A QUANTITATIVE VERSION OF A THEOREM BY JUNGREIS

MICHELLE BUCHER, ROBERTO FRIGERIO, AND CRISTINA PAGLIANTINI

ABSTRACT. A fundamental result by Gromov and Thurston asserts that, if M is a closed hyperbolic n-manifold, then the simplicial volume $\|M\|$ of M is equal to $\mathrm{Vol}(M)/v_n$, where v_n is a constant depending only on the dimension of M. The same result also holds for complete finite-volume hyperbolic manifolds without boundary, while Jungreis proved that the ratio $\mathrm{Vol}(M)/\|M\|$ is strictly smaller than v_n if M is compact with non-empty geodesic boundary. We prove here a quantitative version of Jungreis' result for $n \geq 4$, which bounds from below the ratio $\|M\|/\mathrm{Vol}(M)$ in terms of the ratio $\mathrm{Vol}(\partial M)/\mathrm{Vol}(M)$. As a consequence, we show that a sequence $\{M_i\}$ of compact hyperbolic n-manifolds with geodesic boundary satisfies $\lim_i \mathrm{Vol}(M_i)/\|M_i\| = v_n$ if and only if $\lim_i \mathrm{Vol}(\partial M_i)/\mathrm{Vol}(M_i) = 0$.

We also provide estimates of the simplicial volume of hyperbolic manifolds with geodesic boundary in dimension three.

INTRODUCTION

The simplicial volume is a homotopy invariant of compact manifolds introduced by Gromov in his pioneering work [Gro82]. If M is a connected, compact, oriented manifold with (possibly empty) boundary, then the simplicial volume of M, denoted by $\|M, \partial M\|$, is the infimum of the sum of the absolute values of the coefficients over all singular chains representing the real fundamental cycle of M (see Section 1). If $\partial M = \emptyset$ we denote the simplicial volume of M simply by $\|M\|$. If M is open, the fundamental class and the simplicial volume of M admit analogous definitions in the context of homology of *locally finite* chains, but in this paper we will restrict our attention to compact manifolds: unless otherwise stated, henceforth every manifold is assumed to be compact. Observe that the simplicial volume of an oriented manifold does not depend on its orientation and that it is straightforward to extend the definition also to nonorientable or disconnected manifolds: if M is connected and nonorientable, then its simplicial volume is equal to one half of the simplicial volume of its orientable double covering, and the simplicial volume of any manifold is the sum of the simplicial volumes of its connected components.

Several vanishing and nonvanishing results for the simplicial volume are available by now, but the exact value of nonvanishing simplicial volumes is known only in a very few cases. A celebrated result by Gromov and Thurston [Gro82, Thu79] implies that, if M is a hyperbolic n-manifold without boundary M, then

$$(1) \qquad \|M\| = \frac{\mathrm{Vol}(M)}{v_n} \ ,$$

where $\mathrm{Vol}(M)$ is the Riemannian volume of M and v_n is the volume of the regular ideal geodesic n-simplex in hyperbolic space. In the closed case, the only other exact computation of nonvanishing simplicial volume is for the product of two closed hyperbolic surfaces or more generally manifolds locally isometric to the product of two hyperbolic planes [BK08].

After replacing $\|M\|$ with $\|M, \partial M\|$, equality (1) holds also when M is the natural compactification of any complete noncompact hyperbolic n-manifold of finite volume without boundary (see *e.g.* [Fra04, FP10, FM11, BBI13]). In this case, every component of ∂M supports a Euclidean structure, so $\|\partial M\| = 0$. In the case when $\|\partial M\| \neq 0$, the only exact computations of the simplicial volume of compact manifolds with nonempty boundary are provided in [BFP15] for products of surfaces with the interval and for compact 3-manifolds obtained by adding 1-handles to Seifert manifolds. Building on these examples, more values for the simplicial volume can be obtained by surgery or by taking connected sums or amalgamated sums over submanifolds with amenable fundamental group (see *e.g.* [Gro82, Kue03, BBF+14]). However, the exact value of the simplicial volume is not known for any compact n-manifold M such that $\|\partial M\| \neq 0$, $n \geq 4$, and for any hyperbolic n-manifold with non-empty geodesic boundary, $n \geq 3$.

Hyperbolic manifolds with geodesic boundary. Jungreis proved in [Jun97] that, if M is an n-dimensional hyperbolic manifold with nonempty geodesic boundary, $n \geq 3$, then

$$(2) \qquad \|M, \partial M\| > \frac{\mathrm{Vol}(M)}{v_n} \ .$$

In Section 4 we provide a quantitative version of Jungreis' result in the case when $n \geq 4$. More precisely, we prove the following:

Theorem 1. *Let $n \geq 4$. Then there exists a constant $\eta_n > 0$ depending only on n such that*

$$\frac{\|M, \partial M\|}{\mathrm{Vol}(M)} \geq \frac{1}{v_n} + \eta_n \cdot \frac{\mathrm{Vol}(\partial M)}{\mathrm{Vol}(M)} \ .$$

It is well-known that $\|M, \partial M\| = \mathrm{Vol}(M)/v_2 = \mathrm{Vol}(M)/\pi$ for every hyperbolic surface with geodesic boundary M, so Theorem 1 cannot be true in dimension 2. The 3-dimensional case is still open.

Theorem 1 states that, if $n \geq 4$, then $\mathrm{Vol}(M)/\|M, \partial M\|$ cannot approach v_n unless the $(n-1)$-dimensional volume of ∂M is small with respect to the

volume of M. On the other hand, it is known that $\mathrm{Vol}(M)/\|M, \partial M\|$ indeed approaches v_n if $\mathrm{Vol}(\partial M)/\mathrm{Vol}(M)$ is small. In fact, the following result is proved in [FP10] for $n \geq 3$: for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\frac{\mathrm{Vol}(\partial M)}{\mathrm{Vol}(M)} < \delta \quad \Longrightarrow \quad \frac{\mathrm{Vol}(M)}{\|M, \partial M\|} \geq v_n - \varepsilon$$

for every hyperbolic n-manifold M with nonempty geodesic boundary.

Note that in particular, the ratio between $\|M, \partial M\|$ and $\mathrm{Vol}(M)$ does not depend only on the dimension of M. Putting together this result with Theorem 1, we obtain, for $n \geq 4$, a complete characterization of hyperbolic n-manifolds with geodesic boundary whose simplicial volume is close to the bound given by inequality (2):

Corollary 2. *Let $n \geq 4$, and let M_i be a sequence of hyperbolic n-manifolds with nonempty geodesic boundary. Then*

$$\lim_{i \to \infty} \frac{\mathrm{Vol}(M_i)}{\|M_i, \partial M_i\|} = v_n \quad \Longleftrightarrow \quad \lim_{i \to \infty} \frac{\mathrm{Vol}(\partial M_i)}{\mathrm{Vol}(M_i)} = 0 \ .$$

The 3-dimensional case. Let M be a boundary irreducible aspherical 3-manifold. In [BFP15, Theorem 1.4] we proved the following sharp lower bound for the simplicial volume of M in terms of the simplicial volume of ∂M:

$$(3) \qquad\qquad \|M, \partial M\| \geq \frac{5}{4}\|\partial M\| \ .$$

Every hyperbolic manifold with geodesic boundary is aspherical and boundary irreducible. Therefore, even if Theorem 1 is still open in dimension 3, inequality (3) may be exploited to show that, if $\mathrm{Vol}(\partial M)$ is big with respect to $\mathrm{Vol}(M)$, then indeed the simplicial volume of M is bounded away from $\mathrm{Vol}(M)/v_3$. In the same spirit, by combining combinatorial and geometric arguments, in Section 5 we prove the following result, where

$$G = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \ldots \approx 0.915965$$

is Catalan's constant.

Theorem 3. *Let M be a hyperbolic 3-manifold with nonempty geodesic boundary. Then*

$$\|M, \partial M\| \geq \frac{\mathrm{Vol}(M)}{v_3} + \frac{v_3 - G}{2(3v_3 - 2G)}\left(7\|\partial M\| - 4\frac{\mathrm{Vol}(M)}{v_3}\right) \ .$$

At the moment, no precise computation of the simplicial volume of hyperbolic 3-manifold with nonempty geodesic boundary is known. Let us briefly introduce some families of examples for which the bound provided by Theorem 3 is sharper than both Jungreis' bound (2) and bound (3).

For every $g \geq 2$ let $\overline{\mathcal{M}}_g$ be the set of hyperbolic 3-manifolds M with connected geodesic boundary such that $\chi(\partial M) = 2 - 2g$ (so ∂M, if orientable, is the closed orientable surface of genus g). Recall that for every 3-manifold

with boundary M the equality $\chi(\partial M) = 2\chi(M)$ holds, and in particular $\chi(\partial M)$ is even. Therefore, the union $\bigcup_{g \geq 2} \overline{\mathcal{M}}_g$ coincides with the set of hyperbolic 3-manifolds with connected geodesic boundary.

For every $g \geq 2$ we denote by $\mathcal{M}_g$ the set of 3-manifolds with boundary M that admit an ideal triangulation by g tetrahedra and have Euler characteristic $\chi(M) = 1 - g$ (see Section 6 for the definition of ideal triangulation). Every element of $\mathcal{M}_g$ has connected boundary and supports a hyperbolic structure with geodesic boundary (which is unique by Mostow rigidity), hence $\mathcal{M}_g \subseteq \overline{\mathcal{M}}_g$ (see Proposition 6.1). Furthermore, Miyamoto proved in [Miy94] that elements of $\mathcal{M}_g$ are exactly the ones having the smallest volume among the elements of $\overline{\mathcal{M}}_g$. In particular, $\mathcal{M}_g$ is nonempty for every $g \geq 2$. The eight elements of $\mathcal{M}_2$ are exactly the smallest hyperbolic manifolds with nonempty geodesic boundary [KM91, Miy94].

As a consequence of Corollary 2, the simplicial volume and the Riemannian volume of hyperbolic 3-manifolds with nonempty geodesic boundary are not related by a universal proportionality constant. Nevertheless, it is reasonable to expect that these invariants are closely related to each other. Therefore, we make here the following conjecture:

Conjecture 4. For $g \geq 2$, the elements of $\mathcal{M}_g$ are exactly the ones having the smallest simplicial volume among the elements of $\overline{\mathcal{M}}_g$. Moreover, the eight elements of $\mathcal{M}_2$ are the hyperbolic manifolds with nonempty geodesic boundary having the smallest simplicial volume.

Together with Miyamoto's results about volumes of hyperbolic manifolds with geodesic boundary [Miy94], Theorem 3 implies the following (see Section 6):

Corollary 5. If $M \in \overline{\mathcal{M}}_2$, then $\|M, \partial M\| \geq 6.461 \approx 1.615 \cdot \|\partial M\|$. If $M \in \overline{\mathcal{M}}_3$, then $\|M, \partial M\| \geq 10.882 \approx 1.360 \cdot \|\partial M\|$. If $M \in \overline{\mathcal{M}}_4$, then $\|M, \partial M\| \geq 15.165 \approx 1.264 \cdot \|\partial M\|$.

As we will see in Section 6, the corollary shows that Theorem 3 indeed improves Jungreis' inequality (2) and bound (3) in some cases. More precisely we will show that if $M \in \mathcal{M}_2 \cup \mathcal{M}_3 \cup \mathcal{M}_4$, the bounds provided by Theorem 3 and Corollary 5 coincide, and are sharper than the bounds provided by inequalities (2) and (3), while if $M \in \mathcal{M}_g$, $g \geq 5$, then the sharpest known bound for $\|M, \partial M\|$ is provided by inequality (3).

1. SIMPLICIAL VOLUME

Let X be a topological space and $Y \subseteq X$ a (possibly empty) subspace of X. Let R be a normed ring. Henceforth we confine ourself to $R = \mathbb{R}, \mathbb{Q}$ or $\mathbb{Z}$, where each of these rings is endowed with the norm given by the absolute value. For $i \in \mathbb{N}$ we denote by $C_i(X; R)$ the module of singular i-chains over R, i.e. the R-module freely generated by the set $S_i(X)$ of singular i-simplices in X, and we set as usual $C_i(X, Y; R) = C_i(X; R)/C_i(Y; R)$. Notice that we

will identify $C_i(X, Y; R)$ with the free R-module generated by $S_i(X) \setminus S_i(Y)$. In particular, for $z \in C_i(X, Y; R)$, it will be understood from the equality $z = \sum_{k=1}^{n} a_k \sigma_k$ that $\sigma_k \neq \sigma_h$ for $k \neq h$, and $\sigma_k \notin S_i(Y)$ for every k. We denote by $H_*(X, Y; R)$ the singular homology of the pair (X, Y) with coefficients in R, *i.e.* the homology of the complex $(C_*(X, Y; R), d_*)$, where d_* is the usual differential.

We endow the R-module $C_i(X, Y; R)$ with the L^1-norm defined by

$$\left\| \sum_{\sigma} a_\sigma \sigma \right\|_R = \sum_{\sigma} |a_\sigma| \ ,$$

where σ ranges over the simplices in $S_i(X) \setminus S_i(Y)$. We denote simply by $\|\cdot\|$ the norm $\|\cdot\|_{\mathbb{R}}$. The norm $\|\cdot\|_R$ descends to a seminorm on $H_*(X, Y; R)$, which is still denoted by $\|\cdot\|_R$ and is defined as follows: if $\alpha \in H_i(X, Y; R)$, then

$$\|\alpha\|_R = \inf \{ \|\beta\|_R, \ \beta \in C_i(X, Y; R), \ d\beta = 0, \ [\beta] = \alpha \} \ .$$

The *real* singular homology module $H_*(X, Y; \mathbb{R})$ and the seminorm $\|\cdot\|_{\mathbb{R}}$ will be simply denoted by $H_*(X, Y)$ and $\|\cdot\|$ respectively.

If M is a connected oriented n-manifold with (possibly empty) boundary ∂M, then we denote by $[M, \partial M]_R$ the fundamental class of the pair $(M, \partial M)$ with coefficients in R. The following definition is due to Gromov [Gro82, Thu79]:

Definition 1.1. The *simplicial volume* of M is

$$\|M, \partial M\| = \|[M, \partial M]_{\mathbb{R}}\| = \|[M, \partial M]_{\mathbb{R}}\|_{\mathbb{R}} \ .$$

The *rational*, respectively *integral*, simplicial volume of M is defined as $\|M, \partial M\|_{\mathbb{Q}} = \|[M, \partial M]_{\mathbb{Q}}\|_{\mathbb{Q}}$, respectively $\|M, \partial M\|_{\mathbb{Z}} = \|[M, \partial M]_{\mathbb{Z}}\|_{\mathbb{Z}}$.

Just as in the real case, the rational and the integral simplicial volume may be defined also when M is disconnected or nonorientable. Of course we have the inequalities $\|M, \partial M\| \leq \|M, \partial M\|_{\mathbb{Q}} \leq \|M, \partial M\|_{\mathbb{Z}}$. Using that $\mathbb{Q}$ is dense in $\mathbb{R}$, it may be shown in fact that $\|M, \partial M\| = \|M, \partial M\|_{\mathbb{Q}}$ and we provide here a complete proof of this fact.

Proposition 1.2. *For every n-manifold M, the real and rational simplicial volumes are equal,*

$$\|M, \partial M\| = \|M, \partial M\|_{\mathbb{Q}} \ .$$

Proof. We have to show that $\|M, \partial M\|_{\mathbb{Q}} \leq \|M, \partial M\|$. Let $\varepsilon > 0$ be fixed, and let $z = \sum_{i=1}^{k} a_i \sigma_i$ be a real fundamental cycle for M such that $\|z\| = \sum_{i=1}^{k} |a_i| \leq \|M, \partial M\| + \varepsilon$. We set

$$H_{\mathbb{R}} = \left\{ (x_1, \ldots, x_k) \in \mathbb{R}^k \mid \sum_{i=1}^{k} x_i \sigma_i \text{ is a relative cycle} \right\} \subseteq \mathbb{R}^k \ .$$

Of course, $H_{\mathbb{R}}$ is a linear subspace of $\mathbb{R}^k$. Since $H_{\mathbb{R}}$ is defined by a system of equations with integral coefficients, if $H_{\mathbb{Q}} = H_{\mathbb{R}} \cap \mathbb{Q}^k$, then $H_{\mathbb{Q}}$ is dense

in $H_\mathbb{R}$. As a consequence, we may find sequences of rational coefficients $\{\alpha_i^j\}_{j\in\mathbb{N}} \subseteq \mathbb{Q}$, $i = 1,\dots,k$ such that $z^j = \sum_{i=1}^k \alpha_i^j \sigma_i$ is a rational cycle for every $j \in \mathbb{N}$, and $\lim_j \alpha_i^j = a_i$ for every $i = 1,\dots,k$. This implies in particular that $\lim_j \|z^j\|_\mathbb{Q} = \|z\|$, so we are left to show that the z^j's may be chosen among the representatives of the rational fundamental class of M.

Let $\lambda_j \in \mathbb{Q}$ be defined by $[z^j] = \lambda_j \cdot [M, \partial M]$ (such a λ_j exists because $[M, \partial M]$ lies in the image of $H_n(M, \partial M; \mathbb{Q})$ in $H_n(M, \partial M; \mathbb{R})$ under the change of coefficients homomorphism). The Universal Coefficient Theorem provides a real cocyle $\varphi\colon C_n(M, \partial M; \mathbb{R}) \to \mathbb{R}$ such that $\varphi(z) = 1$. Observe that $\varphi(z^j) = \lambda_j$, so from $\lim_j \alpha_i^j = a_i$ we deduce that $\lim_j \lambda_j = \lim_j \varphi(z^j) = \varphi(z) = 1$. For large j we may thus define $w^j = \lambda_j^{-1} \cdot z^j \in C_n(M, \partial M; \mathbb{Q})$, and by construction w^j represents the rational fundamental class of M. Finally, we have

$$\lim_j \|w^j\|_\mathbb{Q} = \lim_j \frac{\|z^j\|_\mathbb{Q}}{\lambda_j} = \|z\| \leq \|M, \partial M\| + \varepsilon \ ,$$

which finishes the proof of the proposition. $\qquad\square$

On the contrary, the inequality $\|M, \partial M\| \leq \|M, \partial M\|_\mathbb{Z}$ is not an equality in general, for instance $\|S^1\| = 0$ but $\|S^1\|_\mathbb{Z} \geq 1$. The integral simplicial volume does not behave as nicely as the rational or real simplicial volume. For example, it follows from the definition that $\|M\|_\mathbb{Z} \geq 1$ for every manifold M. Therefore, the integral simplicial volume cannot be multiplicative with respect to finite coverings (otherwise it should vanish on manifolds that admit finite nontrivial self-coverings, as S^1). Nevertheless, we will use integral cycles extensively, as they admit a clear geometric interpretation in terms of pseudomanifolds (see Section 2). In order to follow this strategy, we need the following obvious consequence of the equality $\|M, \partial M\|_\mathbb{Q} = \|M, \partial M\|$.

Lemma 1.3. *Let M be connected and oriented, and let $\varepsilon > 0$ be given. Then, there exists an integral cycle $z \in C_n(M, \partial M; \mathbb{Z})$ such that*

$$\frac{\|z\|_\mathbb{Z}}{d} \leq \|M, \partial M\| + \varepsilon \ ,$$

where $[z] = d \cdot [M, \partial M]_\mathbb{Z}$ and $d > 0$ is an integer.

Moreover, the boundary of a fundamental cycle for M is equal to the sum of one fundamental cycle for each component of ∂M, so we also have:

$$(4) \qquad\qquad \|\partial M\| \leq \frac{\|\partial z\|_\mathbb{Z}}{d} \ .$$

Remark 1.4. The statements and the proofs of Proposition 1.2 and Lemma 1.3 hold more generally after replacing the fundamental class $[M, \partial M]_\mathbb{Q}$ by any rational homology class. In other words, for every $i \in \mathbb{N}$ the change of coefficients map $H_i(M, \partial M; \mathbb{Q}) \to H_i(M, \partial M; \mathbb{R})$ is norm-preserving.

Finally, let us list some elementary properties of the simplicial volume which will be needed later.

Proposition 1.5 ([Gro82])**.** *Let M, N be connected oriented manifolds of the same dimension, and suppose that either M, N are both closed, or they both have nonempty boundary. Let $f : N \to M$ be a map of degree d. Then*

$$\|N, \partial N\| \geq |d| \cdot \|M, \partial M\| \ .$$

The following well-known result describes the simplicial volume of closed surfaces. In fact, the same statement also holds for connected surfaces with boundary.

Proposition 1.6 ([Gro82])**.** *Let S be a closed surface. Then*

$$\|S\| = \max\{0, -2\chi(S)\} \ .$$

2. Representing cycles via pseudomanifolds

This section is devoted to recall the notion of pseudomanifold and the interpretation of integral cycles by means of pseudomanifolds.

Pseudomanifolds. Let $n \in \mathbb{N}$. An *n-dimensional pseudomanifold P* consists of a finite number of copies of the standard n-simplex, a choice of pairs of $(n-1)$-dimensional faces of n-simplices such that each face appears in at most one of these pairs, and an affine identification between the faces of each pair. We allow pairs of distinct faces in the same n-simplex. It is *orientable* if orientations on the simplices of P may be chosen in such a way that the affine identifications between the paired faces (endowed with the induced orientations) are all orientation-reversing. A face which does not belong to any pair of identified faces is a *boundary* face.

We denote by $|P|$ the *topological realization* of P, *i.e.* the quotient space of the union of the simplices by the equivalence relation generated by the identification maps. We say that P is connected if $|P|$ is. We denote by $\partial|P|$ the image in $|P|$ of the boundary faces of P, and we say that P is *without boundary* if $\partial|P| = \emptyset$.

A *k-dimensional face* of $|P|$ is the image in $|P|$ of a k-dimensional face of a simplex of P. Usually, we refer to 1-dimensional, respectively 0-dimensional faces of P and $|P|$ as to *edges*, respectively *vertices* of P and $|P|$.

It is well-known that, if P is an n-dimensional pseudomanifold, $n \geq 3$, then $|P|$ does not need to be a manifold. However, in the 3-dimensional orientable case, singularities may occur only at vertices (and it is not difficult to construct examples where they indeed occur). Let us be more precise, and state the following well-known result (see *e.g.* [Hat02, pages 108-109]):

Lemma 2.1. *Let P be an orientable n-dimensional pseudomanifold, and let $V_k \subseteq |P|$ be the union of the k-dimensional faces of $|P|$. Then $|P| \setminus V_{n-3}$ is an orientable manifold. In particular, if P is an orientable 2-dimensional pseudomanifold without boundary, then $|P|$ is an orientable surface without boundary.*

Moreover, the boundary of the topological realization of an n-dimensional pseudomanifold P is naturally the topological realization of an (orientable) $(n-1)$-pseudomanifold ∂P without boundary, and an orientation of P canonically induces an orientation on ∂P. In particular, if P is orientable and 3-dimensional, then $\partial |P|$ is a finite union of orientable closed surfaces.

The pseudomanifold associated to an integral cycle. Let M be an oriented connected n-dimensional manifold with (possibly empty) boundary ∂M. It is well-known that every integral relative cycle on $(M, \partial M)$ can be represented by a map from a suitable pseudomanifold to M. Let us describe this procedure in detail in the case we are interested in, *i.e.* in the case of n-dimensional integral cycles (see also [Hat02, pages 108-109]).

Let $z = \sum_{i=1}^{k} \varepsilon_i \sigma_i$ be an n-dimensional relative cycle in $C_n(M, \partial M; \mathbb{Z})$, where σ_i is a singular n-simplex on M, and $\varepsilon_i = \pm 1$ for every i (note that here we do not assume that $\sigma_i \neq \sigma_j$ for $i \neq j$). We construct an n-pseudomanifold associated to z as follows. Let us consider k distinct copies $\Delta_1^n, \ldots, \Delta_k^n$ of the standard n-simplex Δ^n. For every i we fix an identification between Δ_i^n and Δ^n, so that we may consider σ_i as defined on Δ_i^n. For every $i = 1, \ldots, k$, $j = 0, \ldots, n$, we denote by F_j^i the j-th face of Δ_i^n, and by $\partial_j^i \colon \Delta^{n-1} \to F_j^i \subseteq \Delta_i^n$ the usual face inclusion. We say that the distinct faces F_j^i and $F_{j'}^{i'}$ form a *canceling pair* if $\sigma_i|_{F_j^i} = \sigma_{i'}|_{F_{j'}^{i'}}$ and $(-1)^j \varepsilon_i + (-1)^{j'} \varepsilon_{i'} = 0$. This is equivalent to say that, when computing the boundary ∂z of z, the pair of $(n-1)$-simplices arising from the restrictions of σ_i and $\sigma_{i'}$ to F_j^i and $F_{j'}^{i'}$ cancel each other.

Let us define a pseudomanifold P as follows. The simplices of P are $\Delta_1^n, \ldots, \Delta_k^n$, and we identify the faces belonging to a maximal collection of canceling pairs (note that such a family is not uniquely determined). If F_i^j, $F_{i'}^{j'}$ are paired faces, we identify them via the affine diffeomorphism $\partial_{i'}^{j'} \circ (\partial_i^j)^{-1} \colon F_i^j \to F_{i'}^{j'}$. We observe that P is orientable: in fact, we can define an orientation on P by endowing Δ_i^n with the standard orientation of Δ^n if $\varepsilon_i = 1$, and with the reverse orientation if $\varepsilon_i = -1$.

By construction, the maps $\sigma_1, \ldots, \sigma_k$ glue up to a well-defined continuous map $f \colon |P| \to M$. For every $i = 1, \ldots, k$, let $\hat{\sigma}_i \colon \Delta^n \to |P|$ be the singular simplex obtained by composing the identification $\Delta^n \cong \Delta_i^n$ with the quotient map with values in $|P|$, and let us set $z_P = \sum_{i=1}^{k} \varepsilon_i \hat{\sigma}_i$. Then the chain z_P is a relative cycle in $C_n(|P|, \partial |P|; \mathbb{Z})$ and the map f_* induced by $f \colon (|P|, \partial |P|) \to (M, \partial M)$ on integral singular chains sends z_P to $f_*(z_P) = z$.

By Lemma 1.3, the simplicial volume of a connected oriented n-manifold can be computed from *integral* cycles. By exploiting Thurston's straightening procedure for simplices, in Proposition 3.3 we will show that such efficient cycles may be represented by n-pseudomanifolds with additional useful properties.

3. Geometric properties of straight cycles

The *straightening procedure* for simplices was introduced by Thurston in [Thu79], in order to bound from below the simplicial volume of hyperbolic manifolds. The universal covering of a hyperbolic n-manifold with geodesic boundary is a convex subset of the hyperbolic space $\mathbb{H}^n$, and the support of any straight simplex is just the image of a geodesic simplex of $\mathbb{H}^n$ via the universal covering projection. As a consequence, to compute the simplicial volume of a hyperbolic manifold with geodesic boundary we may restrict to considering only cycles supported by (projections of) geodesic simplices.

Geodesic simplices. Let $\overline{\mathbb{H}^n} = \mathbb{H}^n \cup \partial\mathbb{H}^n$ be the usual compactification of the hyperbolic space $\mathbb{H}^n$. We recall that every pair of points of $\overline{\mathbb{H}^n}$ is connected by a unique geodesic segment (which has infinite length if any of its endpoints lies in $\partial\mathbb{H}^n$). A subset in $\overline{\mathbb{H}^n}$ is *convex* if whenever it contains a pair of points it also contains the geodesic segment connecting them. The *convex hull* of a set A is defined as usual as the intersection of all convex sets containing A. A *(geodesic) k-simplex* Δ in $\overline{\mathbb{H}^n}$ is the convex hull of $k+1$ points in $\overline{\mathbb{H}^n}$, called *vertices*. We say that a k-simplex is:

- *ideal* if all its vertices lie in $\partial\mathbb{H}^n$,
- *regular* if every permutation of its vertices is induced by an isometry of $\mathbb{H}^n$,
- *degenerate* if it is contained in a $(k-1)$-dimensional subspace of $\overline{\mathbb{H}^n}$.

As above, we denote by v_n the volume of the regular ideal simplex in $\overline{\mathbb{H}^n}$. The following result characterizes hyperbolic geodesic simplices of maximal volume, and plays a fundamental role in the study of the simplicial volume of hyperbolic manifolds:

Theorem 3.1 ([HM81, Pey02]). *Let Δ be an n-simplex in $\overline{\mathbb{H}^n}$. Then $\mathrm{Vol}(\Delta) \leqslant v_n$, with equality if and only if Δ is ideal and regular.*

Let Δ be a nondegenerate geodesic n-simplex, and let E be an $(n-2)$-dimensional face of Δ. The *dihedral angle* $\alpha(\Delta, E)$ of Δ at E is defined as follows: let p be a point in $E \cap \mathbb{H}^n$, and let $H \subseteq \mathbb{H}^n$ be the unique 2-dimensional geodesic plane which intersects E orthogonally in p. We set $\alpha(\Delta, E)$ to be equal to the angle in p of the polygon $\Delta \cap H$ of $H \cong \mathbb{H}^2$. Observe that this definition is independent of p.

From the computation of the dihedral angle of the regular ideal geodesic n-simplex, together with the fact that geodesic simplices of almost maximal volume are close in shape to regular ideal simplices, one deduces:

Lemma 3.2. *Let $n \geqslant 4$. Then, there exists $\varepsilon_n > 0$, depending only on n, such that the following condition holds: if $\Delta \subseteq \overline{\mathbb{H}^n}$ is a geodesic n-simplex such that $\mathrm{Vol}(\Delta) \geqslant (1 - \varepsilon_n)v_n$ and α is the dihedral angle of Δ at any of its $(n-2)$-faces, then*

$$2 < \frac{\pi}{\alpha} < 3 \ .$$

We refer the reader to [FFM12, Lemma 2.16] for a proof.

Geometric straightening. Let us come back to the definition of straightening for simplices in hyperbolic manifolds. Henceforth we denote by M a hyperbolic manifold with geodesic boundary. As usual, we also assume that M is oriented.

The universal covering $\widetilde{M}$ of M is a convex subset of $\mathbb{H}^n$ bounded by a countable family of disjoint geodesic hyperplanes (see *e.g.* [Koj90]). If $\sigma \colon \Delta^k \to \widetilde{M}$ is a singular k-simplex, then we may define the simplex $\widetilde{\mathrm{str}}_k(\sigma)$ as follows: set $\widetilde{\mathrm{str}}_k(\sigma)(v) = \sigma(v)$ on every vertex v of Δ^k, and extend using barycentric coordinates (see [Rat94, Chapter 11]) or by an inductive cone construction (which exploits the fact that any pair of points in $\widetilde{M}$ is joined by a unique geodesic, that continuously depends on its endpoints – see *e.g.* [FP10, Section 3.1] for full details). The image of $\widetilde{\mathrm{str}}_k(\sigma)$ is the geodesic simplex spanned by the vertices of σ. Hence, we define a map $\widetilde{\mathrm{str}}_* \colon C_*(\widetilde{M}, \partial\widetilde{M}; R) \to C_*(\widetilde{M}, \partial\widetilde{M}; R)$. Being $\pi_1(M)$-invariant, the map $\widetilde{\mathrm{str}}_*$ induces a map

$$\mathrm{str}_* \colon C_*(M, \partial M; R) \to C_*(M, \partial M; R)$$

which is homotopic to the identity. Simplices that lie in the image of str_* are called *straight*.

Recall that, by Lemma 1.3, the simplicial volume of a connected oriented n-manifold can be computed from *integral* cycles. Using the straightening procedure we show that such cycles may be represented by n-pseudomanifolds with additional properties.

Proposition 3.3. *Suppose that M is a hyperbolic n-manifold with geodesic boundary ∂M. Let $\varepsilon > 0$ be fixed. Then, there exists a relative integral cycle $z \in C_n(M, \partial M; \mathbb{Z})$ with associated pseudomanifold P such that the following conditions hold:*

(1) $[z] = d \cdot [M, \partial M]_{\mathbb{Z}}$ in $H_n(M, \partial M; \mathbb{Z})$ for some integer $d > 0$, and

$$\frac{\|z\|_{\mathbb{Z}}}{d} \leq \|M, \partial M\| + \varepsilon \ ;$$

(2) every singular simplex appearing in z is straight;
(3) every simplex of P has at most one $(n-1)$-dimensional boundary face;
(4) if $n = 3$, then every simplex of P without 2-dimensional boundary faces has at most two edges contained in $\partial|P|$ and every simplex of P has at most three edges in $\partial|P|$.

Proof. By Lemma 1.3 we may choose an integral cycle z' satisfying condition (1), and set $z = \mathrm{str}_n(z') \in C_n(M, \partial M; \mathbb{Z})$. As usual, we understand that no simplex appearing in z is supported in ∂M (otherwise, we may just remove it from z without modifying the class of z in $C_n(M, \partial M; \mathbb{Z})$ and decreasing $\|z\|_{\mathbb{Z}}$). Point (1) descends from the fact that the straightening operator is norm nonincreasing and homotopic to the identity, while point (2) is obvious.

Let σ be a straight n-simplex of z. Let $\widetilde{\sigma}$ be a fixed lift of σ to $\widetilde{M}$. If there exists a component of $\partial\widetilde{M}$ containing $m + 1$ vertices of $\widetilde{\sigma}$, then $\widetilde{\sigma}$ has an m-dimensional face supported on $\partial\widetilde{M}$. In particular, the assumption that σ is not supported on ∂M implies that no component of $\partial\widetilde{M}$ contains all the vertices of $\widetilde{\sigma}$. Hence,

(i) If σ has two faces on ∂M, then the vertices of $\widetilde{\sigma}$ are contained in the same connected component of $\partial\widetilde{M}$, a contradiction.

(ii) Suppose that σ has at least three edges on ∂M. If $n = 3$, the union of the corresponding edges of $\widetilde{\sigma}$ is connected, so at least three vertices of $\widetilde{\sigma}$ lie on the same connected component of $\partial\widetilde{M}$, and at least one 2-face of σ is supported on ∂M.

If four edges of σ lie on ∂M and $n = 3$, then as before, the union of the corresponding edges of $\widetilde{\sigma}$ is connected. But the vertices of these four edges of the 3-simplex are all the vertices of the 3-simplex, which all lie on the same connected component of $\partial\widetilde{M}$, a contradiction.

Now points (3) and (4) immediately descend from (i) and (ii). □

Volume form. Let $\sigma\colon \Delta^n \to M$ be a smooth n-simplex, and let ω be the volume form of M. We set

$$\mathrm{Vol}_{\mathrm{alg}}(\sigma) = \int_{\Delta^n} \sigma^*(\omega) \ .$$

Since straight simplices are smooth, the map

$$C_n(M, \partial M; \mathbb{R}) \to \mathbb{R}, \qquad \sum_{i=1}^n a_i\sigma_i \mapsto \sum_{i=1}^n a_i \mathrm{Vol}_{\mathrm{alg}}(\mathrm{str}_n(\sigma_i))$$

is well-defined. This map is a relative cocycle that represents the volume coclass on M (see *e.g.* [FP10, Section 4] for the details). Therefore, if $z = \sum_{i=1}^h a_i\sigma_i \in C_n(M, \partial M; \mathbb{Z})$ is an integral cycle supported by straight simplices such that $[z] = d \cdot [M, \partial M]_{\mathbb{Z}}$, then

$$(5) \qquad \sum_{i=1}^h a_i \mathrm{Vol}_{\mathrm{alg}}(\sigma_i) = d \cdot \mathrm{Vol}(M) \ .$$

Let us rewrite z as follows:

$$z = \sum_{i=1}^N \varepsilon_i\sigma_i \ ,$$

where $\varepsilon_i = \pm 1$ for every $i = 1, \ldots, N$. Note that we do not assume that $\sigma_i \neq \sigma_j$ for $i \neq j$. Let P be the pseudomanifold associated to z, and recall that the simplices $\Delta_1^n, \ldots, \Delta_N^n$ of P are in bijection with the σ_i's. An identification of Δ_i^n with the standard n-simplex is fixed for every $i = 1, \ldots, N$, so that we may consider σ_i as a map defined on Δ_i^n. We set

$$(6) \qquad \mathrm{Vol}_{\mathrm{alg}}(\Delta_i^n) = \varepsilon_i \mathrm{Vol}_{\mathrm{alg}}(\sigma_i) \ ,$$

and we say that Δ_i^n is *positive* (resp. *degenerate, negative*) if $\mathrm{Vol}_{\mathrm{alg}}(\Delta_i^n) > 0$ (resp. $\mathrm{Vol}_{\mathrm{alg}}(\Delta_i^n) = 0$, $\mathrm{Vol}_{\mathrm{alg}}(\Delta_i^n) < 0$). Equation (5) may now be rewritten as follows:

$$(7) \qquad \sum_{i=1}^{N} \mathrm{Vol}_{\mathrm{alg}}(\Delta_i^n) = d \cdot \mathrm{Vol}(M) \ .$$

If $\widetilde{\sigma}_i$ is any lift of σ_i to $\widetilde{M} \subseteq \mathbb{H}^n$, then Δ_i^n is degenerate if and only if the image of $\widetilde{\sigma}_i$ is. Since $|\mathrm{Vol}_{\mathrm{alg}}(\Delta_i^n)|$ is just the volume of the image of $\widetilde{\sigma}_i$, by Theorem 3.1 we have

$$|\mathrm{Vol}_{\mathrm{alg}}(\Delta_i^n)| \leq v_n \ .$$

If Δ_i^n is nondegenerate and F is an $(n-2)$-face of Δ_i^n, then we define the angle of Δ_i^n at F as the angle of the image of $\widetilde{\sigma}_i$ at $\widetilde{\sigma}_i(F)$.

Lemma 3.4. *Let F be an $(n-2)$-face of ∂P, and let $\Delta_{i_1}^n, \ldots, \Delta_{i_k}^n$ be the simplices of P that contain F (taken with multiplicities). For every $j = 1, \ldots, k$ we also suppose that $\mathrm{Vol}_{\mathrm{alg}}(\Delta_{i_j}^n) > 0$, so in particular $\Delta_{i_j}^n$ is nondegenerate, and has a well-defined angle α_{i_j} at F. Then*

$$\sum_{j=1}^{k} \alpha_{i_j} = \pi \ .$$

Proof. Up to choosing suitable lifts $\widetilde{\sigma}_{i_j}$ of the σ_{i_j}'s, we may glue the $\widetilde{\sigma}_{i_j}$'s in order to develop the union of the $\Delta_{i_j}^n$'s into $\widetilde{M} \subseteq \mathbb{H}^n$. Since the $(n-1)$-faces of ∂P sharing F are developed into two adjacent $(n-1)$-geodesic simplices in $\partial \widetilde{M}$, this implies at once that a suitable algebraic sum of the α_{i_j}'s is equal either to 0 or to π. In order to conclude it is sufficient to show that the condition $\mathrm{Vol}_{\mathrm{alg}}(\Delta_{i_j}^n) > 0$ implies that all the signs in this algebraic sum are positive (this implies in particular that the sum is itself positive, whence equal to π).

To prove the last statement, it is sufficient to check that, if $\Delta_{i_{j_1}}^n$, $\Delta_{i_{j_2}}^n$ are adjacent in P along their common $(n-1)$-face V and the lifts $\widetilde{\sigma}_{i_{j_1}}$, $\widetilde{\sigma}_{i_{j_2}}$ coincide on V, then the images of $\widetilde{\sigma}_{i_{j_1}}$ and $\widetilde{\sigma}_{i_{j_2}}$ lie on different sides of $\widetilde{\sigma}_{i_{j_1}}(V) = \widetilde{\sigma}_{i_{j_2}}(V)$. Let us set for simplicity $j_1 = 1$ and $j_2 = 2$, and for $j = 1, 2$ let $\varepsilon'_{i_j} = 1$ if V is the k-th face of $\Delta_{i_j}^n$ and k is even, and $\varepsilon'_{i_j} = -1$ otherwise. It is easily checked that the images of $\widetilde{\sigma}_{i_1}$ and $\widetilde{\sigma}_{i_2}$ lie on different sides of $\widetilde{\sigma}_{i_1}(V) = \widetilde{\sigma}_{i_1}(V)$ if and only if the quantities

$$\varepsilon'_{i_1} \mathrm{Vol}_{\mathrm{alg}}(\sigma_{i_1}) \,, \qquad \varepsilon'_{i_2} \mathrm{Vol}_{\mathrm{alg}}(\sigma_{i_2})$$

have opposite sign. However, since V corresponds to a canceling pair, we have $\varepsilon_{i_1} \varepsilon'_{i_1} + \varepsilon_{i_2} \varepsilon'_{i_2} = 0$, so the conclusion follows from the positivity of $\mathrm{Vol}_{\mathrm{alg}}(\Delta_{i_1}^n)$ and $\mathrm{Vol}_{\mathrm{alg}}(\Delta_{i_2}^n)$. $\qquad \square$

4. Proof of Theorem 1

Throughout this section we suppose that $\dim M = n \geq 4$. The idea of the proof is as follows: Lemma 3.2 implies that no dihedral angle of a geodesic n-simplex of almost maximal volume can be a submultiple of π. Together with Lemma 3.4, this implies that any fundamental cycle M must contain simplices whose support has small volume (that is, smaller than $(1 - \varepsilon_n)v_n$). In fact, the weights of these simplices in any fundamental cycle may be bounded from below by the simplicial volume of the boundary of M, and this will finally yield the estimate needed in Theorem 1. Let us now provide the detailed computations.

Let $I = \{1, \ldots, N\}$ and let

$$z = \sum_{i \in I} \varepsilon_i \sigma_i$$

be an integral n-cycle satisfying the conditions of Proposition 3.3, where $\varepsilon_i = \pm 1$ for every $i \in I$. Let P be a pseudomanifold associated to z, and let Δ_i^n, $\mathrm{Vol}_{\mathrm{alg}}(\Delta_i^n)$ be defined as in the previous section.

We choose ε_n as in Lemma 3.2, we say that the simplex Δ_i^n is *small* if and only if $\mathrm{Vol}_{\mathrm{alg}}(\Delta_i^n) \leq (1 - \varepsilon_n)v_n$, and we set

$$I_{\mathrm{small}} = \{i \in I \mid \Delta_i^n \text{ is small}\}, \qquad N_{\mathrm{small}} = \#I_{\mathrm{small}} \ .$$

Lemma 4.1. *We have*

$$N_{\mathrm{small}} \geq \frac{d}{n+1}\|\partial M\| \ .$$

Proof. We start by showing that every $(n-2)$-face of $\partial|P|$ is contained in at least one small n-simplex Δ_i^n of P, with $i \in I_{\mathrm{small}}$, corresponding to some σ_i. Indeed, let F be an $(n-2)$-face of $\partial|P|$ and let $\Delta_{i_1}^n, \ldots, \Delta_{i_k}^n$ be the n-simplices of P containing F. Suppose by contradiction that $\mathrm{Vol}_{\mathrm{alg}}(\Delta_{i_j}^n) \geq (1 - \varepsilon_n)v_n$ for every $j = 1, \ldots, k$. Let σ_{i_j} be the straight simplex corresponding to $\Delta_{i_j}^n$. Our assumptions imply that the dihedral angle α_{i_j} of $\sigma_{i_j}(\Delta_{i_j}^n)$ at F is well-defined. Moreover, Lemma 3.4 gives

$$\sum_{j=1}^k \alpha_{i_j} = \pi \ ,$$

which contradicts Lemma 3.2.

Of course, a small simplex could have several $(n-2)$-faces in the boundary, but since an n-simplex has exactly $(n+1)n/2$ faces of codimension two, we can bound the number of small simplices by the number of $(n-2)$-dimensional faces in $\partial|P|$,

$$N_{\mathrm{small}} \geq \frac{2}{(n+1)n}\sharp\{(n-2)-\text{faces in } \partial|P|\}.$$

An $(n-1)$-simplex has exactly n faces of codimension one. Moreover, since ∂P is an $(n-1)$-dimensional pseudomanifold without boundary, every $(n-$

2)-face of $\partial|P|$ is shared by exactly two $(n-1)$-simplices, so the number of $(n-2)$-faces of $\partial|P|$ is equal to $(n/2)c(\partial P)$, where $c(\partial P)$ is the number of $(n-1)$-simplices of ∂P. So inequality (4), i.e.

$$c(\partial P) = \|\partial z\|_{\mathbb{Z}} \geq d \cdot \|\partial M\| \ ,$$

concludes the proof of the lemma. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

To conclude the proof of Theorem 1, note that by equation (7) we have

$$d \cdot \mathrm{Vol}(M) = \sum_{i \in I} \mathrm{Vol}_{\mathrm{alg}}(\Delta_i^n) = \sum_{i \in I_{\mathrm{small}}} \mathrm{Vol}_{\mathrm{alg}}(\Delta_i^n) + \sum_{i \in I \setminus I_{\mathrm{small}}} \mathrm{Vol}_{\mathrm{alg}}(\Delta_i^n)$$

$$\leq N_{\mathrm{small}}(1 - \varepsilon_n)v_n + (N - N_{\mathrm{small}})v_n = (N - N_{\mathrm{small}} \cdot \varepsilon_n)v_n \ .$$

Putting together this inequality with Lemma 4.1 and the inequality $N = \|z\|_{\mathbb{Z}} \leq d(\|M, \partial M\| + \varepsilon)$ we get

$$d \cdot \mathrm{Vol}(M) \leq \left(d(\|M, \partial M\| + \varepsilon) - \frac{d \cdot \varepsilon_n}{n+1}\|\partial M\| \right) v_n \ .$$

As ε is arbitrary, after dividing each side of this inequality by $d \cdot \mathrm{Vol}(M)$ and reordering, we get

$$\frac{\|M, \partial M\|}{\mathrm{Vol}(M)} \geq \frac{1}{v_n} + \frac{\varepsilon_n \cdot \|\partial M\|}{(n+1)\mathrm{Vol}(M)} = \frac{1}{v_n} + \frac{\varepsilon_n \mathrm{Vol}(\partial M)}{(n+1)v_{n-1}\mathrm{Vol}(M)} \ ,$$

which finishes the proof of Theorem 1.

5. Proof of Theorem 3

In order to provide lower bounds on the simplicial volume of hyperbolic 3-manifolds with geodesic boundary we first need to analyze some properties of volumes of hyperbolic 3-simplices. An essential tool for computing such volumes is the *Lobachevsky function* $L \colon \mathbb{R} \to \mathbb{R}$ defined by the formula

$$L(\theta) = -\int_0^\theta \log|2\sin u|\, du \ .$$

In a nondegenerate ideal 3-simplex, opposite sides subtend isometric angles, the sum of the angles of any triple of edges sharing a vertex is equal to π and the simplex is determined up to isometry by its dihedral angles. The following result is proved by Milnor in [Thu79, Chapter 7], and plays a fundamental role in the computation of volumes of hyperbolic 3-simplices.

Proposition 5.1. *Let Δ be a nondegenerate ideal simplex with angles α, β, γ. Then*

$$\mathrm{Vol}(\Delta) = L(\alpha) + L(\beta) + L(\gamma) \ .$$

Moreover,

$$\mathrm{Vol}(\Delta) \leq 3L(\pi/3) = v_3 \approx 1.014942 \ ,$$

where the equality holds if and only if $\alpha = \beta = \gamma = \pi/3$ (i.e. Δ is regular).

We say that a nondegenerate geodesic simplex with nonideal vertices Δ is:

- *1-obtuse* if it has at least one nonacute dihedral angle,
- *2-obtuse* if there exist two edges of Δ which share a vertex and subtend nonacute dihedral angles,
- *3-obtuse* if there exists a face F of Δ such that each edge of F subtends a nonacute dihedral angle.

Lemma 5.2. *There do not exist* 3*-obtuse geodesic simplices. Moreover, if* Δ *is a* 2*-obtuse geodesic simplex, then*

$$\mathrm{Vol}(\Delta) \leq \frac{v_3}{2} \ .$$

Proof. Let F be a face of a nondegenerate geodesic simplex Δ. Let H be the geodesic plane containing F and let $\pi \colon \mathbb{H}^3 \to H$ denote the nearest point projection. Let $v \in \overline{\mathbb{H}^n}$ be the vertex of Δ not contained in H.

For every edge e of F, the geodesic line containing e divides H into two regions. Note that the angle at e is acute if and only if the projection $\pi(v)$ of the last vertex point belongs to the region containing F. Consider the three geodesic lines containing the three edges of F. Since no point in H can simultaneously be contained in the region of H bounded by each of these geodesics and not containing F, it follows that Δ cannot be 3-obtuse.

Suppose now that two of the edges of F subtend nonacute dihedral angles and consider the four regions of H delimited by the two corresponding geodesics. Denote by v_0 the vertex of F given as the intersection of these two geodesics. Note that $\pi(v)$ belongs to the region opposite to the region containing F. Denote by r the reflection along H. Set $v' = r(v)$ and $\Delta' = r(\Delta)$. The convex hull of Δ and Δ' is equal to the convex hull of F, v and v'. Let $\widehat{\Delta}$ be the geodesic simplex with vertices v, v' and the two vertices of F opposite to v_0. Since v_0 belongs to $\widehat{\Delta}$ (see Figure 1) it follows that $\Delta \cup \Delta' \subset \widehat{\Delta}$ and hence

$$v_3 \geq \mathrm{Vol}(\widehat{\Delta}) \geq \mathrm{Vol}(\Delta \cup \Delta') = 2\mathrm{Vol}(\Delta) \ ,$$

which finishes the proof of the lemma. $\qquad\qquad\qquad\qquad\qquad\square$

Recall that

$$G = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \ldots \approx 0.915965$$

is Catalan's constant.

Lemma 5.3. *If* Δ *is a* 1*-obtuse geodesic simplex, then*

$$\mathrm{Vol}(\Delta) \leq G \ .$$

Proof. Suppose first that Δ is a 1-obtuse ideal geodesic simplex. Let α, β, γ be its three dihedral angles with $\alpha \geq \pi/2$ and $\beta + \gamma = \pi - \alpha$. Using Proposition 5.1, one can readily show that when $\alpha \geq \pi/2$ is fixed, the maximum volume $\mathrm{Vol}(\Delta) = L(\alpha) + L(\beta) + L(\gamma)$ is attained at $\beta = \gamma = (\pi - \alpha)/2$. Another easy computation based on Proposition 5.1 implies
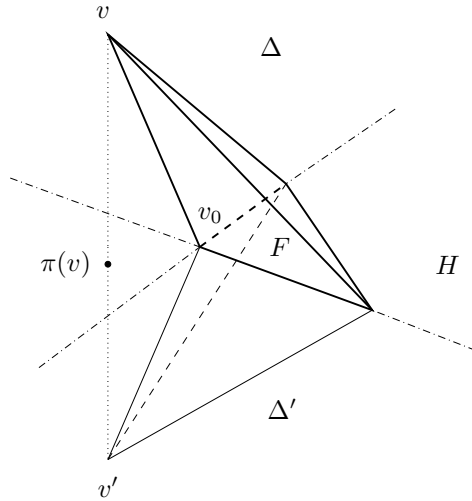
Figure 1. A 2-obtuse simplex Δ and its mirror copy Δ'.

that, under the assumption that $\alpha \geq \pi/2$, the quantity $L(\alpha) + 2L((\pi - \alpha)/2)$ attains its maximum at $\alpha = \pi/2$. Therefore, we may conclude that

$$\mathrm{Vol}(\Delta) \leq L(\pi/2) + 2L(\pi/4) = G \ ,$$

where the last equality is proved in [Thu79, Chapter 7].

Let now Δ be a 1-obtuse nonideal geodesic simplex. The lemma will follow once we exhibit a 1-obtuse ideal geodesic simplex $\overline{\Delta}$ with $\Delta \subset \overline{\Delta}$. Let v_1, v_2 be the vertices on the edge e subtending the nonacute angle. Two of the vertices of $\overline{\Delta}$ will be the two endpoints w_1, w_2 of the geodesic through v_1, v_2. Let v, v' be the two remaining vertices of Δ and denote by F, F' the two faces of Δ opposite to v' and v respectively. Let w, respectively w', be vertices on the boundary of the hyperplane containing F, resp. F', and such that the convex hull of v_1, v_2, w, resp. v_1, v_2, w', contains v, resp. v'. (For example, pick w, resp. w', as the endpoint of the geodesic through v_1 and v, resp. v'.) Let $\overline{\Delta}$ be the ideal geodesic simplex with vertices w_1, w_2, w, w'. As it contains all the vertices of Δ, the simplex Δ is indeed contained in $\overline{\Delta}$. Furthermore, it is still 1-obtuse as its dihedral angle on the edge with endpoints w_1, w_2 is equal to the dihedral angle of Δ at the edge with endpoints v_1, v_2. $\qquad\square$

Proof of Theorem 3. Let z be the integral cycle provided by Proposition 3.3, let P be the associated pseudomanifold, and let $\Delta_1^3, \ldots, \Delta_N^3$ be the simplices of P. In equation (6) a well-defined algebraic volume $\mathrm{Vol}_{\mathrm{alg}}(\Delta_i^3)$ is associated to every Δ_i^3, in such a way that the equality

$$(8) \qquad\qquad d \cdot \mathrm{Vol}(M) = \sum_{i=1}^{N} \mathrm{Vol}_{\mathrm{alg}}(\Delta_i^3)$$

holds. We also say that Δ_i^n is 1-, 2- or 3-obtuse if the corresponding geodesic simplex in $\mathbb{H}^n$ is (by Lemma 5.2, the last possibility cannot hold in fact).

Let Ω_i, $i = 0, \ldots, 4$, be the set of simplices of P having exactly i boundary 2-faces. We denote by t_i the number of elements of Ω_i. By Proposition 3.3 we have $\Omega_2 = \Omega_3 = \Omega_4 = \emptyset$, so that

$$ t_2 = t_3 = t_4 = 0, \qquad \|z\|_{\mathbb{Z}} = t_0 + t_1 = N \ . $$

We denote by $t_{1,n}$ the number of nonpositive simplices in Ω_1 (*i.e.* simplices with nonpositive volume), and by $t_{1,1}$ (resp. $t_{1,2}$) the number of 1-obtuse (resp. 2-obtuse) positive simplices in Ω_1. Moreover, we say that an edge e of the 2-dimensional pseudomanifold ∂P is *nice* if e is the edge of a simplex in Ω_0. We also say that an edge of ∂P is *bad* if it is not nice. An edge of the topological realization $\partial |P|$ is *nice* if it is the image of at least one nice edge in ∂P. An edge in $\partial |P|$ is *bad* if it is not nice. Notice that bad edges in $\partial |P|$ are *not* the image of bad edges in ∂P, since a nice edge in $\partial |P|$ will be the image of a certain number of nice edges in ∂P and necessarily two bad edges of ∂P corresponding to the two (possibly nondistinct) tetrahedra having as 2-faces the 2-faces in $\partial |P|$ containing the original nice edge. We denote by E_{bad} (resp. E_{nice}) the number of bad (resp. nice) edges of $\partial |P|$.

Lemma 5.4. *We have*

$$ 3t_{1,n} + 2t_{1,2} + t_{1,1} \geq E_{\text{bad}} \ . $$

Proof. Let e be a bad edge of $\partial |P|$, let T_{i_1}, T_{i_2} be the triangles of ∂P adjacent to e, and let $\Delta_{i_1}^3$ (resp. $\Delta_{i_2}^3$) be the simplex of P containing T_{i_1} (resp. T_{i_2}). It is easy to show that if F_{i_1} (resp. F_{i_2}) is the 2-face of $\Delta_{i_1}^3$ (resp. $\Delta_{i_2}^3$) such that $e = F_{i_1} \cap T_{i_1} = F_{i_2} \cap T_{i_2}$, then F_{i_1}, F_{i_2} are glued to each other in $|P|$ (see [BFP15, Lemma 4.4]). Also suppose that $\Delta_{i_1}^3$ and $\Delta_{i_2}^3$ are both positive and let α_{i_j} be the dihedral angle of $\Delta_{i_j}^3$ at e. By Lemma 3.4 we have $\alpha_{i_1} + \alpha_{i_2} = \pi$. As a consequence, either $\Delta_{i_1}^3$ or $\Delta_{i_2}^3$ (or both in case $\alpha_{i_1} = \alpha_{i_2} = \pi/2$) has a nonacute angle along e. We have thus shown that at every bad edge of $\partial |P|$ there is (at least) one incident simplex that either is nonpositive or has a nonacute angle at an edge of its boundary face. Since we know that no simplex of P can be 3-obtuse, the conclusion follows from an obvious double counting argument. $\qquad \square$

Proposition 5.5. *We have*

$$ \|M, \partial M\| + \varepsilon \geq \frac{\text{Vol}(M)}{v_3} + \left(1 - \frac{G}{v_3} \right) \frac{E_{\text{bad}}}{d} \ . $$

Proof. Since $v_3 \geq 3(v_3 - G)$ and $v_3/2 \geq 2(v_3 - G)$, by equation (8) and Lemmas 5.2, 5.3 and 5.4 we have

$$d \cdot \mathrm{Vol}(M) \leq (t_0 + t_1 - t_{1,1} - t_{1,2} - t_{1,n})v_3 + Gt_{1,1} + t_{1,2}\frac{v_3}{2}$$

$$= (t_0 + t_1)v_3 - (v_3 - G)t_{1,1} - t_{1,2}\frac{v_3}{2} - t_{1,n}v_3$$

$$\leq (t_0 + t_1)v_3 - (v_3 - G)(t_{1,1} + 2t_{1,2} + 3t_{1,n})$$

$$\leq (t_0 + t_1)v_3 - (v_3 - G)E_{\mathrm{bad}} .$$

Now the conclusion follows from the inequality $t_0 + t_1 = \|z\|_{\mathbb{Z}} \leq d(\|M, \partial M\| + \varepsilon)$ (see Proposition 3.3). $\square$

Proposition 5.6. *We have*

$$\|M, \partial M\| + \varepsilon \geq \frac{7}{4}\|\partial M\| - \frac{E_{\mathrm{bad}}}{2d} .$$

Proof. Recall that $\partial|P|$ is the union of a finite number of closed orientable surfaces, and that the pseudomanifold P comes equipped with a degree d map $f \colon (|P|, \partial|P|) \to (M, \partial M)$. Therefore, we have that $\|\partial|P|\| \geq d \cdot \|\partial M\|$. Since $\partial|P|$ is decomposed into t_1 triangles, this implies that

$$(9) \qquad t_1 \geq d \cdot \|\partial M\| .$$

For the number E_{nice} of nice edges of $\partial|P|$ we have the obvious equality $E_{\mathrm{nice}} = (3/2)t_1 - E_{\mathrm{bad}}$. By definition, every nice edge is contained in a simplex in Ω_0, and by point (4) of Proposition 3.3 any such simplex has at most 2 edges on $\partial|P|$, so

$$(10) \qquad t_0 \geq \frac{E_{\mathrm{nice}}}{2} = \frac{3}{4}t_1 - \frac{E_{\mathrm{bad}}}{2} .$$

Together with Proposition 3.3, Inequalities (9) and (10) imply that

$$d(\|M, \partial M\| + \varepsilon) \geq t_0 + t_1 \geq \frac{7}{4}t_1 - \frac{E_{\mathrm{bad}}}{2} \geq \frac{7d}{4} \cdot \|\partial M\| - \frac{E_{\mathrm{bad}}}{2} ,$$

which finishes the proof of the proposition. $\square$

We are now ready to prove Theorem 3. In fact, if we set $k_0 = E_{\mathrm{bad}}/(2d)$, then putting together Propositions 5.5 and 5.6 we get

$$\|M, \partial M\| + \varepsilon \geq \max\left\{ \frac{\mathrm{Vol}(M)}{v_3} + 2\left(1 - \frac{G}{v_3}\right)k_0, \frac{7}{4} \cdot \|\partial M\| - k_0 \right\} ,$$

whence

$$\|M, \partial M\| + \varepsilon \geq \min_{k \geq 0} \max\left\{ \frac{\mathrm{Vol}(M)}{v_3} + 2\left(1 - \frac{G}{v_3}\right)k, \frac{7}{4} \cdot \|\partial M\| - k \right\} .$$

If $(7/4)\|\partial M\| \leq \mathrm{Vol}(M)/v_3$, then the statement of Theorem 3 is an obvious consequence of Jungreis' inequality (2). Otherwise, the right-hand side of the inequality above is equal to

$$\frac{\mathrm{Vol}(M)}{v_3} + \frac{v_3 - G}{2(3v_3 - 2G)}\left(7\|\partial M\| - 4\frac{\mathrm{Vol}(M)}{v_3} \right)$$
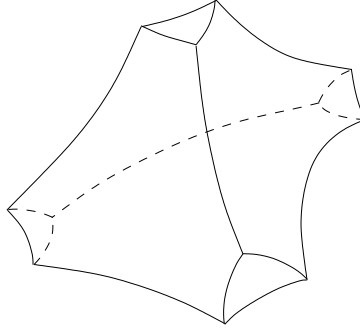
Figure 2. A truncated tetrahedron.

which finishes the proof of Theorem 3 since ε is arbitrary.

6. Small hyperbolic manifolds with geodesic boundary

We start by recalling some results from [FMP03] and [Miy94]. An ideal triangulation of a 3-manifold M is a homeomorphism between M and $|P| \setminus V(|P|)$, where P is a 3-pseudomanifold and $V(|P|)$ is a regular open neighbourhood of the vertices of $|P|$. In other words, it is a realization of M as the space obtained by gluing some topological *truncated tetrahedra*, *i.e.* tetrahedra with neighbourhoods of the vertices removed (see Figure 2).

As in the introduction, let $\mathcal{M}_g$, $g \geq 2$, be the class of 3-manifolds with boundary M that admit an ideal triangulation by g tetrahedra and have Euler characteristic $\chi(M) = 1 - g$ (so $\chi(\partial M) = 2 - 2g$). We also denote by $\overline{\mathcal{M}}_g$ the set of hyperbolic 3-manifolds M with connected geodesic boundary such that $\chi(\partial M) = 2 - 2g$. For $g \geq 2$, let $\Delta_g \subseteq \mathbb{H}^3$ be the regular truncated tetrahedron of dihedral angle $\pi/(3g)$ (see *e.g.* [Koj90, FP04] for the definition of hyperbolic truncated tetrahedron). It is proved in [KM91] that

$$\mathrm{Vol}(\Delta_g) = 8L\left(\frac{\pi}{4}\right) - 3\int_0^{\frac{\pi}{3g}} \mathrm{arccosh}\left(\frac{\cos t}{2\cos t - 1}\right)\, \mathrm{d}t$$

(11)

$$= 4G - 3\int_0^{\frac{\pi}{3g}} \mathrm{arccosh}\left(\frac{\cos t}{2\cos t - 1}\right)\, \mathrm{d}t \ .$$

The following result lists some known properties of manifolds belonging to $\mathcal{M}_g$. The last point implies that $\mathcal{M}_g$ coincides with the set of the elements of $\overline{\mathcal{M}}_g$ of smallest volume.

Proposition 6.1 ([FMP03, Miy94]). *Let $g \geq 2$. Then:*

- (1) *the set $\mathcal{M}_g$ is nonempty;*
- (2) *every element of $\mathcal{M}_g$ admits a hyperbolic structure with geodesic boundary (which is unique up to isometry by Mostow Rigidity Theorem);*

(3) *the boundary of every element of $\mathcal{M}_g$ is connected, so $\mathcal{M}_g \subseteq \overline{\mathcal{M}}_g$;*
(4) *if $M \in \mathcal{M}_g$, then M decomposes into the union of g copies of Δ_g, so in particular $\mathrm{Vol}(M) = g\mathrm{Vol}(\Delta_g)$;*
(5) *if $M \in \overline{\mathcal{M}}_g$, then $\mathrm{Vol}(M) \geq g\mathrm{Vol}(\Delta_g)$.*

Items (2) and (3) and (4) are proved in [FMP03], items (1) and (5) in [Miy94].

Proposition 6.2. *Fix $g \geq 2$. Then all the elements of $\mathcal{M}_g$ share the same simplicial volume.*

Proof. Take $M_1, M_2 \in \mathcal{M}_g$, and let us consider the universal coverings $\widetilde{M}_1$ and $\widetilde{M}_2$. Both $\widetilde{M}_1$ and $\widetilde{M}_2$ are obtained as the union in $\mathbb{H}^3$ of a countable family of copies of Δ_g, which are adjacent along their hexagonal faces, and this easily implies that $\widetilde{M}_1$ and $\widetilde{M}_2$ are isometric to each other. Since the isometry group of $\widetilde{M}_1$ is discrete, this fact can be used to show that M_1 and M_2 are commensurable, *i.e.* there exists a hyperbolic 3-manifold with geodesic boundary M' that is the total space of finite coverings $p_1 \colon M' \to M_1$ and $p_2 \colon M' \to M_2$ (see [Fri06, Lemma 2.4]). Since the Riemannian volume and the simplicial volume are multiplicative with respect to finite coverings, this implies in turn that $\|M_1, \partial M_1\|/\mathrm{Vol}(M_1) = \|M_2, \partial M_2\|/\mathrm{Vol}(M_2)$, which finishes the proof since $\mathrm{Vol}(M_1) = \mathrm{Vol}(M_2)$. $\square$

Let us prove Corollary 5 and see that for $M \in \mathcal{M}_2 \cup \mathcal{M}_3 \cup \mathcal{M}_4$, the bounds provided by the corollary are indeed sharper than Jungreis' inequality (2) and inequality (3).

- If $M \in \overline{\mathcal{M}}_2$, then $\|\partial M\| = 4$ and $\mathrm{Vol}(M) \geq 2\mathrm{Vol}(\Delta_2) \approx 6.452$. Applying Theorem 3 we get

$$\|M, \partial M\| \geq 6.461 \approx 1.615 \cdot \|\partial M\|.$$

  Also observe that, if $M \in \mathcal{M}_2$, then $\mathrm{Vol}(M)/v_3 \approx 6.357$.
- If $M \in \overline{\mathcal{M}}_3$, then $\|\partial M\| = 8$ and $\mathrm{Vol}(M) \geq 3\mathrm{Vol}(\Delta_3) \approx 10.429$. Applying Theorem 3 we get

$$\|M, \partial M\| \geq 10.882 \approx 1.360 \cdot \|\partial M\|.$$

  Also observe that, if $M \in \mathcal{M}_3$, then $\mathrm{Vol}(M)/v_3 \approx 10.274$.
- If $M \in \overline{\mathcal{M}}_4$, then $\|\partial M\| = 12$ and $\mathrm{Vol}(M) \geq 4\mathrm{Vol}(\Delta_4) \approx 14.238$. Applying Theorem 3 we get

$$\|M, \partial M\| \geq 15.165 \approx 1.264 \cdot \|\partial M\|.$$

If $M \in \mathcal{M}_4$, then $\mathrm{Vol}(M)/v_3 \approx 14.097$.

Finally, let us take $M \in \mathcal{M}_g$, $g \geq 5$ and show that the bound

$$\|M, \partial M\| \geq \frac{5}{4}\|\partial M\|$$

proved in [BFP15] is sharper than the ones given by inequality (2) and Theorem 3. Note that it is sufficient to show that

$$\frac{5}{4}\|\partial M\| > \frac{7\|\partial M\|(v_3 - G) + 2\mathrm{Vol}(M)}{2(3v_3 - 2G)} > \frac{\mathrm{Vol}(M)}{v_3} \ .$$

Using that $\|\partial M\| = 4(g-1)$ and $\mathrm{Vol}(M) = g\mathrm{Vol}(\Delta_g)$, after some straightforward algebraic manipulations the first inequality and the second inequality may be rewritten respectively as follows:

$$\left(1 - \frac{1}{g}\right)(v_3 + 4G) > \mathrm{Vol}(\Delta_g) \ , \qquad 7\left(1 - \frac{1}{g}\right)v_3 > \mathrm{Vol}(\Delta_g) \ .$$

We know from equation (11) that $\mathrm{Vol}(\Delta_g) < 4G$. Therefore, for every $g \geq 5$ we have

$$\left(1 - \frac{1}{g}\right)(v_3 + 4G) \geq \frac{4}{5}(v_3 + 4G) > 4G > \mathrm{Vol}(\Delta_g)$$

and

$$7\left(1 - \frac{1}{g}\right)v_3 \geq \frac{28}{5}v_3 > 4G > \mathrm{Vol}(\Delta_g) \ ,$$

whence the conclusion.

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Section de Mathématiques, 2-4 rue du Lièvre, Case postale 64, 1211 Genève 4, Suisse
  E-mail address: michelle.bucher-karlsson@unige.ch

Dipartimento di Matematica, Università di Pisa, Largo B. Pontecorvo 5, 56127 Pisa, Italy
  E-mail address: frigerio@dm.unipi.it

Department Mathematik, ETH Zentrum, Rämistrasse 101, 8092 Zürich
  E-mail address: cristina.pagliantini@math.ethz.ch