & b_1 & 0 & 0 & 0 \\ a_{1,2} + \beta_2 a_{1,3} + \beta_3 a_{1,4} & \tilde{a}_{1,1} & b_2 & 0 & 0 & 0 \\ a_{1,3} + \beta_2 a_{1,4} & a_{1,2} + \beta_2 a_{1,3} + \beta_3 a_{1,4} & 0 & b_2 & 0 & 0 \\ a_{1,4} & a_{1,3} + \beta_2 a_{1,4} & 0 & 0 & b_2 & 0 \\ 0 & a_{1,4} & 0 & 0 & 0 & b_2 \end{bmatrix} = \\
&= \det \begin{bmatrix} \tilde{a}_{1,0} & 0 & b_0 & 0 & 0 & 0 \\ \tilde{a}_{1,1} & \tilde{a}_{1,0} & b_1 & 0 & 0 & 0 \\ 0 & \tilde{a}_{1,1} & b_2 & 0 & 0 & 0 \\ a_{1,3} + \beta_2 a_{1,4} & a_{1,2} + \beta_2 a_{1,3} + \beta_3 a_{1,4} & 0 & b_2 & 0 & 0 \\ a_{1,4} & a_{1,3} + \beta_2 a_{1,4} & 0 & 0 & b_2 & 0 \\ 0 & a_{1,4} & 0 & 0 & 0 & b_2 \end{bmatrix} = b_2^3 \mathrm{Res}(\tilde{p}_1, q, \rho_1).
\end{aligned}
$$

The last matrix is obtained from the previous one by adding to its first column a suitable multiple of the third column.

□

□

3.3 An optimal property of the polynomial $\mathfrak{u}$

If we consider the auxiliary variable u defined by relation

$$
u|\mathbf{r}| = \mu, \tag{18}
$$

then the Keplerian integrals introduced in (2) can be viewed as polynomials in the variables $\rho, \dot{\rho}, u$ by writing u in place of $\mu/|\mathbf{r}|$. In particular, we obtain

$$
\mathbf{L} = (|\dot{\mathbf{r}}|^2 - u)\mathbf{r} - (\dot{\mathbf{r}} \cdot \mathbf{r})\dot{\mathbf{r}}, \qquad \mathcal{E} = \frac{1}{2}|\dot{\mathbf{r}}|^2 - u.
$$

We observe that, for all $\rho, \dot{\rho}, u$,

$$
\mathbf{c} \cdot \mathbf{L} = 0, \qquad \mu^2|\mathbf{L}|^2 = u^2|\mathbf{r}|^2 + 2\mathcal{E}|\mathbf{c}|^2; \tag{19}
$$

the second relation generalizes the classical formula relating eccentricity, energy and angular momentum.

The polynomial system

$$
\mathbf{c}_1 = \mathbf{c}_2, \qquad \mu\mathbf{L}_1 = \mu\mathbf{L}_2, \qquad \mathcal{E}_1 = \mathcal{E}_2, \qquad u_1^2|\mathbf{r}_1|^2 = \mu^2, \qquad u_2^2|\mathbf{r}_2|^2 = \mu^2, \tag{20}
$$

with unknowns $\rho_1, \rho_2, \dot{\rho}_1, \dot{\rho}_2, u_1, u_2$, is generically not consistent, see Corollary 2 at the end of this section. Next we show that if we drop the dependence between u_j and ρ_j given by (18), that is we eliminate the last two equations from (20), then we obtain a consistent polynomial system. Moreover, the univariate polynomial $\mathfrak{u}$ of degree 9 introduced in [7] has the minimum degree among the polynomials in ρ_2 that are obtained by elimination of variables from

$$\mathbf{c}_1 - \mathbf{c}_2, \qquad \mu\mathbf{L}_1 - \mu\mathbf{L}_2, \qquad \mathcal{E}_1 - \mathcal{E}_2. \tag{21}$$

Let

$$I \subseteq \mathbb{R}[\rho_1, \rho_2, \dot\rho_1, \dot\rho_2, u_1, u_2]$$

be the ideal of the polynomial ring in the variables $\rho_1, \rho_2, \dot\rho_1, \dot\rho_2, u_1, u_2$, with real coefficients, generated by the seven polynomials in (21). We recall that a set $\{\mathfrak{g}_1, \dots, \mathfrak{g}_n\}$, with $n \in \mathbb{N}$, is a Groebner basis of a polynomial ideal I for a fixed monomial order $\succ$ if and only if the leading term (for that order) of any element of I is divisible by the leading term of one $\mathfrak{g}_j$, see [2]. The main result of this section is the following.

Theorem 1. *For a generic choice of the data $\mathcal{A}_j, \mathbf{q}_j, \dot{\mathbf{q}}_j$, $j = 1, 2$, we can find a set of polynomials*

$$\{\mathfrak{g}_1, \dots, \mathfrak{g}_6\} \subset \mathbb{R}[\rho_1, \rho_2, \dot\rho_1, \dot\rho_2, u_1, u_2]$$

that is a Groebner basis of the ideal I for the lexicographic order

$$\dot\rho_1 \succ \dot\rho_2 \succ u_1 \succ u_2 \succ \rho_1 \succ \rho_2, \tag{22}$$

and such that

$$\mathfrak{g}_6 = \mathfrak{u},$$

where $\mathfrak{u}$ is the polynomial defined in (11).

Proof. Assuming

$$\mathbf{D}_1 \times \mathbf{D}_2 \neq \mathbf{0}, \qquad \mathbf{e}_1^\rho \times \mathbf{e}_2^\rho \neq \mathbf{0},$$

we consider the following set of generators of the ideal I:

$$\begin{aligned}
\mathfrak{q}_1 &= (\mathbf{c}_1 - \mathbf{c}_2) \cdot \mathbf{D}_1 \times \mathbf{D}_2, \\
\mathfrak{q}_2 &= (\mathbf{c}_1 - \mathbf{c}_2) \cdot \mathbf{D}_1 \times (\mathbf{D}_1 \times \mathbf{D}_2), \\
\mathfrak{q}_3 &= (\mathbf{c}_1 - \mathbf{c}_2) \cdot \mathbf{D}_2 \times (\mathbf{D}_1 \times \mathbf{D}_2), \\
\mathfrak{q}_4 &= \mu(\mathbf{L}_1 - \mathbf{L}_2) \cdot \mathbf{e}_1^\rho \times \mathbf{e}_2^\rho, \\
\mathfrak{q}_5 &= \mu(\mathbf{L}_1 - \mathbf{L}_2) \cdot \mathbf{D}_1, \\
\mathfrak{q}_6 &= \mu(\mathbf{L}_1 - \mathbf{L}_2) \cdot \mathbf{D}_2, \\
\mathfrak{q}_7 &= \mathcal{E}_1 - \mathcal{E}_2.
\end{aligned}$$

The first three polynomials have the form

$$\begin{aligned}
\mathfrak{q}_1 &= q, \\
\mathfrak{q}_2 &= |\mathbf{D}_1 \times \mathbf{D}_2|^2 \dot\rho_1 - \mathbf{J} \cdot \mathbf{D}_1 \times (\mathbf{D}_1 \times \mathbf{D}_2), \\
\mathfrak{q}_3 &= |\mathbf{D}_1 \times \mathbf{D}_2|^2 \dot\rho_2 - \mathbf{J} \cdot \mathbf{D}_2 \times (\mathbf{D}_1 \times \mathbf{D}_2),
\end{aligned}$$

with $q = q(\rho_1, \rho_2)$, $\mathbf{J} = \mathbf{J}(\rho_1, \rho_2)$ defined in (6), (5) respectively. The other generators can be written as

$$\begin{aligned}
\mathfrak{q}_4 &= -(\mathbf{D}_1 \cdot \mathbf{e}_2^\rho) u_1 - (\mathbf{D}_2 \cdot \mathbf{e}_1^\rho) u_2 + \mathfrak{f}_4, \\
\mathfrak{q}_5 &= (\mathbf{D}_1 \cdot \mathbf{r}_2) u_2 + \mathfrak{f}_5, \\
\mathfrak{q}_6 &= -(\mathbf{D}_2 \cdot \mathbf{r}_1) u_1 + \mathfrak{f}_6, \\
\mathfrak{q}_7 &= -u_1 + u_2 + \mathfrak{f}_7,
\end{aligned}$$

for some polynomials $\mathfrak{f}_j = \mathfrak{f}_j(\rho_1, \rho_2, \dot\rho_1, \dot\rho_2)$, $j = 4 \dots 7$. Set

$$A = \mathbf{D}_1 \cdot \mathbf{e}_2^\rho + \mathbf{D}_2 \cdot \mathbf{e}_1^\rho = (\mathbf{q}_1 - \mathbf{q}_2) \cdot \mathbf{e}_1^\rho \times \mathbf{e}_2^\rho.$$

Assuming the three terms

$$A, \quad \mathbf{D}_2 \cdot \mathbf{e}_1^\rho, \quad \mathbf{D}_1 \cdot \mathbf{e}_2^\rho$$

do not vanish, we can substitute the generators $\mathfrak{q}_4, \dots, \mathfrak{q}_7$ with the polynomials

$$\begin{aligned}
\mathfrak{p}_4 &= (\mathbf{D}_1 \cdot \mathbf{e}_2^\rho)\mathfrak{q}_7 - \mathfrak{q}_4 = Au_2 + \mathfrak{a}_1, \\
\mathfrak{p}_5 &= -(\mathbf{D}_2 \cdot \mathbf{e}_1^\rho)\mathfrak{q}_7 - \mathfrak{q}_4 = Au_1 + \mathfrak{a}_2, \\
\mathfrak{p}_6 &= (\mathbf{D}_1 \cdot \mathbf{r}_2)\mathfrak{p}_4 - A\mathfrak{q}_5, \\
\mathfrak{p}_7 &= (\mathbf{D}_2 \cdot \mathbf{r}_1)\mathfrak{p}_5 + A\mathfrak{q}_6,
\end{aligned}$$

where

$$\mathfrak{a}_1 = (\mathbf{D}_1 \cdot \mathbf{e}_2^\rho)\,\mathfrak{f}_7 - \mathfrak{f}_4, \qquad \mathfrak{a}_2 = -(\mathbf{D}_2 \cdot \mathbf{e}_1^\rho)\,\mathfrak{f}_7 - \mathfrak{f}_4.$$

We note that the monomials containing u_1, u_2 cancel out in $\mathfrak{p}_6$, $\mathfrak{p}_7$. Using relations $\mathfrak{q}_2 = \mathfrak{q}_3 = 0$, we can also eliminate $\dot\rho_1$, $\dot\rho_2$ from $\mathfrak{p}_6$, $\mathfrak{p}_7$: we call $\tilde{\mathfrak{p}}_6$, $\tilde{\mathfrak{p}}_7$ the polynomials obtained in this way, that can be written as

$$\tilde{\mathfrak{p}}_6 = \mathfrak{p}_6 + \mathfrak{b}_2\mathfrak{q}_2 + \mathfrak{b}_3\mathfrak{q}_3, \qquad \tilde{\mathfrak{p}}_7 = \mathfrak{p}_7 + \mathfrak{c}_2\mathfrak{q}_2 + \mathfrak{c}_3\mathfrak{q}_3$$

for some polynomials $\mathfrak{b}_j, \mathfrak{c}_j$, $j = 2, 3$ in the variables $\rho_1, \rho_2, \dot\rho_1, \dot\rho_2$.
We can prove that

$$\tilde{\mathfrak{p}}_6 = -(\mathbf{D}_1 \cdot \mathbf{e}_2^\rho)p_1, \qquad \tilde{\mathfrak{p}}_7 = (\mathbf{D}_2 \cdot \mathbf{e}_1^\rho)p_2, \tag{23}$$

where p_1, p_2 are the bivariate polynomials defined in (10). We show the first relation in (23), the computations for the second being similar. We have

$$\begin{aligned}
\mathfrak{p}_6 &= (\mathbf{D}_1 \cdot \mathbf{e}_2^\rho)(\mathbf{D}_1 \cdot \mathbf{r}_2)(\mathcal{E}_1 - \mathcal{E}_2) - \mu(\mathbf{L}_1 - \mathbf{L}_2) \cdot \big[(\mathbf{r}_1 - \mathbf{r}_2) \times \big(\mathbf{D}_1 \times (\mathbf{e}_1^\rho \times \mathbf{e}_2^\rho)\big)\big] \\
&= -(\mathbf{D}_1 \cdot \mathbf{e}_2^\rho)\big[(\mathcal{E}_1 - \mathcal{E}_2)(\mathbf{r}_1 \times \mathbf{r}_2 \cdot \mathbf{e}_1^\rho) + \mu(\mathbf{L}_1 - \mathbf{L}_2) \cdot (\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{e}_1^\rho\big],
\end{aligned}$$

where we used the relations

$$A = (\mathbf{r}_1 - \mathbf{r}_2) \cdot \mathbf{e}_1^\rho \times \mathbf{e}_2^\rho, \qquad \mathbf{D}_1 \cdot \mathbf{r}_1 = 0, \qquad \mathbf{D}_1 \cdot \mathbf{e}_1^\rho = 0.$$

We conclude by noting that

$$\boldsymbol{\xi} = (\mathcal{E}_1 - \mathcal{E}_2)\mathbf{r}_1 \times \mathbf{r}_2 + \mu(\mathbf{L}_1 - \mathbf{L}_2) \times (\mathbf{r}_1 - \mathbf{r}_2).$$

Let us now consider the elimination ideal

$$J = \langle \mathfrak{q}_1, \tilde{\mathfrak{p}}_6, \tilde{\mathfrak{p}}_7 \rangle = \langle q, p_1, p_2 \rangle,$$

in $\mathbb{R}[\rho_1, \rho_2]$. The ideal

$$\tilde J = \langle q, \tilde p_1, \tilde p_2 \rangle,$$

with the polynomials $\tilde p_j$ defined in (14), coincides with J, in fact

$$\tilde p_j = p_j + d_j q, \qquad j = 1, 2$$

for some polynomials $d_j = d_j(\rho_1, \rho_2)$. In particular, we have

$$V(J) = V(\tilde J),$$

where the variety $V(K)$ of a polynomial ideal $K \in \mathbb{R}[\rho_1, \rho_2]$ is the set

$$V(K) = \{(\rho_1, \rho_2) \in \mathbb{C}^2 : p(\rho_1, \rho_2) = 0, \ \forall p \in K\}.$$

The ideal

$$\tilde{J}_1 = \langle \tilde{p}_1, \tilde{p}_2 \rangle,$$

fulfills

$$\tilde{J}_1 \subseteq \tilde{J}, \tag{24}$$

so that

$$V(\tilde{J}_1) \supseteq V(\tilde{J}). \tag{25}$$

Indeed, we shall show that

$$V(\tilde{J}_1) = V(\tilde{J}).$$

Let us introduce the polynomial

$$\mathfrak{v} := \mathrm{Res}(\tilde{p}_1, \tilde{p}_2, \rho_1) = \tilde{a}_{1,1}\tilde{a}_{2,0} - \tilde{a}_{1,0}\tilde{a}_{2,1}.$$

We need the following results.

Lemma 2. *For a generic choice of the data $\mathcal{A}_j, \mathbf{q}_j, \dot{\mathbf{q}}_j, j = 1, 2$, the polynomials $\mathfrak{u}, \mathfrak{v}$ have 9 distinct solutions in $\mathbb{C}$.*

Proof. We show this property for $\mathfrak{u}$; the proof for $\mathfrak{v}$ is analogous. Let

$$\mathfrak{u}(\rho_2) = \sum_{j=0}^{9} c_j \rho_2^j,$$

for some coefficients $c_j \in \mathbb{R}$ depending on the data. First we show that, for a generic choice of the data, the rank of the Jacobian matrix

$$\frac{\partial(c_0, \ldots, c_9)}{\partial(\mathcal{A}_1, \mathcal{A}_2, \mathbf{q}_1, \dot{\mathbf{q}}_1, \mathbf{q}_2, \dot{\mathbf{q}}_2)}$$

is maximal, that is 10. To check this property it suffices to show that the rank is maximal for a particular choice of the data. In fact, if the rank were smaller than 10 in an open set, then by the analytic dependence of the coefficients c_j on the data it would not be maximal at any point. We made this check using the symbolic computation software *Maple 18* with the following data:

$$\mathcal{A}_1 = \big(2\arctan(1/2), 0, 1, 1\big), \qquad \mathcal{A}_2 = \big(2\arctan(1/2), 2\arctan(1/2), 1, 1\big),$$
$$\mathbf{q}_1 = (1, 0, 0), \qquad \dot{\mathbf{q}}_1 = (0, 1/2, 0), \qquad \mathbf{q}_2 = (0, 1, 0), \qquad \dot{\mathbf{q}}_2 = (-1/2, 0, 0).$$

Moreover, by a well known property of polynomials, $\mathfrak{u}$ is *square-free* (i.e. without multiple roots) for a generic choice of the coefficients c_j. This fact, together with the maximal rank property showed above, concludes the proof of the lemma.

□

□

Lemma 3. *For a generic choice of the data $\mathcal{A}_j, \mathbf{q}_j, \dot{\mathbf{q}}_j, j = 1, 2$, we have*

$$\gcd(\tilde{a}_{1,1}, \tilde{a}_{2,1}) = 1, \tag{26}$$

where $\tilde{a}_{1,1}$, $\tilde{a}_{2,1}$ are the univariate polynomials defined in (15), (16).

Proof. We give a proof similar to the one of Lemma 2. Let us write

$$\tilde{a}_{11}(\rho_2) = \sum_{j=0}^{3} c_{1,j}\rho_2^j, \qquad \tilde{a}_{21}(\rho_2) = \sum_{j=0}^{4} c_{2,j}\rho_2^j,$$

for some coefficients $c_{i,j}$ depending on the data. We can show that the Jacobian matrix

$$\frac{\partial(c_{1,0},\dots,c_{1,3},c_{2,0},\dots,c_{2,4})}{\partial(\mathcal{A}_1,\mathcal{A}_2,\mathbf{q}_1,\dot{\mathbf{q}}_1,\mathbf{q}_2,\dot{\mathbf{q}}_2)}$$

has generically maximal rank, i.e. 9, by checking that the rank is maximal for the data of Lemma 2. To conclude we use the fact that for a generic choice of the coefficients $c_{i,j}$ relation (26) holds true.

□

□

By Lemma 3 we can find two univariate polynomials β, γ in the variable ρ_2 such that

$$\beta\tilde{a}_{1,1} + \gamma\tilde{a}_{2,1} = 1. \tag{27}$$

Let us introduce

$$\mathfrak{w} = \beta\tilde{p}_1 + \gamma\tilde{p}_2 = \rho_1 + \mathfrak{z}(\rho_2), \tag{28}$$

where

$$\mathfrak{z} = \beta\tilde{a}_{1,0} + \gamma\tilde{a}_{2,0}.$$

Lemma 4. *The polynomial ideal*

$$\tilde{J}_2 = \langle\mathfrak{w},\mathfrak{v}\rangle$$

is equal to $\tilde{J}_1$.

Proof. From the definition of $\mathfrak{w}$ and from relation

$$\mathfrak{v} = \tilde{a}_{1,1}\tilde{p}_2 - \tilde{a}_{2,1}\tilde{p}_1 \tag{29}$$

we have $\tilde{J}_2 \subseteq \tilde{J}_1$. On the other hand, we can easily invert relations (28), (29) and, using (27), we obtain

$$\tilde{p}_1 = \tilde{a}_{1,1}\mathfrak{w} + \gamma\mathfrak{v}, \qquad \tilde{p}_2 = \tilde{a}_{2,1}\mathfrak{w} - \beta\mathfrak{v},$$

so that the other inclusion $\tilde{J}_1 \subseteq \tilde{J}_2$ holds true.

□

□

Lemmata 2 and 4 imply that $V(\tilde{J}_1)$ has 9 distinct points. In fact, for each root ρ_2 of $\mathfrak{v}$, we find from $\mathfrak{w} = 0$ a unique ρ_1 such that $(\rho_1,\rho_2) \in V(\tilde{J}_1)$. On the other hand, since $\tilde{J} = J$ we have $V(\tilde{J}) = V(J)$ and generically $V(J)$ has 9 distinct points too. We can prove it by using Theorem 1 in [7] and Lemma 2 for the polynomial $\mathfrak{u}$. Then from (25) we have that

$$V(\tilde{J}_1) = V(\tilde{J}). \tag{30}$$

In particular, the polynomials $\mathfrak{v}$ and $\mathfrak{u}$ coincide up to a constant factor.

Now we prove that $\tilde{J}_1$ is indeed equal to $\tilde{J}$. Let us take $h \in \tilde{J}$. Making the division by $\mathfrak{w}$ we obtain

$$h(\rho_1,\rho_2) = h_1(\rho_1,\rho_2)\big(\rho_1 + \mathfrak{z}(\rho_2)\big) + \mathfrak{r}(\rho_2) \tag{31}$$

for some polynomials $h_1, \mathfrak{r}$. The remainder $\mathfrak{r}$ depends only on ρ_2 because $\mathfrak{w}$ is linear in ρ_1. From (24) and (31) we have that $\mathfrak{r} \in \tilde{J}$. Using relation (30) and the fact that $\mathfrak{u}$ is generically square-free we obtain that $\mathfrak{u}$ divides $\mathfrak{r}$ (polynomial division), which together with (31) implies that $h \in \tilde{J}_1$. We conclude that

$$\tilde{J}_1 = \tilde{J}.$$

The polynomials $\mathfrak{g}_1, \ldots, \mathfrak{g}_6$, with

$$\mathfrak{g}_1 = \mathfrak{q}_2, \qquad \mathfrak{g}_2 = \mathfrak{q}_3, \qquad \mathfrak{g}_3 = \mathfrak{p}_4, \qquad \mathfrak{g}_4 = \mathfrak{p}_5, \qquad \mathfrak{g}_5 = \mathfrak{w}, \qquad \mathfrak{g}_6 = \mathfrak{u},$$

form a Groebner basis of the ideal I for the lexicographic order (22). To show this we can simply check that the leading monomials of each pair $(\mathfrak{g}_i, \mathfrak{g}_j)$, with $1 \leq i < j \leq 6$, are relatively prime (see [2, Chap.2]). This concludes the proof of the theorem.

□

□

From the definition of Groebner basis we immediately obtain the following

Corollary 1. *The polynomial* $\mathfrak{u}$ *has the minimum degree among the univariate polynomials in the variable* ρ_2 *belonging to the ideal* I.

As a consequence of the computations in the proof of Theorem 1 we also obtain

Corollary 2. *The polynomial system (20) is generically not consistent. The same result holds true by removing from (20) only one of the two equations* $u_j^2|\mathbf{r}_j|^2 = \mu^2$, $j = 1, 2$.

Proof. We show that the system

$$\mathfrak{g}_j = 0, \quad j = 1 \ldots 6, \qquad u_2^2|\mathbf{r}_2|^2 - \mu^2 = 0 \tag{32}$$

is generically not consistent, where $\mathfrak{g}_j$ are the polynomials in the statement of Theorem 1. By using equations $\mathfrak{g}_1 = \mathfrak{g}_2 = \mathfrak{g}_3 = \mathfrak{g}_5 = 0$ we can obtain from $u_2^2|\mathbf{r}_2|^2 = \mu^2$ another univariate polynomial, say $\hat{\mathfrak{u}}$ in the variable ρ_2. Then $\mathfrak{u}$ and $\hat{\mathfrak{u}}$ have a common root in $\mathbb{C}$ (i.e. are compatible) if and only if

$$\mathrm{Res}(\mathfrak{u}, \hat{\mathfrak{u}}, \rho_2) = 0. \tag{33}$$

Assume there is an open set in the space of the data $\mathcal{A}_j, \mathbf{q}_j, \dot{\mathbf{q}}_j, j = 1, 2$ such that equation (33) holds. Since the left-hand side of (33) is an analytic function of the data, then this equation holds on the whole data set. Therefore, to conclude it is enough to check that equations (32) are not compatible for a particular choice of the data, e.g. as in Lemma 2. In a similar way we can prove that the system

$$\mathfrak{g}_j = 0, \quad j = 1 \ldots 6, \qquad u_1^2|\mathbf{r}_1|^2 - \mu^2 = 0$$

is generically not consistent.

□

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3.4 Compatibility conditions and covariance of the solutions

In this section we discuss how to discard some of the solutions computed with the method described in Section 3 on the base of the full two-body dynamics. Given a pair of attributables $\mathbf{A} = (\mathcal{A}_1, \mathcal{A}_2)$ at epochs $\bar{t}_1, \bar{t}_2$ with covariance matrices $\Gamma_{\mathcal{A}_1}, \Gamma_{\mathcal{A}_2}$, we call $\mathbf{R} = (\rho_1, \dot{\rho}_1, \rho_2, \dot{\rho}_2)$ one of the solutions of the equation

$$\mathbf{\Phi}(\mathbf{R}; \mathbf{A}) = \mathbf{0}, \tag{34}$$

with

$$\mathbf{\Phi}(\mathbf{R};\mathbf{A}) = \left(\begin{array}{c} \mathbf{c}_1 - \mathbf{c}_2 \\ \mathbf{\Xi} \cdot \mathbf{e}_1^\rho \end{array} \right),$$

where

$$\mathbf{\Xi} = \frac{1}{2}(|\dot{\mathbf{r}}_2|^2 - |\dot{\mathbf{r}}_1|^2)\mathbf{r}_1 \times \mathbf{r}_2 - (\dot{\mathbf{r}}_1 \cdot \mathbf{r}_1)\dot{\mathbf{r}}_1 \times (\mathbf{r}_1 - \mathbf{r}_2) + (\dot{\mathbf{r}}_2 \cdot \mathbf{r}_2)\dot{\mathbf{r}}_2 \times (\mathbf{r}_1 - \mathbf{r}_2),$$

which corresponds to the vector $\boldsymbol{\xi}$ defined in (9) after eliminating $\dot{\rho}_1, \dot{\rho}_2$ by (7). We can repeat what follows for each solution of (34). The notation is similar to the one in [5].

Let us introduce the difference vector

$$\mathbf{\Delta}_{a,\ell} = (\Delta_a, \Delta_\ell),$$

with

$$\Delta_a = a_1 - a_2, \qquad \Delta_\ell = \big[\ell_1 - \big(\ell_2 + n(a_2)(\tilde{t}_1 - \tilde{t}_2)\big) + \pi\big](\mathrm{mod}\ 2\pi) - \pi,$$

where $n(a) = \sqrt{\mu}a^{-3/2}$ is the mean motion and $\tilde{t}_i = \bar{t}_i - \rho_i/c$, $i = 1, 2$. Note that here we consider the difference of the two mean anomalies at the same epoch $\tilde{t}_1$ in a way that it is a smooth function at each integer multiple of 2π. We introduce the map

$$(\mathcal{A}_1, \mathcal{A}_2) = \mathbf{A} \mapsto \mathbf{\Psi}(\mathbf{A}) = (\mathcal{A}_1, \mathcal{R}_1, \mathbf{\Delta}_{a,\ell}),$$

giving the orbit $(\mathcal{A}_1, \mathcal{R}_1)$ in attributable coordinates at epoch $\tilde{t}_1$ together with the vector $\mathbf{\Delta}_{a,\ell}$ which is not constrained by equation (34). Introducing the matrices

$$\frac{\partial \mathbf{\Psi}}{\partial \mathbf{A}} = \left[\begin{array}{cc} I & 0 \\ \dfrac{\partial \mathcal{R}_1}{\partial \mathcal{A}_1} & \dfrac{\partial \mathcal{R}_1}{\partial \mathcal{A}_2} \\ \dfrac{\partial \mathbf{\Delta}_{a,\ell}}{\partial \mathcal{A}_1} & \dfrac{\partial \mathbf{\Delta}_{a,\ell}}{\partial \mathcal{A}_2} \end{array} \right] \qquad \text{and} \qquad \Gamma_{\mathbf{A}} = \left[\begin{array}{cc} \Gamma_{\mathcal{A}_1} & 0 \\ 0 & \Gamma_{\mathcal{A}_2} \end{array} \right]$$

we map the covariance of $\mathbf{A}$ to the covariance of $\mathbf{\Psi}(\mathbf{A})$ through the formula

$$\Gamma_{\mathbf{\Psi}(\mathbf{A})} = \frac{\partial \mathbf{\Psi}}{\partial \mathbf{A}}\, \Gamma_{\mathbf{A}} \, \left[\frac{\partial \mathbf{\Psi}}{\partial \mathbf{A}}\right]^T. \tag{35}$$

We can check if there is any solution of (34) fulfilling the compatibility conditions

$$\mathbf{\Delta}_{a,\ell} = \mathbf{0}$$

within a threshold defined by the covariance matrix of the attributables $\Gamma_{\mathbf{A}}$. From (35) we compute the marginal covariance of the vector $\mathbf{\Delta}_{a,\ell}$:

$$\Gamma_{\mathbf{\Delta}_{a,\ell}} = \frac{\partial \mathbf{\Delta}_{a,\ell}}{\partial \mathbf{A}} \Gamma_{\mathbf{A}} \left[\frac{\partial \mathbf{\Delta}_{a,\ell}}{\partial \mathbf{A}}\right]^T.$$

The inverse matrix $C^{\mathbf{\Delta}_{a,\ell}} = \Gamma_{\mathbf{\Delta}_{a,\ell}}^{-1}$ defines a norm $\|\cdot\|_\star$ in the (Δ_a, Δ_ℓ) plane, allowing us to test an identification between the attributables $\mathcal{A}_1, \mathcal{A}_2$: we check whether

$$\|\mathbf{\Delta}_{a,\ell}\|_\star^2 = \mathbf{\Delta}_{a,\ell} C^{\mathbf{\Delta}_{a,\ell}} \mathbf{\Delta}_{a,\ell}^T \le \chi^2_{max},$$

where χ_{max} is a control parameter, to be selected on the basis of large scale tests (see [5, Sect. 5.1]). If a preliminary orbit $(\mathcal{A}_1, \mathcal{R}_1)$ is accepted, from (35) it is also possible to compute its marginal covariance as the 6×6 matrix

$$\Gamma_{(\mathcal{A}_1,\mathcal{R}_1)} = \begin{bmatrix} \Gamma_{\mathcal{A}_1} & \Gamma_{\mathcal{A}_1,\mathcal{R}_1} \\ \Gamma_{\mathcal{R}_1,\mathcal{A}_1} & \Gamma_{\mathcal{R}_1} \end{bmatrix},$$

where

$$\Gamma_{\mathcal{A}_1,\mathcal{R}_1} = \Gamma_{\mathcal{A}_1} \left[\frac{\partial \mathcal{R}_1}{\partial \mathcal{A}_1}\right]^T, \qquad \Gamma_{\mathcal{R}_1} = \frac{\partial \mathcal{R}_1}{\partial \mathbf{A}} \Gamma_{\mathbf{A}} \left[\frac{\partial \mathcal{R}_1}{\partial \mathbf{A}}\right]^T, \qquad \Gamma_{\mathcal{R}_1,\mathcal{A}_1} = \Gamma^T_{\mathcal{A}_1,\mathcal{R}_1}.$$

4 Joining three VSAs

Here we introduce a method to compute preliminary orbits from three VSAs belonging to different nights using the Keplerian integrals (2). In this case the conservation of the angular momentum at the three epochs is enough to obtain a finite number of solutions of the identification problem. In this section indexes $1, 2, 3$ will refer to the mean epochs $\bar{t}_j$ of three VSAs with attributables $\mathcal{A}_j$. We consider the equations:

$$\mathbf{c}_1 = \mathbf{c}_2, \qquad \mathbf{c}_2 = \mathbf{c}_3, \qquad \mathbf{c}_3 = \mathbf{c}_1, \tag{36}$$

that can be written as

$$\mathbf{D}_1\dot{\rho}_1 - \mathbf{D}_2\dot{\rho}_2 = \mathbf{J}_{12}(\rho_1,\rho_2), \qquad \mathbf{D}_2\dot{\rho}_2 - \mathbf{D}_3\dot{\rho}_3 = \mathbf{J}_{23}(\rho_2,\rho_3), \qquad \mathbf{D}_3\dot{\rho}_3 - \mathbf{D}_1\dot{\rho}_1 = \mathbf{J}_{31}(\rho_3,\rho_1),$$

where

$$\begin{aligned} \mathbf{J}_{12}(\rho_1,\rho_2) &= \mathbf{E}_2\rho_2^2 - \mathbf{E}_1\rho_1^2 + \mathbf{F}_2\rho_2 - \mathbf{F}_1\rho_1 + \mathbf{G}_2 - \mathbf{G}_1, \\ \mathbf{J}_{23}(\rho_2,\rho_3) &= \mathbf{E}_3\rho_3^2 - \mathbf{E}_2\rho_2^2 + \mathbf{F}_3\rho_3 - \mathbf{F}_2\rho_2 + \mathbf{G}_3 - \mathbf{G}_2, \\ \mathbf{J}_{31}(\rho_3,\rho_1) &= \mathbf{E}_1\rho_1^2 - \mathbf{E}_3\rho_3^2 + \mathbf{F}_1\rho_1 - \mathbf{F}_3\rho_3 + \mathbf{G}_1 - \mathbf{G}_3. \end{aligned}$$

Equations (36) are redundant, that is, if two of them hold true, then the third equation is also fulfilled. We consider the following projections of equations (36):

$$\begin{aligned} (\mathbf{c}_1 - \mathbf{c}_2)\cdot \mathbf{D}_1 \times \mathbf{D}_2 &= 0, & (37)\\ (\mathbf{c}_1 - \mathbf{c}_2)\cdot \mathbf{D}_1 \times (\mathbf{D}_1 \times \mathbf{D}_2) &= 0, & (38)\\ (\mathbf{c}_2 - \mathbf{c}_3)\cdot \mathbf{D}_2 \times \mathbf{D}_3 &= 0, & (39)\\ (\mathbf{c}_2 - \mathbf{c}_3)\cdot \mathbf{D}_2 \times (\mathbf{D}_2 \times \mathbf{D}_3) &= 0, & (40)\\ (\mathbf{c}_3 - \mathbf{c}_1)\cdot \mathbf{D}_3 \times \mathbf{D}_1 &= 0, & (41)\\ (\mathbf{c}_3 - \mathbf{c}_1)\cdot \mathbf{D}_3 \times (\mathbf{D}_3 \times \mathbf{D}_1) &= 0. & (42) \end{aligned}$$

Proposition 1. *Assume*

$$\mathbf{D}_1 \times \mathbf{D}_2 \cdot \mathbf{D}_3 \neq 0. \tag{43}$$

Then the system of equations (37)–(42) is equivalent to (36).

Proof. Assuming that (41), (42) are fulfilled, to prove that $\mathbf{c}_3 = \mathbf{c}_1$ we only need to show that the projection of this equation onto a vector $\mathbf{v}$, such that $\mathbf{D}_3 \times \mathbf{D}_1, \mathbf{D}_3 \times (\mathbf{D}_3 \times \mathbf{D}_1), \mathbf{v}$ are linearly independent, holds true. We denote by

$$\Pi_{12} = \langle \mathbf{D}_1 \times \mathbf{D}_2, \mathbf{D}_1 \times (\mathbf{D}_1 \times \mathbf{D}_2)\rangle, \qquad \Pi_{23} = \langle \mathbf{D}_2 \times \mathbf{D}_3, \mathbf{D}_2 \times (\mathbf{D}_2 \times \mathbf{D}_3)\rangle$$

the planes passing through the origin generated by the vectors within the brackets. If relation (43) holds, then we have

$$\Pi_{12} \cap \Pi_{23} = \langle \mathbf{D}_1 \times \mathbf{D}_2 \rangle,$$

i.e. the intersection of the two planes is the straight line generated by the vector $\mathbf{v} = \mathbf{D}_1 \times \mathbf{D}_2$. Moreover, we have

$$(\mathbf{D}_1 \times \mathbf{D}_2) \cdot (\mathbf{D}_3 \times \mathbf{D}_1) \times \big(\mathbf{D}_3 \times (\mathbf{D}_3 \times \mathbf{D}_1)\big) = |\mathbf{D}_3 \times \mathbf{D}_1|^2 \mathbf{D}_1 \times \mathbf{D}_2 \cdot \mathbf{D}_3,$$

which does not vanish by (43). Therefore, from (37)–(40) we obtain $(\mathbf{c}_1 - \mathbf{c}_2) \cdot \mathbf{v} = (\mathbf{c}_2 - \mathbf{c}_3) \cdot \mathbf{v} = 0$, that yield $(\mathbf{c}_3 - \mathbf{c}_1) \cdot \mathbf{v} = 0$. In a similar way we can prove that $\mathbf{c}_1 = \mathbf{c}_2$, $\mathbf{c}_2 = \mathbf{c}_3$, provided that (37)–(42) hold.

□

□

Equations (37), (39), (41) depend only on the radial distances. In fact, they correspond to the system

$$\mathbf{J}_{12} \cdot \mathbf{D}_1 \times \mathbf{D}_2 = 0, \qquad \mathbf{J}_{23} \cdot \mathbf{D}_2 \times \mathbf{D}_3 = 0, \qquad \mathbf{J}_{31} \cdot \mathbf{D}_3 \times \mathbf{D}_1 = 0, \tag{44}$$

which can be written as

$$\begin{aligned} q_3 &= a_3\rho_2^2 + b_3\rho_1^2 + c_3\rho_2 + d_3\rho_1 + e_3 = 0, & (45)\\ q_1 &= a_1\rho_3^2 + b_1\rho_2^2 + c_1\rho_3 + d_1\rho_2 + e_1 = 0, & (46)\\ q_2 &= a_2\rho_1^2 + b_2\rho_3^2 + c_2\rho_1 + d_2\rho_3 + e_2 = 0, & (47) \end{aligned}$$

where

$$\begin{gathered} a_3 = \mathbf{E}_2 \cdot \mathbf{D}_1 \times \mathbf{D}_2, \qquad b_3 = -\mathbf{E}_1 \cdot \mathbf{D}_1 \times \mathbf{D}_2,\\ c_3 = \mathbf{F}_2 \cdot \mathbf{D}_1 \times \mathbf{D}_2, \qquad d_3 = -\mathbf{F}_1 \cdot \mathbf{D}_1 \times \mathbf{D}_2,\\ e_3 = (\mathbf{G}_2 - \mathbf{G}_1) \cdot \mathbf{D}_1 \times \mathbf{D}_2, \end{gathered}$$

and the other coefficients a_j, b_j, c_j, d_j, e_j, for $j = 1, 2$, have similar expressions, obtained by cycling the indexes. To eliminate ρ_1, ρ_3 from (44) we first compute the resultant

$$r = \mathrm{Res}(q_3, q_2, \rho_1),$$

which depends only on ρ_2, ρ_3. Then we compute the resultant

$$\mathfrak{q} = \mathrm{Res}(r, q_1, \rho_3),$$

which is a univariate polynomial of degree 8 in the variable ρ_2. Therefore, provided that (43) holds, to get the solutions of (36) first we search for the roots $\bar{\rho}_2$ of $\mathfrak{q}(\rho_2)$, then we compute the corresponding values $\bar{\rho}_3$ from system $r(\rho_3, \bar{\rho}_2) = q_1(\rho_3, \bar{\rho}_2) = 0$, and finally the corresponding values $\bar{\rho}_1$ from system $q_3(\rho_1, \bar{\rho}_2) = q_2(\bar{\rho}_3, \rho_1) = 0$. Since the unknowns ρ_j represent distances we can discard triples $(\bar{\rho}_1, \bar{\rho}_2, \bar{\rho}_3)$ where some ρ_j is non-positive. From equations (38), (40), (42) we can write the radial velocities $\dot{\rho}_j$ as functions of pairs of radial distances:

$$\begin{aligned} \dot{\rho}_2 &= \frac{\mathbf{J}_{12}(\rho_1, \rho_2) \cdot \mathbf{D}_1 \times (\mathbf{D}_1 \times \mathbf{D}_2)}{|\mathbf{D}_1 \times \mathbf{D}_2|^2},\\ \dot{\rho}_3 &= \frac{\mathbf{J}_{23}(\rho_2, \rho_3) \cdot \mathbf{D}_2 \times (\mathbf{D}_2 \times \mathbf{D}_3)}{|\mathbf{D}_2 \times \mathbf{D}_3|^2},\\ \dot{\rho}_1 &= \frac{\mathbf{J}_{31}(\rho_3, \rho_1) \cdot \mathbf{D}_3 \times (\mathbf{D}_3 \times \mathbf{D}_1)}{|\mathbf{D}_3 \times \mathbf{D}_1|^2}. \end{aligned}$$

Remark 2. *A simple way to discard triples $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$, before making the computation described in this section, is to use the intersection criterion introduced in [7] to discard pairs of attributables. More precisely, we can apply this criterion three times, i.e. we check for each $j = 1, 2, 3$ whether the conic Q_j, defined by $q_j = 0$ (see equations (45), (46), (47)), intersects the square $\mathcal{R} = [\rho_{min}, \rho_{max}] \times [\rho_{min}, \rho_{max}]$ for some fixed $\rho_{max} > \rho_{min} > 0$. If this criterion fails in one of these cases we discard the selected triple. For more details see the appendix in [7].*

4.1 Solutions with zero angular momentum

A particular solution of system (36) can be obtained by searching for values of $\rho_j, \dot{\rho}_j$ such that

$$\mathbf{c}_j(\rho_j, \dot{\rho}_j) = \mathbf{0}, \qquad j = 1, 2, 3.$$

Relation $\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{0}$ implies that there exists $\lambda \in \mathbb{R}$ such that

$$\dot{\rho}\mathbf{e}^\rho + \rho\boldsymbol{\eta} + \dot{\mathbf{q}} = \lambda(\rho\mathbf{e}^\rho + \mathbf{q}), \tag{48}$$

with $\boldsymbol{\eta} = \dot{\alpha}\cos\delta\mathbf{e}^\alpha + \dot{\delta}\mathbf{e}^\delta$. Setting $\sigma = \dot{\rho} - \lambda\rho$ we can write (48) as

$$\sigma\mathbf{e}^\rho + \rho\boldsymbol{\eta} - \lambda\mathbf{q} = -\dot{\mathbf{q}}. \tag{49}$$

We introduce the vector

$$\mathbf{u} = \mathbf{q} - (\mathbf{q}\cdot\mathbf{e}^\rho)\mathbf{e}^\rho - \frac{1}{\eta^2}(\mathbf{q}\cdot\boldsymbol{\eta})\boldsymbol{\eta},$$

which is orthogonal to both $\mathbf{e}^\rho, \boldsymbol{\eta}$, where $\eta = |\boldsymbol{\eta}|$ is called the proper motion. Thus, we can write (49) as

$$[\sigma - \lambda(\mathbf{q}\cdot\mathbf{e}^\rho)]\mathbf{e}^\rho + \Big[\rho - \frac{\lambda}{\eta^2}(\mathbf{q}\cdot\boldsymbol{\eta})\Big]\boldsymbol{\eta} - \lambda\mathbf{u} = -\dot{\mathbf{q}}.$$

Since $\{\mathbf{e}^\rho, \boldsymbol{\eta}, \mathbf{u}\}$ is generically an orthogonal basis of $\mathbb{R}^3$, we find

$$\lambda = \frac{1}{|\mathbf{u}|^2}(\dot{\mathbf{q}}\cdot\mathbf{u}), \qquad \rho = \frac{1}{\eta^2}(\lambda\mathbf{q} - \dot{\mathbf{q}})\cdot\boldsymbol{\eta}, \qquad \dot{\rho} = \lambda\rho + (\lambda\mathbf{q} - \dot{\mathbf{q}})\cdot\mathbf{e}^\rho.$$

In particular, we obtain the value

$$\rho = \frac{1}{\eta^2}\Big(\frac{1}{|\mathbf{u}|^2}(\dot{\mathbf{q}}\cdot\mathbf{u})(\mathbf{q}\cdot\boldsymbol{\eta}) - \dot{\mathbf{q}}\cdot\boldsymbol{\eta}\Big)$$

for the radial distance, corresponding to a solution with zero angular momentum.

4.2 Compatibility conditions and covariance of the solutions

We discuss how to discard solutions of (36) in a way similar to Section 3.4. Given a triple of attributables $\mathbf{A} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ with covariance matrices $\Gamma_{\mathcal{A}_1}, \Gamma_{\mathcal{A}_2}, \Gamma_{\mathcal{A}_3}$, we call $\mathbf{R} = (\rho_1, \dot{\rho}_1, \rho_2, \dot{\rho}_2, \rho_3, \dot{\rho}_3)$ one of the solutions of the equation

$$\boldsymbol{\Phi}(\mathbf{R}; \mathbf{A}) = \mathbf{0}, \tag{50}$$

with

$$\boldsymbol{\Phi}(\mathbf{R}; \mathbf{A}) = \begin{pmatrix} (\mathbf{c}_1 - \mathbf{c}_2)\cdot\mathbf{D}_1 \times (\mathbf{D}_1 \times \mathbf{D}_2) \\ (\mathbf{c}_1 - \mathbf{c}_2)\cdot\mathbf{D}_1 \times \mathbf{D}_2 \\ (\mathbf{c}_2 - \mathbf{c}_3)\cdot\mathbf{D}_2 \times (\mathbf{D}_2 \times \mathbf{D}_3) \\ (\mathbf{c}_2 - \mathbf{c}_3)\cdot\mathbf{D}_2 \times \mathbf{D}_3 \\ (\mathbf{c}_3 - \mathbf{c}_1)\cdot\mathbf{D}_3 \times (\mathbf{D}_3 \times \mathbf{D}_1) \\ (\mathbf{c}_3 - \mathbf{c}_1)\cdot\mathbf{D}_3 \times \mathbf{D}_1 \end{pmatrix}.$$

We can repeat what follows for each solution of (50).

Let us introduce the difference vectors

$$\begin{aligned}
\mathbf{\Delta}_{12} &= \big(a_1 - a_2, [\omega_1 - \omega_2 + \pi](\text{mod } 2\pi) - \pi, \big[\ell_1 - \big(\ell_2 + n(a_2)(\tilde{t}_1 - \tilde{t}_2)\big) + \pi\big](\text{mod } 2\pi) - \pi\big),\\
\mathbf{\Delta}_{32} &= \big(a_3 - a_2, [\omega_3 - \omega_2 + \pi](\text{mod } 2\pi) - \pi, \big[\ell_3 - \big(\ell_2 + n(a_2)(\tilde{t}_3 - \tilde{t}_2)\big) + \pi\big](\text{mod } 2\pi) - \pi\big),
\end{aligned}$$

where the third component is the difference of the two mean anomalies referring to epoch $\tilde{t}_i = \bar{t}_i - \rho_i/c$, and $n(a) = \sqrt{\mu}a^{-3/2}$ is the mean motion. Here the difference of two angles is computed in a way that it is a smooth function at each integer multiple of 2π. We introduce the map

$$(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3) = \mathbf{A} \mapsto \mathbf{\Psi}(\mathbf{A}) = (\mathcal{A}_2, \mathcal{R}_2, \mathbf{\Delta}_{12}, \mathbf{\Delta}_{32}),$$

giving the orbit $(\mathcal{A}_2, \mathcal{R}_2)$ in attributable coordinates at epoch $\tilde{t}_2$ together with the vectors $\mathbf{\Delta}_{12}$, $\mathbf{\Delta}_{32}$, which are not constrained by the angular momentum integrals. We want to check if there is any solution of (50) fulfilling the compatibility conditions

$$\mathbf{\Delta}_{12} = \mathbf{\Delta}_{32} = \mathbf{0}$$

within a threshold defined by the covariance matrix of the attributables

$$\Gamma_{\mathbf{A}} = \begin{bmatrix} \Gamma_{\mathcal{A}_1} & 0 & 0 \\ 0 & \Gamma_{\mathcal{A}_2} & 0 \\ 0 & 0 & \Gamma_{\mathcal{A}_3} \end{bmatrix}.$$

We map the covariance of $\mathbf{A}$ to the covariance of $\mathbf{\Psi}(\mathbf{A})$ through

$$\Gamma_{\mathbf{\Psi}(\mathbf{A})} = \frac{\partial \mathbf{\Psi}}{\partial \mathbf{A}}\, \Gamma_{\mathbf{A}}\, \left[\frac{\partial \mathbf{\Psi}}{\partial \mathbf{A}}\right]^T,$$

where

$$\frac{\partial \mathbf{\Psi}}{\partial \mathbf{A}} = \begin{bmatrix} 0 & I & 0 \\ \dfrac{\partial \mathcal{R}_2}{\partial \mathcal{A}_1} & \dfrac{\partial \mathcal{R}_2}{\partial \mathcal{A}_2} & \dfrac{\partial \mathcal{R}_2}{\partial \mathcal{A}_3} \\ \dfrac{\partial \mathbf{\Delta}_{12}}{\partial \mathcal{A}_1} & \dfrac{\partial \mathbf{\Delta}_{12}}{\partial \mathcal{A}_2} & \dfrac{\partial \mathbf{\Delta}_{12}}{\partial \mathcal{A}_3} \\ \dfrac{\partial \mathbf{\Delta}_{32}}{\partial \mathcal{A}_1} & \dfrac{\partial \mathbf{\Delta}_{32}}{\partial \mathcal{A}_2} & \dfrac{\partial \mathbf{\Delta}_{32}}{\partial \mathcal{A}_3} \end{bmatrix}.$$

The matrices $\frac{\partial \mathcal{R}_2}{\partial \mathcal{A}_j}$, $j = 1, 2, 3$, can be computed from the relation

$$\frac{\partial \mathbf{R}}{\partial \mathbf{A}}(\mathbf{A}) = -\left[\frac{\partial \mathbf{\Phi}}{\partial \mathbf{R}}(\mathbf{R}(\mathbf{A}), \mathbf{A})\right]^{-1} \frac{\partial \mathbf{\Phi}}{\partial \mathbf{A}}(\mathbf{R}(\mathbf{A}), \mathbf{A}).$$

The marginal covariance matrix for the vector $\mathbf{\Delta} = (\mathbf{\Delta}_{12}, \mathbf{\Delta}_{32})$ is given by the block

$$\Gamma_{\mathbf{\Delta}} = \begin{bmatrix} \Gamma_{\mathbf{\Delta}_{12}} & \Gamma_{\mathbf{\Delta}_{12}, \mathbf{\Delta}_{32}} \\ \Gamma_{\mathbf{\Delta}_{32}, \mathbf{\Delta}_{12}} & \Gamma_{\mathbf{\Delta}_{32}} \end{bmatrix}$$

of $\Gamma_{\mathbf{\Psi}(\mathbf{A})}$, where

$$\begin{aligned}
\Gamma_{\mathbf{\Delta}_{12}} &= \frac{\partial \mathbf{\Delta}_{12}}{\partial \mathbf{A}} \Gamma_{\mathbf{A}} \left[\frac{\partial \mathbf{\Delta}_{12}}{\partial \mathbf{A}}\right]^T, &\qquad \Gamma_{\mathbf{\Delta}_{12}, \mathbf{\Delta}_{32}} &= \frac{\partial \mathbf{\Delta}_{12}}{\partial \mathbf{A}} \Gamma_{\mathbf{A}} \left[\frac{\partial \mathbf{\Delta}_{32}}{\partial \mathbf{A}}\right]^T, \\
\Gamma_{\mathbf{\Delta}_{32}} &= \frac{\partial \mathbf{\Delta}_{32}}{\partial \mathbf{A}} \Gamma_{\mathbf{A}} \left[\frac{\partial \mathbf{\Delta}_{32}}{\partial \mathbf{A}}\right]^T, &\qquad \Gamma_{\mathbf{\Delta}_{32}, \mathbf{\Delta}_{12}} &= \Gamma^T_{\mathbf{\Delta}_{12}, \mathbf{\Delta}_{32}}.
\end{aligned}$$

The inverse matrix $C^{\mathbf{\Delta}} = \Gamma_{\mathbf{\Delta}}^{-1}$ defines a norm $\| \cdot \|_\star$ allowing us to test an identification between the attributables $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$: we check whether

$$\|\mathbf{\Delta}\|_\star^2 = \mathbf{\Delta} C^{\mathbf{\Delta}} \mathbf{\Delta}^T \le \chi^2_{max}, \tag{51}$$

where χ_{max} is a control parameter.

For each orbit, solution of (50), fulfilling condition (51) we can also define a covariance matrix Γ_2 for the attributable coordinates $(\mathcal{A}_2, \mathcal{R}_2)$:

$$\Gamma_2 = \left[\begin{array}{cc} \Gamma_{\mathcal{A}_2} & \Gamma_{\mathcal{A}_2,\mathcal{R}_2} \\ \Gamma_{\mathcal{R}_2,\mathcal{A}_2} & \Gamma_{\mathcal{R}_2} \end{array} \right],$$

where $\Gamma_{\mathcal{A}_2}$ is given and

$$\Gamma_{\mathcal{A}_2,\mathcal{R}_2} = \Gamma_{\mathcal{A}_2} \left[\frac{\partial \mathcal{R}_2}{\partial \mathcal{A}_2} \right]^T, \qquad \Gamma_{\mathcal{R}_2} = \frac{\partial \mathcal{R}_2}{\partial \mathbf{A}} \Gamma_{\mathbf{A}} \left[\frac{\partial \mathcal{R}_2}{\partial \mathbf{A}} \right]^T, \qquad \Gamma_{\mathcal{R}_2,\mathcal{A}_2} = \Gamma^T_{\mathcal{A}_2,\mathcal{R}_2}.$$

5 Numerical tests

In this section we test the methods described in Sections 3, 4 using the observations of two main belt asteroids: (450003) and 2014 YW_{11}. In Table 1 we list three tracklets composed by four observations (right ascension α, declination δ) of asteroid (450003), collected with the Pan-STARRS telescope (PS1). We also display the observational biases due to systematic errors in the star catalog for which we have correction tables (see [3]).

tr	obs	α (deg)	bias (α)	δ (deg)	bias (δ)	date (UTC)
1	1	350.66612	−0.044	4.05939	0.132	2015 07 28.56903
	2	350.66973	−0.044	4.06026	0.132	2015 07 28.58174
	3	350.67334	−0.044	4.06114	0.132	2015 07 28.59445
	4	350.67698	−0.044	4.06201	0.132	2015 07 28.60688
2	1	355.75182	−0.028	3.71542	0.074	2015 08 21.50571
	2	355.75278	−0.028	3.71413	0.074	2015 08 21.51835
	3	355.75375	−0.028	3.71282	0.074	2015 08 21.53096
	4	355.75470	−0.028	3.71151	0.074	2015 08 21.54360
3	1	356.33105	−0.051	0.05804	0.188	2015 09 12.39647
	2	356.33029	−0.051	0.05559	0.188	2015 09 12.40823
	3	356.32952	−0.051	0.05312	0.188	2015 09 12.41999
	4	356.32875	−0.051	0.05064	0.188	2015 09 12.43174

Table 1: The three selected tracklets, each composed by four observations of asteroid (450003). The angles α and δ are given in degrees, their biases in arcseconds.

In Table 2 we show the approximated values of the components of the three attributables computed from the tracklets in Table 1. The observational bias is applied to the values of α, δ listed above.

We compare the preliminary orbits obtained by Gauss' method with the methods described in this paper by means of the least squares solution, computed with all the tracklets of Table 1, and of its covariance matrix. In Table 3 we show the results of this comparison. The labels G_2, G_3 refer to the orbits obtained with Gauss' method using different observations from Table 1: for

att	α (deg)	δ (deg)	$\dot{\alpha}$ (arcsec/d)	$\dot{\delta}$ (arcsec/d)	date (TDT)
1	350.67152	4.06066	1031.73966	248.94682	2015 07 28.58881
2	355.75328	3.71346	273.91459	-371.71349	2015 08 21.52544
3	356.32992	0.05430	-234.37136	-755.55043	2015 09 12.41490

Table 2: Attributables computed from the three tracklets in Table 1.

	a (au)	e	I	Ω	ω	ℓ	norm
G_2	2.13208	0.32946	4.79407	177.16888	155.00135	2.11089	3622.9
L_2	2.14785	0.33138	4.90092	177.00134	157.19614	1.18703	4786.1
G_3	2.06025	0.30304	4.91806	176.87697	156.93193	1.46197	10957.8
L_3	2.05587	0.31248	4.66792	176.87899	155.66710	1.88742	1540.3
LS_3	2.09738	0.31910	4.83036	176.87550	156.85541	1.38240	//

Table 3: Preliminary orbits at the date 2015 8 20.84305 (TDT), obtained with Gauss' method and with the identification methods described in Sections 3, 4. The last line contains the least squares solution used for the comparison. The angles are given in degrees. In the last column we list the values of the norms defined in (52).

G_2 we use observations $1, 4$ of tracklet 1 and observation 1 of tracklet 2; for G_3 we use observation 1 of each tracklet. The labels L_2, L_3 refer to the methods described in Sections 3, 4. For L_2 we use attributables $1, 2$ listed in Table 2; for L_3 we use all the attributables in this table. Let $\mathscr{E}_{G_2}$, $\mathscr{E}_{L_2}$, $\mathscr{E}_{G_3}$, $\mathscr{E}_{L_3}$ be the preliminary orbits computed with the different methods. The label LS_3 in the last line of Table 3 refers to the least squares orbit computed from $\mathscr{E}_{G_3}$. We call $\mathscr{E}_{LS_3}$ the corresponding least squares orbit and Γ_{LS_3} the related covariance matrix. All the preliminary orbits are propagated to the mean epoch of the observations used to compute the least squares solution $\mathscr{E}_{LS_3}$. The norms displayed in the last column of Table 3 are defined as

$$
\begin{aligned}
|\mathscr{E}_{G_2}| &= \Delta_{G_2} \cdot C_{LS_3} \Delta_{G_2}, \quad |\mathscr{E}_{L_2}| = \Delta_{L_2} \cdot C_{LS_3} \Delta_{L_2}, \\
|\mathscr{E}_{G_3}| &= \Delta_{G_3} \cdot C_{LS_3} \Delta_{G_3}, \quad |\mathscr{E}_{L_3}| = \Delta_{L_3} \cdot C_{LS_3} \Delta_{L_3},
\end{aligned} \tag{52}
$$

where

$$
\Delta_{G_2} = \mathscr{E}_{G_2} - \mathscr{E}_{LS_3}, \qquad \Delta_{L_2} = \mathscr{E}_{L_2} - \mathscr{E}_{LS_3}, \qquad \Delta_{G_3} = \mathscr{E}_{G_3} - \mathscr{E}_{LS_3}, \qquad \Delta_{L_3} = \mathscr{E}_{L_3} - \mathscr{E}_{LS_3},
$$

and $C_{LS_3} = \Gamma_{LS_3}^{-1}$ is the normal matrix corresponding to $\mathscr{E}_{LS_3}$. For this test case the norm $|\mathscr{E}_{L_3}|$ is smaller than $|\mathscr{E}_{G_3}|$ by one order of magnitude, while the norms $|\mathscr{E}_{G_2}|$ and $|\mathscr{E}_{L_2}|$ are comparable. However, we observe that the role of the linkage of two VSAs is simply to test the compatibility of pairs of tracklets and discard a large number of them. When we join a third tracklet to a pair of linked VSAs we compute from scratch a preliminary orbit.

As a second test, we consider three tracklets composed by four observations of asteroid 2014 YW_{11} made by PS1. In this case the first tracklet is very far apart in time from the other two (see Tables 4, 5), therefore Gauss' method can not be used. In Table 6 we compare the preliminary orbits $\mathscr{E}_{L_2}$, $\mathscr{E}_{L_3}$ with the least squares solution $\mathscr{E}_{LS_3}$, where the latter is computed starting from $\mathscr{E}_{L_3}$. From these results, the Keplerian integrals methods described in this paper appear well suited to identify VSAs with a large time separation.

tr	obs	α (deg)	bias (α)	δ (deg)	bias (δ)	date (UTC)
1	1	130.53486	0.119	20.56497	0.094	2012 02 13.30727
	2	130.53097	0.119	20.56493	0.094	2012 02 13.32181
	3	130.52707	0.119	20.56489	0.094	2012 02 13.33638
	4	130.52318	0.119	20.56483	0.094	2012 02 13.35096
2	1	61.09531	−0.022	29.84245	0.136	2014 12 29.40432
	2	61.09403	−0.022	29.84132	0.136	2014 12 29.41580
	3	61.09269	−0.022	29.84016	0.136	2014 12 29.42728
	4	61.09142	−0.022	29.83898	0.136	2014 12 29.43881
3	1	62.07000	0.009	27.87887	0.124	2015 01 23.33565
	2	62.07199	0.009	27.87816	0.124	2015 01 23.34820
	3	62.07398	0.009	27.87739	0.124	2015 01 23.36075
	4	62.07598	0.009	27.87666	0.124	2015 01 23.37329

Table 4: The three selected tracklets, each composed by four observations of asteroid 2014 YW_{11}. The angles α and δ are given in degrees, their biases in arcseconds.

att	α (deg)	δ (deg)	$\dot{\alpha}$ (arcsec/d)	$\dot{\delta}$ (arcsec/d)	date (TDT)
1	130.52898	20.56488	-962.06502	-11.39645	2012 02 13.32987
2	61.09336	29.84070	-407.39435	-362.76023	2014 12 29.42233
3	62.07298	27.87775	571.09302	-212.16315	2015 01 23.35525

Table 5: Attributables computed from the three tracklets in Table 4.

	a (au)	e	I	Ω	ω	ℓ	norm
L_2	2.19793	0.15470	4.95738	328.99987	103.96920	239.71121	52915.1
L_3	2.19479	0.14983	4.96004	328.99346	105.99094	238.85008	363.4
LS_3	2.19507	0.14899	4.96545	328.94870	106.08163	238.83817	//

Table 6: Preliminary orbits at the date 2013 11 9.48155 (TDT), obtained with the methods described in Sections 3, 4. The angles are given in degrees. In the last column we list the values of the norms defined in (52).

6 Conclusions

The methods currently in use to deal with the linkage problem (statistical ranging, systematic ranging, sampling of the admissible region, kd-tree algorithm) can be used successfully to link VSAs over relatively short time spans, but they can not be employed when the time separation between the tracklets is large. The need of improving the efficiency of the existing identification methods is shown by the large database of unidentified tracklets of asteroid observations available at the MPC website. In this paper we used elimination theory to show that the polynomial equation of degree 9 introduced in [7] has the minimum degree among the univariate polynomial equations that are consequence of the conservation laws of Kepler's problem, provided that we drop the dependence between the inverse of the heliocentric distance and the topocentric distance. We also introduced a method to join triples of VSAs leading to a polynomial equation of degree 8. Being based on the conservation laws of the two-body motion, the Keplerian integrals methods can be used also with VSAs observed at different apparitions. Moreover, the related equations can be fast and accurately solved by numerical methods, therefore these algorithms are suitable to be used with large database of tracklets. We think that the Keplerian integral methods are a significant addition to the suite of already available methods. In particular, they will be useful in an orbit determination pipeline, similar to the ones described in [12, Chap.11].

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