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On the computation of preliminary orbits for Earth satellites with radar observations

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ABSTRACT

We introduce a new method to perform preliminary orbit determination for satellites on low Earth orbits (LEO). This method works with tracks of radar observations: each track is composed by $n \geq 4$ topocentric position vectors per pass of the satellite, taken at very short time intervals. We assume very accurate values for the range ρ, while the angular positions (i.e. the line of sight, given by the pointing of the antenna) are less accurate. We wish to correct the errors in the angular positions already in the computation of a preliminary orbit. With the information contained in a pair of radar tracks, using the laws of the two-body dynamics, we can write eight equations in eight unknowns. The unknowns are the components of the topocentric velocity orthogonal to the line of sight at the two mean epochs of the tracks, and the corrections $\boldsymbol{\Delta}$ to be applied to the angular positions. We take advantage of the fact that the components of $\boldsymbol{\Delta}$ are typically small. We show the results of some tests, performed with simulated observations, and compare this method with Gibbs' and the Keplerian integrals methods.

Key words: methods: analytical – methods: numerical – surveys – celestial mechanics.

1 INTRODUCTION

We investigate the preliminary orbit determination problem for a satellite of the Earth using radar observations collected by an instrument with given technical specifications, and with a fixed observation scheduling. Assume we collect the following data for the observed object:

$$(t_j, \rho_j, \alpha_j, \delta_j), \qquad j = 1 \ldots 4, \tag{1}$$

where the triples $(\rho_j, \alpha_j, \delta_j)$ represent topocentric spherical coordinates of the object at epochs t_j of the observations. Typically α_j, δ_j are the values of right ascension and declination. We shall call a *radar track* the set of observations in (1).

The following assumptions will be made on the data composing the tracks. The time difference $t_{j+1} - t_j$ between consecutive observations is $\Delta t = 10$ s. The range data ρ_j are very precise: the statistical error in the range is given by its rms σ_ρ, which is 10 m. On the other hand we assume that the angles α_j, δ_j are not precisely determined: their rms σ_α, σ_δ are supposed to be 0.2 degrees.

Given a radar track we can compute by interpolation the following data:

$$(\bar{t}, \bar{\alpha}, \bar{\delta}, \rho, \dot{\rho}, \ddot{\rho}). \tag{2}$$

Here $\bar{t}$, $\bar{\alpha}$ and $\bar{\delta}$ are the mean values of the epoch and the angles, and ρ, $\dot{\rho}$, $\ddot{\rho}$ are the values of a function $\rho(t)$ and its derivatives at $t = \bar{t}$, where $\rho(t)$ is given by a quadratic fit with the (t_j, ρ_j) data. For low Earth orbits (LEO) these assumptions imply that the interpolated values of $\dot{\alpha}$, $\dot{\delta}$ are very badly accurate, to the point that their value can be of the same order of the errors, therefore they are practically undetermined.

We use topocentric spherical coordinates and velocities

$$(\rho, \alpha, \delta, \dot{\rho}, \dot{\alpha}, \dot{\delta})$$

for the orbit that we want to compute. By the above considerations, given a vector-like equation (2) obtained by a radar track, we can keep the values of ρ, $\dot{\rho}$ and consider as unknowns the quantities $(\Delta\alpha, \Delta\delta, \dot{\alpha}, \dot{\delta})$, with

$$\alpha = \bar{\alpha} + \Delta\alpha, \quad \delta = \bar{\delta} + \Delta\delta,$$

where $\Delta\alpha$, $\Delta\delta$ are small deviations from the mean values $\bar{\alpha}$, $\bar{\delta}$.

To search for the values of the unknowns we need to use additional data: we can try to use the data of two radar tracks, together with a dynamical model, to compute one or more preliminary orbits. This is a *linkage problem*, see Milani & Gronchi (2010).

In this paper, we propose a new method for the linkage, which takes advantage of the smallness of $\Delta\alpha$, $\Delta\delta$, that we call *infinitesimal angles*. We write the equations for preliminary orbits by using the five algebraic integrals of Kepler's problem, Lambert's equation for elliptic motion (see Section A) and the projection of the equations of motion along the line of sight.

Moreover, we perform some tests to compare this method with Gibbs' method, using only one radar track, and with the

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Keplerian integrals (KI) method, which solves a linkage problem using $(\bar{\alpha}, \bar{\delta}, \rho, \dot{\rho})$ at two mean epochs (see Taff & Hall 1977; Farnocchia et al. 2010; Gronchi, Farnocchia & Dimare 2011).

The paper is organized as follows. First we introduce some notation and recall the basic results on Kepler's motion which are relevant for this work (see Sections 2–4). The equations for the linkage problem are presented in Section 5, and in Sections 6, 7 we show two different ways to compute the solutions. In Section 9, we present the results of some numerical tests, including a comparison with the already known methods recalled in Section 8. Finally, in Section A, we recall the proof of Lambert's theorem for elliptic orbits and give a geometrical interpretation of the results. Moreover, we show a method to correct the observations of a radar track so that they correspond to points in the same plane.

2 THE EQUATIONS OF MOTION

Let us denote by $\boldsymbol{e}^\rho$ the unit vector corresponding to the line of sight, and by $\boldsymbol{q}$ the geocentric position of the observer. Then the geocentric position of the observed body is $\boldsymbol{r} = \boldsymbol{q} + \rho\boldsymbol{e}^\rho$, where ρ is the range. Using the right ascension α and the declination δ as coordinates we have

$$\boldsymbol{e}^\rho = (\cos\delta\cos\alpha, \cos\delta\sin\alpha, \sin\delta).$$

We assume the observed body is moving according to Newton's equations

$$\ddot{\boldsymbol{r}} = -\frac{\mu}{|\boldsymbol{r}|^3}\boldsymbol{r}. \tag{3}$$

We introduce the moving frame $\{\boldsymbol{e}^\rho, \hat{\boldsymbol{v}}, \hat{\boldsymbol{n}}\}$, depending on the epoch t, where $\hat{\boldsymbol{v}} = \frac{d}{ds}\boldsymbol{e}^\rho$, regarding $\boldsymbol{e}^\rho$ as function of the arc-length s, and $\hat{\boldsymbol{n}} = \boldsymbol{e}^\rho \times \hat{\boldsymbol{v}}$. By projecting equation (3) on these vectors we obtain

$$\begin{cases} \ddot{\rho} - \rho\eta^2 + \ddot{\boldsymbol{q}}\cdot\boldsymbol{e}^\rho = -\dfrac{\mu}{|\boldsymbol{r}|^3}(\boldsymbol{r}\cdot\boldsymbol{e}^\rho) \\ 2\dot{\rho}\eta + \rho\dot{\eta} + \ddot{\boldsymbol{q}}\cdot\hat{\boldsymbol{v}} = -\dfrac{\mu}{|\boldsymbol{r}|^3}(\boldsymbol{r}\cdot\hat{\boldsymbol{v}}) \\ \kappa\eta^2\rho + \ddot{\boldsymbol{q}}\cdot\hat{\boldsymbol{n}} = -\dfrac{\mu}{|\boldsymbol{r}|^3}(\boldsymbol{r}\cdot\hat{\boldsymbol{n}}), \end{cases}$$

where $\eta = \sqrt{\dot{\alpha}^2\cos^2\delta + \dot{\delta}^2}$ is the proper motion and $\kappa = \frac{\mathrm{d}}{\mathrm{d}s}\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{n}}$. For later use we introduce the notation

$$\mathcal{K} = \left(\ddot{\boldsymbol{r}} + \frac{\mu}{|\boldsymbol{r}|^3}\boldsymbol{r}\right)\cdot\boldsymbol{e}^\rho = \ddot{\rho} - \rho\eta^2 + \ddot{\boldsymbol{q}}\cdot\boldsymbol{e}^\rho + \frac{\mu}{|\boldsymbol{r}|^3}(\boldsymbol{r}\cdot\boldsymbol{e}^\rho).$$

3 THE TWO-BODY INTEGRALS

We write below (see also Gronchi et al. 2011) the expressions of the first integrals of Kepler's problem, i.e. the angular momentum $\boldsymbol{c}$, the energy $\mathcal{E}$ and the Laplace-Lenz vector $\boldsymbol{L}$, in the variables $\rho, \alpha, \delta, \dot{\rho}, \xi, \zeta$, with

$$\xi = \rho\dot{\alpha}\cos\delta, \quad \zeta = \rho\dot{\delta}. \tag{4}$$

We have

$$\boldsymbol{c} = \boldsymbol{A}\xi + \boldsymbol{B}\zeta + \mathbf{C},$$
$$\mathcal{E} = \frac{1}{2}|\dot{\boldsymbol{r}}|^2 - \frac{\mu}{|\boldsymbol{r}|},$$
$$\mu\boldsymbol{L}(\rho,\dot{\rho}) = \dot{\boldsymbol{r}}\times\boldsymbol{c} - \mu\frac{\boldsymbol{r}}{|\boldsymbol{r}|} = \left(|\dot{\boldsymbol{r}}|^2 - \frac{\mu}{|\boldsymbol{r}|}\right)\boldsymbol{r} - (\dot{\boldsymbol{r}}\cdot\boldsymbol{r})\dot{\boldsymbol{r}},$$

where

$$\boldsymbol{A} = \boldsymbol{r}\times\boldsymbol{e}^\alpha, \quad \boldsymbol{B} = \boldsymbol{r}\times\boldsymbol{e}^\delta, \quad \boldsymbol{C} = \boldsymbol{r}\times\dot{\boldsymbol{q}} + \dot{\rho}\,\boldsymbol{q}\times\boldsymbol{e}^\rho,$$

with

$$\boldsymbol{e}^\alpha = \frac{1}{\cos\delta}\frac{\partial\boldsymbol{e}^\rho}{\partial\alpha}, \qquad \boldsymbol{e}^\delta = \frac{\partial\boldsymbol{e}^\rho}{\partial\delta},$$

and

$$\dot{\boldsymbol{r}} = \xi\boldsymbol{e}^\alpha + \zeta\boldsymbol{e}^\delta + (\dot{\rho}\boldsymbol{e}^\rho + \dot{\boldsymbol{q}}),$$
$$|\dot{\boldsymbol{r}}|^2 = \xi^2 + \zeta^2 + 2\dot{\boldsymbol{q}}\cdot\boldsymbol{e}^\alpha\xi + 2\dot{\boldsymbol{q}}\cdot\boldsymbol{e}^\delta\zeta + |\dot{\rho}\boldsymbol{e}^\rho + \dot{\boldsymbol{q}}|^2,$$
$$\dot{\boldsymbol{r}}\cdot\boldsymbol{r} = \boldsymbol{q}\cdot\boldsymbol{e}^\alpha\xi + \boldsymbol{q}\cdot\boldsymbol{e}^\delta\zeta + (\dot{\rho}\boldsymbol{e}^\rho + \dot{\boldsymbol{q}})\cdot\boldsymbol{r}.$$

We introduce the notation

$$q^\alpha = \boldsymbol{q}\cdot\boldsymbol{e}^\alpha, \quad q^\delta = \boldsymbol{q}\cdot\boldsymbol{e}^\delta, \quad \dot{q}^\alpha = \dot{\boldsymbol{q}}\cdot\boldsymbol{e}^\alpha, \quad \dot{q}^\delta = \dot{\boldsymbol{q}}\cdot\boldsymbol{e}^\delta.$$

Note that $\xi^2 + \zeta^2 = \rho^2\eta^2$.

4 LAMBERT'S EQUATION

Lambert's theorem for elliptic motion gives the following relation for the orbital elements of a body on a Keplerian orbit at epochs t_1, t_2:

$$n(t_2 - t_1) = \beta - \gamma - (\sin\beta - \sin\gamma) + 2k\pi. \tag{5}$$

Here, $k \in \mathbb{N}$ is the number of revolutions in the time interval $[t_1, t_2]$, $n = n(a)$ is the mean motion, where $a = -\mu/(2\mathcal{E})$ (the energy is the same at the two epochs), and the angles β, γ are defined by

$$\sin^2\frac{\beta}{2} = \frac{r_1 + r_2 + d}{4a}, \qquad \sin^2\frac{\gamma}{2} = \frac{r_1 + r_2 - d}{4a}, \tag{6}$$

and

$$0 \le \beta - \gamma \le 2\pi,$$

with r_1, r_2 the distances from the centre of force, and d the length of the chord joining the two positions of the body at epochs t_1, t_2. For a fixed number of revolutions we have four different choices for the pairs (β, γ), see Section A and Battin (1987) for the details.

5 LINKAGE

We wish to link two sets of radar data of the form (2), with mean epochs $\bar{t}_i$, $i = 1, 2$, and compute one or more preliminary orbits. In the following we use labels 1, 2 for the quantities introduced in Sections 2–4 according to the epoch.

Let us denote by $\mathcal{L}$ the expression defining Lambert's equation. More precisely, $\mathcal{L} = 0$ is one of the possible cases occurring in (5) with $t_i = \tilde{t}_i = \bar{t}_i - \rho_i/c$, where c is the velocity of light (correction by aberration), see Section A. Moreover, let us define $\boldsymbol{v}_2 = \boldsymbol{e}_2^\rho \times \boldsymbol{q}_2$. We consider the system

$$(\boldsymbol{c}_1 - \boldsymbol{c}_2, \mathcal{E}_1 - \mathcal{E}_2, \mathcal{K}_1, \mathcal{K}_2, (\boldsymbol{L}_1 - \boldsymbol{L}_2)\cdot\boldsymbol{v}_2, \mathcal{L}) = \boldsymbol{0} \tag{7}$$

of eight equations in the eight unknowns $(\boldsymbol{X}, \boldsymbol{\Delta})$, with

$$\boldsymbol{X} = (\xi_1, \zeta_1, \xi_2, \zeta_2), \qquad \boldsymbol{\Delta} = (\Delta\alpha_1, \Delta\delta_1, \Delta\alpha_2, \Delta\delta_2).$$

Note that the unknowns are divided into two sets so that $\boldsymbol{\Delta}$ is the vector of infinitesimal angles. To solve system (7) we first compute $\boldsymbol{X}$ as function of $\boldsymbol{\Delta}$ using four of these equations, then we substitute $\boldsymbol{X}(\boldsymbol{\Delta})$ into the remaining equations and search for solutions of the resulting non-linear system by applying Newton–Raphson's method. Taking advantage of the assumed smallness of the solutions $\boldsymbol{\Delta}$, we can use $\boldsymbol{\Delta} = \boldsymbol{0}$ as starting guess.

6 COMPUTING $X(\boldsymbol{\Delta})$

We describe below two ways to compute $\boldsymbol{X}$ as function of $\boldsymbol{\Delta}$ using some of the equations of system (7). One approach uses linear equations, see Section 6.1, while the equations for the other are quadratic, see Section 6.2.

6.1 Linear equations

Substituting $2\mathcal{E}_1 + \rho_1\mathcal{K}_1 - 2\mathcal{E}_2 - \rho_2\mathcal{K}_2$ in place of $\mathcal{E}_1 - \mathcal{E}_2$ in (7) we obtain an equivalent system and the equation

$$2\mathcal{E}_1 + \rho_1\mathcal{K}_1 = 2\mathcal{E}_2 + \rho_2\mathcal{K}_2 \tag{8}$$

is linear in the variables $\boldsymbol{X} = (\xi_1, \zeta_1, \xi_2, \zeta_2)$. Using equation (8) and the conservation of the angular momentum we obtain a linear system in the variables $\boldsymbol{X}$:

$$\mathbf{M}\boldsymbol{X} = \boldsymbol{V}. \tag{9}$$

Here

$$\mathbf{M} = \begin{bmatrix} A_{11} & B_{11} & -A_{21} & -B_{21} \\ A_{12} & B_{12} & -A_{22} & -B_{22} \\ A_{13} & B_{13} & -A_{23} & -B_{23} \\ \dot{q}_1^{\alpha} & \dot{q}_1^{\delta} & -\dot{q}_2^{\alpha} & -\dot{q}_2^{\delta} \end{bmatrix},$$

where A_{ij}, B_{ij} are the components of $\boldsymbol{A}_i$, $\boldsymbol{B}_i$, and $\dot{q}_i^{\alpha} = \dot{\boldsymbol{q}}_i \cdot \boldsymbol{e}_i^{\alpha}$, $\dot{q}_i^{\delta} = \dot{\boldsymbol{q}}_i \cdot \boldsymbol{e}_i^{\delta}$, for $i = 1, 2$. Moreover

$$\boldsymbol{V} = (C_{21} - C_{11}, C_{22} - C_{12}, C_{23} - C_{13}, D_2 - D_1)^{\mathrm{T}},$$

where C_{ij} are the components of $\boldsymbol{C}_i$ and

$$D_i = \frac{1}{2}\left(\rho_i^2\eta_i^2 + |\dot{\rho}_i\boldsymbol{e}_i^{\rho} + \dot{\boldsymbol{q}}_i|^2\right) - \frac{\mu}{|\boldsymbol{r}_i|},$$

with η_i^2 expressed as function of $(\Delta\alpha_i, \Delta\delta_i)$ by using the equations $\mathcal{K}_i = 0$, $i = 1, 2$, that is using relation

$$\eta^2 = \frac{1}{\rho}\left(\ddot{\rho} + \ddot{\boldsymbol{q}}\cdot\boldsymbol{e}^{\rho} + \frac{\mu}{|\boldsymbol{r}|^3}(\boldsymbol{r}\cdot\boldsymbol{e}^{\rho})\right)$$

at the two epochs $\bar{t}_1, \bar{t}_2$. We can write $\boldsymbol{X}$ as function of $\boldsymbol{\Delta}$ by solving system (9). Let us call M_{hj} the components of $\mathbf{M}$, and V_h the components of $\boldsymbol{V}$. The solutions of (9) are given by

$$\xi_i = \frac{|\mathbf{M}_{2i-1}|}{|\mathbf{M}|}, \quad \zeta_i = \frac{|\mathbf{M}_{2i}|}{|\mathbf{M}|}, \quad i = 1, 2, \tag{10}$$

where $\mathbf{M}_k$ has components

$$M_{hj}^{(k)} = \begin{cases} M_{hj} & \text{if } k \neq j \\ V_h & \text{if } k = j \end{cases}$$

and $|\mathbf{M}|$, $|\mathbf{M}_k|$ represent the determinants of $\mathbf{M}$, $\mathbf{M}_k$.

6.2 Quadratic equations

The orbits at epochs $\tilde{t}_i = \bar{t}_i - \rho_i/c$, $i = 1, 2$, computed with the solution $\boldsymbol{X}$ of system (9), do not necessarily share the same energy $\mathcal{E}$. This can produce some problems in the linear algorithm described above, especially when solving Lambert's equation, where the right-hand sides of (6) may become greater than 1 during the iterations of Newton–Raphson's method. We can force the orbits to share the same energy by solving the first 4 equations in (7), that are quadratic in the variable $\boldsymbol{X}$. By introducing the vector

$$\boldsymbol{Y} = (\xi_1, \zeta_1, \xi_2),$$

we can write the conservation of the angular momentum as the linear system

$$\mathbf{N}\boldsymbol{Y} = \boldsymbol{W}. \tag{11}$$

Here,

$$\mathbf{N} = \begin{bmatrix} A_{11} & B_{11} & -A_{21} \\ A_{12} & B_{12} & -A_{22} \\ A_{13} & B_{13} & -A_{23} \end{bmatrix}$$

and

$$\boldsymbol{W} = \zeta_2\boldsymbol{W}^{(1)} + \boldsymbol{W}^{(0)},$$

where

$$\boldsymbol{W}^{(1)} = (B_{21}, B_{22}, B_{23})^{\mathrm{T}},$$

$$\boldsymbol{W}^{(0)} = (C_{21} - C_{11}, C_{22} - C_{12}, C_{23} - C_{13})^{\mathrm{T}}.$$

We solve system (11). Let us call N_{hj} the components of $\mathbf{N}$ and $W_h, W_h^{(0)}, W_h^{(1)}$ the components of $\boldsymbol{W}, \boldsymbol{W}^{(0)}, \boldsymbol{W}^{(1)}$. The solutions of (11) are functions of ζ_2, $\boldsymbol{\Delta}$, and are given by

$$\tilde{\xi}_1 = \frac{|\mathbf{N}_1|}{|\mathbf{N}|}, \qquad \tilde{\zeta}_1 = \frac{|\mathbf{N}_2|}{|\mathbf{N}|}, \qquad \tilde{\xi}_2 = \frac{|\mathbf{N}_3|}{|\mathbf{N}|},$$

where $\mathbf{N}_k$ has components

$$N_{hj}^{(k)} = \begin{cases} N_{hj} & \text{if } k \neq j \\ W_h & \text{if } k = j \end{cases}.$$

From the conservation of energy we can find ζ_2 as function of $\boldsymbol{\Delta}$. We write

$$F_2\zeta_2^2 + F_1\zeta_2 + F_0 = 0, \tag{12}$$

with

$$F_2 = \frac{1}{|\mathbf{N}|^2}(|\mathbf{N}_1^{(1)}|^2 + |\mathbf{N}_2^{(1)}|^2 - |\mathbf{N}_3^{(1)}|^2) - 1$$

$$F_1 = \frac{2}{|\mathbf{N}|^2}(|\mathbf{N}_1^{(1)}||\mathbf{N}_1^{(0)}| + |\mathbf{N}_2^{(1)}||\mathbf{N}_2^{(0)}| - |\mathbf{N}_3^{(1)}||\mathcal{N}_3^{(0)}|)$$
$$+ \frac{2}{|\mathbf{N}|}(\dot{q}_1^{\alpha}|\mathbf{N}_1^{(1)}| + \dot{q}_1^{\delta}|\mathbf{N}_2^{(1)}| - \dot{q}_2^{\alpha}|\mathbf{N}_3^{(1)}| - \dot{q}_2^{\delta}|\mathbf{N}|)$$

$$F_0 = \frac{1}{|\mathbf{N}|^2}(|\mathbf{N}_1^{(0)}|^2 + |\mathbf{N}_2^{(0)}|^2 - |\mathbf{N}_3^{(0)}|^2)$$
$$+ \frac{2}{|\mathbf{N}|}(\dot{q}_1^{\alpha}|\mathbf{N}_1^{(0)}| + \dot{q}_1^{\delta}|\mathbf{N}_2^{(0)}| - \dot{q}_2^{\alpha}|\mathbf{N}_3^{(0)}|)$$
$$+ \mathfrak{D}_1 - \mathfrak{D}_2,$$

where $\mathbf{N}_k^{(\ell)}$, $k = 1, 2, 3$, $\ell = 0, 1$, has components

$$N_{hj}^{(k,\ell)} = \begin{cases} N_{hj} & \text{if } k \neq j \\ W_h^{(\ell)} & \text{if } k = j \end{cases},$$

and

$$\mathfrak{D}_i = 2D_i - \rho_i^2\eta_i^2, \qquad i = 1, 2.$$

Therefore we have

$$\xi_1(\boldsymbol{\Delta}) = \tilde{\xi}_1(\zeta_2(\boldsymbol{\Delta}), \boldsymbol{\Delta}),$$
$$\zeta_1(\boldsymbol{\Delta}) = \tilde{\zeta}_1(\zeta_2(\boldsymbol{\Delta}), \boldsymbol{\Delta}),$$
$$\xi_2(\boldsymbol{\Delta}) = \tilde{\xi}_2(\zeta_2(\boldsymbol{\Delta}), \boldsymbol{\Delta}),$$

where $\zeta_2(\boldsymbol{\Delta})$ is a solution of (12). Note that we can have up to two acceptable expressions for $\boldsymbol{X}(\boldsymbol{\Delta})$.

7 COMPUTING $\boldsymbol{\Delta}$

We introduce the vector

$$\boldsymbol{G} = (\mathcal{K}_1, \mathcal{K}_2, (\boldsymbol{L}_1 - \boldsymbol{L}_2)\cdot\boldsymbol{v}_2, \mathcal{L}).$$

To select the relevant expressions of $\mathcal{L}$ we need to guess the value of k in equation (5). We can do this by assuming $\boldsymbol{\Delta} = \boldsymbol{0}$ and computing the possible orbits according to the linear or quadratic equations for $\boldsymbol{X}(\boldsymbol{\Delta})$. In both cases we obtain two possible values for the number of revolutions k: with the linear equations we can have two different values of k at the two epochs $\tilde{t}_1, \tilde{t}_2$; with the quadratic equations we may obtain two orbits with different k at the same epoch, say $\tilde{t}_1$, but from conservation of energy we have the same values at $\tilde{t}_2$.

By substituting the possible expressions of $\boldsymbol{X}(\boldsymbol{\Delta})$, coming from either the linear or the quadratic equations, we obtain the reduced system

$$\mathcal{G}(\boldsymbol{\Delta}) = \boldsymbol{G}(\boldsymbol{X}(\boldsymbol{\Delta}), \boldsymbol{\Delta}) = \boldsymbol{0}. \tag{13}$$

Since the unknowns in $\boldsymbol{\Delta}$ are small, we can try to apply Newton–Raphson's method with $\boldsymbol{\Delta} = \boldsymbol{0}$ as starting guess. Thus we try to compute an approximation for $\boldsymbol{\Delta}$ by the iterative formula

$$\boldsymbol{\Delta}_{k+1} = \boldsymbol{\Delta}_k - \left[\frac{\partial\mathcal{G}}{\partial\boldsymbol{\Delta}}(\boldsymbol{\Delta}_k)\right]^{-1}\mathcal{G}(\boldsymbol{\Delta}_k), \qquad \boldsymbol{\Delta}_0 = \boldsymbol{0}. \tag{14}$$

Equations (14) are linear, and are defined by (13) and by the Jacobian matrix

$$\frac{\partial\mathcal{G}}{\partial\boldsymbol{\Delta}}(\boldsymbol{\Delta}_k) = \frac{\partial\boldsymbol{G}}{\partial\boldsymbol{X}}(\boldsymbol{X}_k, \boldsymbol{\Delta}_k)\frac{\partial\boldsymbol{X}}{\partial\boldsymbol{\Delta}}(\boldsymbol{\Delta}_k) + \frac{\partial\boldsymbol{G}}{\partial\boldsymbol{\Delta}}(\boldsymbol{X}_k, \boldsymbol{\Delta}_k),$$

with $\boldsymbol{X}_k = \boldsymbol{X}(\boldsymbol{\Delta}_k)$.

Note, that at each iteration the number of solutions can be doubled, but if we impose the value of $\boldsymbol{\Delta}_{k+1}$ to be close to $\boldsymbol{\Delta}_k$ then we can usually avoid bifurcations.

The computation of the Jacobian matrix $\frac{\partial\mathcal{G}}{\partial\boldsymbol{\Delta}}$ is described below, enhancing the differences between the linear and the quadratic case.

7.1 The derivatives $\frac{\partial \boldsymbol{G}}{\partial \boldsymbol{X}}$

$$\frac{\partial\mathcal{K}_1}{\partial\boldsymbol{X}} = -\frac{2}{\rho_1}(\xi_1, \zeta_1, 0, 0)$$

$$\frac{\partial\mathcal{K}_2}{\partial\boldsymbol{X}} = -\frac{2}{\rho_2}(0, 0, \xi_2, \zeta_2).$$

We observe that

$$\boldsymbol{L}_2\cdot\boldsymbol{v}_2 = -\frac{1}{\mu}(\dot{\boldsymbol{r}}_2\cdot\boldsymbol{r}_2)(\dot{\boldsymbol{r}}_2\cdot\boldsymbol{v}_2).$$

Thus we have

$$\begin{aligned}
\frac{\partial}{\partial\xi_1}[(\boldsymbol{L}_1 - \boldsymbol{L}_2)\cdot\boldsymbol{v}_2] &= \frac{2}{\mu}(\xi_1 + \dot{\boldsymbol{q}}_1\cdot\boldsymbol{e}_1^{\alpha})(\boldsymbol{r}_1\cdot\boldsymbol{v}_2)\\
&\quad - \frac{1}{\mu}[(\boldsymbol{q}_1\cdot\boldsymbol{e}_1^{\alpha})(\dot{\boldsymbol{r}}_1\cdot\boldsymbol{v}_2) + (\dot{\boldsymbol{r}}_1\cdot\boldsymbol{r}_1)(\boldsymbol{e}_1^{\alpha}\cdot\boldsymbol{v}_2)]\\
\frac{\partial}{\partial\zeta_1}[(\boldsymbol{L}_1 - \boldsymbol{L}_2)\cdot\boldsymbol{v}_2] &= \frac{2}{\mu}(\zeta_1 + \dot{\boldsymbol{q}}_1\cdot\boldsymbol{e}_1^{\delta})(\boldsymbol{r}_1\cdot\boldsymbol{v}_2)\\
&\quad - \frac{1}{\mu}[(\boldsymbol{q}_1\cdot\boldsymbol{e}_1^{\delta})(\dot{\boldsymbol{r}}_1\cdot\boldsymbol{v}_2) + (\dot{\boldsymbol{r}}_1\cdot\boldsymbol{r}_1)(\boldsymbol{e}_1^{\delta}\cdot\boldsymbol{v}_2)]\\
\frac{\partial}{\partial\xi_2}[(\boldsymbol{L}_1 - \boldsymbol{L}_2)\cdot\boldsymbol{v}_2] &= \frac{1}{\mu}[(\boldsymbol{q}_2\cdot\boldsymbol{e}_2^{\alpha})(\dot{\boldsymbol{r}}_2\cdot\boldsymbol{v}_2) + (\dot{\boldsymbol{r}}_2\cdot\boldsymbol{r}_2)(\boldsymbol{e}_2^{\alpha}\cdot\boldsymbol{v}_2)]\\
\frac{\partial}{\partial\zeta_2}[(\boldsymbol{L}_1 - \boldsymbol{L}_2)\cdot\boldsymbol{v}_2] &= \frac{1}{\mu}[(\boldsymbol{q}_2\cdot\boldsymbol{e}_2^{\delta})(\dot{\boldsymbol{r}}_2\cdot\boldsymbol{v}_2) + (\dot{\boldsymbol{r}}_2\cdot\boldsymbol{r}_2)(\boldsymbol{e}_2^{\delta}\cdot\boldsymbol{v}_2)].
\end{aligned}$$

For Lambert's equation, the derivatives are given by

$$\frac{\partial\mathcal{L}}{\partial\boldsymbol{X}} = \frac{\partial n}{\partial\boldsymbol{X}}(\tilde{t}_1 - \tilde{t}_2) + \frac{\partial(\beta - \sin\beta)}{\partial\boldsymbol{X}} - \frac{\partial(\gamma - \sin\gamma)}{\partial\boldsymbol{X}},$$

$$\frac{\partial n}{\partial\boldsymbol{X}} = -\frac{3}{2\mu}\sqrt{-2\mathcal{E}_1}\frac{\partial(2\mathcal{E}_1)}{\partial\boldsymbol{X}},$$

$$\frac{\partial(\beta - \sin\beta)}{\partial\boldsymbol{X}} = (1 - \cos\beta)\frac{\partial\beta}{\partial\boldsymbol{X}} = 2\sqrt{\frac{\Gamma_+}{1 - \Gamma_+}}\frac{\partial\Gamma_+}{\partial\boldsymbol{X}},$$

$$\frac{\partial(\gamma - \sin\gamma)}{\partial\boldsymbol{X}} = (1 - \cos\gamma)\frac{\partial\gamma}{\partial\boldsymbol{X}} = 2\sqrt{\frac{\Gamma_-}{1 - \Gamma_-}}\frac{\partial\Gamma_-}{\partial\boldsymbol{X}},$$

with

$$\Gamma_+ = \sin^2\frac{\beta}{2} = -\frac{r_1 + r_2 + d}{2\mu}\mathcal{E}_1,$$

$$\Gamma_- = \sin^2\frac{\gamma}{2} = -\frac{r_1 + r_2 - d}{2\mu}\mathcal{E}_1.$$

In the expression for $\frac{\partial n}{\partial\boldsymbol{X}}$ we use the energy $\mathcal{E}_1$ at epoch $\tilde{t}_1$. We could as well choose $\mathcal{E}_2$ at epoch $\tilde{t}_2$: this choice is arbitrary in the linear case, in fact computing $\boldsymbol{X}(\boldsymbol{\Delta})$ with the linear algorithm, we generally have $\mathcal{E}_1(\xi_1(\boldsymbol{\Delta}), \zeta_1(\boldsymbol{\Delta})) \neq \mathcal{E}_2(\xi_2(\boldsymbol{\Delta}), \zeta_2(\boldsymbol{\Delta}))$.

Since r_1, r_2, d do not depend on $\boldsymbol{X}$, we have

$$\frac{\partial\Gamma_+}{\partial\boldsymbol{X}} = -\frac{r_1 + r_2 + d}{2\mu}\frac{\partial\mathcal{E}_1}{\partial\boldsymbol{X}},$$

$$\frac{\partial\Gamma_-}{\partial\boldsymbol{X}} = -\frac{r_1 + r_2 - d}{2\mu}\frac{\partial\mathcal{E}_1}{\partial\boldsymbol{X}},$$

with

$$\frac{\partial\mathcal{E}_1}{\partial\boldsymbol{X}} = ((\xi_1 + \boldsymbol{q}_1\cdot\boldsymbol{e}_1^{\alpha}), (\zeta_1 + \boldsymbol{q}_1\cdot\boldsymbol{e}_1^{\delta}), 0, 0).$$

7.2 The derivatives $\frac{\partial \boldsymbol{X}}{\partial \boldsymbol{\Delta}}$

To compute derivatives with respect to $\boldsymbol{\Delta}$ we use as intermediate variables the unit vectors $\boldsymbol{e}_j^{\rho}, \boldsymbol{e}_j^{\alpha}, \boldsymbol{e}_j^{\delta}, j = 1, 2$. To this aim we introduce the vector

$$\boldsymbol{E} = \begin{bmatrix}\boldsymbol{E}_1\\ \boldsymbol{E}_2\end{bmatrix}, \qquad \text{where} \quad \boldsymbol{E}_j = \begin{pmatrix}\boldsymbol{e}_j^{\rho}\\ \boldsymbol{e}_j^{\alpha}\\ \boldsymbol{e}_j^{\delta}\end{pmatrix}.$$

Its derivatives with respect to $\boldsymbol{\Delta}$ are given by

$$\frac{\partial\boldsymbol{E}}{\partial\boldsymbol{\Delta}} = \begin{bmatrix}\frac{\partial\boldsymbol{E}_1}{\partial(\alpha_1,\delta_1)} & \boldsymbol{0}\\ \boldsymbol{0} & \frac{\partial\boldsymbol{E}_2}{\partial(\alpha_2,\delta_2)}\end{bmatrix},$$

where

$$\frac{\partial\boldsymbol{E}_j}{\partial(\alpha_j, \delta_j)} = \begin{bmatrix}\cos\delta_j\boldsymbol{e}_j^{\alpha} & \boldsymbol{e}_j^{\delta}\\ \boldsymbol{e}_j^{\perp} & \boldsymbol{0}\\ -\sin\delta_j\boldsymbol{e}_j^{\alpha} & -\boldsymbol{e}_j^{\rho}\end{bmatrix}$$

and $\boldsymbol{e}_j^{\perp} = -(\cos\alpha_j, \sin\alpha_j, 0)^{\mathrm{T}}$.

Moreover, we need to compute $\frac{\partial\boldsymbol{X}}{\partial\boldsymbol{E}}$. We describe the different procedures for the linear and quadratic methods.

7.2.1 The derivatives $\frac{\partial X}{\partial E}$, linear case

Using equation (10), we only need to compute

$$\frac{\partial \xi_h}{\partial \boldsymbol{E}} = \frac{1}{|\mathbf{M}|}\frac{\partial|\mathbf{M}_{2h-1}|}{\partial \boldsymbol{E}} - \frac{|\mathbf{M}_{2h-1}|}{|\mathbf{M}|^2}\frac{\partial|\mathbf{M}|}{\partial \boldsymbol{E}},$$

$$\frac{\partial \zeta_h}{\partial \boldsymbol{E}} = \frac{1}{|\mathbf{M}|}\frac{\partial|\mathbf{M}_{2h}|}{\partial \boldsymbol{E}} - \frac{|\mathbf{M}_{2h}|}{|\mathbf{M}|^2}\frac{\partial|\mathbf{M}|}{\partial \boldsymbol{E}}.$$

We take advantage of the following relation, valid for any matrix $\mathbf{A}$ of order n with coefficients a_{ij} depending on a variable x:

$$\frac{\mathrm{d}}{\mathrm{d}x}|\mathbf{A}| = \sum_{h=1}^{n}|\mathbf{B}_h|,$$

where $\mathbf{B}_h$ has coefficients $b_{ij}^{(h)}$, with

$$b_{ij}^{(h)} = \begin{cases} a_{ij} & h \neq j \\ \frac{\mathrm{d}}{\mathrm{d}x}a_{ij} & h = j \end{cases}.$$

7.2.2 The derivatives $\frac{\partial X}{\partial E}$, quadratic case

From the implicit function theorem applied to equation (12) we obtain

$$\frac{\partial \zeta_2}{\partial \boldsymbol{E}} = -\left[\frac{1}{(2F_2\zeta_2 + F_1)}\left(\frac{\partial F_2}{\partial \boldsymbol{E}}\zeta_2^2 + \frac{\partial F_1}{\partial \boldsymbol{E}}\zeta_2 + \frac{\partial F_0}{\partial \boldsymbol{E}}\right)\right]\Bigg|_{\zeta_2=\zeta_2^{(i)}(\boldsymbol{E})}.$$

Let us define

$$\xi_1(\boldsymbol{E}) = \tilde{\xi}_1(\zeta_2(\boldsymbol{E}), \boldsymbol{E}),$$

$$\zeta_1(\boldsymbol{E}) = \tilde{\zeta}_1(\zeta_2(\boldsymbol{E}), \boldsymbol{E}),$$

$$\xi_2(\boldsymbol{E}) = \tilde{\xi}_2(\zeta_2(\boldsymbol{E}), \boldsymbol{E}).$$

We have

$$\frac{\partial \xi_1}{\partial \boldsymbol{E}} = \frac{1}{|\mathbf{N}|}\left(\frac{\partial|\mathbf{N}_1|}{\partial \zeta_2}\frac{\partial \zeta_2}{\partial \boldsymbol{E}} + \frac{\partial|\mathbf{N}_1|}{\partial \boldsymbol{E}}\right) - \frac{|\mathbf{N}_1|}{|\mathbf{N}|^2}\frac{\partial|\mathbf{N}|}{\partial \boldsymbol{E}},$$

$$\frac{\partial \zeta_1}{\partial \boldsymbol{E}} = \frac{1}{|\mathbf{N}|}\left(\frac{\partial|\mathbf{N}_2|}{\partial \zeta_2}\frac{\partial \zeta_2}{\partial \boldsymbol{E}} + \frac{\partial|\mathbf{N}_2|}{\partial \boldsymbol{E}}\right) - \frac{|\mathbf{N}_2|}{|\mathbf{N}|^2}\frac{\partial|\mathbf{N}|}{\partial \boldsymbol{E}},$$

$$\frac{\partial \xi_2}{\partial \boldsymbol{E}} = \frac{1}{|\mathbf{N}|}\left(\frac{\partial|\mathbf{N}_3|}{\partial \zeta_2}\frac{\partial \zeta_2}{\partial \boldsymbol{E}} + \frac{\partial|\mathbf{N}_3|}{\partial \boldsymbol{E}}\right) - \frac{|\mathbf{N}_3|}{|\mathbf{N}|^2}\frac{\partial|\mathbf{N}|}{\partial \boldsymbol{E}}.$$

7.3 The derivatives $\frac{\partial \mathrm{G}}{\partial \Delta}$

As in Section 7.2, we compute the derivatives of $\mathcal{G}$ with respect to $\boldsymbol{E}$ and multiply the result by $\frac{\partial \boldsymbol{E}}{\partial \Delta}$. We have

$$\frac{\partial \mathcal{K}_j}{\partial \boldsymbol{e}_j^{\rho}} = \ddot{\boldsymbol{q}}_j + \mu\frac{\boldsymbol{q}_j}{|\boldsymbol{r}_j|^3}\left(1 - 3\rho_j\frac{(\boldsymbol{r}_j\cdot\boldsymbol{e}_j^{\rho})}{|\boldsymbol{r}_j|^2}\right), \quad j = 1, 2$$

and

$$\frac{\partial \mathcal{K}_j}{\partial \boldsymbol{e}_j^{\alpha}} = \frac{\partial \mathcal{K}_j}{\partial \boldsymbol{e}_j^{\delta}} = 0, \quad \frac{\partial \mathcal{K}_1}{\partial \boldsymbol{E}_2} = \frac{\partial \mathcal{K}_2}{\partial \boldsymbol{E}_1} = \mathbf{0}.$$

$$\frac{\partial}{\partial \boldsymbol{e}_1^{\rho}}[(\boldsymbol{L}_1 - \boldsymbol{L}_2)\cdot\boldsymbol{v}_2] = \frac{1}{\mu}\left[(\boldsymbol{r}_1\cdot\boldsymbol{v}_2)\left(2\dot{\rho}_1\dot{\boldsymbol{q}}_1 + \mu\rho_1\frac{\boldsymbol{q}_1}{|\boldsymbol{r}_1|^3}\right) + \left(|\dot{\boldsymbol{r}}_1|^2 - \frac{\mu}{|\boldsymbol{r}_1|}\right)\rho_1\boldsymbol{v}_2 - (\dot{\boldsymbol{r}}_1\cdot\boldsymbol{v}_2)(\dot{\rho}_1\boldsymbol{q}_1 + \rho_1\dot{\boldsymbol{q}}_1) - (\dot{\boldsymbol{r}}_1\cdot\boldsymbol{r}_1)\dot{\rho}_1\boldsymbol{v}_2\right],$$

$$\frac{\partial}{\partial \boldsymbol{e}_1^{\alpha}}[(\boldsymbol{L}_1 - \boldsymbol{L}_2)\cdot\boldsymbol{v}_2] = \frac{\xi_1}{\mu}[2(\boldsymbol{r}_1\cdot\boldsymbol{v}_2)\dot{\boldsymbol{q}}_1 - (\dot{\boldsymbol{r}}_1\cdot\boldsymbol{v}_2)\boldsymbol{q}_1 - (\dot{\boldsymbol{r}}_1\cdot\boldsymbol{r}_1)\boldsymbol{v}_2],$$

$$\frac{\partial}{\partial \boldsymbol{e}_1^{\delta}}[(\boldsymbol{L}_1 - \boldsymbol{L}_2)\cdot\boldsymbol{v}_2] = \frac{\zeta_1}{\mu}[2(\boldsymbol{r}_1\cdot\boldsymbol{v}_2)\dot{\boldsymbol{q}}_1 - (\dot{\boldsymbol{r}}_1\cdot\boldsymbol{v}_2)\boldsymbol{q}_1 - (\dot{\boldsymbol{r}}_1\cdot\boldsymbol{r}_1)\boldsymbol{v}_2],$$

$$\frac{\partial}{\partial \boldsymbol{e}_2^{\rho}}[(\boldsymbol{L}_1 - \boldsymbol{L}_2)\cdot\boldsymbol{v}_2] = -\boldsymbol{L}_1\times\boldsymbol{q}_2 + \frac{1}{\mu}[(\dot{\rho}_2\boldsymbol{q}_2 + \rho_2\dot{\boldsymbol{q}}_2)(\dot{\boldsymbol{r}}_2\cdot\boldsymbol{v}_2) + (\dot{\boldsymbol{r}}_2\cdot\boldsymbol{r}_2)\boldsymbol{q}_2\times\dot{\boldsymbol{q}}_2],$$

$$\frac{\partial}{\partial \boldsymbol{e}_2^{\alpha}}[(\boldsymbol{L}_1 - \boldsymbol{L}_2)\cdot\boldsymbol{v}_2] = \frac{1}{\mu}[\xi_2(\dot{\boldsymbol{r}}_2\cdot\boldsymbol{v}_2) + \zeta_2(\dot{\boldsymbol{r}}_2\cdot\boldsymbol{r}_2)]\boldsymbol{q}_2,$$

$$\frac{\partial}{\partial \boldsymbol{e}_2^{\delta}}[(\boldsymbol{L}_1 - \boldsymbol{L}_2)\cdot\boldsymbol{v}_2] = \frac{1}{\mu}[\zeta_2(\dot{\boldsymbol{r}}_2\cdot\boldsymbol{v}_2) - \xi_2(\dot{\boldsymbol{r}}_2\cdot\boldsymbol{r}_2)]\boldsymbol{q}_2.$$

For Lambert's equation we have

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{E}} = \frac{\partial n}{\partial \boldsymbol{E}}(\tilde{t}_1 - \tilde{t}_2) + \frac{\partial(\beta - \sin\beta)}{\partial \boldsymbol{E}} - \frac{\partial(\gamma - \sin\gamma)}{\partial \boldsymbol{E}},$$

with

$$\frac{\partial n}{\partial \boldsymbol{E}_1} = -\frac{3}{2\mu}\sqrt{-2\mathcal{E}_1}\frac{\partial(2\mathcal{E}_1)}{\partial \boldsymbol{E}_1}, \quad \frac{\partial n}{\partial \boldsymbol{E}_2} = \mathbf{0},$$

$$\frac{\partial(\beta - \sin\beta)}{\partial \boldsymbol{E}} = 2\sqrt{\frac{\Gamma_+}{1 - \Gamma_+}}\frac{\partial \Gamma_+}{\partial \boldsymbol{E}},$$

$$\frac{\partial(\gamma - \sin\gamma)}{\partial \boldsymbol{E}} = 2\sqrt{\frac{\Gamma_-}{1 - \Gamma_-}}\frac{\partial \Gamma_-}{\partial \boldsymbol{E}}.$$

Moreover

$$\frac{\partial \Gamma_+}{\partial \boldsymbol{E}_1} = -\frac{2\mathcal{E}_1}{4\mu}\left(\frac{\partial r_1}{\partial \boldsymbol{E}_1} + \frac{\partial d}{\partial \boldsymbol{E}_1}\right) + \frac{\Gamma_+}{2\mathcal{E}_1}\frac{\partial(2\mathcal{E}_1)}{\partial \boldsymbol{E}_1},$$

$$\frac{\partial \Gamma_+}{\partial \boldsymbol{E}_2} = -\frac{2\mathcal{E}_1}{4\mu}\left(\frac{\partial r_2}{\partial \boldsymbol{E}_2} + \frac{\partial d}{\partial \boldsymbol{E}_2}\right),$$

$$\frac{\partial \Gamma_-}{\partial \boldsymbol{E}_1} = -\frac{2\mathcal{E}_1}{4\mu}\left(\frac{\partial r_1}{\partial \boldsymbol{E}_1} - \frac{\partial d}{\partial \boldsymbol{E}_1}\right) + \frac{\Gamma_-}{2\mathcal{E}_1}\frac{\partial(2\mathcal{E}_1)}{\partial \boldsymbol{E}_1},$$

$$\frac{\partial \Gamma_-}{\partial \boldsymbol{E}_2} = -\frac{2\mathcal{E}_1}{4\mu}\left(\frac{\partial r_2}{\partial \boldsymbol{E}_2} - \frac{\partial d}{\partial \boldsymbol{E}_2}\right),$$

with

$$\frac{\partial(2\mathcal{E}_1)}{\partial \boldsymbol{E}_1} = \left(2\dot{\rho}_1\dot{\boldsymbol{q}}_1 + 2\mu\rho_1\frac{\boldsymbol{q}_1}{r_1^3},\ 2\xi_1\dot{\boldsymbol{q}}_1,\ 2\zeta_1\dot{\boldsymbol{q}}_1\right),$$

$$\frac{\partial r_1}{\partial \boldsymbol{E}_1} = \left(\frac{\rho_1\boldsymbol{q}_1}{r_1}, \mathbf{0}, \mathbf{0}\right), \quad \frac{\partial r_2}{\partial \boldsymbol{E}_2} = \left(\frac{\rho_2\boldsymbol{q}_2}{r_2}, \mathbf{0}, \mathbf{0}\right),$$

$$\frac{\partial d}{\partial \boldsymbol{E}_1} = \frac{\rho_1}{d}(\boldsymbol{q}_1 - \mathbf{r}_2, \mathbf{0}, \mathbf{0}), \qquad \frac{\partial d}{\partial \boldsymbol{E}_2} = \frac{\rho_2}{d}(\boldsymbol{q}_2 - \mathbf{r}_1, \mathbf{0}, \mathbf{0}).$$

8 ALTERNATIVE KNOWN METHODS

We recall below two already known methods that can be used in place of the one described in Sections 5–7, with the available data. An important difference is that these methods do not provide corrections to the angles α, δ.

8.1 Gibbs' method

From three position vectors of an observed body at the same pass we can compute an orbit using Gibbs' method, see Herrick (1976, chapter 8). We recall below the formulas of this method. Given the position vectors $\boldsymbol{r}_j$ at times t_j, with $j = 1, 2, 3$, we have

$$\dot{\boldsymbol{r}}_2 = -d_1\,\boldsymbol{r}_1 + d_2\,\boldsymbol{r}_2 + d_3\,\boldsymbol{r}_3,$$

where

$$d_j = G_j + H_j r_j^{-3}, \quad j = 1, 2, 3,$$

$$G_1 = \frac{t_{32}^2}{t_{21}\,t_{32}\,t_{31}}\,,\; G_3 = \frac{t_{21}^2}{t_{21}\,t_{32}\,t_{31}}\,,\; G_2 = G_1 - G_3,$$
$$H_1 = \mu\, t_{32}/12\,,\; H_3 = \mu t_{21}/12\,,\; H_2 = H_1 - H_3.$$

Here, $t_{ij} = t_i - t_j$, $r_j = |\boldsymbol{r}_j|$. Before applying Gibbs' method we can use the algorithm of Section A2 to correct the data so that they correspond to coplanar geocentric position vectors.

8.2 Keplerian integrals

From two radar tracks, we can obtain by interpolation the values of $(\bar{\alpha}, \bar{\delta}, \rho, \dot{\rho})$ at epochs $\bar{t}_1, \bar{t}_2$. If we wish to determine the values of the unknowns $\dot{\alpha}, \dot{\delta}$, or equivalently of ξ, ζ defined by (4), we can use the KI method, see Taff & Hall (1977), Farnocchia et al. (2010), Gronchi et al. (2011). This method uses the equations

$$\boldsymbol{c}_1 = \boldsymbol{c}_2, \qquad \mathcal{E}_1 = \mathcal{E}_2,$$

which can be explicitly solved, giving at most two solutions.

Table 1. Keplerian elements of the test orbit at epoch $\bar{t}_1$ = 54127.1550347 MJD. Distances are expressed in km, angles in degrees.

a	e	I	Ω	ω	ℓ
7818.10	0.066	65.81	216.25	357.16	202.09

Table 2. rms of the errors added to the radar tracks.

rms	α, δ (deg)	ρ (m)
Case 1	0.2	0
Case 2	0.04	2
Case 3	0.06	3
Case 4	0.1	5
Case 5	0.2	10

Table 4. Relative error for the orbital elements computed with IAL at epoch $\bar{t}_1$.

	IAL	rms
$\delta a/a$	2.9×10^{-7}	Case 1
	-2.4×10^{-5}	Case 2
	1.2×10^{-1}	Case 3
$\delta e/e$	1.8×10^{-5}	Case 1
	-2.0×10^{-3}	Case 2
	3.3×10^{-1}	Case 3
$\delta I/I$	1.7×10^{-6}	Case 1
	-5.4×10^{-4}	Case 2
	-2.6×10^{-2}	Case 3
$\delta\Omega/\Omega$	-1.5×10^{-8}	Case 1
	2.8×10^{-6}	Case 2
	-8.1×10^{-3}	Case 3
$\delta\omega/\omega$	-7.4×10^{-6}	Case 1
	6.1×10^{-4}	Case 2
	-6.1×10^{-1}	Case 3
$\delta\ell/\ell$	1.5×10^{-5}	Case 1
	-1.3×10^{-3}	Case 2
	-7.7×10^{-1}	Case 3

9 NUMERICAL TESTS

We have performed some numerical tests with simulated objects, without the J_2 effect, but adding errors to the observations. Here, we describe the results for only one simulated object, whose orbital elements are displayed in Table 1 for epoch $\bar{t}_1$.

We produce two tracks of simulated observations of this object with a two-body propagation and add to the data of the tracks a Gaussian error, with zero mean and the standard deviations listed in Table 2. In particular we consider the case where we add no error to the range ρ (Case 1 in the table). The data that we obtain by interpolation from the modified radar tracks are displayed in Table 3 for the simulated object. Indeed Case 1 is peculiar, in fact we interpolate the available values of α, δ and we use the exact values of $\rho, \dot{\rho}, \ddot{\rho}$, that we can compute from the given orbit.

First we compute an orbit at epoch $\bar{t}_1$ using the infinitesimal angles method with the linear equations (IAL) introduced in Section 6.1. In Table 4, we show the (signed) relative error for each orbital element computed by IAL with respect to the corresponding known values. This computation is successful only for a reduced noise level, Cases 1 and 2 in Table 2. In particular, in Case 1, where there is no error in $\rho, \dot{\rho}, \ddot{\rho}$, IAL is able to correct the errors in α,

Table 3. Data interpolated from the radar tracks of the test object at epochs $\bar{t}_1$ = 54127.1550348 MJD and $\bar{t}_2$ = 54127.5821181 MJD, using the different noise levels of Table 2.

Epoch	Data	Case 1	Case 2	Case 3	Case 4	Case 5
	$\bar{\alpha}$ (deg)	51.175 32	51.209 94	51.205 62	51.196 96	51.175 32
	$\bar{\delta}$ (deg)	−5.466 10	−5.431 48	−5.435 81	−5.444 46	−5.466 10
$\bar{t}_1$	ρ (km)	1984.410 270 20	1984.409 360 97	1984.408 906 36	1984.407 997 13	1984.405 724 06
	$\dot{\rho}$ (km d^{-1})	−73 313.626 74	−73 295.547 60	−73 286.508 03	−73 268.428 89	−73 223.231 05
	$\ddot{\rho}$ (km d^{-2})	116 444 362.776	116449505.035	116 477 961.145	116 534 873.370	116 677 153.912
	$\bar{\alpha}$ (deg)	264.303 69	264.272 53	264.276 42	264.284 21	264.303 69
	$\bar{\delta}$ (deg)	−66.769 98	−66.801 15	−66.797 25	−66.789 46	−66.769 98
$\bar{t}_2$	ρ (km)	1893.534 107 62	1893.536 356 46	1893.537 480 88	1893.539 729 72	1893.545 351 82
	$\dot{\rho}$ (km d^{-1})	−323 586.568 97	−323 638.420 16	−323 664.345 76	−323 716.196 96	−323 845.824 95
	$\ddot{\rho}$ (km d^{-2})	123 666 888.648	123 507 436.811	123 396 401.795	123 174 331.770	122 619 156.710

Table 5. Comparison of relative errors for the orbital elements computed with G, KI and IAQ at epoch $\tilde{t}_1$.

	G	KI	IAQ	rms
	5.8×10^{-2}	-1.9×10^{-4}	2.8×10^{-7}	Case 1
$\delta a/a$	2.6×10^{-2}	-3.1×10^{-4}	-9.3×10^{-6}	Case 4
	5.8×10^{-2}	-4.2×10^{-4}	7.9×10^{-7}	Case 5
	-9.2×10^{-1}	5.5×10^{-3}	1.8×10^{-5}	Case 1
$\delta e/e$	-4.4×10^{-1}	5.8×10^{-3}	-8.4×10^{-3}	Case 4
	-9.2×10^{-1}	8.2×10^{-3}	-1.9×10^{-2}	Case 5
	6.6×10^{-2}	5.7×10^{-4}	1.4×10^{-6}	Case 1
$\delta I/I$	3.4×10^{-2}	3.9×10^{-4}	-3.1×10^{-3}	Case 4
	6.6×10^{-2}	5.7×10^{-4}	-7.2×10^{-3}	Case 5
	6.0×10^{-3}	-9.8×10^{-6}	4.7×10^{-8}	Case 1
$\delta\Omega/\Omega$	3.1×10^{-3}	8.3×10^{-6}	2.3×10^{-4}	Case 4
	6.0×10^{-3}	-9.7×10^{-6}	5.8×10^{-4}	Case 5
	-5.7×10^{-2}	-4.4×10^{-4}	-7.7×10^{-6}	Case 1
$\delta\omega/\omega$	-2.8×10^{-3}	3.6×10^{-5}	7.2×10^{-4}	Case 4
	-6.0×10^{-2}	7.7×10^{-5}	9.9×10^{-4}	Case 5
	8.7×10^{-2}	1.0×10^{-3}	1.6×10^{-5}	Case 1
$\delta\ell/\ell$	-3.1×10^{-3}	2.8×10^{-5}	-1.7×10^{-3}	Case 4
	9.1×10^{-2}	1.4×10^{-6}	-2.6×10^{-3}	Case 5

δ and to recover the orbital elements of the known orbit. However, this method shows some limitations: for Case 3 the computed orbit displayed in Table 4 is very different from the known one, and for larger values of the observational errors either we could not compute an orbit or the results were wrong.

In Table 5, we show the relative error for the orbital elements at epoch $\tilde{t}_1$ computed by the method of Gibbs (G), by the KI and by the infinitesimal angles with the quadratic equations (IAQ) introduced in Section 6.2. Here we consider the noise levels corresponding to Cases 1, 4, and 5 in Table 2. In Case 1, IAQ is also able to correct the errors in α, δ and to recover the orbital elements of the known orbit. For Case 4 and Case 5, IAQ obtains a better value of the semimajor axis a, and slightly worse values of the other elements, if compared with KI. To be consistent, for KI in Case 1 we use the exact values of ρ, $\dot{\rho}$. The results with Gibbs' method are not very good. This also occurs by applying the corrections to the data explained in Section A2. On the other hand this method uses only part of the information: here, we use the three vectors $(t_j, \rho_j, \alpha_j, \delta_j)$ of the first track at epochs t_j, with $j = 1, 2, 4$.

10 CONCLUSIONS

We have introduced a new method to compute preliminary orbits of Earth satellites using radar observations. This consists in solving system (7) and we considered two different possible approaches, denoted by IAL and IAQ. The comparison of this new method with already existing ones has been performed for some test cases. From the results of Section 9, we conclude that IAQ works much better than IAL and Gibbs' method and it shows some advantages with respect to KI. Large-scale tests should be done to check the performance of IAQ, possibly with real data. We plan to investigate the case which includes the J_2 effect in the equations: this is essential to link radar tracks of LEO orbits after several revolutions.

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APPENDIX A: COMPLEMENTARY RESULTS

A1 Lambert's equations for elliptic motion

In this section, following Whittaker (1989) and Plummer (1918), we summarize the steps to derive Lambert's equation for elliptic motion under a Newtonian force and we give a geometric interpretation of the result. Indeed we obtain four distinct equations per number of revolutions of the observed body. Note that, dealing with radar observations of space debris, the time between two distinct arcs of observations usually covers several revolutions.

Theorem 1. (Lambert 1761) In the elliptic motion under the Newtonian gravitational attraction, the time $\Delta t = t_2 - t_1$ spent to describe any arc (without multiple revolutions) from the initial position P_1 to the final position P_2 depends only on the semimajor axis a, on the sum $r = r_1 + r_2$ of the two distances $r_1 = |P_1 - F|$, $r_2 = |P_2 - F|$ from the centre of force F, and the length d of the chord joining P_1 and P_2. More precisely we have

$$n\Delta t = \beta - \gamma - (\sin\beta - \sin\gamma),$$

where $n = n(a)$ is the mean motion, and the angles β, γ are defined by

$$\sin^2\frac{\beta}{2} = \frac{r+d}{4a}, \qquad \sin^2\frac{\gamma}{2} = \frac{r-d}{4a},$$

and

$$0 \leq \beta - \gamma \leq 2\pi. \tag{A1}$$

Proof. We can assume, without loss of generality, that the positions of the points P_1, P_2 are defined by two values E_1, E_2 of the eccentric anomalies such that $0 \leq E_2 - E_1 \leq 2\pi$.

The difference of Kepler's equations at the two epochs gives

$$n\Delta t = E_2 - E_1 - e(\sin E_2 - \sin E_1),$$

where e is the orbital eccentricity. From elementary geometrical relations we obtain

$$\frac{r}{a} = 2\left(1 - e\cos\frac{E_1+E_2}{2}\cos\frac{E_2-E_1}{2}\right),$$

$$\frac{d}{a} = 2\sin\frac{E_2 - E_1}{2}\sqrt{1 - e^2\cos^2\frac{E_1 + E_2}{2}}.$$

It follows that

$$\frac{r+d}{2a} = 1 - \cos\left(\frac{E_2 - E_1}{2} + \arccos\left(e\cos\frac{E_2 + E_1}{2}\right)\right), \tag{A2}$$

$$\frac{r-d}{2a} = 1 - \cos\left(-\frac{E_2 - E_1}{2} + \arccos\left(e\cos\frac{E_2 + E_1}{2}\right)\right). \tag{A3}$$

In particular, for a real elliptical orbit to be possible the given scalar quantities must satisfy the relations $r \geq d$ and $4a - r \geq d$. If we define

$$\beta_0 = 2\arcsin\left(\sqrt{\frac{r+d}{4a}}\right), \qquad \gamma_0 = 2\arcsin\left(\sqrt{\frac{r-d}{4a}}\right),$$

then, using relation

$$1 - \cos\theta = 2\sin^2\frac{\theta}{2}, \qquad \theta \in \mathbb{R},$$

and setting

$$\beta = \frac{E_2 - E_1}{2} + \arccos\left(e\cos\frac{E_2 + E_1}{2}\right), \tag{A4}$$

$$\gamma = -\frac{E_2 - E_1}{2} + \arccos\left(e\cos\frac{E_2 + E_1}{2}\right), \tag{A5}$$

we find that the pairs

$$(\beta, \gamma) = (\beta_0, \gamma_0),\ (\beta_0, -\gamma_0),\ (2\pi - \beta_0, -\gamma_0),\ (2\pi - \beta_0, \gamma_0) \tag{A6}$$

satisfy equations (A2), (A3). Up to addition of the same integer multiple of 2π to both β and γ, the pairs (A6) are the only ones fulfilling (A2), (A3) and (A1). From (A4), (A5) we obtain

$$\beta - \gamma = E_2 - E_1, \qquad \cos\frac{\beta + \gamma}{2} = e\cos\frac{E_2 + E_1}{2},$$

that yields

$$n\Delta t = \beta - \gamma - (\sin\beta - \sin\gamma).$$

In fact

$$\sin\beta - \sin\gamma = 2\sin\frac{\beta - \gamma}{2}\cos\frac{\beta + \gamma}{2}$$
$$= 2e\sin\frac{E_2 - E_1}{2}\cos\frac{E_2 + E_1}{2} = e(\sin E_2 - \sin E_1).$$

The pairs (β, γ) given in equation (A6) correspond to four geometrically distinct possible paths from the initial to the final position, see Fig. A1. Given the points P_1, P_2 and the attracting focus F, for a fixed value a of the semimajor axis, we find two different ellipses passing through P_1 and P_2. They share the attracting focus F, but not the second focus (F_* and F_{**} in the figure). For each ellipse we have two possible arcs from P_1 to P_2, with different orientation, clockwise and counter-clockwise. The four cases are discussed in Plummer (1909), Plummer (1918), and are distinguished on the basis of the abscissa of the intercept Q of the straight line through P_1 and P_2, on the axis passing through the foci of one of the ellipses, measured from its centre.

In Plummer (1918) the four cases are also distinguished using the region $\mathcal{R}$ whose border is formed by the arc and the chord joining P_1 and P_2, see Fig. A2. We use this criterion for the classification given below.

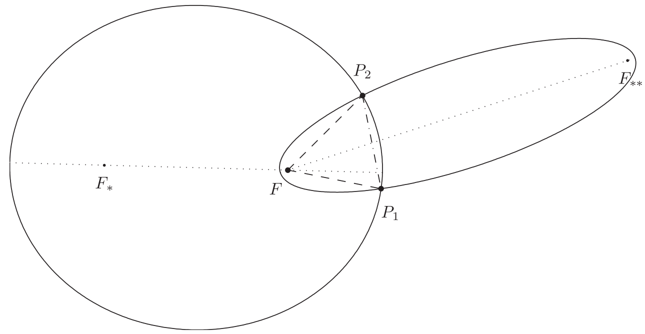

Figure A1. The four cases occurring in Lambert's theorem.

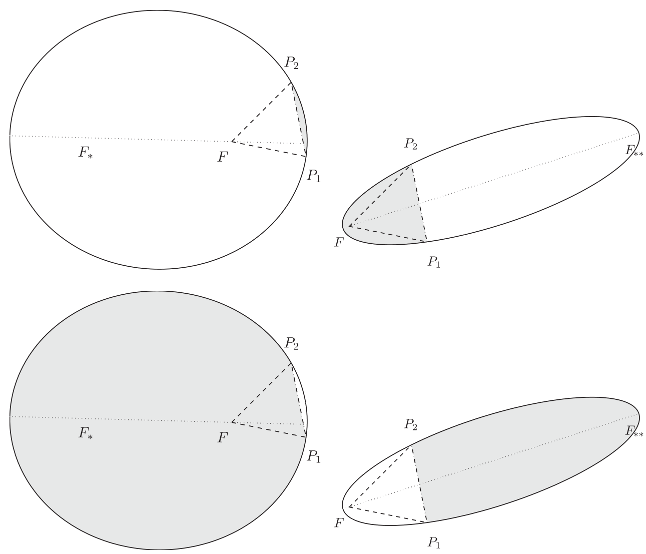

Figure A2. The region $\mathcal{R}$ (filled in grey) corresponding to the four cases.

For a complete list of the equations coming from Lambert's theorem, that need to be considered in our problem, we have to take into account the possible occurrence of multiple revolutions along the orbit. Denoting by n the mean motion, the following expressions for Δt are obtained:[1]

(i) $\Delta t = T_1 + 2k\pi/n$, when the arc covers k revolutions and $\mathcal{R}$ contains neither of the foci;
(ii) $\Delta t = T_2 + 2k\pi/n$, when the arc covers k revolutions, $\mathcal{R}$ contains the attracting focus F but not the other one;
(iii) $\Delta t = -T_1 + 2(k+1)\pi/n$, when the arc covers k revolutions and $\mathcal{R}$ contains both foci;
(iv) $\Delta t = -T_2 + 2(k+1)\pi/n$, when the arc covers k revolutions, $\mathcal{R}$ does not contain the attracting focus F but contains the other one,

where T_1, T_2 are given by

$$nT_1 = \beta_0 - \gamma_0 - (\sin\beta_0 - \sin\gamma_0),$$

$$nT_2 = \beta_0 + \gamma_0 - (\sin\beta_0 + \sin\gamma_0).$$

The four cases above can be summarized in the equation

$$n\Delta t = \beta - \gamma - (\sin\beta - \sin\gamma) + 2k\pi, \quad k \in \mathbb{N},$$

[1] Here the region $\mathcal{R}$ is defined ignoring multiple revolutions.

where the angles β, γ are defined by

$$\sin^2\frac{\beta}{2} = \frac{r+d}{4a}, \qquad \sin^2\frac{\gamma}{2} = \frac{r-d}{4a},$$

and

$$0 \leq \beta - \gamma \leq 2\pi.$$

We also observe that in Prussing (1979) there is a geometrical interpretation for the angles β, γ.

A2 Corrections to the observations

We describe a procedure that could be used to correct the angular positions of a track by a pure geometrical argument.

Assume we have the geocentric position vectors $\boldsymbol{r}_j = \rho_j \boldsymbol{e}^\rho_j + \boldsymbol{q}_j$, $j = 1\ldots 4$, with $\boldsymbol{q}_j$ the geocentric positions of the observer, from the radar observations of the celestial body. The vectors $\boldsymbol{r}_j$ would be coplanar, if the orbit were perfectly Keplerian. In general this holds only approximately, due to the observational errors and to the perturbations which should be added to Kepler's motion. We wish to correct these position vectors and define coplanar vectors $\boldsymbol{r}'_j$, which are slightly different from $\boldsymbol{r}_j$ and keep the measured value ρ_j of the topocentric radial distances. In the attempt to define a good approximation of the plane of this idealized *Kepler* motion, we compute the minimum of the function

$$\boldsymbol{\nu} \mapsto Q(\boldsymbol{\nu}) = \sum_{j=1}^{4} (\boldsymbol{r}_j \cdot \boldsymbol{\nu})^2$$

with the constraint $|\boldsymbol{\nu}| = 1$. We obtain the equation

$$\sum_{j=1}^{4} (\boldsymbol{r}_j \cdot \boldsymbol{\nu})\boldsymbol{r}_j - \lambda\boldsymbol{\nu} = \boldsymbol{0}, \tag{A7}$$

with the Lagrange multiplier $\lambda \in \mathbb{R}$, and consider the solution $\boldsymbol{\nu}_{\min}$ of equation (A7) relative to the minimum eigenvalue $\lambda_{\min}$. We take $\boldsymbol{e}^\nu = \boldsymbol{\nu}_{\min}$ as the direction of the Kepler motion plane, denoted by Π_ν. Then, for each $j = 1, \ldots, 4$, we rotate the vectors $\boldsymbol{\rho}_j = \rho_j \boldsymbol{e}^\rho_j$ into a vector $\mathcal{R}\boldsymbol{\rho}_j$ as follows (see Fig. A3).

Since we want to minimize the change in the line of sight, i.e. the observation direction $\boldsymbol{e}^\rho_j$, we rotate the latter around the axis orthogonal to the plane generated by $\boldsymbol{e}^\nu$, $\boldsymbol{e}^\rho_j$ to reach the plane Π_ν. In this way we draw a geodesic arc on the sphere with radius ρ_j, centred at the observer position defined by $\boldsymbol{q}_j$. This arc joins the position of the observed body with the plane Π_ν.

To describe this procedure in coordinates we introduce the angles $\theta_j, \phi_j \in [0, \pi]$ defined by

$$\cos\theta_j = \frac{\boldsymbol{e}^\nu \cdot \boldsymbol{q}_j}{q_j}, \quad \cos\phi_j = \boldsymbol{e}^\nu \cdot \boldsymbol{e}^\rho_j.$$

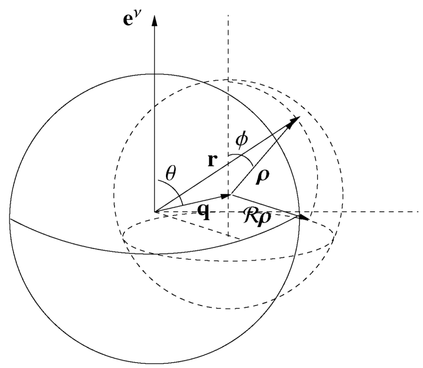

Figure A3. Sketch of the correction of the line of sight. Here we skip the index *j*.

The rotated vector $\mathcal{R}\boldsymbol{\rho}_j$ can be expressed as the linear combination

$$\mathcal{R}\boldsymbol{\rho}_j = A_j \boldsymbol{e}^\nu + B_j \boldsymbol{e}^\rho_j,$$

with $B_j \geq 0$ (since we do not want to rotate the line of sight by more than 90°).

Now set the following conditions:

(i) $|\mathcal{R}\boldsymbol{\rho}_j| = \rho_j$,
(ii) $[\boldsymbol{q}_j + \mathcal{R}\boldsymbol{\rho}_j] \cdot \boldsymbol{e}^\nu = 0.$

We obtain

$$A_j^2 + B_j^2 + 2A_jB_j\cos\phi_j = \rho_j^2,$$
$$q_j\cos\theta_j + A_j + B_j\cos\phi_j = 0.$$

From the second equation we obtain

$$A_j = -q_j\cos\theta_j - B_j\cos\phi_j,$$

that substituted into the first yields

$$B_j = \frac{1}{\sin\phi_j}\sqrt{\rho_j^2 - q_j^2\cos^2\theta_j},$$

so that

$$A_j = -\left(q_j\cos\theta_j + \cot\phi_j\sqrt{\rho_j^2 - q_j^2\cos^2\theta_j}\right).$$

We observe that this procedure works provided

$$\rho_j \geq q_j\cos\theta_j, \qquad \text{for } j = 1, \ldots, 4.$$

This paper has been typeset from a TEX/LATEX file prepared by the author.