

On the Hilbert quasi-polynomials of non-standard graded rings

Let $R := \mathbb{K}[x_1, \dots, x_k]$, I be an homogeneous ideal of R . The numerical function $H_{R/I} : \mathbb{N} \rightarrow \mathbb{N}$ defined by $H_{R/I}(d) := \dim_{\mathbb{K}}((R/I)_d)$ is called *Hilbert function of R/I* . It is eventually a polynomial, HP_R .

$R = \mathbb{Q}[x_1, \dots, x_5]$, **graded by $[1, 1, 1, 1, 1]$ (standard grading)**

$$HP_R = 1/24x^4 + 5/12x^3 + 35/24x^2 + 25/12x + 1$$

R **graded by $W = [1, 2, 3, 4, 6]$. $HP_R^W(n) = P_{n \bmod 12}(n)$**

$$\begin{aligned} HP_R^W = [& P_0(x) = 1/3456x^4 + 1/108x^3 + 5/48x^2 + 1/2x + 1, \\ & P_1(x) = 1/3456x^4 + 1/108x^3 + 19/192x^2 + 43/108x + 1705/3456, \\ & P_2(x) = 1/3456x^4 + 1/108x^3 + 5/48x^2 + 25/54x + 125/216, \\ & P_3(x) = 1/3456x^4 + 1/108x^3 + 19/192x^2 + 5/12x + 75/128, \\ & P_4(x) = 1/3456x^4 + 1/108x^3 + 5/48x^2 + 13/27x + 20/27, \\ & P_5(x) = 1/3456x^4 + 1/108x^3 + 19/192x^2 + 41/108x + 1001/3456, \\ & P_6(x) = 1/3456x^4 + 1/108x^3 + 5/48x^2 + 1/2x + 7/8, \\ & P_7(x) = 1/3456x^4 + 1/108x^3 + 19/192x^2 + 43/108x + 1705/3456, \\ & P_8(x) = 1/3456x^4 + 1/108x^3 + 5/48x^2 + 25/54x + 19/27, \\ & P_9(x) = 1/3456x^4 + 1/108x^3 + 19/192x^2 + 5/12x + 75/128, \\ & P_{10}(x) = 1/3456x^4 + 1/108x^3 + 5/48x^2 + 13/27x + 133/216, \\ & P_{11}(x) = 1/3456x^4 + 1/108x^3 + 19/192x^2 + 41/108x + 1001/3456] \end{aligned}$$

- $d = lcm(1, 2, 3, 4, 6) = 12$, $\delta_3 = \delta_4 = 1$, $\delta_0 = 12$
- $\delta_2 = 2 = lcm(2)$ ($\{2, 4, 6\}$)
- $\delta_1 = 6 = lcm(2, 2, 3, 2)$ ($\{2, 4\}, \{2, 6\}, \{3, 6\}, \{4, 6\}$)