INTEGRAL FOLIATED SIMPLICIAL VOLUME OF ASPHERICAL MANIFOLDS

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Abstract. Simplicial volumes measure the complexity of fundamental cycles of manifolds. In this article, we consider the relation between simplicial volume and two of its variants – the stable integral simplicial volume and the integral foliated simplicial volume. The definition of the latter depends on a choice of a measure preserving action of the fundamental group on a probability space.

We show that integral foliated simplicial volume is monotone with respect to weak containment of measure preserving actions and yields upper bounds on (integral) homology growth.

Using ergodic theory we prove that simplicial volume, integral foliated simplicial volume and stable integral simplicial volume coincide for closed hyperbolic 3-manifolds and closed aspherical manifolds with amenable residually finite fundamental group (being equal to zero in the latter case).

However, we show that integral foliated simplicial volume and the classical simplicial volume do *not* coincide for hyperbolic manifolds of dimension at least 4.

1. Introduction

Simplicial volume is a homotopy invariant of manifolds, measuring the complexity of singular fundamental cycles with $\mathbb{R}$-coefficients. The *simplicial volume* of an oriented closed connected n-manifold M is defined as

$$\|M\| := \inf\bigl\{|c|_1 \bigm| c \in C_n(M;\mathbb{R}),\ \partial c = 0,\ [c] = [M]_\mathbb{R}\bigr\} \in \mathbb{R}_{\geqslant 0},$$

where $|c|_1$ denotes the ℓ^1-norm of the singular chain c with respect to the basis of all singular n-simplices in M. Despite its topological definition, simplicial volume carries geometric information and allows for a rich interplay between topological and geometric properties of manifolds [10, 21].

A long-standing purely topological problem on simplicial volume was formulated by Gromov [11, p. 232][13, 3.1. (e) on p. 769]:

Question 1.1. Let M be an oriented closed connected aspherical manifold. Does $\|M\| = 0$ imply $\chi(M) = 0$? Does $\|M\| = 0$ imply the vanishing of L^2-Betti numbers of M?

A possible strategy to answer Question 1.1 in the affirmative is to replace simplicial volume by a suitable integral approximation, and then to use a

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Poincaré duality argument to bound (L^2)-Betti numbers in terms of this integral approximation. Finally, one should relate the integral approximation to the simplicial volume on aspherical manifolds.

One instance of such an integral approximation is stable integral simplicial volume. The *stable integral simplicial volume* of an oriented closed connected manifold M is defined as

$$\|M\|_{\mathbb{Z}}^{\infty} := \inf\Big\{\frac{1}{d}\cdot\|\overline{M}\|_{\mathbb{Z}} \;\Big|\; d \in \mathbb{N} \text{ and } \overline{M} \to M \text{ is a } d\text{-sheeted covering}\Big\},$$

where the integral simplicial volume $\|\overline{M}\|_{\mathbb{Z}}$ is defined like the ordinary simplicial volume but using $\mathbb{Z}$-fundamental cycles.

Another instance of this strategy is integral foliated simplicial volume. A definition of integral foliated simplicial volume and a corresponding L^2-Betti number estimate was suggested by Gromov [12, p. 305f] and confirmed by Schmidt [33]. *Integral foliated simplicial volume* $\lvert M \rvert = \inf_\alpha \lvert M \rvert^{\alpha}$ is defined in terms of fundamental cycles with twisted coefficients in $L^\infty(X,\mathbb{Z})$, where $\alpha = \pi_1(M) \curvearrowright X$ is a probability space with a measure preserving action of the fundamental group of M (see Section 2 for the exact definition).

For all oriented closed connected manifolds M these simplicial volumes fit into the sandwich [23]

$$\|M\| \leqslant \lvert M \rvert \leqslant \|M\|_{\mathbb{Z}}^{\infty}.$$

This leads to the following fundamental problems [23, Question 1.7, Question 7.2, Question 4.21]:

Question 1.2. Do simplicial volume and integral foliated simplicial volume coincide for aspherical manifolds?

Question 1.3. Do integral foliated simplicial volume and stable integral simplicial volume coincide for aspherical manifolds with residually finite fundamental group?

Question 1.4. How does integral foliated simplicial volume depend on the action on the probability space? If it does, is it an interesting dynamical invariant of the action? Which role is played by the Bernoulli shift?

In this article, we contribute (partial) solutions to these questions. On the one hand, we show that integral foliated simplicial volume is compatible with weak containment of measure preserving actions. In combination with results from ergodic theory this shows that integral foliated simplicial volume and stable integral simplicial volume coincide in various cases, e.g., for amenable residually finite or free fundamental group. Moreover, we give a geometric proof of the fact that stable integral simplicial volume, integral foliated simplicial volume, and simplicial volume all are zero for aspherical manifolds with amenable fundamental group.

On the other hand, we show that integral foliated simplicial volume and classical simplicial volume do *not* coincide for hyperbolic manifolds of dimension at least 4. Since hyperbolic manifolds are aspherical, this answers Question 1.2 in the negative (even when restricting to manifolds with residually finite fundamental group).

We now describe these results in more detail.

1.1. **Integral foliated simplicial volume and weak containment.** The measure preserving actions used to define integral foliated simplicial volume are organized into a hierarchy by means of weak containment (see Section 3.1 for the definitions). Integral foliated simplicial volume is compatible with this hierarchy in the following sense (Theorem 3.3):

Theorem 1.5 (monotonicity of integral foliated simplicial volume)**.** *Let M be an oriented closed connected manifold with fundamental group Γ, and let $\alpha = \Gamma \curvearrowright (X, \mu)$ and $\beta = \Gamma \curvearrowright (Y, \nu)$ be free non-atomic standard Γ-spaces with $\alpha \prec \beta$ (i.e., α is weakly contained in β). Then*

$$|M|^{\beta} \leqslant |M|^{\alpha}.$$

In combination with results from ergodic theory, we obtain the following consequences for integral foliated simplicial volume: Bernoulli shift spaces give the maximal (whence bad) value among free measure preserving actions (Corollary 3.4). If the fundamental group satisfies the universality property EMD* (Definition 3.5) from ergodic theory, then integral foliated simplicial volume and stable integral simplicial volume coincide (Corollary 3.6). For instance, this applies to free fundamental groups (Corollary 3.8) and to residually finite amenable fundamental groups (Corollary 3.7).

1.2. **Integral foliated simplicial volume and bounds on homology.** A *chain* in a group Γ is a descending sequence $\Gamma = \Gamma_0 > \Gamma_1 > \Gamma_2 > \dots$ of finite index subgroups. We associate to a chain a measure preserving action on the coset tree, i.e., the inverse limit of the Γ/Γ_i (see Subsection 2.1). We denote the torsion subgroup of a finitely generated abelian group A by $\operatorname{tors} A$; it is a finite abelian group.

Theorem 1.6 (homology bounds)**.** *Let $n \in \mathbb{N}$. Let M be an oriented closed connected n-manifold with fundamental group Γ, let $(\Gamma_i)_i$ be a chain of Γ, and let $M_i \to M$ be the finite covering associated to Γ_i. Let α be the standard Γ-action on the coset tree of $(\Gamma_i)_i$. Then for every $k \in \mathbb{N}$ and for every principal ideal domain R we have (where rk_R denotes the R-dimension of the free part of finitely generated R-modules)*

$$\limsup_{i\to\infty} \frac{\log|\operatorname{tors} H_k(M_i;\mathbb{Z})|}{[\Gamma:\Gamma_i]} \leq \log(n+1)\cdot 2^{n+1}\cdot |M|^{\alpha};$$
$$\limsup_{i\to\infty} \frac{\operatorname{rk}_R H_k(M_i;R)}{[\Gamma:\Gamma_i]} \leq |M|^{\alpha}.$$

We note that by Lück's approximation theorem

$$\limsup_{i\to\infty} \frac{\dim_{\mathbb{Q}} H_k(M_i;\mathbb{Q})}{[\Gamma:\Gamma_i]} = \lim_{i\to\infty} \frac{\dim_{\mathbb{Q}} H_k(M_i;\mathbb{Q})}{[\Gamma:\Gamma_i]}$$

equals the k-th L^2-Betti number of M provided $(\Gamma_i)_i$ is a residual chain, which means that $\Gamma_i < \Gamma$ is normal and the intersection of all Γ_i is the trivial group.

1.3. **Closed hyperbolic manifolds.** Let us now recall what is known about the three types of simplicial volume that we are dealing with for closed hyperbolic manifolds.

A celebrated result by Gromov and Thurston states that the simplicial volume of a closed hyperbolic manifold is equal to the Riemannian volume divided by the volume v_n of the regular ideal geodesic n-simplex in the hyperbolic space $\mathbb{H}^n$ [10, 34]. Notice that the regular ideal geodesic n-simplex is unique up to isometry.

For closed hyperbolic surfaces, it is known that simplicial volume, stable integral simplicial volume, and integral foliated simplicial volume all coincide [23, Example 6.2].

For closed hyperbolic 3-manifolds, the integral foliated simplicial volume is equal to the simplicial volume [23, Theorem 1.1], but the exact relation with stable integral simplicial volume was unknown. In Section 3.3 we complete the picture in dimension 3, showing that the three considered flavours of simplicial volume are equal (Corollary 3.11).

Theorem 1.7. *Let M be an oriented closed connected hyperbolic 3-manifold. Then*
$$\|M\| = \mathopen{|\!|} M \mathclose{|\!|} = \|M\|_{\mathbb{Z}}^{\infty}.$$

This result basically answers a question [8, Question 6.5] in the affirmative that was originally stated for *stable complexity* rather than stable integral simplicial volume (but these very close notions basically play the same role in all applications). The proof is based on Theorem 1.5. Indeed, the fundamental group of a closed hyperbolic 3-manifold has property EMD* (see Definition 3.5). Hence, the equality of integral foliated simplicial volume and stable integral simplicial volume for EMD* groups yield the conclusion.

Theorem 1.7 admits the following geometric interpretation. In their proof of the Ehrenpreis conjecture [16], Kahn and Markovic showed that every closed orientable hyperbolic surface S has a finite covering that decomposes into pairs of pants whose boundary curves have length arbitrarily close to an arbitrarily big constant $R > 0$. Theorem 1.7 provides a sort of 3-dimensional version of this result. Namely, Theorem 1.7 is equivalent to the fact that, for every $\varepsilon > 0$ and $R \gg 0$, every closed hyperbolic 3-manifold M has a finite covering $\widehat{M}$ admitting an integral fundamental cycle $z_{\widehat{M}}$ with the following property: if N is the number of singular simplices appearing in $z_{\widehat{M}}$, then at least $(1-\varepsilon)N$ simplices of $z_{\widehat{M}}$ are ε-close in shape to a regular simplex with edge length bigger than R.

For closed hyperbolic manifolds of dimension at least 4, the stable integral simplicial volume is *not* equal to the simplicial volume. More precisely, Francaviglia, Frigerio, and Martelli proved that the ratio between stable integral simplicial volume and simplicial volume is uniformly strictly bigger than 1 [8, Theorem 2.1].

In Section 5, we generalize this result to integral foliated simplicial volume (Theorem 5.1), as vaguely suggested by Francaviglia, Frigerio, and Martelli [8, Question 6.4]:

Theorem 1.8. *For all $n \in \mathbb{N}_{\geqslant 4}$ there is a constant $C_n \in \mathbb{R}_{<1}$ with the following property: For all oriented closed connected hyperbolic n-manifolds M*

we have

$$\|M\| \leqslant C_n \cdot \lvert M \rvert .$$

For the proof, we notice that in dimension at least 4, the dihedral angle of the regular ideal simplex does not divide 2π and in the same spirit as for stable integral simplicial volume [8] we show that foliated integral cycles cannot be used to produce efficient fundamental cycles computing the simplicial volume. Indeed, every fundamental cycle contains simplices with volume significantly smaller than v_n or there are overlappings producing loss of volume. To prove our statement we need to carefully estimate this loss of volume.

1.4. **Closed amenable manifolds.** We will now refer to an oriented closed connected manifold with amenable fundamental group as *closed amenable manifold*. It is well-known that the simplicial volume of closed amenable manifolds vanishes [10, 15]. This result relies on bounded cohomology techniques that cannot be exploited in the context of integral coefficients.

For finite fundamental groups, integral foliated simplicial volume and stable integral simplicial are equal (and non-zero) [23, Corollary 6.3]. Moreover, if a manifold splits off an S^1-factor, then the integral foliated simplicial volume vanishes [33, Chapter 5.2]. Sauer introduced an invariant related to the integral foliated simplicial volume and provided an upper bound of this invariant in terms of the minimal volume; moreover, for closed amenable aspherical manifolds this invariant vanishes [31, Section 3].

Theorem 1.9. *Let M be an oriented closed connected aspherical manifold of non-zero dimension with amenable fundamental group Γ. Let $\alpha = \Gamma \curvearrowright (X, \mu)$ be a free standard Γ-space. Then*

$$\lvert M \rvert = \lvert M \rvert^{\alpha} = 0.$$

The first statement of the next theorem is Corollary 3.7. The second statement about vanishing is a combination of Theorem 1.9 applied to the action of Γ on its profinite completion (cf. Subsection 2.1) and the fact that $\lvert M \rvert^{\widehat{\Gamma}} = \|M\|_{\mathbb{Z}}^{\infty}$ (Theorem 2.6).

Theorem 1.10. *Let M be an oriented closed connected manifold with residually finite amenable fundamental group Γ. Then*

$$\lvert M \rvert = \lvert M \rvert^{\widehat{\Gamma}} = \|M\|_{\mathbb{Z}}^{\infty}.$$

where $\widehat{\Gamma}$ denotes the profinite completion of Γ. If, in addition, M is aspherical, then

$$\lvert M \rvert = \lvert M \rvert^{\widehat{\Gamma}} = \|M\|_{\mathbb{Z}}^{\infty} = 0.$$

By Theorem 1.9 applied to the action of Γ on the coset tree associated to a Farber chain (cf. Subsection 2.1) and by Theorem 1.6 we obtain:

Theorem 1.11. *Let M be an oriented closed connected aspherical manifold with amenable fundamental group Γ. Let $(\Gamma_i)_i$ be a Farber chain of Γ, and let $M_i \to M$ be the finite covering associated to Γ_i. For every integer $k \geq 0$*

and for every principal ideal domain R we have

$$\limsup_{i\to\infty} \frac{\log|\operatorname{tors} H_k(M_i;\mathbb{Z})|}{[\Gamma:\Gamma_i]} = 0;$$
$$\limsup_{i\to\infty} \frac{\operatorname{rk}_R H_k(M_i;R)}{[\Gamma:\Gamma_i]} = 0.$$

Using different methods, A. Kar, P. Kropholler and N. Nikolov recently proved a more general form of the above theorem for simplicial complexes that are not necessarily aspherical but whose k-th homology of the universal covering vanishes [17]. Earlier, the above statement was shown for residual chains and under the assumption that Γ has a normal infinite elementary amenable subgroup by Lück [27] and for general amenable fundamental groups and residual chains by Sauer [32].

Note that the middle equation in the above statement is a well known result; it follows from Lück's approximation theorem [26] and the vanishing of L^2-Betti numbers of amenable groups by Cheeger and Gromov [5].

In view of the original, motivating problem about simplicial volume and the Euler characteristic it would be interesting to know the answer to the following question:

Question 1.12. Let M be an oriented closed connected aspherical manifold with $\|M\| = 0$. Does this imply $|M| = 0$? If $\pi_1(M)$ is residually finite, does this imply $\|M\|_{\mathbb{Z}}^{\infty} = 0$?

1.5. **Analogies between integral foliated simplicial volume and cost.** We draw an analogy between integral foliated simplicial volume and cost. The latter invariant was introduced by Gaboriau [9].

Let $\alpha = \Gamma \curvearrowright (X,\mu)$ be a standard Γ-action, and let M be an n-dimensional closed aspherical manifold with $\pi_1(M) = \Gamma$. Then $|M|^{\alpha}$ can be regarded as an invariant of α. There is no direct relation between $|M|^{\alpha}$ and the cost of α, nor between $|M|$ and the cost of Γ which is defined as the infimum of the costs of all free standard Γ-actions. However, there are similarities. The cost of Γ can be thought of as an ergodic-theoretic version of the rank of Γ (i.e., the minimal number of generators or, equivalently, the minimal number of Γ-orbits in a Caley graph of Γ). The integral foliated simplicial volume can be thought of as an ergodic-theoretic version of the minimal number of Γ-orbits of n-simplices in a simplicial model of the classifying space of Γ. Theorem 1.5 was inspired by an analogous theorem for the cost by Abert and Weiss [2]. Gaboriau's fixed price problem asks whether the costs of two free standard Γ-actions always coincide. The following analog is also a more specific instance of Question 1.4.

Question 1.13 (analog of fixed price problem)**.** Let M be an oriented closed connected aspherical manifold with fundamental group Γ. Let α and β be free standard Γ-actions. Does $|M|^{\alpha} = |M|^{\beta}$ hold?

Organization of this article. In Section 2, we recall the exact definition of integral foliated simplicial volume. The behaviour of integral foliated simplicial volume with respect to weak containment and applications thereof are studied in Section 3. The homology bounds by integral foliated simplicial volume are discussed in Section 4. The higher-dimensional hyperbolic case is treated in Section 5, the amenable aspherical case in Section 6.

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2. Integral foliated simplicial volume

Integral foliated simplicial volume mixes the rigidity of integral coefficients with the flexibility of probability spaces [12, p. 305f][33]. In the following, we recall the exact definition and collect some notation and terminology. More background on integral foliated simplicial volume and its basic relations with simplicial volume and stable integral simplicial volume can be found in the literature [23].

2.1. **Probability measure preserving actions.** A *standard Borel space* is a measurable space that is isomorphic to a Polish space with its Borel σ-algebra. A *standard Borel probability space* is a standard Borel space endowed with a probability measure. More information on the convenient category of standard Borel spaces can be found in the book by Kechris [18].

Let Γ be a countable group. A *standard* Γ*-space* is a standard Borel probability space (X, μ) together with a measurable μ-preserving (left) Γ-action. If $\alpha = \Gamma \curvearrowright (X, \mu)$ is a standard Γ-space, then we denote the action of $g \in \Gamma$ on $x \in X$ also by $g^\alpha(x)$. Standard Γ-spaces $\alpha = \Gamma \curvearrowright (X, \mu)$ and $\beta = \Gamma \curvearrowright (Y, \nu)$ are *isomorphic*, $\alpha \cong_\Gamma \beta$, if there exist probability measure preserving Γ-equivariant measurable maps $X \longrightarrow Y$ and $Y \longrightarrow X$ defined on subsets of full measure that are mutually inverse up to null sets. A standard Γ-space is *(essentially) free* or *ergodic* if the Γ-action is free on a subset of full measure or ergodic respectively. We describe two important examples of standard Γ-spaces: Bernoulli-shifts and profinite actions coming from chains of subgroups.

Let B be a standard Borel probability space. The *Bernoulli shift of* Γ *with base* B is the standard Borel space B^Γ with the product probability measure and the left translation action of Γ. If Γ is an infinite countable group and B is non-trivial, then the Bernoulli shift B^Γ is essentially free and mixing (thus ergodic) [29, p. 58].

A *chain* in a group Γ is a descending sequence $\Gamma = \Gamma_0 > \Gamma_1 > \Gamma_2 > \dots$ of finite index subgroups. The *coset tree* X of the chain is the inverse limit

$$X = \varprojlim_{i \in \mathbb{N}} \Gamma/\Gamma_i$$

of the finite Γ-spaces Γ/Γ_i. Further, the profinite space X carries a Γ-invariant Borel probability measure μ that is characterized by its pushforward to every Γ/Γ_i being the normalized counting measure. The Γ-action on the standard Γ-space (X, μ) is ergodic [1, Chapter 3]. If the chain consists of normal subgroups whose intersection is trivial (a so-called *residual* *chain*), then the Γ-action on X is essentially free. One calls the chain *Farber* if the Γ-action on (X, μ) is essentially free; this notion also admits a group-theoretic characterization [6, (0-1) in Theorem 0.3.].

More generally, instead of taking the inverse limit over a chain of subgroups, one can also take an inverse limit over a system of subgroups, directed by inclusion. The *profinite completion* $\widehat{\Gamma}$ is defined as

$$\widehat{\Gamma} := \varprojlim_{\Lambda \in S} \Gamma/\Lambda$$

where S is the directed system of all finite index subgroups of Γ. Then $\widehat{\Gamma}$ is a profinite group. The unique Borel probability measure μ that is pushed forward to the normalized counting measures on the finite quotients is the normalized Haar measure of $\widehat{\Gamma}$. Similarly to the case of coset trees, one sees that the left translation action of Γ on $\widehat{\Gamma}$ is ergodic; this action is essentially free if and only if Γ is residually finite.

2.2. **Parametrized fundamental cycles.**

Definition 2.1 (parametrized fundamental cycles)**.** Let M be an oriented closed connected n-manifold with fundamental group Γ and universal covering $\widetilde{M} \longrightarrow M$.

- If $\alpha = \Gamma \curvearrowright (X, \mu)$ is a standard Γ-space, then we equip $L^\infty(X, \mu, \mathbb{Z})$ with the right Γ-action
$$\begin{aligned} L^\infty(X,\mu,\mathbb{Z}) \times \Gamma &\longrightarrow L^\infty(X,\mu,\mathbb{Z}) \\ (f,g) &\longmapsto g^\alpha(f) := \bigl(x \mapsto f(g^\alpha(x))\bigr). \end{aligned}$$
and we write i_M^α for the change of coefficients homomorphism
$$\begin{aligned} i_M^\alpha \colon C_*(M;\mathbb{Z}) \cong \mathbb{Z} \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M};\mathbb{Z}) &\longrightarrow L^\infty(X,\mu,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M};\mathbb{Z}) =: C_*(M;\alpha) \\ 1 \otimes c &\longmapsto 1 \otimes c \end{aligned}$$
induced by the inclusion $\mathbb{Z} \hookrightarrow L^\infty(X, \mu, \mathbb{Z})$ as constant functions.
- If $\alpha = \Gamma \curvearrowright (X, \mu)$ is a standard Γ-space, then
$$\begin{aligned} [M]^\alpha := H_n(i_M^\alpha)([M]_\mathbb{Z}) &\in H_n\bigl(M;\alpha\bigr) \\ &= H_n\bigl(L^\infty(X,\mu,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M};\mathbb{Z})\bigr) \end{aligned}$$
is the *α-parametrized fundamental class of M*. All cycles in the chain complex $C_*(M;\alpha) = L^\infty(X,\mu,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M};\mathbb{Z})$ representing $[M]^\alpha$ are called *α-parametrized fundamental cycles of M*.

Integral foliated simplicial volume is the infimum of ℓ^1-norms over all parametrized fundamental cycles:

Definition 2.2 (integral foliated simplicial volume)**.** Let M be an oriented closed connected n-manifold with fundamental group Γ, and let $\alpha = \Gamma \curvearrowright (X, \mu)$ be a standard Γ-space.

- Let $c = \sum_{j=1}^k f_j \otimes \sigma_j \in C_*(M;\alpha)$ be a chain in *reduced form*, i.e., the singular simplices $\sigma_1, \dots, \sigma_k$ on $\widetilde{M}$ satisfy $\pi \circ \sigma_j \neq \pi \circ \sigma_\ell$ for all $j, \ell \in \{1, \dots, k\}$ with $j \neq \ell$ (where $\pi \colon \widetilde{M} \longrightarrow M$ is the universal covering map). Then we define

$$|c|^{\alpha} := \left|\sum_{j=1}^{k} f_j \otimes \sigma_j\right|^{\alpha} := \sum_{j=1}^{k} \int_X |f_j| \, d\mu \in \mathbb{R}_{\geqslant 0}.$$

(Clearly, all reduced forms of a given chain lead to the same ℓ^1-norm because the probability measure is Γ-invariant.)

- The *α-parametrized simplicial volume of M*, denoted by $|M|^{\alpha}$, is the infimum of the ℓ^1-norms of all α-parametrized fundamental cycles of M.
- The *integral foliated simplicial volume of M*, denoted by $|M|$, is the infimum of all $|M|^{\alpha}$ over all isomorphism classes of standard Γ-spaces α.

Remark 2.3. Notice that oriented closed connected manifolds have countable fundamental groups. If M is an oriented closed connected manifold with fundamental group Γ and if α and β are standard Γ-spaces with $\alpha \cong_{\Gamma} \beta$, then $|M|^{\alpha} = |M|^{\beta}$. Moreover, if Γ is a countable group, then the class of isomorphism classes of standard Γ-spaces indeed forms a set [33, Remark 5.26].

Remark 2.4. Let M be an oriented closed connected manifold with fundamental group Γ and let $\alpha = \Gamma \curvearrowright (X, \mu)$ be a standard Γ-space. Let $D \subset \widetilde{M}$ be a (set-theoretical, strict) fundamental domain for the Γ-action on $\widetilde{M}$ by deck transformations. Let $c = \sum_{j=1}^{k} f_j \otimes \sigma_j \in C_*(M; \alpha)$ be a chain where the (not necessarily distinct!) singular simplices $\sigma_1, \dots, \sigma_k$ all have their 0-vertex in D, and where $f_1, \dots, f_k \in L^{\infty}(X, \mu, \mathbb{Z})$. Then

$$|c|^{\alpha} = \left|\sum_{j=1}^{k} f_j \otimes \sigma_j\right|^{\alpha} = \left|\sum_{j=1}^{k} f_j \otimes \sigma_j\right|^{(X,\mu)},$$

where $|\cdot|^{(X,\mu)}$ is the corresponding ℓ^1-norm on $L^{\infty}(X, \mu, \mathbb{Z}) \otimes_{\mathbb{Z}} C_*(\widetilde{M}; \mathbb{Z})$. I.e., for chains that do not contain different singular simplices from the same Γ-orbit, we can compute the ℓ^1-norm also in the non-equivariant chain complex. This will be convenient below when considering potentially different actions on the *same* probability space.

For the sake of completeness, we also describe the relation between parametrized fundamental cycles and locally finite fundamental cycles of the universal covering.

Lemma 2.5 (parametrized fundamental cycles yield locally finite fundamental cycles)**.** *Let M be an oriented closed connected n-manifold with fundamental group Γ, let $\widetilde{M}$ be its universal covering and let $\alpha = \Gamma \curvearrowright (X, \mu)$ be a standard Γ-space. Moreover, let $c = \sum_{j=1}^{k} f_j \otimes \sigma_j \in C_n(M; \alpha)$ be an α-parametrized fundamental cycle of M. Then for μ-a.e. $x \in X$ the chain*

$$c_x := \sum_{\gamma \in \Gamma} \sum_{j=1}^{k} f_j(\gamma^{-1} \cdot x) \cdot \gamma \cdot \sigma_j$$

is a well-defined locally finite fundamental cycle of $\widetilde{M}$*.*

Proof. Let $B(\alpha,\mathbb{Z})$ denote the abelian group of all (strictly) bounded, measurable, everywhere defined functions of type $X \longrightarrow \mathbb{Z}$, and let $N(\alpha,\mathbb{Z}) \subset B(\alpha,\mathbb{Z})$ be the subgroup of μ-a.e. vanishing functions. Then $L^\infty(X,\mu,\mathbb{Z}) = B(\alpha,\mathbb{Z})/N(\alpha,\mathbb{Z})$ as $\mathbb{Z}\Gamma$-modules, where we equip $B(\alpha,\mathbb{Z})$ and $N(\alpha,\mathbb{Z})$ with the obvious right Γ-actions.

For $x \in X$ there is a well-defined evaluation chain map

$$\begin{aligned} \varphi_x \colon B(\alpha,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M};\mathbb{Z}) &\longrightarrow C_*^{\mathrm{lf}}(\widetilde{M};\mathbb{Z}) \\ f \otimes \sigma &\longmapsto \sum_{\gamma\in\Gamma} f(\gamma^{-1}\cdot x)\cdot\gamma\cdot\sigma; \end{aligned}$$

notice that the sum on the right hand side indeed is locally finite because Γ acts properly discontinuously on $\widetilde{M}$ by deck transformations.

Let $c_\mathbb{Z} \in C_n(M;\mathbb{Z}) \cong \mathbb{Z}\otimes_{\mathbb{Z}\Gamma} C_n(\widetilde{M};\mathbb{Z})$ be a fundamental cycle of M. Then we can view $c_\mathbb{Z}$ (via constant functions) as a chain in $B(\alpha,\mathbb{Z})\otimes_{\mathbb{Z}\Gamma} C_n(\widetilde{M};\mathbb{Z})$ and $\varphi_x(c_\mathbb{Z})$ is the transfer of $c_\mathbb{Z}$ to $\widetilde{M}$ and thus is a locally finite fundamental cycle of $\widetilde{M}$.

If $c \in C_n(M;\alpha)$ is an α-parametrized fundamental cycle, then c can be represented by a chain $c' \in B(\alpha,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_n(\widetilde{M};\mathbb{Z})$ such that there exist $b \in B(\alpha,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_{n+1}(\widetilde{M};\mathbb{Z})$ and $z \in N(\alpha,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_n(\widetilde{M};\mathbb{Z})$ with [23, Remark 4.20]

$$c' = c_\mathbb{Z} + \partial b + z.$$

Therefore, for μ-a.e. $x \in X$ we have

$$c_x = \varphi_x(c') = \varphi_x(c_\mathbb{Z} + \partial b) = \varphi_z(c_\mathbb{Z}) + \partial\varphi_x(b),$$

which is a locally finite fundamental cycle of $\widetilde{M}$. □

2.3. Relation between integral foliated simplicial volume and stable integral simplicial volume.

Actions on coset trees and the profinite completion provide the link between integral foliated simplicial volume and stable integral simplicial volume.

Theorem 2.6. *Let M be an oriented closed connected manifold with fundamental group Γ. Then*

$$|M| \leqslant |M|^{\widehat{\Gamma}} = \|M\|_\mathbb{Z}^\infty.$$

More specifically, if $(\Gamma_i)_i$ is a chain of Γ and $\alpha = \Gamma \curvearrowright (X,\mu)$ the corresponding action on the coset tree, then

$$|M|^\alpha = \lim_{i\to\infty} \frac{\|M_i\|_\mathbb{Z}}{[\Gamma:\Gamma_i]}.$$

where $M_i \to M$ is the covering associated to $\Gamma_i \subset \Gamma = \pi_1(M)$.

Proof. The first statement is proved by Löh and Pagliantini [23, Theorem 6.6], and the proof [23, Remark 6.7] also shows that

$$|M|^\alpha = \inf_{i\to\infty} \frac{\|M_i\|_\mathbb{Z}}{[\Gamma:\Gamma_i]}.$$

But the infimum is actually a limit: Let $(a(i))_i$ be a sequence in $\mathbb{N}$ converging to ∞ such that

$$\liminf_{i\to\infty} \frac{\|M_i\|_\mathbb{Z}}{[\Gamma : \Gamma_i]} = \lim_{i\to\infty} \frac{\|M_{a(i)}\|_\mathbb{Z}}{[\Gamma : \Gamma_{a(i)}]}.$$

The standard Γ-action on the coset tree associated to the chain $(\Gamma_{a(i)})_i$ is clearly isomorphic to the one on the coset tree associated to the chain $(\Gamma_i)_i$. Hence by the aforementioned result of Löh and Pagliantini we obtain that

$$| M |^{\alpha} = \inf_{i\in\mathbb{N}} \frac{\|M_{a(i)}\|_\mathbb{Z}}{[\Gamma : \Gamma_{a(i)}]} = \liminf_{i\to\infty} \frac{\|M_i\|_\mathbb{Z}}{[\Gamma : \Gamma_i]}.$$

One argues analogously for the limit superior. This concludes the proof. $\square$

3. Integral foliated simplicial volume and weak containment

Probability measure preserving actions are organized into a hierarchy by means of weak containment. We recall the notion of weak containment and its main properties in Section 3.1. In Section 3.2, we will prove monotonicity of integral foliated simplicial volume with respect to weak containment. Some simple consequences of this monotonicity are discussed in Section 3.3.

3.1. Weak containment.

We first recall Kechris's notion of weak containment [19, 20] and its relation with the weak topology on the space of actions on a given standard Borel probability space.

Definition 3.1 (weak containment)**.** Let Γ be a countable group, and let $\alpha = \Gamma \curvearrowright (X,\mu)$ and $\beta = \Gamma \curvearrowright (Y,\nu)$ be standard Γ-spaces. Then α *is weakly contained in* β if the following holds: For all $\varepsilon \in \mathbb{R}_{>0}$, all finite subsets $F \subset \Gamma$, all $m \in \mathbb{N}$, and all Borel sets $A_1, \dots, A_m \subset X$ there exist Borel subsets $B_1, \dots, B_m \subset Y$ with

$$\forall_{\gamma\in F}\,\forall_{j,k\in\{1,\dots,m\}} \quad \bigl|\mu(\gamma^\alpha(A_j)\cap A_k) - \nu(\gamma^\beta(B_j)\cap B_k)\bigr| < \varepsilon.$$

In this case, we write $\alpha \prec \beta$. We call α and β *weakly equivalent* if $\alpha \prec \beta$ and $\beta \prec \alpha$.

For example, if the standard Γ-space α is a factor of a standard Γ-space β, then $\alpha \prec \beta$ holds.

We will use the following characterization of weak containment:

Proposition 3.2 (weak containment vs. weak closure [19, Proposition 10.1])**.** *Let Γ be a countable group and let $\alpha = \Gamma \curvearrowright (X,\mu)$ and $\beta = \Gamma \curvearrowright (Y,\nu)$ be non-atomic standard Γ-spaces. Then $\alpha \prec \beta$ if and only if α lies in the closure of*

$$\bigl\{\gamma \in A(\Gamma, X, \mu) \bigm| \gamma \cong_\Gamma \beta\bigr\}$$

in $A(\Gamma, X, \mu)$ with respect to the weak topology.

Here, $A(\Gamma, X, \mu)$ denotes the set of all μ-preserving actions of Γ on the standard Borel probability space (X, μ) by Borel isomorphisms. The weak topology on $A(\Gamma, X, \mu)$ is defined as follows: The set $\operatorname{Aut}(X,\mu)$ of Borel automorphisms of (X,μ) carries a weak topology with respect to the family of all evaluation maps $\varphi \mapsto \varphi(A)$ associated with Borel subsets $A \subset X$. I.e., if $\varphi \in \operatorname{Aut}(X,\mu)$, then the family of sets of the type

$$\bigl\{\psi \in \operatorname{Aut}(X,\mu) \bigm| \forall_{j\in\{1,\dots,m\}}\ \mu\bigl(\varphi(A_j) \mathbin{\triangle} \psi(A_j)\bigr) < \delta\bigr\}$$

where $\delta \in \mathbb{R}_{>0}$, $m \in \mathbb{N}$, and $A_1, \dots, A_m \subset X$ are Borel subsets is an open neighbourhood basis of the weak topology on $\operatorname{Aut}(X,\mu)$ [19, Chapter 1(B)]. Viewing $A(\Gamma, X, \mu)$ as a subset of the product $\operatorname{Aut}(X,\mu)^\Gamma$ then induces a topology on $A(\Gamma, X, \mu)$, which is also called *weak topology*.

3.2. **Monotonicity of integral foliated simplicial volume under weak containment.** We now prove the following monotonicity result:

Theorem 3.3 (monotonicity of integral foliated simplicial volume)**.** *Let M be an oriented closed connected manifold with fundamental group Γ, and let $\alpha = \Gamma \curvearrowright (X,\mu)$ and $\beta = \Gamma \curvearrowright (Y,\nu)$ be free non-atomic standard Γ-spaces with $\alpha \prec \beta$. Then*

$$|M|^\beta \leqslant |M|^\alpha.$$

Proof. Notice that in the case of finite fundamental group Γ, every free standard Γ-space α satisfies $|M|^\alpha = 1/|\Gamma| \cdot \|\widetilde{M}\|_\mathbb{Z}$ [23, Proposition 4.26 and Example 4.5]. Therefore, we now focus on the infinite case.

Let $n := \dim M$, let $c \in C_n(M;\alpha)$ be an α-parametrized fundamental cycle of M, and let $\varepsilon \in \mathbb{R}_{>0}$. Taking the infimum over all such fundamental cycles and all such ε shows that it is sufficient to prove $|M|^\beta \leqslant |c|^\alpha + \varepsilon$. To this end, we show that there exists a standard Γ-space $\gamma \in A(\Gamma, X, \mu)$ with $\gamma \cong_\Gamma \beta$ and $|M|^\beta = |M|^\gamma \leqslant |c|^\alpha + \varepsilon$.

As a first step, we write c suitably in reduced form. Because c is an α-parametrized fundamental cycle of M there exist an integral fundamental cycle $z \in C_n(M;\mathbb{Z})$ and a chain $b \in C_{n+1}(M;\alpha)$ with

$$c = z + \partial b \in C_n(M;\alpha);$$

here, we view $C_*(M;\mathbb{Z})$ as a subcomplex of $C_*(M;\alpha)$ via the inclusion of $\mathbb{Z} \hookrightarrow L^\infty(X,\mu,\mathbb{Z})$ as constant functions. Let $D \subset \widetilde{M}$ be a (set-theoretical, strict) fundamental domain for the deck transformation action of Γ on $\widetilde{M}$. We can then write

$$z = \sum_{\sigma \in S} a_\sigma \otimes \sigma \in C_n(M;\alpha),$$
$$b = \sum_{\tau \in T} f_\tau \otimes \tau \in C_{n+1}(M;\alpha),$$

where $S \subset \operatorname{map}(\Delta^n, \widetilde{M})$, $T \subset \operatorname{map}(\Delta^{n+1}, \widetilde{M})$ are finite subsets of singular simplices whose 0-vertex lies in D, and where $f_\tau \in L^\infty(X,\mu,\mathbb{Z})$ are essentially bounded measurable functions and $a_\sigma \in \mathbb{Z} \subset L^\infty(X,\mu,\mathbb{Z})$ are constant functions. Without loss of generality, we may assume that the f_τ are represented as bounded (and not only essentially bounded) functions and that at least one of the f_τ is not constant 0. We then obtain in $C_*(M;\alpha)$

$$\begin{aligned} c &= \sum_{\sigma \in S} a_\sigma \otimes \sigma + \partial\Bigl(\sum_{\tau \in T} f_\tau \otimes \tau\Bigr) \\ &= \sum_{\sigma \in S} a_\sigma \otimes \sigma + \sum_{j=1}^{n+1} \sum_{\tau \in T} (-1)^j \cdot f_\tau \otimes (\tau \circ i_j) + \sum_{\tau \in T} g_\tau^\alpha(f_\tau) \otimes \tau_0; \end{aligned}$$

here, for $j \in \{0, \dots, n+1\}$, we write $i_j \colon \Delta^n \longrightarrow \Delta^{n+1}$ for the inclusion of the j-th face, and for $\tau \in T$, we let $g_\tau \in \Gamma$ be the unique element satisfying

$$g_\tau^{-1} \cdot (\tau \circ i_0)(e_0) = g_\tau^{-1} \cdot \tau(e_1) \in D,$$

and we set $\tau_0 := g_\tau^{-1} \cdot \tau$. Hence, in the above representation of c all singular simplices have their 0-vertex in D. In view of Remark 2.4 we therefore have

$$|c|^{\alpha} = \left| \sum_{\sigma \in S} a_\sigma \otimes \sigma + \sum_{j=1}^{n+1} \sum_{\tau \in T} (-1)^j \cdot f_\tau \otimes (\tau \circ i_j) + \sum_{\tau \in T} g_\tau^{\alpha}(f_\tau) \otimes \tau_0 \right|^{(X,\mu)}.$$

As next step we bring the characterization of weak containment via the weak topology (Proposition 3.2) into play. We choose a finite Borel partition $X = A_1 \sqcup \dots \sqcup A_m$ of X that is finer than the (finite) set

$$\bigl\{ f_\tau^{-1}(k) \subset X \bigm| k \in \mathbb{Z}, \tau \in T \bigr\},$$

and we consider

$$\delta := \frac{\varepsilon}{m \cdot \sum_{\tau \in T} \|f_\tau\|_\infty} \in \mathbb{R}_{>0}$$

as well as the finite set

$$F := \{ g_\tau^{-1} \mid \tau \in T \} \subset \Gamma.$$

By Proposition 3.2 there is a standard Γ-space $\gamma \in A(\Gamma, X, \mu)$ with $\gamma \cong_\Gamma \beta$ and

$$\forall_{g \in F} \quad \forall_{j \in \{1, \dots, m\}} \quad \mu\bigl(g^{\alpha}(A_j) \mathbin{\triangle} g^{\gamma}(A_j)\bigr) < \delta.$$

Finally, we consider the chain $c' \in C_n(M; \gamma)$ that is represented by the chain

$$\sum_{\sigma \in S} a_\sigma \otimes \sigma + \sum_{j=1}^{n+1} \sum_{\tau \in T} (-1)^j \cdot f_\tau \otimes (\tau \circ i_j) + \sum_{\tau \in T} g_\tau^{\gamma}(f_\tau) \otimes \tau_0$$

from $L^\infty(X, \mu, \mathbb{Z}) \otimes_{\mathbb{Z}} C_n(\widetilde{M}; \mathbb{Z})$. Then the same calculation as in the first step shows that

$$c' = z + \partial \biggl(\sum_{\tau \in T} f_\tau \otimes \tau \biggr)$$

holds in $C_*(M; \gamma)$ and that

$$|c'|^{\gamma} = \left| \sum_{\sigma \in S} a_\sigma \otimes \sigma + \sum_{j=1}^{n+1} \sum_{\tau \in T} (-1)^j \cdot f_\tau \otimes (\tau \circ i_j) + \sum_{\tau \in T} g_\tau^{\gamma}(f_\tau) \otimes \tau_0 \right|^{(X,\mu)}.$$

In particular, c' is a γ-parametrized fundamental cycle and

$$\begin{aligned}
&\bigl| \boldsymbol{|} c \boldsymbol{|}^{\alpha} - \boldsymbol{|} c' \boldsymbol{|}^{\gamma} \bigr| \\
&= \Biggl| \Bigl\boldsymbol{|} \sum_{\sigma\in S} a_\sigma \otimes \sigma + \sum_{j=1}^{n+1} \sum_{\tau\in T} (-1)^j \cdot f_\tau \otimes (\tau \circ i_j) + \sum_{\tau\in T} g^{\alpha}_{\tau}(f_\tau) \otimes \tau_0 \Bigr\boldsymbol{|}^{(X,\mu)} \\
&\quad - \Bigl\boldsymbol{|} \sum_{\sigma\in S} a_\sigma \otimes \sigma + \sum_{j=1}^{n+1} \sum_{\tau\in T} (-1)^j \cdot f_\tau \otimes (\tau \circ i_j) + \sum_{\tau\in T} g^{\gamma}_{\tau}(f_\tau) \otimes \tau_0 \Bigr\boldsymbol{|}^{(X,\mu)} \Biggr| \\
&\leqslant \Bigl\boldsymbol{|} \sum_{\sigma\in S} a_\sigma \otimes \sigma + \sum_{j=1}^{n+1} \sum_{\tau\in T} (-1)^j \cdot f_\tau \otimes (\tau \circ i_j) + \sum_{\tau\in T} g^{\alpha}_{\tau}(f_\tau) \otimes \tau_0 \\
&\quad - \Bigl(\sum_{\sigma\in S} a_\sigma \otimes \sigma + \sum_{j=1}^{n+1} \sum_{\tau\in T} (-1)^j \cdot f_\tau \otimes (\tau \circ i_j) + \sum_{\tau\in T} g^{\gamma}_{\tau}(f_\tau) \otimes \tau_0 \Bigr) \Bigr\boldsymbol{|}^{(X,\mu)} \\
&\leqslant \sum_{\tau\in T} \bigl\| g^{\alpha}_{\tau}(f_\tau) - g^{\gamma}_{\tau}(f_\tau) \bigr\|_\infty.
\end{aligned}$$

In the second step we used the (reverse) triangle inequality for $\boldsymbol{|} \cdot \boldsymbol{|}^{(X,\mu)}$; in the third step we used the definition of the ℓ^1-norm on $L^\infty(X,\mu,\mathbb{Z}) \otimes_{\mathbb{Z}} C_n(\widetilde{M};\mathbb{Z})$ and the triangle inequality.

For each $\tau \in T$, we can write

$$f_\tau = \sum_{j=1}^{m} a_{\tau,j} \cdot \chi_{A_j} \in L^\infty(X,\mu,\mathbb{Z})$$

with certain $a_{\tau,1},\dots,a_{\tau,m} \in \mathbb{Z}$. Hence,

$$\begin{aligned}
\bigl\| g^{\alpha}_{\tau}(f_\tau) - g^{\gamma}_{\tau}(f_\tau) \bigr\|_\infty &\leqslant \sum_{j=1}^{m} |a_{\tau,j}| \cdot \bigl\| g^{\alpha}_{\tau}(\chi_{A_j}) - g^{\gamma}_{\tau}(\chi_{A_j}) \bigr\|_\infty \\
&\leqslant \sum_{j=1}^{m} |a_{\tau,j}| \cdot \bigl\| \chi_{(g_\tau^{-1})^{\alpha}(A_j)} - \chi_{(g_\tau^{-1})^{\gamma}(A_j)} \bigr\|_\infty \\
&= \sum_{j=1}^{m} |a_{\tau,j}| \cdot \mu\bigl((g_\tau^{-1})^{\alpha}(A_j) \triangle (g_\tau^{-1})^{\gamma}(A_j) \bigr) \\
&\leqslant m \cdot \|f_\tau\|_\infty \cdot \delta.
\end{aligned}$$

Therefore, we have

$$\bigl| \boldsymbol{|} c \boldsymbol{|}^{\alpha} - \boldsymbol{|} c' \boldsymbol{|}^{\gamma} \bigr| \leqslant m \cdot \sum_{\tau\in T} \|f_\tau\|_\infty \cdot \delta \leqslant \varepsilon.$$

In particular, $\boldsymbol{|} M \boldsymbol{|}^{\beta} = \boldsymbol{|} M \boldsymbol{|}^{\gamma} \leqslant \boldsymbol{|} c' \boldsymbol{|}^{\gamma} \leqslant \boldsymbol{|} c \boldsymbol{|}^{\alpha} + \varepsilon$, as desired. □

3.3. **Consequences.** In the following, we will combine Theorem 3.3 with known results from ergodic theory on weak containment. In particular, we will consider Bernoulli shifts and the profinite completion of residually finite groups.

Corollary 3.4 (maximality of Bernoulli shifts)**.** *Let M be an oriented closed connected manifold with infinite fundamental group Γ and let B be a non-trivial standard Borel probability space. Then*

$$\lvert M \rvert^{\alpha} \leqslant \lvert M \rvert^{B^{\Gamma}}$$

holds for all free standard Γ-spaces α.

Proof. This follows directly from Theorem 3.3 and the fact that any free standard Γ-space weakly contains all non-trivial Bernoulli shifts [2]. □

Let us recall a measurable density notion by Kechris [20], related to the profinite completion:

Definition 3.5 (Property EMD*)**.** An infinite countable group Γ has property EMD* if any ergodic standard Γ-space is weakly contained in the profinite completion $\widehat{\Gamma}$ of Γ.

Corollary 3.6 (profinite completion and stable integral simplicial volume)**.** *Let M be an oriented closed connected manifold with fundamental group Γ. If Γ has* EMD**, then for all ergodic standard Γ-spaces α we have*

$$\lvert M \rvert^{\alpha} \geqslant \lvert M \rvert^{\widehat{\Gamma}} = \|M\|_{\mathbb{Z}}^{\infty},$$

and hence $\lvert M \rvert = \|M\|_{\mathbb{Z}}^{\infty}$.

Proof. If Γ has EMD* and α is an ergodic standard Γ-space, then we obtain from Theorem 3.3 that

$$\lvert M \rvert^{\alpha} \geqslant \lvert M \rvert^{\widehat{\Gamma}} = \|M\|_{\mathbb{Z}}^{\infty}.$$

On the other hand, we know that integral foliated simplicial volume can be computed by using ergodic standard Γ-spaces [23, Proposition 4.17] and that $\lvert M \rvert \leqslant \|M\|_{\mathbb{Z}}^{\infty}$ (Theorem 2.6). Hence, $\lvert M \rvert = \|M\|_{\mathbb{Z}}^{\infty}$, as desired. □

Corollary 3.7 (amenable fundamental groups)**.** *Let M be an oriented closed connected manifold with residually finite amenable fundamental group Γ. Then*

$$\lvert M \rvert^{\alpha} = \lvert M \rvert^{\widehat{\Gamma}} = \|M\|_{\mathbb{Z}}^{\infty}$$

holds for all free standard Γ-spaces α. In particular, $\lvert M \rvert = \|M\|_{\mathbb{Z}}^{\infty}$.

Proof. If Γ is finite, then this is a straightforward calculation [23, Proposition 4.15, Corollary 4.27].

If Γ is infinite, then all free standard Γ-spaces are weakly equivalent [7, 19]; in particular, they are weakly equivalent to the profinite completion $\widehat{\Gamma}$. Now the claim follows from Theorem 3.3 and Theorem 2.6. □

The case of aspherical manifolds with amenable fundamental group will be discussed in Section 6.

Corollary 3.8 (free fundamental groups)**.** *Let M be an oriented closed connected manifold with free fundamental group Γ. Then*

$$\lvert M \rvert = \lvert M \rvert^{\widehat{\Gamma}} = \|M\|_{\mathbb{Z}}^{\infty}.$$

Proof. If Γ is trivial, then this chain of equalities clearly holds [23, Example 4.5]. Moreover, it is known that free groups of non-zero rank satisfy EMD* [20]. Hence, we can apply Corollary 3.6. □

Remark 3.9. If M is an oriented closed connected manifold with free fundamental group Γ of rank r, then $\|M\| = 0$ [22, p. 76]. However, we will now see that if $r \geqslant 2$, then

$$\boldsymbol{|} M \boldsymbol{|} = \|M\|_{\mathbb{Z}}^{\infty} > 0.$$

Looking at the classifying map $M \longrightarrow B\Gamma$ shows that [25, Theorem 1.35(1), Example 1.36]

$$b_1^{(2)}(M) = b_1^{(2)}(\widetilde{M}, \Gamma) \geqslant b_1^{(2)}(\Gamma) = r - 1 > 0.$$

Hence, the fact that integral foliated simplicial volume (and also stable integral simplicial volume) provide an upper bound for L^2-Betti numbers [33, Corollary 5.28] implies that also $\boldsymbol{|} M \boldsymbol{|} = \|M\|_{\mathbb{Z}}^{\infty} > 0$.

In case of rank $r = 1$, then there are examples with vanishing stable integral simplicial volume (e.g., S^1), but also examples with non-vanishing stable integral simplicial volume (e.g., $(S^1 \times S^{n-1}) \# (S^2 \times S^{n-2})$ for all $n \geqslant 2$, as one can see using the formula of L^2-Betti numbers for connected sums [25, Theorem 1.35(6)] and the mentioned upper bound by Schmidt).

Let us now consider the case of hyperbolic 3-manifolds.

Proposition 3.10. *The fundamental group of a virtually fibered closed hyperbolic* 3*-manifold has property* EMD**.*

Proof. This result has already been noticed by Kechris [20] and Bowen and Tucker-Drob [4]. For the sake of completeness, we include a proof. First of all, notice that for residually finite groups property EMD* is equivalent to property MD [35, Theorem 1.4], another universal property related to profinite completion due to Kechris [20]. Hence, we deduce that surface groups have property MD [4, Theorem 1.4]: Indeed, using as normal subgroup of a surface group the kernel of its abelianization map, Lubotzky and Shalom [24, Theorem 2.8] showed that a surface group satisfies the hypotheses for the MD-inheritance result [4, Theorem 1.4].

Let us now consider the fundamental group Γ of a closed hyperbolic 3-manifold that fibers over S^1, i.e., the semidirect product of a surface group Λ and $\mathbb{Z}$. Taking as normal subgroup of Γ the surface group Λ we may again apply MD-inheritance [4, Theorem 1.4] to conclude that Γ has property MD, and hence property EMD*. For residually finite groups property EMD* is preserved by passing from a finite index subgroup to the ambient group, hence yielding the conclusion. □

Corollary 3.11 (hyperbolic 3-manifolds)**.** *Let M be an oriented closed connected hyperbolic* 3*-manifold. Then*

$$\|M\| = \boldsymbol{|} M \boldsymbol{|} = \|M\|_{\mathbb{Z}}^{\infty}.$$

Proof. Agol's virtual fiber theorem [3] and Proposition 3.10 give that the fundamental group of *every* closed hyperbolic 3-manifold has property EMD*. Hence, Corollary 3.6 implies that integral foliated simplicial volume and stable integral simplicial volume are equal for closed hyperbolic 3-manifolds. On the other hand, it is known that $\|M\| = \boldsymbol{|} M \boldsymbol{|}$ in this case [23, Theorem 1.1]. □

Notice that the corresponding result for higher-dimensional manifolds does not hold (Section 5).

Remark 3.12. An example of group which does not satisfy property EMD* is $\mathrm{SL}(n,\mathbb{Z})$ for $n > 2$ [4].

4. Homology bounds via integral foliated simplicial volume

Theorem 1.6, which we prove in this section, can be quickly deduced from Theorem 2.6 and suitable estimates for torsion and rank in terms of integral simplicial volume:

Proof of Theorem 1.6. Every oriented closed connected n-manifold N satisfies the torsion homology bound [32, Theorem 3.2]

$$\log(|\operatorname{tors} H_k(N;\mathbb{Z})|) \leq \log(n+1)\binom{n+1}{k+1}\|N\|_{\mathbb{Z}}$$

in every degree k. In particular, by Theorem 2.6 we obtain for the tower of finite coverings associated to the chain $(\Gamma_i)_i$ of $\Gamma = \pi_1(M)$ that

$$\begin{aligned}\limsup_{i\to\infty} \frac{\log(|\operatorname{tors} H_k(M_i;\mathbb{Z})|)}{[\Gamma:\Gamma_i]} &\leq \log(n+1)2^{n+1} \cdot \lim_{i\to\infty} \frac{\|M_i\|_{\mathbb{Z}}}{[\Gamma:\Gamma_i]} \\ &= \log(n+1)2^{n+1} \cdot |M|^{\alpha},\end{aligned}$$

where n denotes the dimension of M. Starting from the Betti number estimate in Lemma 4.1 below we similarly obtain

$$\limsup_{i\to\infty} \frac{\operatorname{rk}_R H_k(M_i;R)}{[\Gamma:\Gamma_i]} \leqslant |M|^{\alpha}$$

for all principal ideal domains R. □

Poincaré duality allows to bound Betti numbers in terms of integral simplicial volume. For the sake of completeness, we give a simple proof of this fact. For simplicity, we consider only principal ideal domains as coefficients.

Lemma 4.1. *Let R be a principal ideal domain and let M be an oriented closed connected n-manifold. Then for all $k \in \mathbb{N}$ we have*

$$\operatorname{rk}_R H_k(M;R) \leqslant \|M\|_{\mathbb{Z}}.$$

Proof. Let $c = \sum_{j=1}^m a_j \cdot \sigma_j \in C_n(M;\mathbb{Z})$ be a fundamental cycle of M in reduced form with $|c|_{1,\mathbb{Z}} = \|M\|_{\mathbb{Z}}$ and let $k \in \mathbb{N}$. Then the Poincaré duality map

$$\begin{aligned}\cdot \cap [M]_{\mathbb{Z}} \colon H^{n-k}(M;R) &\longrightarrow H_k(M;R) \\ [f] &\longmapsto [f \cap c] = (-1)^{k\cdot(n-k)} \cdot \Bigl[\sum_{j=1}^m a_j \cdot f({}_{n-k}\lfloor\sigma) \cdot \sigma\rfloor_k\Bigr]\end{aligned}$$

is surjective. In particular, $H_k(M;R)$ is a quotient of a submodule of a free R-module of rank at most m. So, $\operatorname{rk}_R H_k(M;R) \leqslant m \leqslant |c|_{1,\mathbb{Z}} = \|M\|_{\mathbb{Z}}$. □

5. Integral foliated simplicial volume of higher-dimensional hyperbolic manifolds

This section is devoted to the proof of Theorem 1.8:

Theorem 5.1. *For all $n \in \mathbb{N}_{\geqslant 4}$ there is a $C_n \in \mathbb{R}_{<1}$ with the following property: For all oriented closed connected hyperbolic n-manifolds M we have*

$$\|M\| \leqslant C_n \cdot |\!|\!| M |\!|\!| \, .$$

5.1. **Setup.** As usual, we denote by $\Gamma \cong \pi_1(M)$ the automorphism group of the universal covering $\pi \colon \widetilde{M} \to M$. We also fix a standard Γ-space X, and we denote by α the action of Γ on X. Let $c \in C_n(M, \alpha) = L^\infty(X, \mu, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_n(\widetilde{M}; \mathbb{Z})$ be a fundamental cycle for M. We rewrite c in a convenient form for the computations we are going to carry out. Namely, for a suitable finite set $I = \{1, \dots, h\}$ of indices we have

$$c = \sum_{i \in I} \varepsilon_i \chi_{A_i} \otimes \sigma_i \ , \tag{1}$$

where:

(i) the first vertex of each σ_i lies in a fixed (set-theoretical, strict) fundamental domain D for the Γ-action on $\widetilde{M}$ by deck transformations;
(ii) $\varepsilon_i \in \{1, -1\}$ for every $i \in I$;
(iii) for every $i, j \in I$, either $A_i = A_j$, or $A_i \cap A_j = \emptyset$;
(iv) if $A_i = A_j$ and $\sigma_i = \sigma_j$, then $\varepsilon_i = \varepsilon_j$.

We set

$$\beta_i = \varepsilon_i \mu(A_i) \in \mathbb{R} \ ,$$

so

$$|\!|\!| c |\!|\!|^\alpha = \sum_{i \in I} |\beta_i| \ ,$$

and we recall that the chain

$$c_{\mathbb{R}} = \sum_{i \in I} \beta_i (\pi \circ \sigma_i) = \sum_{i \in I} \varepsilon_i \mu(A_i)(\pi \circ \sigma_i) \ \in C_n(M, \mathbb{R})$$

is a real fundamental cycle for M [33, Remark 5.23][23, Proposition 4.6].

A crucial role in our argument will be played by the locally finite chain

$$c_x = \sum_{i \in I} \sum_{\gamma \in \Gamma} \varepsilon_i \chi_{A_i}(\gamma^{-1} x) \cdot \gamma \sigma_i \in C_n^{\mathrm{lf}}(\widetilde{M}; \mathbb{Z})$$

defined for μ-a.e. $x \in X$, which is a locally finite fundamental cycle of $\widetilde{M}$ (Lemma 2.5).

Observe that, as a consequence of our choices, if $\gamma \sigma_i = \gamma' \sigma_j$ for some $\gamma, \gamma' \in \Gamma$ and $i, j \in I$, then $\gamma = \gamma'$ and $\sigma_i = \sigma_j$ (but possibly $i \neq j$). If this is the case and $\gamma^{-1}(x) \in A_i \cap A_j$, then the singular simplex $\gamma \sigma_i = \gamma \sigma_j$ appears with multiplicities in c_x. However, we will keep track of the fact that these singular simplices arise from distinct summands in (1). In this spirit, the chain c_x should be considered just as a sum of signed simplices (i.e., a locally finite chain whose coefficients lie in $\{\pm 1\}$), possibly with repetitions. To be precise we say that the pair $(\gamma, i) \in \Gamma \times I$ *appears in* c_x if $\gamma^{-1}(x) \in A_i$. Note that it could happen that $\gamma\sigma_i = \gamma\sigma_j$ for some $i \neq j$, but (γ, i) appears in c_x, while (γ, j) does not.

By the very definitions, for every $i \in I, \gamma \in \Gamma$ we have

$$\mu\left(\{x \in X \mid (\gamma, i) \text{ appears in } c_x\}\right) = \mu(\gamma A_i) = \mu(A_i) = |\beta_i| \ . \tag{2}$$

Moreover, we will assume that c is straight as explained below.

5.2. **Straight simplices.** Recall that a geodesic p-simplex in $\mathbb{H}^n$ is just the convex hull of $(p+1)$ (ordered) points lying in $\overline{\mathbb{H}}^n = \mathbb{H}^n \cup \partial\mathbb{H}^n$. Such a simplex is *ideal* if all its vertices belong to $\partial\mathbb{H}^n$, and it is *degenerate* if its vertices lie on a $(p-1)$-dimensional hyperbolic subspace of $\overline{\mathbb{H}}^n$. If $\sigma \colon \Delta^p \to \mathbb{H}^n$ is a singular simplex, then we denote by $\mathrm{str}_p(\sigma)$ the straight simplex associated to σ, i.e., the barycentric parameterization of the unique geodesic simplex having the same vertices as σ. It is well known that the straightening map can be linearly extended to an $\mathrm{Isom}(\mathbb{H}^n)$-equivariant chain map $\mathrm{str}_* \colon C_*(\mathbb{H}^n, \mathbb{Z}) \to C_*(\mathbb{H}^n, \mathbb{Z})$, which is $\mathrm{Isom}(\mathbb{H}^n)$-equivariantly homotopic to the identity [30, §11.6]. As a consequence, the norm non-increasing map

$$\mathrm{id} \otimes_{\mathbb{Z}\Gamma} \mathrm{str}_* \colon C_*(M, \alpha) \to C_*(M, \alpha)$$

induces the identity in homology. Therefore, henceforth we will assume that each σ_i in our parametrized foliated fundamental cycle c is straight.

We define the *algebraic volume* of σ_i by setting

$$\mathrm{vol}_{\mathrm{alg}}(\sigma_i) = \varepsilon_i \int_{\sigma_i} \omega_{\widetilde{M}} \ ,$$

where $\omega_{\widetilde{M}}$ is the volume form of $\widetilde{M} = \mathbb{H}^n$. Since $c_{\mathbb{R}}$ is a real fundamental cycle for M we have the equality

$$\mathrm{vol}(M) = \sum_{i \in I} |\beta_i| \mathrm{vol}_{\mathrm{alg}}(\sigma_i) \ . \tag{3}$$

The fact that parametrized integral cycles cannot be used to produce efficient cycles descends from the following observation: in dimension greater than 3, the dihedral angle of the regular ideal simplex does not divide 2π. As a consequence, for a.e. $x \in X$ the integral (locally finite) cycle c_x must contain a certain quantity of small simplices. This implies in turn that c_x cannot project onto an efficient fundamental cycle of M. In the case when c_x is Γ-equivariant, this fact is precisely stated and proved by Francaviglia, Frigerio, and Martelli [8], and implies that the ratio between the (stable) integral simplicial volume and the ordinary simplicial volume of M is strictly bigger than one. However, c_x is not Γ-equivariant in general, so more work is needed in our context.

Let us first recall the statements from the equivariant case [8] that we will be using later on. A *ridge* of a geodesic simplex (or of a singular simplex) is a face of the simplex of codimension 2. If Δ is a nondegenerate geodesic n-simplex in $\mathbb{H}^n$ and E is a ridge Δ, then the *dihedral angle* $\alpha(\Delta, E)$ of Δ at E is defined in the following way: let p be a point in $E \cap \mathbb{H}^n$, and let $H \subseteq \mathbb{H}^n$ be the unique 2-dimensional geodesic plane which intersects E orthogonally in p. We define $\alpha(\Delta, E)$ as the angle in p of the polygon $\Delta \cap H$ of $H \cong \mathbb{H}^2$. It is easily seen that this is well-defined (i.e., independent of p). For every $n \geqslant 3$, we denote the dihedral angle of the ideal regular n-dimensional simplex at any of its ridges by α_n.

It is readily seen by intersecting the simplex with a horosphere centered at any vertex that α_n equals the dihedral angle of the regular *Euclidean* $(n-1)$-dimensional simplex at any of its $(n-3)$-dimensional faces, so $\alpha_n = \arccos \frac{1}{n-1}$. In particular, we have $\alpha_3 = \arccos \frac{1}{2} = \frac{\pi}{3}$. Since $\frac{2\pi}{6} < \arccos \frac{1}{3} < \frac{2\pi}{5}$ and $\frac{2\pi}{5} < \arccos \frac{1}{n} < \frac{2\pi}{4}$ for every $n \geqslant 4$, the real number $\frac{2\pi}{\alpha_n}$ is an integer if and only if $n = 3$. For $n \in \mathbb{N}_{\geqslant 4}$, we write $k_n \in \mathbb{N}$ for the unique integer satisfying

$$k_n \alpha_n < 2\pi < (k_n + 1)\alpha_n;$$

then $k_n = 5$ if $n = 4$ and $k_n = 4$ if $n \geqslant 5$. If the volume of a geodesic simplex is close to v_n, then the simplex must be close in shape to the regular ideal one, so one gets the following:

Lemma 5.2 ([8, Lemma 3.16]). *Let $n \geqslant 4$. Then, there exist $a_n > 0$ and $\varepsilon_n > 0$, such that the following condition holds: if a geodesic n-simplex Δ satisfies $\mathrm{vol}(\Delta) \geqslant (1 - \varepsilon_n) v_n$ and if α is the dihedral angle of Δ at any of its ridges, then*

$$\frac{2\pi}{k_n + 1}(1 + a_n) < \alpha < \frac{2\pi}{k_n}(1 - a_n).$$

5.3. **The incenter and inradius of a simplex.** Consider a nondegenerate geodesic k-simplex $\Delta \subseteq \mathbb{H}^n$, and let $H(\Delta) \subseteq \mathbb{H}^n$ be the unique hyperbolic subspace of dimension k containing Δ. We are going to recall the definition of *inradius* $r(\Delta)$ of Δ, which is due to Luo [28].

For every point $p \in \Delta$ we denote by $r_\Delta(p)$ the radius of the maximal k-ball of $H(\Delta)$ centered in p and contained in Δ, and we set

$$r(\Delta) := \sup_{p \in \Delta} r_\Delta(p) \ \in \ (0, +\infty] \ .$$

Since the volume of any k-simplex is smaller than v_k and the volume of k-balls diverges as the radius diverges, there exists a constant $r_k > 0$ such that $r_\Delta(p) \leqslant r_k$ for every $p \in \Delta$, so $r(\Delta) \in (0, +\infty)$. Moreover, there is a unique point $p \in \Delta$ with $r_\Delta(p) = r(\Delta)$ [28][8, Lemma 3.15]. Such a point is denoted by the symbol $\mathrm{inc}(\Delta)$, and it is called the *incenter* of Δ.

The following lemma shows that, in big simplices, the incenter of a face is uniformly distant from any other non-incident face.

Lemma 5.3 ([8, Lemma 3.15]). *Let $n \geqslant 3$. There exist $\varepsilon_n > 0$ and $\delta_n > 0$ such that the following holds for every geodesic n-simplex $\Delta \subseteq \mathbb{H}^n$ with $\mathrm{vol}(\Delta) \geqslant v_n(1 - \varepsilon_n)$: let e be any face of Δ and e' another face of Δ that does not contain e; then*

$$d(\mathrm{inc}(e), e') > 2\delta_n.$$

In particular, if e, e' are distinct ridges of Δ, then

$$B(\mathrm{inc}(e), \delta_n) \cap B(\mathrm{inc}(e'), \delta_n) = \emptyset.$$

Henceforth we fix constants $\varepsilon_n > 0$, $a_n > 0$ and $\delta_n > 0$ such that the statements of Lemmas 5.2 and 5.3 hold. We also set

$$\eta_n = \mathrm{vol}(B(p, \delta_n)) \ ,$$

where $B(p, \delta_n)$ is any ball of radius δ_n in $\mathbb{H}^n$; notice that η_n is independent of the point p.

5.4. **Proof of Theorem 5.1.** Taking the infimum over all standard Γ-spaces X and over all the fundamental cycles $c \in C_n(M, \alpha)$ shows that Theorem 1.8 will be a consequence of the following:

Theorem 5.4. *Let $n \in \mathbb{N}$, $n \geqslant 4$, and*

$$C_n := \max \left\{ 1 - \frac{\varepsilon_n}{12}, 1 - \frac{\eta_n}{3v_n}, 1 - \frac{a_n \eta_n}{2v_n} \right\} < 1.$$

Then, for every oriented closed connected hyperbolic n-manifold M with fundamental group Γ, every standard Γ-space $\alpha = \Gamma \curvearrowright (X, \mu)$, and every fundamental cycle $c \in C_n(M, \alpha)$, the following inequality holds:

$$\|M\| \leqslant C_n \cdot |c|^{\alpha}.$$

The rest of this section is devoted to the proof of Theorem 5.4. We will now keep the notation and assumptions introduced in Section 5.1.

Definition 5.5. We say that the simplex σ_i is *big* if $\mathrm{vol}_{\mathrm{alg}}(\sigma_i) > (1 - \varepsilon_n) v_n$, and *small* otherwise. We also set

$$I_b = \{ i \in I \mid \sigma_i \text{ is big} \}, \quad I_s = I \setminus I_b ,$$

and

$$c_b = \sum_{i \in I_b} \varepsilon_i \chi_{A_i} \otimes \sigma_i , \qquad c_s = \sum_{i \in I_s} \varepsilon_i \chi_{A_i} \otimes \sigma_i ,$$

so that $c = c_b + c_s$. Also observe that we have $|c|^{\alpha} = |c_b|^{\alpha} + |c_s|^{\alpha}$. (Of course, c_b and c_s need not be cycles.)

The following result says that a cycle is efficient only if its small simplices have a small weight.

Proposition 5.6. *Suppose that*

$$|c_s|^{\alpha} \geqslant \frac{|c|^{\alpha}}{12} .$$

Then

$$|c|^{\alpha} \geqslant \frac{\|M\|}{1 - \varepsilon_n / 12} .$$

Proof. By (3) we have

$$\begin{aligned}
\|M\| &= \frac{\mathrm{vol}(M)}{v_n} = \frac{\sum_{i \in I} |\beta_i| \mathrm{vol}_{\mathrm{alg}}(\sigma_i)}{v_n} \\
&= \frac{\sum_{i \in I_b} |\beta_i| \mathrm{vol}_{\mathrm{alg}}(\sigma_i)}{v_n} + \frac{\sum_{i \in I_s} |\beta_i| \mathrm{vol}_{\mathrm{alg}}(\sigma_i)}{v_n} \\
&\leqslant \frac{\sum_{i \in I_b} |\beta_i| v_n}{v_n} + \frac{\sum_{i \in I_s} |\beta_i| (1 - \varepsilon_n) v_n}{v_n} = \sum_{i \in I} |\beta_i| - \varepsilon_n \sum_{i \in I_s} |\beta_i| \\
&= |c|^{\alpha} - \varepsilon_n \sum_{i \in I_s} |\beta_i| = |c|^{\alpha} - \varepsilon_n |c_s|^{\alpha} \leqslant |c|^{\alpha} \left(1 - \frac{\varepsilon_n}{12} \right) . \qquad \square
\end{aligned}$$

Therefore, we are now left to consider the case when $|c_s|^{\alpha}$ is small.

5.4.1. *Full ridges.* In the sequel we work with Δ-complexes. These are variations of simplicial complexes in which distinct simplices may share more than one face and faces of the same simplex may be identified [14, §2.1]. To the real cycle $c_\mathbb{R}$ there is associated the finite Δ-complex P which is defined as follows. We take one copy Δ_i of the n-dimensional standard simplex Δ^n, and we identify the $(n-1)$-dimensional faces $F_1 \subseteq \Delta_{i_1}, \ldots, F_s \subseteq \Delta_{i_s}$ of these simplices if $\pi \circ \sigma_{i_1} \circ \partial_{j_1} = \cdots = \pi \circ \sigma_{i_s} \circ \partial_{j_s}$, where ∂_{j_l} is the usual affine identification between the $(n-1)$-dimensional standard simplex and the face F_l of Δ_{i_l}. Observe that, after identifying Δ_j with the standard simplex Δ^n for every j, the singular simplices $\pi \circ \sigma_i$ glue into a continuous map

$$f \colon P \to M \ .$$

Let now e be a ridge of P, and observe that the set of vertices of e is endowed with a well-defined order. We denote by $\widetilde{e}$ the unique lift of $f(e)$ to $\mathbb{H}^n$ having the first vertex in D, so $\widetilde{e}$ is naturally a singular $(n-2)$-simplex. We shall often denote by $\widetilde{e}$ also the image of such a simplex, since this will not produce any confusion. If S is a finite set, we denote by $|S|$ the cardinality of S. For every $x \in X$ we set

$$\begin{aligned}
\Omega(e,x) &= \{(\gamma,i) \,|\, (\gamma,i) \text{ appears in } c_x \text{ and } \widetilde{e} \text{ is a ridge of } \gamma\sigma_i\} \subseteq \Gamma \times I \ , \\
\Omega_b(e,x) &= \{(\gamma,i) \in \Omega(e,x) \,|\, \sigma_i \text{ is big}\} \ , \\
N(e,x) &= |\Omega(e,x)| \ , \\
N_b(e,x) &= |\Omega_b(e,x)|
\end{aligned}$$

(observe that, since c_x is locally finite, the sets $\Omega_b(e,x) \subseteq \Omega(e,x)$ are finite). It is immediate to check that the functions $N(e,\cdot) \colon X \to \mathbb{N}$, $N_b(e,\cdot) \colon X \to \mathbb{N}$ are measurable. We now consider the following measurable subsets of X:

$$\begin{aligned}
F(e) &= \{x \in X \,|\, N_b(e,x) \geqslant k_n + 1\} \ . \\
NF(e) &= \{x \in X \,|\, 1 \leqslant N(e,x) \, , \ N_b(e,x) \leqslant k_n\} \ ,
\end{aligned}$$

and we say that e is *x-full* if $x \in F(e)$, and *x-non-full* if $x \in NF(e)$. Observe that, thanks to Lemma 5.2, if e is x-full, then the big simplices appearing in c_x must produce some overlapping around $\widetilde{e}$. On the other hand, if e is not x-full, then at least one small simplex appears around $\widetilde{e}$ in c_x.

Henceforth we denote by E the set of $(n-2)$-dimensional (simplicial) faces of P, and for every $i \in I$ we denote by $E(\sigma_i)$ the set of $(n-2)$-dimensional (singular) faces of σ_i, i.e., the set of ridges of σ_i.

Let us now fix $i \in I$. We want to measure the fact that σ_i may produce overlappings in c_x for some $x \in X$. To this aim, we fix a ridge τ of σ_i. If $e_\tau \in E$ is the $(n-2)$-dimensional face corresponding to τ, then there exists a unique element $\gamma_\tau \in \Gamma$ such that $\gamma_\tau \cdot \tau = \widetilde{e}_\tau$ and we set

$$\begin{aligned}
F(\sigma_i,\tau) = F(\sigma_i,\tau,e_\tau) &= \{x \in F(e_\tau) \,|\, (\gamma_\tau, i) \in \Omega(e_\tau,x)\} \ \subseteq X \ , \\
NF(\sigma_i,\tau) = NF(\sigma_i,\tau,e_\tau) &= \{x \in NF(e_\tau) \,|\, (\gamma_\tau, i) \in \Omega(e_\tau,x)\} \ \subseteq X \ .
\end{aligned}$$

Observe that, according to our definitions,

$$F(\sigma_i,\tau) = \gamma_\tau A_i \cap F(e_\tau) \, , \quad NF(\sigma_i,\tau) = \gamma_\tau A_i \cap NF(e_\tau) \ .$$

On the other hand, for later convenience we set

$$F(\sigma_i, \tau, e) = \emptyset = NF(\sigma_i, \tau, e) \ ,$$

for every $e \in E \setminus \{e_\tau\}$. So

$$F(\sigma_i, \tau) = \bigcup_{e \in E} F(\sigma_i, \tau, e) \ , \qquad NF(\sigma_i, \tau) = \bigcup_{e \in E} NF(\sigma_i, \tau, e) \ .$$

Of course, for every $x \in X$ such that (γ_τ, i) appears in c_x we have that e_τ is either x-full or x-non-full, so by the very definitions (and by Equation (2)) we have that

$$|\beta_i| = \mu(F(\sigma_i, \tau)) + \mu(NF(\sigma_i, \tau)) \ .$$

By summing over the ridges of σ_i we then get

$$\frac{n(n+1)}{2} |\beta_i| = \sum_{\tau \in E(\sigma_i)} \big(\mu(F(\sigma_i, \tau)) + \mu(NF(\sigma_i, \tau))\big) \ . \tag{4}$$

Lemma 5.7. *We have*

$$\frac{n(n+1)}{2} \left| c_b \right|^\alpha \leqslant \sum_{i \in I_b} \sum_{\tau \in E(\sigma_i)} \mu\big(F(\sigma_i, \tau)\big) + \sum_{e \in E} 5\mu\big(NF(e)\big) \ .$$

Proof. From Equation (4) we get

$$\begin{aligned} \frac{n(n+1)}{2} \left| c_b \right|^\alpha &= \frac{n(n+1)}{2} \sum_{i \in I_b} |\beta_i| \\ &= \sum_{i \in I_b} \sum_{\tau \in E(\sigma_i)} \big(\mu(F(\sigma_i, \tau)) + \mu(NF(\sigma_i, \tau))\big) \ . \end{aligned}$$

Recall now that $k_n \leqslant 5$ for every $n \in \mathbb{N}$; so, by definition, around any x-non-full ridge $\widetilde{e}$ in $\mathbb{H}^n$ at most five big simplices of c_x may appear. As a consequence for every $e \in E$ we have that

$$\sum_{i \in I_b} \sum_{\tau \in E(\sigma_i)} \mu(NF(\sigma_i, \tau, e)) \leqslant 5\mu(NF(e)),$$

so

$$\sum_{i \in I_b} \sum_{\tau \in E(\sigma_i)} \mu(NF(\sigma_i, \tau)) = \sum_{e \in E} \sum_{i \in I_b} \sum_{\tau \in E(\sigma_i)} \mu(NF(\sigma_i, \tau, e)) \leqslant 5 \sum_{e \in E} \mu(NF(e)) \ ,$$

and this concludes the proof. □

The following proposition shows that, if the total weight of big simplices of c is large, then there must be many overlappings in c_x for x in a subset of large measure.

Proposition 5.8. *Suppose that*

$$\left| c_s \right|^\alpha \leqslant \frac{\left| c \right|^\alpha}{12} \ .$$

Then

$$\sum_{i \in I_b} \sum_{\tau \in E(\sigma_i)} \mu(F(\sigma_i, \tau)) \geqslant 5 \left| c \right|^\alpha \ .$$

Proof. Let $e \in E$ and take $x \in NF(e)$. By Lemma 5.2, at least one small simplex must appear around $\widetilde{e}$ in c_x. In other words, we have that

$$NF(e) \subseteq \bigcup_{i \in I_s} \bigcup_{\tau \in E(\sigma_i)} NF(\sigma_i, \tau, e) \ ,$$

so

$$\mu(NF(e)) \leqslant \sum_{i \in I_s} \sum_{\tau \in E(\sigma_i)} \mu(NF(\sigma_i, \tau, e)) \ .$$

By summing over $e \in E$ we obtain

$$\begin{aligned} \sum_{e \in E} \mu(NF(e)) &\leqslant \sum_{e \in E} \sum_{i \in I_s} \sum_{\tau \in E(\sigma_i)} \mu(NF(\sigma_i, \tau, e)) = \sum_{i \in I_s} \sum_{\tau \in E(\sigma_i)} \mu(NF(\sigma_i, \tau)) \\ &\leqslant \frac{n(n+1)}{2} \sum_{i \in I_s} |\beta_i| = \frac{n(n+1)}{2} \lvert c_s \rvert^{\alpha} \ , \end{aligned}$$

where the second inequality follows from (4). Using this inequality and Lemma 5.7 we then get that

$$\begin{aligned} \frac{n(n+1)}{2} \lvert c \rvert^{\alpha} &= \frac{n(n+1)}{2} (\lvert c_b \rvert^{\alpha} + \lvert c_s \rvert^{\alpha}) \\ &\leqslant \sum_{i \in I_b} \sum_{\tau \in E(\sigma_i)} \mu(F(\sigma_i, \tau)) + \sum_{e \in E} 5 \mu(NF(e)) + \frac{n(n+1)}{2} \lvert c_s \rvert^{\alpha} \\ &\leqslant \sum_{i \in I_b} \sum_{\tau \in E(\sigma_i)} \mu(F(\sigma_i, \tau)) + 5 \frac{n(n+1)}{2} \lvert c_s \rvert^{\alpha} + \frac{n(n+1)}{2} \lvert c_s \rvert^{\alpha} \\ &= \sum_{i \in I_b} \sum_{\tau \in E(\sigma_i)} \mu(F(\sigma_i, \tau)) + 3n(n+1) \lvert c_s \rvert^{\alpha} \\ &\leqslant \sum_{i \in I_b} \sum_{\tau \in E(\sigma_i)} \mu(F(\sigma_i, \tau)) + \frac{n(n+1)}{4} \lvert c \rvert^{\alpha} \ , \end{aligned}$$

so

$$\sum_{i \in I_b} \sum_{\tau \in E(\sigma_i)} \mu(F(\sigma_i, \tau)) \geqslant \frac{n(n+1)}{4} \lvert c \rvert^{\alpha} \geqslant 5 \lvert c \rvert^{\alpha} \ . \qquad \square$$

Remark 5.9. Suppose that every simplex in c has positive algebraic volume (this is very "likely" if $\lvert c \rvert^{\alpha}$ is close to $\|M\|$). Then for every $x \in X$ all the simplices in c_x have positive algebraic volume. Since c_x has integral coefficients, this readily implies that no overlapping occurs in c_x. In particular, no full ridge appears in any c_x, so $\mu(F(\sigma_i, \tau, e)) = 0$ for every $e \in E$, $i \in I$, $\tau \in E(\sigma_i)$. By Proposition 5.8 and Proposition 5.6 we now have

$$\lvert c_s \rvert^{\alpha} \geqslant \frac{\lvert c \rvert^{\alpha}}{12} \qquad \text{and} \qquad \lvert c \rvert^{\alpha} \geqslant \frac{\|M\|}{1 - \varepsilon_n / 12} \ .$$

This implies (a stronger version of) Theorem 5.4 in the case when the simplices appearing in c all have positive algebraic volume.

5.5. **Generalized chains and local degree.** Let $B(\mathbb{H}^n;\mathbb{R})$ be the real vector space having as a basis the set of measurable bounded subsets of $\mathbb{H}^n$, endowed with the obvious action by Γ. A *generalized chain* is an element of the space

$$\mathbb{R} \otimes_{\mathbb{R}\Gamma} B(\mathbb{H}^n;\mathbb{R}) \ .$$

If $1 \otimes B$ is an indecomposable element in $\mathbb{R} \otimes_{\mathbb{R}\Gamma} B(\mathbb{H}^n;\mathbb{R})$, then we define the *local degree* $\deg_p(1 \otimes B) \in \mathbb{N}$ of $1 \otimes B$ at $p \in M$ as the number of points in the intersection between B and the preimage of p via the universal covering map $\pi \colon \mathbb{H}^n \to M$ (such a number is well-defined because Γ permutes the fiber of p, and it is finite since B is relatively compact and the fiber of p is discrete). It is readily seen that the local degree extends to a well-defined linear map

$$\deg_p \colon \mathbb{R} \otimes_{\mathbb{R}\Gamma} B(\mathbb{H}^n;\mathbb{R}) \to \mathbb{R} \ .$$

Let now $z = \sum_{\lambda \in \Lambda} a_\lambda \otimes B_\lambda$ be a fixed generalized chain. The number

$$\mathrm{vol}(z) = \sum_{\lambda \in \Lambda} a_\lambda \, \mathrm{vol}(B_\lambda)$$

is well-defined, and a double-counting argument shows that

$$\mathrm{vol}(z) = \int_M \deg_p(z) \, dp$$

(observe that the map $\deg_\cdot(z) \colon M \to \mathbb{R}$ is measurable, since each B_λ is Borel). Let us set

$$I_{\mathrm{pos}} = \{ i \in I \mid \mathrm{vol}_{\mathrm{alg}}(\sigma_i) \geqslant 0 \} \ ,$$

so that $I_b \subseteq I_{\mathrm{pos}}$. Since $c_{\mathbb{R}}$ is a real fundamental cycle and the boundary of the image of each σ_i is a null set, we easily have that

$$\deg_p \Bigg(\sum_{i \in I_{\mathrm{pos}}} |\beta_i| \, \mathrm{im}(\sigma_i) - \sum_{i \in I \setminus I_{\mathrm{pos}}} |\beta_i| \, \mathrm{im}(\sigma_i) \Bigg) = 1$$

for almost every $p \in M$.

In particular, if we set

$$\overline{z}_{\mathrm{pos}} = \sum_{i \in I_{\mathrm{pos}}} |\beta_i| \otimes \mathrm{im}(\sigma_i) \ ,$$

then we deduce that

$$\deg_p(\overline{z}_{\mathrm{pos}}) \geqslant 1 \qquad (5)$$

for almost every $p \in M$. We now need to refine this estimate in order to show that overlappings in c_x have a cost in terms of the ℓ^1-norm of the fundamental cycle $c_{\mathbb{R}}$.

Take $e \in E$. If $F(e) \neq \emptyset$, then $\widetilde{e} \subseteq \mathbb{H}^n$ is a ridge of a big geodesic simplex. Since big simplices are nondegenerate, also $\widetilde{e}$ is nondegenerate. In particular, the incenter $\mathrm{inc}(\widetilde{e})$ is well-defined. We denote by $\mathrm{inc}(e) \in M$ the image of $\mathrm{inc}(\widetilde{e})$ in M.

Recall that, if σ_i is a big simplex and $\tau \in E(\sigma_i)$, then there exists a unique element $\gamma_\tau \in \Gamma$ such that $\gamma_\tau \cdot \tau = \widetilde{e}_\tau$, where $\widetilde{e}_\tau$ is the preferred lift of the $(n-2)$-dimensional face $e_\tau \in E$ corresponding to τ. We consider the following generalized chains:

$$\overline{z}_- = \sum_{i \in I_b} \sum_{\tau \in E(\sigma_i)} \mu(F(\sigma_i, \tau)) \otimes ((\gamma_\tau \cdot \mathrm{im}(\sigma_i)) \cap B(\mathrm{inc}(\widetilde{e}_\tau), \delta_n))$$

(observe that γ_τ and $\mathrm{inc}(\widetilde{e}_\tau)$ are defined whenever $\mu(F(\sigma_i, \tau)) \neq 0$),

$$\overline{z}_+ = \sum_{e \in E} \mu(F(e)) \otimes B(\mathrm{inc}(\widetilde{e}), \delta_n)$$

(again, $\mathrm{inc}(\widetilde{e})$ is defined whenever $\mu(F(e)) \neq 0$), and we finally set

$$\overline{z} = \overline{z}_{\mathrm{pos}} - \overline{z}_- + \overline{z}_+ \ . \tag{6}$$

Roughly speaking, the generalized chain $\overline{z}$ is obtained by removing from the big simplices in $\overline{z}_{\mathrm{pos}}$ the portions that cover small balls around the incenters of full ridges, and adding back (a suitable weighted sum of) such small balls. We will show that the degree of $\overline{z}$ at almost every point of M is still at least 1. However, when there are many overlappings, the absolute value of the volume of $\overline{z}_-$ is substantially bigger than the volume of $\overline{z}_+$. These two facts imply that the volume of $\overline{z}_{\mathrm{pos}}$ must be considerably bigger than $\mathrm{vol}(M)$. Therefore, the sum of the coefficients appearing in $\overline{z}_{\mathrm{pos}}$ must be bigger than $\|M\|$, and this implies in turn the desired bound from below for $|c|^\alpha$.

In order to pursue this strategy, we need to introduce the notion of *generalized parametrized chain*. Let $B(\mathbb{H}^n; \mathbb{Z})$ be the free abelian group having as a basis the set of measurable bounded subsets of $\mathbb{H}^n$, endowed with the obvious action by Γ. By definition, a generalized parametrized chain is an element of the space

$$L^\infty(X, \mu, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} B(\mathbb{H}^n; \mathbb{Z}) \ .$$

The integration map $\int_X \colon L^\infty(X, \mu, \mathbb{Z}) \to \mathbb{R}$ and the inclusion $i \colon B(\mathbb{H}^n; \mathbb{Z}) \to B(\mathbb{H}^n; \mathbb{R})$ induce a well-defined homomorphism

$$\theta := \int_X \otimes\, i \colon L^\infty(X, \mu, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} B(\mathbb{H}^n; \mathbb{Z}) \to \mathbb{R} \otimes_{\mathbb{R}\Gamma} B(\mathbb{H}^n; \mathbb{R}) \ .$$

If $\widetilde{z}$ is a generalized parametrized chain and $p \in M$, then we set

$$\deg_p(\widetilde{z}) = \deg_p(\theta(\widetilde{z})) \ .$$

Moreover, if $\widetilde{z} = \sum_{j \in J} f_j \otimes B_j$, then for every $x \in X$ we can consider the locally finite formal sum

$$\widetilde{z}_x = \sum_{j \in J} \sum_{\gamma \in \Gamma} f_j(\gamma^{-1} x) \gamma(B_j) \ ,$$

and we may define the local degree of such a sum at $\widetilde{p} \in \mathbb{H}^n$ by setting

$$\deg_{\widetilde{p}}(\widetilde{z}_x) = \sum_{j \in J} \sum_{\gamma \in \Gamma} f_j(\gamma^{-1} x) \chi_{\gamma(B_j)}(\widetilde{p}) \ .$$

Lemma 5.10. *Let $\widetilde{z}$ be a generalized parametrized chain. Then*

$$\deg_p(\widetilde{z}) = \int_X \deg_{\widetilde{p}}(\widetilde{z}_x) \, d\mu(x) \in \mathbb{R}$$

for every $p \in M$ and $\widetilde{p} \in \mathbb{H}^n$ such that $\pi(\widetilde{p}) = p$.

Proof. By linearity, we may assume that $\widetilde{z} = \chi_A \otimes B$ for some measurable subsets $A \subseteq X$, $B \subseteq \mathbb{H}^n$. Moreover, if $k = |(B \cap \pi^{-1}(p))|$, then there exist elements $\gamma_1, \ldots, \gamma_k$ in Γ such that $\chi_{\gamma(B)}(\widetilde{p}) = \chi_B(\gamma^{-1}(\widetilde{p})) = 1$ if and only if $\gamma = \gamma_h$ for some $h \in \{1, \ldots, k\}$. Therefore,

$$\deg_{\widetilde{p}}(\widetilde{z}_x) = \sum_{\gamma \in \Gamma} \chi_A(\gamma^{-1}x)\chi_{\gamma(B)}(\widetilde{p}) = \sum_{h=1}^{k} \chi_A(\gamma_h^{-1}x) = \sum_{h=1}^{k} \chi_{\gamma_h \cdot A}(x) \ ,$$

and

$$\int_X \deg_{\widetilde{p}}(\widetilde{z}_x)\, d\mu(x) = \sum_{h=1}^{k} \int_X \chi_{\gamma_h \cdot A}(x)\, d\mu(x) = k\mu(A) = \deg_p(\widetilde{z}) \ . \qquad \square$$

Let us now return to the study of our parametrized cycle $c = \sum_{i \in I} \varepsilon_i \chi_{A_i} \otimes \sigma_i$. From now on $\bar{z}$ denotes the generalized chain in (6).

Proposition 5.11. *For almost every $p \in M$, we have $\deg_p(\overline{z}) \geqslant 1$.*

Proof. We set

$$\begin{aligned}
\widetilde{z}_{\text{pos}} &= \sum_{i \in I_{\text{pos}}} \chi_{A_i} \otimes \operatorname{im}(\sigma_i) \ , \\
\widetilde{z}_{-} &= \sum_{i \in I_b} \sum_{\tau \in E(\sigma_i)} \chi_{F(\sigma_i,\tau)} \otimes ((\gamma_\tau \cdot \operatorname{im}(\sigma_i)) \cap B(\operatorname{inc}(\widetilde{e}_\tau), \delta_n)) \ , \\
\widetilde{z}_{+} &= \sum_{e \in E} \chi_{F(e)} \otimes B(\operatorname{inc}(\widetilde{e}), \delta_n) \ ,
\end{aligned}$$

and

$$\widetilde{z} = \widetilde{z}_{\text{pos}} - \widetilde{z}_{-} + \widetilde{z}_{+} \ .$$

It follows from our choices that $\theta(\widetilde{z}) = \overline{z}$, so $\deg_p(\widetilde{z}) = \deg_p(\overline{z})$ for every point $p \in M$. Therefore, by Lemma 5.10 we are left to show that

$$\deg_{\widetilde{p}}(\widetilde{z}_x) \geqslant 1$$

for almost every $\widetilde{p} \in \mathbb{H}^n$ and almost every $x \in X$.

Let $i \in I_b$. Recall that

$$F(\sigma_i, \tau) = \gamma_\tau A_i \cap F(e_\tau) \ .$$

Therefore, for every $\gamma \in \Gamma$ and every $x \in X$ we have

$$\chi_{F(\sigma_i,\tau)}(\gamma^{-1}(x)) = \chi_{\gamma F(\sigma_i,\tau)}(x) = \chi_{\gamma\gamma_\tau A_i \cap \gamma F(e_\tau)}(x) \ .$$

Then, for every $x \in X$ and $\widetilde{p} \in \mathbb{H}^n$ we have

$$\begin{aligned}
& \sum_{\gamma \in \Gamma} \sum_{\tau \in E(\sigma_i)} \chi_{F(\sigma_i,\tau)}(\gamma^{-1}(x)) \chi_{\gamma\gamma_\tau(\operatorname{im}(\sigma_i)) \cap \gamma B(\operatorname{inc}(\widetilde{e}_\tau), \delta_n)}(\widetilde{p}) \\
=\ & \sum_{\gamma \in \Gamma} \sum_{\tau \in E(\sigma_i)} \chi_{\gamma\gamma_\tau A_i \cap \gamma F(e_\tau)}(x) \chi_{\gamma\gamma_\tau(\operatorname{im}(\sigma_i)) \cap \gamma B(\operatorname{inc}(\widetilde{e}_\tau), \delta_n)}(\widetilde{p}) \\
=\ & \sum_{\gamma \in \Gamma} \sum_{\tau \in E(\sigma_i)} \chi_{\gamma A_i \cap \gamma\gamma_\tau^{-1} F(e_\tau)}(x) \chi_{\gamma(\operatorname{im}(\sigma_i)) \cap \gamma\gamma_\tau^{-1} B(\operatorname{inc}(\widetilde{e}_\tau), \delta_n)}(\widetilde{p}) \\
=\ & \sum_{\gamma \in \Gamma} \sum_{\tau \in E(\sigma_i)} \chi_{\gamma A_i \cap \gamma\gamma_\tau^{-1} F(e_\tau)}(x) \chi_{\gamma(\operatorname{im}(\sigma_i)) \cap \gamma B(\operatorname{inc}(\gamma_\tau^{-1}\widetilde{e}_\tau), \delta_n)}(\widetilde{p}) \ .
\end{aligned} \tag{7}$$

Recall now that, since σ_i is big, the δ_n-balls centered at the incenters of the ridges of σ_i are disjoint. As a consequence we have that

$$\begin{aligned}\chi_{\gamma A_i}(x)\chi_{\gamma\,\mathrm{im}(\sigma_i)}(\widetilde{p}) &\geqslant \sum_{\tau\in E(\sigma_i)} \chi_{\gamma A_i}(x)\chi_{\gamma(\mathrm{im}(\sigma_i))\cap\gamma B(\mathrm{inc}(\gamma_\tau^{-1}\widetilde{e}_\tau),\delta_n)}(\widetilde{p}) \\ &\geqslant \sum_{\tau\in E(\sigma_i)} \chi_{\gamma A_i\cap\gamma\gamma_\tau^{-1}F(e_\tau)}(x)\chi_{\gamma(\mathrm{im}(\sigma_i))\cap\gamma B(\mathrm{inc}(\gamma_\tau^{-1}\widetilde{e}_\tau),\delta_n)}(\widetilde{p})\ .\end{aligned}$$

By summing over $\gamma\in\Gamma$ and using (7) we obtain

$$\begin{aligned}&\sum_{\gamma\in\Gamma}\chi_{\gamma A_i}(x)\chi_{\gamma\,\mathrm{im}(\sigma_i)}(\widetilde{p}) \\ \geqslant &\sum_{\gamma\in\Gamma}\sum_{\tau\in E(\sigma_i)} \chi_{F(\sigma_i,\tau)}(\gamma^{-1}(x))\chi_{\gamma\gamma_\tau(\mathrm{im}(\sigma_i))\cap\gamma B(\mathrm{inc}(\widetilde{e}_\tau),\delta_n)}(\widetilde{p})\ .\end{aligned}$$

Finally, by summing over $i\in I_{\mathrm{pos}}$ we get

$$\begin{aligned}&\sum_{i\in I_{\mathrm{pos}}}\sum_{\gamma\in\Gamma}\chi_{\gamma A_i}(x)\chi_{\gamma\,\mathrm{im}(\sigma_i)}(\widetilde{p}) \\ \geqslant &\sum_{i\in I_b}\sum_{\gamma\in\Gamma}\chi_{\gamma A_i}(x)\chi_{\gamma\,\mathrm{im}(\sigma_i)}(\widetilde{p}) \\ \geqslant &\sum_{i\in I_b}\sum_{\gamma\in\Gamma}\sum_{\tau\in E(\sigma_i)} \chi_{F(\sigma_i,\tau)}(\gamma^{-1}(x))\chi_{\gamma\gamma_\tau(\mathrm{im}(\sigma_i))\cap\gamma B(\mathrm{inc}(\widetilde{e}_\tau),\delta_n)}(\widetilde{p})\ ,\end{aligned}$$

which just means that

$$\deg_{\widetilde{p}}(\widetilde{z}_{\mathrm{pos},x}) \geqslant \deg_{\widetilde{p}}(\widetilde{z}_{-,x})\ .$$

Therefore, $\deg_{\widetilde{p}}(\widetilde{z}_x)\geqslant 0$ for every $x\in X$, $\widetilde{p}\in\mathbb{H}^n$. In order to conclude it is sufficient to show that, if x is such that c_x is a fundamental cycle for $\mathbb{H}^n$, then $\deg_{\widetilde{p}}(\widetilde{z}_x)\geqslant 1$ for almost every $\widetilde{p}\in\mathbb{H}^n$. Let us fix such an $x\in X$. Since c_x is a fundamental cycle and the boundary of the image of each (translate of) σ_i is a null set, for almost every $\widetilde{p}\in\mathbb{H}^n$ we have

$$\sum_{i\in I_{\mathrm{pos}}}\sum_{\gamma\in\Gamma}\chi_{A_i}(\gamma^{-1}x)\chi_{\mathrm{im}(\gamma\sigma_i)}(\widetilde{p}) - \sum_{i\in I\setminus I_{\mathrm{pos}}}\sum_{\gamma\in\Gamma}\chi_{A_i}(\gamma^{-1}x)\chi_{\mathrm{im}(\gamma\sigma_i)}(\widetilde{p}) = 1\ ,$$

so

$$\deg_{\widetilde{p}}(\widetilde{z}_{\mathrm{pos},x}) = \sum_{i\in I_{\mathrm{pos}}}\sum_{\gamma\in\Gamma}\chi_{A_i}(\gamma^{-1}x)\chi_{\mathrm{im}(\gamma\sigma_i)}(\widetilde{p}) \geqslant 1\ .$$

Therefore, if $\deg_{\widetilde{p}}(\widetilde{z}_{-,x})=0$, then $\deg_{\widetilde{p}}(\widetilde{z}_x)\geqslant\deg_{\widetilde{p}}(\widetilde{z}_{\mathrm{pos},x})\geqslant 1$, and we are done. Otherwise, we have $\deg_{\widetilde{p}}(\widetilde{z}_{-,x})\geqslant 1$. Therefore, there exist $i\in I_{\mathrm{pos}}$, $\tau\in E(\sigma_i)$ and $\gamma\in\Gamma$ such that

$$\chi_{F(\sigma_i,\tau)}(\gamma^{-1}(x))\chi_{\gamma\gamma_\tau(\mathrm{im}(\sigma_i))\cap\gamma B(\mathrm{inc}(\widetilde{e}_\tau),\delta_n)}(\widetilde{p}) = 1\ ,$$

i.e.,

$$x\in\gamma F(\sigma_i,\tau,e_\tau)\subseteq\gamma F(e_\tau) \qquad \text{and} \qquad \widetilde{p}\in\gamma B(\mathrm{inc}(\widetilde{e}_\tau),\delta_n)\ .$$

This implies at once that

$$\deg_{\widetilde{p}}(\widetilde{z}_{+,x})\geqslant 1\ ,$$

so

$$\deg_{\widetilde{p}}(\widetilde{z}_x) = \big(\deg_{\widetilde{p}}(\widetilde{z}_{\mathrm{pos},x}) - \deg_{\widetilde{p}}(\widetilde{z}_{-,x})\big) + \deg_{\widetilde{p}}(\widetilde{z}_{+,x}) \geqslant 0+1\geqslant 1\ ,$$

whence the conclusion. $\square$

Proposition 5.12. *We have*

$$|c|^{\alpha} v_n - \frac{1+a_n}{k_n+1}\eta_n \sum_{i\in I_b}\sum_{\tau\in E(\sigma_i)} \mu(F(\sigma_i,\tau)) + \eta_n \sum_{e\in E}\mu(F(e)) \geqslant \mathrm{vol}(M)\ . \tag{8}$$

Proof. Of course we have

$$\mathrm{vol}(\overline{z}_{\mathrm{pos}}) = \sum_{i\in I_{\mathrm{pos}}} |\beta_i|\,\mathrm{vol}(\mathrm{im}(\sigma_i)) \leqslant v_n \sum_{i\in I_{\mathrm{pos}}} |\beta_i| \leqslant |c|^{\alpha} v_n$$

and

$$\mathrm{vol}(\overline{z}_+) = \sum_{e\in E}\mu(F(e))\,\mathrm{vol}(B(\mathrm{inc}(\widetilde{e},\delta_n))) = \eta_n \sum_{e\in E}\mu(F(e))\ .$$

Now recall that δ_n was chosen in such a way that, if $i\in I_b$, then the boundary of the ball of radius δ_n centered at the incenter of any ridge τ of $\mathrm{im}(\sigma_i)$ does not meet any $(n-1)$-face of $\mathrm{im}(\sigma_i)$ not containing τ. As a consequence, if $\mu(F(\sigma_i,\tau))\neq 0$, then $\mathrm{vol}((\gamma_\tau\cdot \mathrm{im}(\sigma_i))\cap B(\mathrm{inc}(\widetilde{e}_\tau),\delta_n)) \geqslant \eta_n(1+a_n)/(k_n+1)$. Therefore, we have

$$\begin{aligned}\mathrm{vol}(\overline{z}_-) &= \sum_{i\in I_b}\sum_{\tau\in E(\sigma_i)} \mu(F(\sigma_i,\tau))\,\mathrm{vol}\left((\gamma_\tau\cdot \mathrm{im}(\sigma_i))\cap B(\mathrm{inc}(\widetilde{e}_\tau),\delta_n)\right)\\ &\geqslant \frac{1+a_n}{k_n+1}\eta_n \sum_{i\in I_b}\sum_{\tau\in E(\sigma_i)} \mu(F(\sigma_i,\tau))\ .\end{aligned}$$

Therefore, we get that

$$\mathrm{vol}(\overline{z}) \leqslant |c|^{\alpha} v_n - \frac{1+a_n}{k_n+1}\eta_n \sum_{i\in I_b}\sum_{\tau\in E(\sigma_i)} \mu(F(\sigma_i,\tau)) + \eta_n \sum_{e\in E}\mu(F(e))\ .$$

On the other hand, by Proposition 5.11 we have

$$\mathrm{vol}(M) \leqslant \int_M \deg_p(\overline{z})\,dp = \mathrm{vol}(\overline{z})\ ,$$

and this concludes the proof. $\square$

Proposition 5.13. *Suppose that* $|c_s|^{\alpha} \leqslant |c|^{\alpha}/12$*. Then*

$$\|M\| \leqslant \max\left\{1-\frac{\eta_n}{3v_n},\, 1-\frac{a_n\eta_n}{2v_n}\right\}\cdot |c|^{\alpha}\ .$$

Proof. We divide the proof in two cases. Suppose first that

$$\sum_{e\in E}\mu(F(e)) \leqslant \frac{|c|^{\alpha}}{2}\ .$$

Since $k_n+1\leqslant 6$ and $a_n>0$, from Lemma 5.8 we deduce that

$$\frac{1+a_n}{k_n+1}\sum_{i\in I_b}\sum_{\tau\in E(\sigma_i)} \mu(F(\sigma_i,\tau)) \geqslant \frac{5}{6}|c|^{\alpha}\ .$$

Plugging these inequalities into (8) we obtain

$$\|M\| = \frac{\mathrm{vol}(M)}{v_n} \leqslant |c|^{\alpha} + \frac{\eta_n}{v_n}\left(\frac{|c|^{\alpha}}{2} - \frac{5\,|c|^{\alpha}}{6}\right) = \left(1-\frac{\eta_n}{3v_n}\right)|c|^{\alpha}.$$

Now assume that $\sum_{e\in E}\mu(F(e)) \geqslant |c|^{\alpha}/2$. Recall that around any x-full ridge $\widetilde{e}$ in $\mathbb{H}^n$ at least k_n+1 big simplices of c_x appear. As a consequence for every $e\in E$ we have that

$$\sum_{i\in I_b}\sum_{\tau\in E(\sigma_i)}\mu(F(\sigma_i,\tau,e)) \geqslant (k_n+1)\mu(F(e)),$$

so

$$\frac{1}{k_n+1}\sum_{e\in E}\sum_{i\in I_b}\sum_{\tau\in E(\sigma_i)}\mu(F(\sigma_i,\tau,e)) \geqslant \sum_{e\in E}\mu(F(e)).$$

Substituting this inequality into (8) we get

$$\begin{aligned}\|M\| = \frac{\mathrm{vol}(M)}{v_n} &\leqslant |c|^{\alpha} + \frac{\eta_n}{v_n}\left(\sum_{e\in E}\mu(F(e)) - (1+a_n)\sum_{e\in E}\mu(F(e))\right)\\ &= |c|^{\alpha} - \frac{a_n\eta_n}{v_n}\sum_{e\in E}\mu(F(e)) \leqslant |c|^{\alpha} - \frac{a_n\eta_n\,|c|^{\alpha}}{2v_n}. \qquad \square\end{aligned}$$

The results proved in Propositions 5.6 and 5.13 readily imply Theorem 5.4.

6. Integral foliated simplicial volume of aspherical manifolds with amenable fundamental group

The section is devoted to the proof of Theorem 1.9. Before we start, we remark that we cannot generalize Theorem 1.9 to rationally essential manifolds.

Remark 6.1. The statement of Theorem 1.9 does not hold if we replace "aspherical" by "rationally essential". For example, the oriented closed connected manifold $M := (S^1)^4\#(S^2\times S^2)$ is rationally essential and has amenable residually finite fundamental group, but an L^2-Betti number estimate as in Remark 3.9 shows that $\|M\|^{\infty}_{\mathbb{Z}} = |M| > 0$.

6.1. **Strategy of proof of Theorem 1.9.** We start with a parametrized fundamental cycle c in $L^\infty(X,\mu,\mathbb{Z})\otimes_{\mathbb{Z}\Gamma} C_n(\widetilde{M};\mathbb{Z})$ that comes from an integral fundamental cycle on M. Using the generalized Rokhlin lemma by Ornstein-Weiss from ergodic theory we rewrite c as a sum of chains of the form $\chi_{A_i}\otimes\widetilde{c}_i$ with $A_i\subset X$ and $\widetilde{c}_i\in C_n(\widetilde{M};\mathbb{Z})$. The chains $\widetilde{c}_i$ will have boundaries that are small in norm. We then fill $\partial\widetilde{c}_i$ in the contractible space $\widetilde{M}$ more efficiently by chains $\widetilde{c}'_i$. The norm of the parametrized fundamental cycle resulting from replacing $\widetilde{c}_i$ by $\widetilde{c}'_i$ can be made arbitrarily small.

We will now provide the details for this argument.

6.2. **Setup.** Let $n := \dim M$. The case $n=1$ (i.e., $M\cong S^1$) being trivial, we may assume without loss of generality that $n>1$.

Let $\pi\colon \widetilde{M}\longrightarrow M$ be the universal covering of M and let D be a strict set-theoretical fundamental domain for the (left) deck transformation action of $\Gamma := \pi_1(M)$ on $\widetilde{M}$. We assume that Γ is amenable. Furthermore, let $\alpha = \Gamma \curvearrowright (X,\mu)$ be a free standard Γ-space.

Let $c = \sum_{j=1}^m a_j\cdot\sigma_j \in C_n(M;\mathbb{Z})$ be an integral fundamental cycle of M; we arrange this in such a (non-reduced) way that the coefficients satisfy $a_1, \dots, a_m \in \{-1, 1\}$ (but we still require that the same simplex does not occur with positive and negative sign). We consider the unique lift $\widetilde{c} := \sum_{j=1}^m a_j \cdot \widetilde{\sigma}_j \in C_n(\widetilde{M}; \mathbb{Z})$ satisfying

$$\pi \circ \widetilde{\sigma}_j = \sigma_j \quad \text{and} \quad \widetilde{\sigma}_j(e_0) \in D$$

for all $j \in \{1, \dots, m\}$.

As next step, we define a suitable finite subset $S \subset \Gamma$ that measures the defect of $\widetilde{c}$ from being a cycle: Because c is a cycle we can match any face of $\sigma_1, \dots, \sigma_m$ occurring with positive sign in the expression ∂c to another one of these faces with negative sign. Let $\Sigma_+ \sqcup \Sigma_-$ be such a splitting of the set Σ of faces of $\sigma_1, \dots, \sigma_m$, and let $\widehat{\cdot} \colon \Sigma_+ \longrightarrow \Sigma_-$ be the corresponding matching bijection; then

$$\partial c = \sum_{\tau \in \Sigma_+} (\tau - \widehat{\tau}).$$

For each $\tau \in \Sigma$ we denote the corresponding face of $\widetilde{c}$ by $\widetilde{\tau}$. Then for each $\tau \in \Sigma_+$ there is a unique $g_\tau \in \Gamma$ with

$$\widetilde{\widehat{\tau}} = g_\tau \cdot \widetilde{\tau}.$$

By definition of $\widehat{\tau}$, we then obtain

$$\partial \widetilde{c} = \sum_{\tau \in \Sigma_+} (\widetilde{\tau} - g_\tau \cdot \widetilde{\tau}).$$

Let S be a finite symmetric generating set of Γ that contains $\{g_\tau \mid \tau \in \Sigma_+\}$.

6.3. **Fundamental cycles and the generalized Rokhlin lemma.** We denote the S-boundary of a subset $F \subset \Gamma$ by

$$\partial_S F = \{\gamma \in F \mid \exists_{s \in S}\ \gamma s \notin F\}.$$

Fix $\varepsilon > 0$. By a version [31, Theorem 5.2] of the generalized Rokhlin lemma of Ornstein-Weiss there are finite subsets $F_1, \dots, F_N \subset \Gamma$ and Borel subsets $A_1, \dots, A_N \subset X$ such that

- for every $i \in \{1, \dots, N\}$, the set F_i is (S, ε)-invariant in the sense that $|\partial_S F_i|/|F_i| < \varepsilon$;
- for every $i \in \{1, \dots, N\}$, the sets γA_i with $\gamma \in F_i$ are pairwise disjoint;
- the sets $F_i A_i$ with $i \in \{1, \dots, N\}$ are pairwise disjoint;
- $\mu(R) < \varepsilon$ where $R := X \setminus \bigcup_{i=1}^N F_i A_i$.

Then the following computation holds in $L^\infty(X, \mu, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M}; \mathbb{Z})$:

$$\begin{aligned} 1 \otimes \widetilde{c} &= \left(\sum_{j=1}^m \sum_{i=1}^N \sum_{\gamma \in F_i} a_j \chi_{\gamma A_i} \otimes \widetilde{\sigma}_j \right) + \underbrace{\sum_{j=1}^m a_j \chi_R \otimes \widetilde{\sigma}_j}_{=:E} \\ &= \sum_{i=1}^N \chi_{A_i} \otimes \underbrace{\left(\sum_{j=1}^m \sum_{\gamma \in F_i} a_j \gamma^{-1} \widetilde{\sigma}_j \right)}_{=:\widetilde{c}_i} + E. \end{aligned}$$

In the following, let $i \in \{1, \dots, N\}$. Interchanging summation, we have

$$\widetilde{c}_i = \sum_{\gamma \in F_i} \sum_{j=1}^{m} a_j \gamma^{-1} \widetilde{\sigma}_j = \sum_{\gamma \in F_i} \gamma^{-1} \widetilde{c}.$$

Lemma 6.2 (small boundary in $\widetilde{M}$)**.** *The chain $\widetilde{c}_i$ has small boundary; more precisely, we have*

$$|\partial \widetilde{c}_i|_{1,\mathbb{Z}} \leqslant \varepsilon \cdot |F_i| \cdot |\Sigma_+|.$$

Proof. Because F_i is (S, ε)-invariant, it suffices to show $|\partial \widetilde{c}_i|_{1,\mathbb{Z}} \leqslant |\partial_S F_i| \cdot |\Sigma_+|$. By construction,

$$\begin{aligned} \partial \widetilde{c}_i &= \sum_{\gamma \in F_i} \sum_{\tau \in \Sigma_+} \gamma^{-1} \cdot (\widetilde{\tau} - g_\tau \cdot \widetilde{\tau}) \\ &= \sum_{\gamma \in F_i} \sum_{\tau \in \Sigma_+} \gamma^{-1} \cdot \widetilde{\tau} - \sum_{\gamma \in F_i} \sum_{\tau \in \Sigma_+} \gamma^{-1} \cdot g_\tau \cdot \widetilde{\tau}. \end{aligned}$$

So, the only terms in $\partial \widetilde{c}_i$ that do *not* cancel are of the shape $\pm \gamma^{-1} \cdot \widetilde{\tau}$ with $\tau \in \Sigma_+$ and $\gamma^{-1} \in \partial_S F_i$. Hence, $|\partial \widetilde{c}_i|_{1,\mathbb{Z}} \leqslant |\partial_S F_i| \cdot |\Sigma_+|$. ☐

6.4. **Efficient filling of the boundary.** We will now fill the boundary of $\widetilde{c}_i$ more efficiently in $\widetilde{M}$.

Lemma 6.3 (efficient filling)**.** *There exists a chain $\widetilde{c}'_i \in C_n(\widetilde{M}; \mathbb{Z})$ with*

$$\partial \widetilde{c}'_i = \partial \widetilde{c}_i \quad \textit{and} \quad |\widetilde{c}'_i|_{1,\mathbb{Z}} \leqslant (n+1) \cdot |\partial \widetilde{c}_i|_{1,\mathbb{Z}}.$$

Proof. Because M is aspherical, the universal covering $\widetilde{M}$ is contractible. Then any homotopy between $\mathrm{id}_{\widetilde{M}}$ and a constant map $p \colon \widetilde{M} \longrightarrow \widetilde{M}$ induces a chain homotopy $h_* \colon C_*(\widetilde{M}; \mathbb{Z}) \longrightarrow C_{*+1}(\widetilde{M}; \mathbb{Z})$ between $\mathrm{id}_{C_*(\widetilde{M};\mathbb{Z})}$ and $C_*(p; \mathbb{Z})$ satisfying

$$\|h_k\| \leqslant k + 1$$

(with respect to $|\cdot|_{1,\mathbb{Z}}$) for all $k \in \mathbb{N}$; this follows from the standard construction [14, Theorem 2.10] of h_* by subdividing $\Delta^k \times [0, 1]$ into $k + 1$ simplices of dimension k.

By construction,

$$\widetilde{b}_i := \partial \widetilde{c}_i \in C_{n-1}(\widetilde{M}; \mathbb{Z})$$

is a cycle. Moreover, because $n > 1$ we have $H_{n-1}(\bullet; \mathbb{Z}) \cong 0$ and for any cycle $d \in C_{n-1}(\bullet; \mathbb{Z})$ there exists a chain $z \in C_n(\bullet; \mathbb{Z})$ with $\partial z = d$ and $|z|_{1,\mathbb{Z}} \leqslant |d|_{1,\mathbb{Z}}$ (this follows from a direct computation in $C_*(\bullet; \mathbb{Z})$). Because p factors over a one-point space $\bullet$, we thus find a chain $z_i \in C_n(\widetilde{M}; \mathbb{Z})$ with

$$\partial z_i = C_{n-1}(p; \mathbb{Z})(\widetilde{b}_i) \quad \text{and} \quad |z_i|_{1,\mathbb{Z}} \leqslant \bigl|C_{n-1}(p; \mathbb{Z})(\widetilde{b}_i)\bigr|_{1,\mathbb{Z}} \leqslant |\widetilde{b}_i|_{1,\mathbb{Z}}.$$

We now consider

$$\widetilde{c}'_i := h_{n-1}(\widetilde{b}_i) + z_i \in C_n(\widetilde{M}; \mathbb{Z}).$$

Then the chain homotopy relation $\partial \circ h + h \circ \partial = \mathrm{id} - C_*(p; \mathbb{Z})$ shows that

$$\begin{aligned} \partial \widetilde{c}'_i &= \partial h_{n-1}(\widetilde{b}_i) + \partial z_i \\ &= \widetilde{b}_i - C_{n-1}(p; \mathbb{Z})(\widetilde{b}_i) - h_{n-2} \circ \partial \widetilde{b}_i + \partial z_i \\ &= \widetilde{b}_i = \partial \widetilde{c}_i \end{aligned}$$

and

$$|\widetilde{c}'_i|_{1,\mathbb{Z}} \leqslant n \cdot |\widetilde{b}_i|_{1,\mathbb{Z}} + |z_i|_{1,\mathbb{Z}} \leqslant (n+1) \cdot |\widetilde{b}_i|_{1,\mathbb{Z}},$$

as claimed. ☐

With the $\widetilde{c}'_i$ obtained from the previous lemma, we define a new chain

$$c' := \Bigl(\sum_{i=1}^{N} \chi_{A_i} \otimes \widetilde{c}'_i\Bigr) + E \in L^\infty(X, \mu, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_n(\widetilde{M}; \mathbb{Z}).$$

6.5. **Efficient fundamental cycles.**

Lemma 6.4. *The chain c' is an α-parametrized fundamental cycle of M.*

Proof. Since $\widetilde{M}$ is aspherical and $\partial \widetilde{c}_i = \partial \widetilde{c}'_i$, there are chains $\widetilde{w}_i \in C_{n+1}(\widetilde{M}; \mathbb{Z})$ with $\partial w_i = \widetilde{c}_i - \widetilde{c}'_i$. The claim now follows from the computation

$$\begin{aligned} c' = \Bigl(\sum_{i=1}^{N} \chi_{A_i} \otimes \widetilde{c}'_i\Bigr) + E &= \Bigl(\sum_{i=1}^{N} \chi_{A_i} \otimes \widetilde{c}_i\Bigr) + E + \sum_{i=1}^{N} \chi_{A_i} \otimes \widetilde{\partial w}_i \\ &= \Bigl(\sum_{i=1}^{N} \chi_{A_i} \otimes \widetilde{c}_i\Bigr) + E + \partial\Bigl(\sum_{i=1}^{N} \chi_{A_i} \otimes \widetilde{w}_i\Bigr) \\ &= 1 \otimes \widetilde{c} + \partial\Bigl(\sum_{i=1}^{N} \chi_{A_i} \otimes \widetilde{w}_i\Bigr) \end{aligned}$$

and the fact that $1 \otimes \widetilde{c}$ is an α-parametrized fundamental cycle. ☐

6.6. **Conclusion of the proof.** The previous two lemmas imply the following estimate.

$$\begin{aligned} |M|^\alpha \le |c'|^\alpha &\le \sum_{i=1}^{N} \mu(A_i) |\widetilde{c}'_i|_{1,\mathbb{Z}} + |E|^\alpha \\ &\le \sum_{i=1}^{N} \mu(A_i) |\widetilde{c}'_i|_{1,\mathbb{Z}} + \mu(R) |\widetilde{c}|_{1,\mathbb{Z}} \\ &\le \Bigl(\sum_{i=1}^{N} (n+1) \mu(A_i) |\partial \widetilde{c}_i|_{1,\mathbb{Z}}\Bigr) + \varepsilon |\widetilde{c}|_{1,\mathbb{Z}} \\ &\le |\Sigma_+| \varepsilon (n+1) \Bigl(\sum_{i=1}^{N} \mu(A_i) |F_i|\Bigr) + \varepsilon |\widetilde{c}|_{1,\mathbb{Z}} \\ &= |\Sigma_+| \varepsilon (n+1) \Bigl(\sum_{i=1}^{N} \mu(F_i A_i)\Bigr) + \varepsilon |\widetilde{c}|_{1,\mathbb{Z}} \\ &= |\Sigma_+| \varepsilon (n+1) \mu\Bigl(\bigcup_{i=1}^{N} F_i A_i\Bigr) + \varepsilon |\widetilde{c}|_{1,\mathbb{Z}} \\ &\le \varepsilon\bigl(|\Sigma_+|(n+1) + |\widetilde{c}|_{1,\mathbb{Z}}\bigr). \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the above estimate finishes the proof of Theorem 1.9.

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