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On the plane steady-state flow of a shear-thinning liquid past an obstacle in the singular case

Giovanni P. Galdi and Carlo R. Grisanti

To Professor Hugo Beirão da Veiga on the occasion of his seventieth birthday, with great friendship and appreciation

Abstract. We show existence of strong solutions to the steady-state, two-dimensional exterior problem for a class of shear-thinning liquids –where shear viscosity is a suitable decreasing function of shear rate– for data of arbitrary size. Notice that the analogous problem is, to date, open for liquids governed by the Navier-Stokes equations, where viscosity is constant. Two important features of this work are that, on the one hand and unlike previous contributions by the same authors, the current results do not require non-vanishing of the constant-viscosity part of the stress tensor, and, on the other hand, we allow the shear-thinning contribution to be "arbitrarily small", and, therefore, the model used here can be as "close" as we please to the classical Navier-Stokes one.

1. Introduction

As widely recognized, one of the most significant open questions in the mathematical theory of the Navier–Stokes equations is whether the two-dimensional steady-state problem in an exterior domain, Ω, admits a solution for data of arbitrary size; see [**2**, Chapter XI]. In this respect, of particular physical interest is the case where the only non-zero datum reduces to a prescribed constant velocity field at infinity, v^∞, describing the translational motion with speed $|v^\infty|$ of a cylinder in a viscous liquid that executes a corresponding time-independent flow.

The main difficulty in proving existence for large data is related to the circumstance that, to date, the only a priori estimate valid under these general conditions is for the Dirichlet norm of the velocity field, v:

$$\int_\Omega |\nabla v|^2 \le M\,, \tag{1.1}$$

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C.R.Grisanti is a member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

with M depending only on the data. However, by (1.1) one is not able to provide, so far, any valuable information about whether v converges to the *assigned* v^∞ at large distances, even in a generalized sense; see [**2**, Chapter XI], and the reference therein.

At this point it should be recalled and emphasized that liquids modeled by the Navier–Stokes equations (*Newtonian liquids*) are characterized by the property that the shear viscosity coefficient, μ, is a constant. However, in several important applications, the liquids involved do not have such a property, in that μ is a (typically monotonic) function of the shear rate (*generalized Newtonian*). A representative class of shear-thinning models is furnished by the following viscosity/shear rate relation

$$\mu = \mu_0 + \mu_1 |Dv|^\sigma\,, \quad Dv := \tfrac{1}{2}\big(\nabla v + (\nabla v)^\top\big) \tag{1.2}$$

where μ_0, μ_1, and σ are constant satisfying $\mu_0 \geq 0$, $\mu_1 > 0$, and $\sigma > -1$. The familiar Navier-Stokes (Newtonian) case, is then recovered by taking $\sigma = 0$ in (1.2).

Among others, a remarkable example of generalized Newtonian liquid is blood. In fact, in a wide range of flow conditions, blood shows a *shear-thinning* feature, namely, the coefficient μ is a decreasing function of $|Dv|$. For the model (1.2), this amounts to choose $\sigma \in (-1, 0)$; see, e.g., [**4**] for more details.

Motivated by the above considerations, in the recent paper [**3**] the present authors have investigated the problem of existence of plane, exterior steady-state motions in the case of a shear-thinning liquid. In particular, they have shown that, unlike what currently known for a Navier–Stokes liquid, for a sufficiently large class of shear-thinning liquids the corresponding plane steady-state exterior problem has always a solution for data of unrestricted size (in suitable function class). The model (1.2) with $\mu_0 > 0$ and arbitrary $\mu_1 > 0$, $\sigma \in (-1, 0)$ is a special member of this class. Since σ can be taken *as close as we please* to 0, in a more physical language, we can state that the existence problem is completely solvable provided we make the Newtonian liquid only “slightly” shear-thinning (for instance, by adding to it suitable polymers).

One of the fundamental reasons why the approach in [**3**] was successful is because, in such a case, we can prove the following “global” estimate that is more convenient than (1.1)

$$\int_\Omega |\nabla(v - v^\infty)|^p \leq M\,, \quad \text{for some } p \in (1, 2)\,. \tag{1.3}$$

Actually, being now $p < 2$ (= space dimension), from (1.3) and Sobolev theorem, we can then conclude that $v \to v^\infty$, at least in an appropriate sense.[1]

It should now be remarked that the method used in [**3**], for its success, made substantial use of the fact that the viscosity coefficient μ was a nonlinear “perturbation” of a constant. In other words, the relevant second-order elliptic operator is a (nonlinear) monotone perturbation of the Stokes operator. In terms of the representative model (1.2), this amounts to say that $\mu_0 > 0$.

Objective of this paper is to continue and, to an extent, complete the research carried out in [**3**], by relaxing such a restriction and thus allowing the relevant operator to be “singular”. Again in the case of (1.2), this is equivalent to take $\mu_0 = 0$.

[1] In fact, in [**3**] it is shown how, starting from (1.3), one can eventually prove $v \to v^\infty$ uniformly pointwise.

As expected, removing the above restriction entails a couple of basic problems that we describe next. In the first place, we are no longer able to use in full the regularity result of [**6**] that was crucial to show in [**3**] convergence of the approximate solutions. Yet, we can provide a weaker version of it (Theorem 2.3) that will nevertheless allow us to prove the desired convergence for a class of shear-thinning liquid that are as "close" as we wish to the Navier–Stokes model. With respect to (1.2) the latter means that we can take ($\mu_0 = 0$ and) μ_1 positive, and σ negative and arbitrarily close to 0. In the second place, for the physically significant case where $v^\infty \neq 0$, the proof (even formal) of the fundamental a priori estimate (1.3) is no longer "standard", due to the circumstance that the classical Hopf lift method of v^∞ does not work (Proposition 3.1).

The plan of the work is the following. In Section 1, we recall some more or less standard notation, formulate the basic problem with the corresponding assumptions on the Cauchy stress tensor defining the shear-thinning property of the liquid, and collect some preliminary results, mostly, concerning the regularity of weak solutions. Our method of proof is based on the classical "invading domains" technique. This consists in showing existence on each member of a sequence of increasing bounded subdomains whose union is the whole of Ω, and to prove a suitable bound on the solutions with a constant independent on the diameter of the subdomain. This is exactly the content of Section 2, where we show that solutions to our problem satisfies this property (in a suitable function class) for both cases $v^\infty \neq 0$ (Proposition 3.1) and $v^\infty = 0$ (Proposition 3.2). The reason why we treat these two cases separately, is because in the case $v^\infty = 0$ less restrictions are needed on the constitutive property of the liquid. Finally, in Section 3, we use the results of the previous sections to prove our main results, which establish the existence to the original problem for data of arbitrary size in the cases $v^\infty \neq 0$ (Theorem 4.1) and $v^\infty = 0$ (Theorem 4.2).

2. Notation, formulation of the problem and preliminary results

We indicate by $B(x, R)$ the open ball (circle) of $\mathbb{R}^2$ with center in $x \in \mathbb{R}^2$ and radius R. Here and throughout, Ω denotes a planar exterior domain, that is, the complement of a compact, simply connected set in $\mathbb{R}^2$. Without loss of generality we assume $\mathbb{R}^2 \setminus \Omega \subset B(0, \overline{R})$, for some $\overline{R} > 0$. The boundary of Ω is required to be Lipschitz. For any $R \geq \overline{R}$ we set

$$\Omega_R := \Omega \cap B(0, R).$$

Next, by $\mathbb{R}^{2\times 2}_{sym}$ we denote the set of 2×2 symmetric tensors of order 2. For a given $p \in \mathbb{R},\ p > 1$, the number $p' = \frac{p}{p-1}$ denotes the conjugate exponent. Given a function $\phi \in L^p(A)$ with $A \subset \mathbb{R}^2$, we indicate its L^p norm as $\|\phi\|_{p,A}$. If $A = \Omega$ we will simply write $\|\phi\|_p$. In addition to the usual Sobolev spaces we use the following spaces of vector valued functions $\mathbb{R}^2 \to \mathbb{R}^2$

$$\mathcal{D}(A) = \{\phi \in C_0^\infty(A), \nabla \cdot \phi = 0\}, \quad D^{1,p}(A) = \{\phi \in L^1_{loc}(A) : \nabla\phi \in L^p(A)\}$$

and for a function $\phi \in D^{1,p}(A)$ we set $\|\phi\|_{D^{1,p}(A)} := \|\nabla\phi\|_{p,A}$. We remark that this is not a norm in $D^{1,p}(A)$ but becomes such on the following *homogeneous* Sobolev spaces

$$D_0^{1,p}(A) = \overline{C_0^\infty(A)}, \qquad \mathcal{D}_0^{1,p}(A) = \overline{\mathcal{D}(A)} \qquad \text{in the norm } \|\cdot\|_{D^{1,p}(A)}.$$

We also define the dual space

$$D_0^{-1,p'}(A) := \Big(D_0^{1,p}(A)\Big)'.$$

We wish to emphasize that, throughout the paper, the symbol $\|\cdot\|_{-1,p'}$ stands for the norm in this space and *not* in the dual space $W^{-1,p'}$. The duality pairing between $f \in D_0^{1,p}(\Omega)$ and $g \in D_0^{-1,p'}(\Omega)$ is written as $\langle f, g\rangle$. For details and corresponding properties of the above homogeneous spaces we refer to [**2**].

Finally, by $(f,g)_A$ we mean the usual scalar product in $L^2(A)$, $(f,g)_A = \int_A f(x)g(x)\,dx$. Whenever $A = \Omega$, we shall omit the subscript.

Objective of this paper is to provide existence of solutions to the following boundary-value problem

$$\begin{cases} v\cdot\nabla v + \nabla\pi = \nabla\cdot S(Dv) + f & \text{in } \Omega \\ \nabla\cdot v = 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \\ \lim\limits_{|x|\to\infty} v(x) = v^\infty & \end{cases} \tag{2.1}$$

where S is the Cauchy stress tensor with properties that will be specified further on (see (2.2)–(2.5)), whereas f and $v^\infty \in \mathbb{R}^2$ are given vector quantities.

In order to reach our objective, we need some basic considerations that we shall collect next.

We begin with the following result, that is classical in the framework of the Faedo-Galerkin approximation method, and whose proof can be found, for instance, in [**2**, Lemma VII.2.1]

LEMMA 2.1. *Let A be a connected subset of $\mathbb{R}^2$. Then we can find a sequence $\{\psi_k\} \subset \mathcal{D}(A)$ such that $(\nabla\psi_i, \nabla\psi_j) = \delta_{ij}$ and whose linear hull can approximate any function of $\mathcal{D}(\Omega)$ in the $C^1(\overline{A})$-norm.*

In order to manage the non homogeneous asymptotic condition when $|x| \to \infty$ we write the velocity as $v = u + b$ where b is an appropriate extension of v^∞. To this purpose we need the following classical extension result

LEMMA 2.2. *There exists a function $b \in C^\infty(\overline{\Omega})$ such that $\nabla\cdot b = 0$, $b \equiv 0$ in a neighborhood of $\partial\Omega$, $b \equiv v^\infty$ outside $B(0,\overline{R})$ and*

$$|(u\cdot\nabla b, u)| \le C_b \|\nabla u\|_p^2, \qquad \forall u \in \mathcal{D}(\Omega).$$

For the proof see, e.g., [**3**, Lemma 1].

Concerning the stress tensor S we make the following hypotheses:

$$S : \mathbb{R}^{2\times 2}_{sym} \longrightarrow \mathbb{R}^{2\times 2}_{sym} \text{ is continuous}$$

and there exist $\beta_1, \beta_2 > 0$ such that

$$|S(D)| \le \beta_2 |D|^{p-1} \qquad \forall D \in \mathbb{R}^{2\times 2}_{sym} \tag{2.2}$$

$$\beta_1 |D|^p \le S(D) : D \qquad \forall D \in \mathbb{R}^{2\times 2}_{sym} \tag{2.3}$$

$$(S(D) - S(C)) : (D - C) \ge 0 \qquad \forall C, D \in \mathbb{R}^{2\times 2}_{sym}. \tag{2.4}$$

For some results, especially concerning the regularity of solutions, we further require that for some constant $\alpha > 0$ it holds that

$$(2.5)\quad (S(D) - S(C)) : (D - C) \geq \alpha\,(1 + |C| + |D|)^{p-2}\,|D - C|^2 \quad \forall\, C, D \in \mathbb{R}^{2\times 2}_{sym}.$$

We remark that the typical model of singular viscosity given in (1.2) with $\mu_0 = 0$, namely, $S(D) = |D|^{p-2}D$, satisfies the above condition (see e.g. [**6**, Sec. 2]).

In the proof of our main theorem it is crucial to have some summability of the second derivatives of the velocity field. To this purpose we have the following interior regularity result.

Theorem 2.3. *Let A be an open subset on $\mathbb{R}^2$, $\frac{3}{2} < p < 2$, $f \in D_0^{-1,p'}(A) \cap L^{p'}(A)$ and S satisfying (2.2), (2.3) and (2.5). If $v \in D^{1,p}(A)$ with $\nabla \cdot v = 0$ satisfy the following identity*

$$(v \cdot \nabla\phi, v) = (S(Dv), D\phi) - \langle f, \phi\rangle, \qquad \forall\, \phi \in \mathcal{D}_0^{1,p}(A)$$

then, for any $B \subset\subset A$ it results that $v \in W^{2,s}(B)$ for any $s \in [1, 2)$ and $\|v\|_{2,s,B} \leq \Lambda$ with Λ depending on $s, |B|, \|f\|_{-1,p'}, \|\nabla u\|_{p,A}$ and $\delta := dist(B, \mathbb{R}^2 \setminus A)$. Moreover, Λ is a non-increasing function of δ.

Proof. The proof of this theorem is largely based on that of an analogous one given in [**3**, Theorem 2], which, in turn uses a procedure due to J. Naumann and J. Wolf [**6**, Sec. 2]. It is worth observing that our statement provides an explicit dependence of the $W^{2,s}$ norm on the parameters, which is crucial further on in showing Theorem 4.1. □

3. Approximating solutions

In the current section we shall confine our analysis to show existence of solutions to our original problem when the spatial domain is the *bounded* subdomain Ω_R of Ω (see Sec. 2), under suitable (fictitious) boundary conditions on $\partial B(0, R)$. Precisely, we have the following result that proves, in particular, a *uniform* (in R) bound for the L^p norm of the gradient of the velocity field. Later on, this will allow us to let $R \to \infty$ (along a sequence) and show that the corresponding solutions tend to a solution of the original problem in the whole Ω.

To this end, we shall distinguish the cases $v^\infty \neq 0$ (Proposition 3.1) and $v^\infty = 0$ (Proposition 3.2). The reason for this distinction relies on the fact that the proof of the latter is much simpler than the former and, in addition, holds under more general assumptions, so that we prefer to give it separately.

Proposition 3.1. *Let be $f \in D_0^{-1,p'}(\Omega) \cap L^{p'}(\Omega)$, $\frac{3}{2} < p < 2$, S satisfying (2.2), (2.3), (2.4), $v^\infty \in \mathbb{R} - \{0\}$, b and $\overline{R}$ as in Lemma 2.2, and $\lambda \geq 1$. Then, there exists a $\overline{\beta}_1$ depending on $\|f\|_{-1,p'}, \lambda, \|Db\|_p$ such that for any $R > \overline{R}$, any $\beta_1 \geq \overline{\beta}_1$ and any $\beta_2 \in [\beta_1, \lambda\beta_1]$ there is at least one solution $u \in \mathcal{D}_0^{1,p}(\Omega_R)$ to the following problem*

$$(3.1)\qquad ((u + b) \cdot \nabla\phi, (u + b)) = (S(D(u + b)), D\phi) - \langle f, \phi\rangle \qquad \forall\phi \in \mathcal{D}_0^{1,p}(\Omega_R).$$

Moreover, we can find a constant M depending on f, b, β_1, p and Ω, but not depending on R, such that $\|\nabla u\|_p \leq M$.

Proof. Let ψ_k be the sequence of Lemma 2.1 with Ω_R in place of A. We use the Faedo-Galerkin method, looking for an approximating solution of the form $u_m =$

$\sum_{k=1}^m c_{km}\psi_k$ where the coefficients c_{km} are unknown. The latter are determined by solving the following system of nonlinear algebraic equations in $\mathbb{R}^m$

$$((u_m+b)\cdot\nabla\psi_k,(u_m+b)) = (S(D(u_m+b)),D\psi_k) - \langle f,\psi_k\rangle,\ k=1,\dots,m. \tag{3.2}$$

If we multiply the k-th equation of the system by c_{km} and we sum over k, we get

$$((u_m+b)\cdot\nabla u_m,(u_m+b)) = (S(D(u_m+b)),Du_m) - \langle f,u_m\rangle. \tag{3.3}$$

Since $\nabla\cdot b=0$ and $u_m\in\mathcal{D}(\Omega)$ we show at once

$$(u_m\cdot\nabla u_m,u_m) = (b\cdot\nabla u_m,u_m) = 0.$$

Let us next observe that, by (2.3)

$$(S(D(u_m+b)),D(u_m+b)) \geq \beta_1\|D(u_m+b)\|_p^p \geq \frac{\beta_1}{2}\|Du_m\|_p^p - \beta_1\|Db\|_p^p$$

By the Hölder inequality, (2.2) and the Young inequality, we have

$$\begin{aligned}|(S(D(u_m+b),Db)| &\leq \beta_2\|D(u_m+b)\|_p^{p-1}\|Db\|_p \leq \frac{\beta_1}{16}\|D(u_m+b)\|_p^p\\ &+\frac{\beta_2^p}{p}\left(\frac{16}{p'\beta_1}\right)^{p-1}\|Db\|_p^p \leq \frac{\beta_1}{8}\|Du_m\|_p^p + \left(\frac{\beta_1}{8}+\frac{\beta_2^p}{p}\left(\frac{16}{p'\beta_1}\right)^{p-1}\right)\|Db\|_p^p\end{aligned}$$

Concerning the convective term, we extend the function u_m to 0 in $\Omega\setminus\Omega_R$, and we apply the Korn inequality in the whole Ω with constant K_p (independent of R). Using Lemma 2.2 we can infer

$$|(u_m\cdot\nabla u_m,b)| \leq C_b\|\nabla u_m\|_p^2 \leq C_bK_p^2\|Du_m\|_p^2.$$

Moreover, by Hölder, Korn and Young inequalities

$$|(b\cdot\nabla u_m,b)| \leq K_p\|Du_m\|_p\|b\|_{2p'}^2 \leq \frac{\beta_1}{8}\|Du_m\|_p^p + \frac{1}{p'}\left(\frac{8K_p^p}{p\beta_1}\right)^{\frac{1}{p-1}}\|b\|_{2p'}^{2p'},$$

$$|\langle f,u_m\rangle| \leq K_p\|f\|_{-1,p'}\|Du_m\|_p \leq \frac{\beta_1}{8}\|Du_m\|_p^p + \frac{1}{p'}\left(\frac{8K_p^p}{p\beta_1}\right)^{\frac{1}{p-1}}\|f\|_{-1,p'}^{p'}.$$

By collecting all the above estimates we deduce

$$\begin{aligned}&(S(D(u_m+b)),Du_m) - \langle f,u_m\rangle - ((u_m+b)\cdot\nabla u_m,b)\\ &\geq \frac{\beta_1}{8}\|Du_m\|_p^p - C_bK_p^2\|Du_m\|_p^2 - \left(\frac{9\beta_1}{8}+\frac{\beta_2^p}{p}\left(\frac{16}{p'\beta_1}\right)^{p-1}\right)\|Db\|_p^p\\ &\qquad -\frac{1}{p'}\left(\frac{8K_p^p}{p\beta_1}\right)^{\frac{1}{p-1}}\left(\|f\|_{-1,p'}^{p'}+\|b\|_{2p'}^{2p'}\right).\end{aligned} \tag{3.4}$$

In order to find a solution of the system (3.2) we define a function $P:\mathbb{R}^m\longrightarrow\mathbb{R}^m$ with components $P_k,\ k=1,\dots,m$, defined by

$$P_k(\xi) = (S(D(u_m+b)),D\psi_k) - \langle f,\psi_k\rangle - ((u_m+b)\cdot\nabla\psi_k,u_m+b)$$

with $u_m=\sum_{k=1}^m\xi_k\psi_k$. By inequality (3.4) we thus have that, setting

$$X=\|Du_m\|_p,\quad A=\frac{\beta_1}{8},\quad B=C_bK_p^2,$$

$$C=\left(\beta_1+\frac{\beta_1}{8}+\frac{\beta_2^p}{p}\left(\frac{16}{p'\beta_1}\right)^{p-1}\right)\|Db\|_p^p + \frac{1}{p'}\left(\frac{8K_p^p}{p\beta_1}\right)^{\frac{1}{p-1}}\left(\|f\|_{D_0^{-1,p'}}^{p'}+\|b\|_{2p'}^{2p'}\right)$$

$$P(\xi)\cdot\xi \geq AX^p - BX^2 - C =: \varphi(X).$$

If we compute the derivative of φ,

$$\varphi'(X) = pAX^{p-1} - 2BX = X^{p-1}(pA - 2BX^{2-p}),$$

we may find that in $\overline{X} = \left(\frac{pA}{2B}\right)^{\frac{1}{2-p}}$ the function φ achieves its maximum value

$$\begin{aligned}\varphi(\overline{X}) &= A\left(\frac{pA}{2B}\right)^{\frac{p}{2-p}} - B\left(\frac{pA}{2B}\right)^{\frac{2}{2-p}} - C = \left(\frac{p}{2B}\right)^{\frac{p}{2-p}} A^{\frac{2}{2-p}} \\ &-B\left(\frac{p}{2B}\right)^{\frac{2}{2-p}} A^{\frac{2}{2-p}} - C = A^{\frac{2}{2-p}}\left(\left(\frac{p}{2B}\right)^{\frac{p}{2-p}} - B\left(\frac{p}{2B}\right)^{\frac{2}{2-p}}\right) - C \\ &= A^{\frac{2}{2-p}}\left(\frac{p}{2B}\right)^{\frac{p}{2-p}}\left(1 - B\frac{p}{2B}\right) - C = A^{\frac{2}{2-p}}\left(\frac{p}{2B}\right)^{\frac{p}{2-p}}\left(1 - \frac{p}{2}\right) - C.\end{aligned}$$

The value just obtained depends on many parameters. Let us fix all of them with the exception of β_1 and β_2. We thus conclude

$$\varphi(\overline{X}) = a\beta_1^{\frac{2}{2-p}} - b\beta_1 - c\left(\frac{\beta_2}{\beta_1}\right)^p \beta_1 - d\beta_1^{\frac{1}{1-p}} =: g(\beta_1,\beta_2).$$

It is now necessary to observe that the coefficients β_1 and β_2 are not independent. Indeed, by (2.2) and (2.3) we get that $\beta_1 \leq \beta_2$. Moreover, in the simplest case of the power-law model given in (1.2) with $\mu_0 = 0$, where $S(D) = \beta|D|^{p-2}D$, we have $\beta_1 = \beta_2$. To avoid unnecessary complication in the formulation of the result, we will suppose that the ratio $\frac{\beta_2}{\beta_1}$ is bounded from above by a fixed constant λ. Hence

$$g(\beta_1,\beta_2) \geq a\beta_1^{\frac{2}{2-p}} - \widetilde{b}\beta_1 - d\beta_1^{\frac{1}{1-p}} := h(\beta_1).$$

It is immediately checked that $h(\beta_1) \to \infty$ if $\beta_1 \to \infty$. As a result, there exists $\overline{\beta}_1$ such that $\varphi(\overline{X}) > 0$. However, by a straightforward calculation we easily show that

$$|||\xi||| := \|\sum_{k=1}^{m} \xi_k D\psi_k\|_p$$

is a norm in $\mathbb{R}^m$, and consequently we obtain that $P(\xi)\cdot\xi \geq 0$ on the shell $|||\xi||| = \overline{X}$. This information, along with the Brouwer fixed point theorem (see [**5**, Lemma I.4.3]), allows us to deduce that there exists $\overline{\xi}$ such that $P(\overline{\xi}) = 0$ and $|||\overline{\xi}||| \leq \overline{X}$. It then follows that the system (3.2) has a solution $u_m = \sum_{k=1}^m c_{km}\psi_k$ where $c_{km} = \overline{\xi}_k$ and, in addition,

$$\|Du_m\|_p \leq \overline{X}. \tag{3.5}$$

By (2.2) and (3.5) we also have that

$$\|S(D(u_m+b))\|_{p'} \leq \beta_2\left(\|Du_m\|_p^{p-1} + \|Db\|_p^{p-1}\right) \leq \beta_2\left(\overline{X}^{p-1} + \|Db\|_p^{p-1}\right)$$

hence the sequence $\{S(D(u_m+b))\}$ is bounded in $L^{p'}(\Omega_R)$. Extending to 0 outside Ω_R the functions u_m, by Sobolev and Korn inequalities, we get also that $\|u_m\|_{p^*} \leq c\overline{X}$. Notice that the quantity $c\overline{X}$ is independent of m and R. By the Poincaré inequality we get that the sequence $\{u_m\}$ is also bounded in $W_0^{1,p}(\Omega_R)$. We remark that in this case the bound depends on R but this is not relevant in the present proposition since the domain Ω_R is fixed. This allows us to apply the Rellich-Kodrachov embedding theorem to deduce the compactness of the sequence in $L^q(\Omega_R)$ for any $q \in [1, p^*)$. Let us recall that, since $p > \frac{3}{2}$, we have that $2p' < p^*$.

All the previous considerations are sufficient to extract from $\{u_m\}$ a subsequence (not relabeled) and two funcions $u \in D_0^{1,p}(\Omega_R) \cap L^{p^*}(\Omega_R)$ and $G \in L^{p'}(\Omega_R)$ such that

$$\begin{aligned} &u_m \rightharpoonup u \quad \text{weakly in } D_0^{1,p}(\Omega_R), \quad u_m \rightharpoonup u \quad \text{weakly in } L^{p^*}(\Omega_R), \\ &u_m \to u \quad \text{strongly in } L^{2p'}(\Omega_R),\ S(D(u_m+b)) \rightharpoonup G \quad \text{weakly in } L^{p'}(\Omega_R). \end{aligned} \tag{3.6}$$

Moreover, since $\mathcal{D}_0^{1,p}(\Omega_R)$ is closed in $D_0^{1,p}(\Omega_R)$, actually $u \in \mathcal{D}_0^{1,p}(\Omega_R)$. Now we are ready to pass to the limit $m \to \infty$ in the system (3.2). To this end, we observe that, by $(3.6)_2$ it follows at once that

$$((u_m+b)\cdot\nabla\psi_k, (u_m+b)) \to ((u+b)\cdot\nabla\psi_k, (u+b)) \qquad \forall\, k \in \mathbb{N}, \tag{3.7}$$

$$(S(D(u_m+b)), D\psi_k) \to (G, D\psi_k) \qquad \forall\, k \in \mathbb{N}. \tag{3.8}$$

By (3.7), (3.8) and the density argument of Lemma 2.1 it is easy to prove that

$$((u+b)\cdot\nabla\phi, (u+b)) = (G, D\phi) - \langle f, \phi\rangle, \quad \forall\, \phi \in \mathcal{D}(\Omega_R).$$

Finally, by the definition of the space $\mathcal{D}_0^{1,p}(\Omega_R)$ and the fact that by $(3.6)_2$ $u \in L^{2p'}(\Omega_R)$, with the help of a continuity argument, one shows that

$$((u+b)\cdot\nabla\phi, (u+b)) = (G, D\phi) - \langle f, \phi\rangle, \quad \forall\, \phi \in \mathcal{D}_0^{1,p}(\Omega_R). \tag{3.9}$$

In order to replace G by $S(D(u+b))$ in the previous identity we will use the Minty-Browder trick. We can set $\phi = u_m \in \mathcal{D}_0^{1,p}(\Omega_R)$ in equation (3.9) obtaining

$$((u+b)\cdot\nabla u_m, (u+b)) = (G, Du_m) - \langle f, u_m\rangle, \quad \forall m \in \mathbb{N}. \tag{3.10}$$

Once again, since $u \in L^{2p'}(\Omega_R)$, by $(3.6)_1$, we get

$$((u+b)\cdot\nabla u_m, (u+b)) \to ((u+b)\cdot\nabla u, (u+b))$$

Finally, since $G \in L^{p'}(\Omega_R)$ and $f \in L^{p'}(\Omega_R)$, by $(3.6)_1$ we also show

$$(G, Du_m) \to (G, Du)\,, \qquad \langle f, u_m\rangle \to \langle f, u\rangle.$$

Passing to the limit $m \to \infty$ in identity (3.10) gives us

$$((u+b)\cdot\nabla u, (u+b)) + \langle f, u\rangle = (G, Du)\,. \tag{3.11}$$

Going back to equation (3.3) we shall now consider the convergence of the term $((u_m+b)\cdot\nabla u_m, (u_m+b))$. By $(3.6)_1$ and the strong convergence $(3.6)_2$, we have

$$(u_m+b)\otimes(u_m+b) \to (u+b)\otimes(u+b) \quad \text{strongly in } L^{p'}(\Omega_R)$$

$$((u_m+b)\cdot\nabla u_m, (u_m+b)) \to ((u+b)\cdot\nabla u, (u+b))\,. \tag{3.12}$$

Hence, letting $m \to \infty$ in (3.3) and taking into account (3.12), (3.11) we obtain

$$\lim_{m\to\infty} (S(D(u_m+b)), Du_m) = ((u+b)\cdot\nabla u, (u+b)) + \langle f, u\rangle = (G, Du)\,. \tag{3.13}$$

To bring the Minty-Browder trick to its conclusion, we take an arbitrary function $\phi \in \mathcal{D}_0^{1,p}(\Omega_R)$ and $\epsilon > 0$. By the monotonicity of S (2.4) we have that

$$(S(D(u_m+b)) - S(D(u-\epsilon\phi+b)), Du_m - D(u-\epsilon\phi)) \geq 0.$$

By (3.13), $(3.6)_2$ and $(3.6)_1$ we can pass to the limit $m \to \infty$ to achieve

$$(G - S(D(u-\epsilon\phi+b)), \epsilon D\phi) \geq 0. \tag{3.14}$$

Dividing both sides of (3.14) by ϵ we can pass to the limit as $\epsilon \to 0$ with the aid of (2.2), the continuity of S and the Lebesgue dominated convergence to get

$$(G - S(D(u+b)), D\phi) \geq 0 \qquad \forall\, \phi \in \mathcal{D}_0^{1,p}(\Omega_R).$$

Changing ϕ with $-\phi$ in the above inequality we get $(G, D\phi) = (S(D(u+b)), D\phi)$ and substituting this relation in (3.9) we conclude the proof. □

In the case $v^\infty = 0$, we do not need to introduce the extension field b and we have the following result without any restriction on the coefficients β_i, $i = 1, 2$.

Proposition 3.2. *Let* $\frac{3}{2} < p < 2$, $f \in D_0^{-1,p'}(\Omega) \cap L^{p'}(\Omega)$, *$S$ satisfy* (2.2), (2.3), (2.4) *and* $\widetilde{R} > 0$ *such that* $(\mathbb{R}^2 \setminus \Omega) \subset B(0, \widetilde{R})$. *Then, for any* $R \geq \widetilde{R}$, *there exists a solution* $u \in \mathcal{D}_0^{1,p}(\Omega_R)$ *of the following problem*

$$(u \cdot \nabla\phi, u) = (S(Du), D\phi) - \langle f, \phi \rangle \qquad \forall \phi \in \mathcal{D}_0^{1,p}(\Omega_R).$$

Moreover there exists a constant M depending on f, β_1, p and Ω, but not depending on R, such that $\|\nabla u\|_p \leq M$.

Proof. We use the same notations of Proposition 3.1. The new system is

$$(u_m \cdot \nabla\psi_k, u_m) = (S(Du_m), D\psi_k) - \langle f, \psi_k \rangle, \quad k = 1, \dots, m \tag{3.15}$$

By (2.3) we get

$$(S(Du_m), Du_m) \geq \beta_1 \|Du_m\|_p^p.$$

By the Hölder, Korn and Young inequalities we have

$$|\langle f, u_m \rangle| \leq K_p \|f\|_{D_0^{-1,p'}} \|Du_m\|_p \leq \frac{\beta_1}{2}\|Du_m\|_p^p + \frac{1}{p'}\left(\frac{2K_p^p}{p\beta_1}\right)^{\frac{1}{p-1}} \|f\|^{p'}_{D_0^{-1,p'}}.$$

By the above estimates we obtain that

$$(S(Du_m), Du_m) - \langle f, u_m \rangle \geq \frac{\beta_1}{2}\|Du_m\|_p^p - \frac{1}{p'}\left(\frac{2K_p^p}{p\beta_1}\right)^{\frac{1}{p-1}} \|f\|^{p'}_{D_0^{-1,p'}}.$$

The function $P : \mathbb{R}^m \to \mathbb{R}^m$ becomes

$$P_k(\xi) = (S(Du_m), D\psi_k) - \langle f, \psi_k \rangle, \qquad u_m = \sum_{j=1}^{m} \xi_j \psi_j.$$

Setting

$$X = \|Du_m\|_p, \quad A = \frac{\beta_1}{2}, \quad C = \frac{1}{p'}\left(\frac{2K_p^p}{p\beta_1}\right)^{\frac{1}{p-1}} \|f\|^{p'}_{D_0^{-1,p'}}$$

we have that

$$P(\xi)\cdot\xi \geq AX^p - C \geq 0 \ \text{ if } \ |||\xi||| = \left(\frac{C}{A}\right)^{\frac{1}{p}} = \left(\frac{2}{p'\beta_1}\left(\frac{2K_p^p}{p\beta_1}\right)^{\frac{1}{p-1}} \|f\|^{p'}_{D_0^{-1,p'}}\right)^{\frac{1}{p}} =: \overline{X}.$$

Hence we get that the system (3.15) has a solution u_m with $\|Du_m\|_p \leq \overline{X}$. We remark that no restrictions on the viscosity coefficients are needed to find a solution. The remaining part of the proof is like the one of Proposition 3.1. □

4. Main result

By employing the finding established in the previous section, we are now in a position to show the main result of our paper. To this end, we shall again distinguish the cases $v^\infty \neq 0$ (Theorem 4.1) and $v^\infty = 0$ (Theorem 4.2).

THEOREM 4.1. *Let Ω be an exterior domain of $\mathbb{R}^2$ with Lipschitz boundary and $\frac{3}{2} < p < 2$. Then, for any $f \in D_0^{-1,p'}(\Omega) \cap L^{p'}(\Omega), \lambda \geq 1, v^\infty \in \mathbb{R}^2 - \{0\}$, there exists $\overline{\beta}_1$, such that for any S satisfying conditions* (2.2), (2.3) *and* (2.5) *with $\beta_1 \geq \overline{\beta}_1$ and $\beta_2 \in [\beta_1, \lambda\beta_1]$, the system* (2.1) *admits a solution (v, π) in the sense of distributions. Moreover $v \in \mathcal{D}^{1,p}(\Omega) \cap W^{2,s}_{loc}(\Omega)$ for any $s \in [1, 2)$ and $\pi \in L^{p'}_{loc}(\Omega)$.*

PROOF. Let b and $\overline{R}$ be as in Lemma 2.2, and for any $n \in \mathbb{N}, n \geq \overline{R}$, let be u^n the solution determined in Proposition 3.1 with $\Omega_R \equiv \Omega_n$. We extend u^n to 0 outside Ω_n, thus obtaining a sequence $\{u^n\}$ of functions belonging to $\mathcal{D}^{1,p}_0(\Omega)$. By Proposition 3.1 and the Sobolev inequality it results that

$$\|u^n\|_{p^*} \leq c\|\nabla u^n\|_p \leq cM \tag{4.1}$$

with M independent of n. It is worth remarking that the constant c does not depend on n, since it represents the Sobolev constant in the whole $\mathbb{R}^2$ and not in Ω_n. Hence we can select a subsequence (not relabeled) and find a function $u \in \mathcal{D}^{1,p}_0(\Omega)$ such that

$$u^n \rightharpoonup u \quad \text{weakly in } \mathcal{D}^{1,p}_0(\Omega), \qquad u^n \rightharpoonup u \quad \text{weakly in } L^{p^*}(\Omega). \tag{4.2}$$

In order to show that $v = u + b$ is a solution of the system (2.1), we fix a function $\phi \in \mathcal{D}(\Omega)$ and we test the equations with such a ϕ. Since ϕ has compact support, there exists a bounded open set $K \subset \Omega$ and $\overline{n}$ such that

$$\text{spt}(\phi) \subset K \subset\subset \Omega_n \qquad \forall n \geq \overline{n}.$$

Hence $\phi \in \mathcal{D}^{1,p}_0(\Omega_n)$ and it can be used as a test function in equation (3.1) to get

$$((u^n + b) \cdot \nabla\phi, (u^n + b)) = (S(D(u^n + b)), D\phi) - \langle f, \phi\rangle \qquad \forall n \geq \overline{n}. \tag{4.3}$$

Let us examine the convective term. By the bounds (4.1) and the fact that K is a bounded set we have $\|u^n\|_{1,p,K} \leq c(K)$. Since $2p' < p^*$, the Rellich-Kondrachov theorem ensures that there exists a subsequence (not relabeled) such that

$$u^n \to w \quad \text{strongly in } L^{2p'}(K) \tag{4.4}$$

for a suitable $w \in L^{2p'}(K)$. By the weak convergence (4.2) we have that $\{u^n\}$ converges weakly to u in $L^{2p'}(K)$ and hence it has to be $w = u$ a.e. in K. By the strong convergence (4.4) we immediately have

$$((u^n + b) \cdot \nabla\phi, (u^n + b)) \to ((u + b) \cdot \nabla\phi, (u + b)). \tag{4.5}$$

Before going further in the proof, let us make a brief remark. At this point it is impossible to follow the scheme used in Proposition 3.1 which makes use of the Minty-Browder trick. This is due to the lack of convergence in the analog of equation (3.12). Indeed, as the domain is unbounded, we cannot rely on the same compactness argument to achieve at least one strong convergence in the triple product. We must change our strategy by appealing to interior regularity established in Theorem 2.3. Let us then apply Theorem 2.3 with $s > \frac{2p}{p+2}, A = \Omega$, $B = K$ and $v = u^n + b$ to get that

$$\|u^n + b\|_{2,s,K} \leq \widetilde{\Lambda} \tag{4.6}$$

where $\widetilde{\Lambda}$ does not depend on n since $\|\nabla u^n\|_{p,\Omega}$ is uniformly bounded by (4.1). By the Rellich-Kondrachov theorem we can extract a subsequence (not relabeled) converging strongly in $W^{1,p}(K)$. By the uniqueness of the limit and (4.2) we have that $\{\nabla(u^n + b)\}$ converges strongly to $\nabla(u + b)$ in $L^p(K)$ and, up to a further subsequence,

$$\nabla(u^n + b) \to \nabla(u + b) \qquad \text{a.e. in } K. \tag{4.7}$$

By (2.2) and estimate (4.1), we have that $\|S(D(u^n + b))\|_{p'} \leq \widetilde{M}$ with $\widetilde{M}$ not depending on n. This bound, together with the almost everywhere convergence (4.7) and the continuity of S ensures, by using [**5**, Lemma I.1.3], that $\{S(D(u^n+b))\}$ converges weakly to $S(D(u + b))$ in $L^{p'}(K)$. Since $D\phi \in L^p(K)$ we get that

$$(S(D(u^n + b)), D\phi) \to (S(D(u + b)), D\phi). \tag{4.8}$$

To conclude, we remark that the set K and all the subsequences extracted, depend on ϕ but this is not the case for u which is determined only by the global weak convergences (4.2). Hence, by (4.3), (4.5) and (4.8), we have

$$((u + b) \cdot \nabla\phi, (u + b)) = (S(D(u + b)), D\phi) - \langle f, \phi\rangle \qquad \forall\, \phi \in \mathcal{D}(\Omega). \tag{4.9}$$

It remains to prove $(2.1)_4$. Before doing this, we need again to consider the interior regularity of u. By Theorem 2.3 it does not follow directly that $u + b \in W^{2,s}_{loc}(\Omega)$, since $u + b$ satisfy the identity (4.9) for any smooth test function and not for any function in $\mathcal{D}^{1,p}_0(\Omega)$ as required in the quoted theorem. We cannot use a continuity argument here, since the convective term will not fit, due to the unboundedness of Ω. Nevertheless we can achieve our goal going back to the sequence $\{u^n\}$. Let us fix an arbitrary bounded open set $B \subset\subset \Omega$. By (4.6) we get that there exists $w \in W^{2,s}(B)$ and a subsequence weakly converging to it in $W^{2,s}(B)$. By the choice made for s we have also the weak convergence (up to another subsequence) in $L^{p*}(B)$ hence, by (4.2), $u + b = w$ a.e. in B. It follows that $u + b \in W^{2,s}_{loc}(\Omega)$ and the bound (4.6) holds true for $u + b$ too. To be more precise, let us consider a ball of fixed diameter B which lies outside $B(0, \overline{R})$. Since b is constant on such kind of ball, we get that $\|u\|_{2,s,B} \leq \widetilde{\Lambda}$ and the bound is uniform for all balls in this situation. Since $W^{2,s}(B)$ is embedded in $C^{0,\lambda}(\overline{B})$ with $0 < \lambda < 2 - \frac{2}{s}$ (see [**1**, Lemma 5.17]), we get

$$\|u\|_{C^{0,\lambda}(\overline{B})} \leq \Lambda_1 \tag{4.10}$$

where Λ_1 is uniform with respect to any ball B of the same fixed diameter, and lying outside $B(0, \overline{R})$. We are now able to prove the uniform decay $(2.1)_4$. Since $b(x) = v^\infty$ for any x in $\Omega \setminus B(0, \overline{R})$ it will suffice to prove that $\lim_{|x|\to\infty} u(x) = 0$. By contradiction, let us suppose that there exists $\epsilon > 0$ and a sequence of points x_j such that $\lim_{j\to\infty} |x_j| = +\infty$ and $|u(x_j)| > \epsilon$ for any $j \in \mathbb{N}$. By the Hölder continuity of u and the estimate (4.10), we have that

$$|u(x)| \geq |u(x_j)| - \Lambda_1|x_j - x|^\lambda \geq \frac{\epsilon}{2} \qquad \text{if } |x_j - x| \leq \left(\frac{\epsilon}{2\Lambda_1}\right)^{\frac{1}{\lambda}} := r(\epsilon).$$

It is not restrictive to suppose that $|x_i - x_j| > 2r(\epsilon)$ if $i \neq j$, hence

$$\|u\|^{p^*}_{p^*} \geq \sum_{j=0}^{\infty} \int_{B(x_j, r(\epsilon))} \left(\frac{\epsilon}{2}\right)^{p^*} dx = +\infty$$

that gives the desired contradiction.

It remains only to determine the pressure field. Let $n_0 \in \mathbb{N}$ be such that $(\mathbb{R}^2 \setminus \Omega) \subset B(0, n_0)$. We can define a functional $F_{n_0} : D_0^{1,p}(\Omega_{n_0}) \longrightarrow \mathbb{R}$ in the following way

$$F_{n_0}(\psi) = (S(Du + b), D\psi)_{\Omega_{n_0}} - ((u + b) \cdot \nabla\psi, (u + b))_{\Omega_{n_0}} - \langle f, \psi\rangle_{\Omega_{n_0}}.$$

F_{n_0} is linear and, since $2p' < p^*$, it is bounded. Moreover, observing that Ω_{n_0} is bounded and applying a density argument to (4.9), it vanishes identically on $\mathcal{D}_0^{1,p}(\Omega_{n_0})$. Using [**2**, Theorem III.5.3] we can find a function $\pi_{n_0} \in L^{p'}(\Omega_{n_0})$, determined up to a costant, such that $F_{n_0}(\psi) = (\pi_{n_0}, \nabla\cdot\psi)_{\Omega_{n_0}}$ for any $\psi \in D_0^{1,p}(\Omega_{n_0})$. By means of an iterative argument we can find, for any $n \in \mathbb{N},\ n > n_0$ a function $\pi_n \in L^{p'}(\Omega_n)$ such that, for any $\psi \in D_0^{1,p}(\Omega_n)$

$$(S(Du + b), D\psi)_{\Omega_n} - ((u + b) \cdot \nabla\psi, (u + b))_{\Omega_n} - \langle f, \psi\rangle_{\Omega_n} = (\pi_n, \nabla \cdot \psi)_{\Omega_n}$$

and we can choose the constant in such a way that $\pi_n = \pi_{n-1}$ in Ω_{n-1}. Defining $\pi(x) = \pi_n(x)$ if $x \in \Omega_n$ we obtain that $\pi \in L^{p'}_{loc}(\Omega)$ and

$$(S(Du + b), D\psi) - ((u + b) \cdot \nabla\psi, (u + b)) - \langle f, \psi\rangle = (\pi, \nabla \cdot \psi) \quad \forall\, \psi \in C_0^\infty(\Omega).$$

Setting $v = u + b$ we conclude the proof. □

We shall next consider the case $v^\infty = 0$.

THEOREM 4.2. *Let Ω be an exterior domain in $\mathbb{R}^2$ with Lipschitz boundary and $\frac{3}{2} < p < 2$. Then, for any $f \in D_0^{-1,p'}(\Omega) \cap L^{p'}(\Omega)$ and S satisfying conditions (2.2), (2.3) and (2.5), the system (2.1) with $v^\infty = 0$ admits a solution (v, π) in the sense of distributions. Moreover $v \in \mathcal{D}_0^{1,p}(\Omega) \cap W^{2,s}_{loc}(\Omega)$ for any $s \in [1, 2)$ and $\pi \in L^{p'}_{loc}(\Omega)$.*

PROOF. The proof is entirely analogous to that of Theorem 4.1, by setting $b = 0$ and using Proposition 3.2 instead of Proposition 3.1. □

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UNIVERSITY OF PITTSBURGH - BENEDUM ENGINEERING HALL, PITTSBURGH, PA 15621, USA
E-mail address: `galdi@pitt.edu`

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI PISA - VIA BUONARROTI 1/C, 56127 PISA, ITALY
E-mail address: `grisanti@dma.unipi.it`