A NOTE ON SEMI-CONJUGACY FOR CIRCLE ACTIONS

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Abstract. We define a notion of semi-conjugacy between orientation-preserving actions of a group Γ on the circle, which for fixed point free actions coincides with a classical definition of Ghys. We then show that two circle actions are semi-conjugate if and only if they have the same bounded Euler class. This settles some existing confusion present in the literature.

1. Introduction

A fundamental problem in one-dimensional dynamics is the classification of group actions on the circle. More precisely, denote by $\mathrm{Homeo}^+(S^1)$ the group of orientation-preserving homeomorphisms of the circle. Given a group Γ, we will refer to a homomorphism $\rho : \Gamma \to \mathrm{Homeo}^+(S^1)$ as a *circle action*. One would like to associate to every circle action of Γ a family of invariants which classify the action up to a suitable equivalence relation, ideally up to conjugacy. For the case of a single transformation acting minimally on the circle, this problem was solved by Poincaré around the end of the 19th century, using his theory of rotation number.

In [Ghy87, Ghy01] Étienne Ghys introduced and studied a far reaching generalization of the rotation number, the bounded Euler class of a circle action. For *minimal* actions, i.e. actions for which every orbit is dense, he thereby achieved a complete classification result:

Theorem 1.1 ([Ghy01, Theorem 6.5])**.** *Let $\rho_1, \rho_2 : \Gamma \to \mathrm{Homeo}^+(S^1)$ be minimal circle actions. Then ρ_1 and ρ_2 are conjugate if and only if they have the same bounded Euler class.*

The bounded Euler class is thus a complete conjugation-invariant for minimal actions. For non-minimal actions, this result is not true. Instead, non-minimal actions sharing the same bounded Euler class only satisfy a weaker equivalence relation. In [Ghy87] Ghys introduced the notion of *semi-conjugacy* between circle actions, which generalizes the notion of conjugacy. It was then claimed that the bounded Euler class of a circle action determines the action up to semi-conjugacy:

Theorem 1.2 ([Ghy87, Theorem A1])**.** *Two circle actions $\rho_1, \rho_2 : \Gamma \to \mathrm{Homeo}^+(S^1)$ are semi-conjugate if and only if they have the same bounded Euler class.*

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This implies in particular, that semi-conjugacy is an equivalence relation. However, the definition of semi-conjugacy *as written* is clearly not an equivalence relation (see Remark 2.7 below) and thus needs to be amended. Ghys noticed the problem and provided a fix for it in the subsequent paper [Ghy01], see in particular [Ghy01, Theorem 6.6]. While the latter result is more precise for the dynamical properties of group actions in term of their bounded Euler class, the equivalence of [Ghy87, Theorem A1] is missing from the statement of Theorem 6.6 in [Ghy01]. Moreover, these issues do not seem to have been widely noticed, since references to Theorem 1.2 with the misformulated definition of semi-conjugacy have appeared regularly in the literature. In view of this situation, we found it worthwhile to point out an amended definition of semi-conjugacy, for which Theorem 1.2 holds in full generality, and to provide a complete and self-contained proof of Theorem 1.2 for this definition. Let us emphasize that we do not claim any originality for the ideas entering into the proof.

While various corrected definitions of semi-conjugacy seem to be known among experts, to the best of our knowledge none of them has ever appeared in print. In this note we give three equivalent characterizations of semi-conjugacy. The first characterization as presented in Definition 2.5 below is due to the second author and has the major advantage that it is easily seen to be an equivalence relation. We thus take it as our definition. Two alternative characterizations due respectively to Maxime Wolff [Wol] and the first named author [Buc08] will be discussed in Subsection 5.B below.

We would like to emphasize that for fixed point free actions Ghys' original definition agrees with our definition (see Corollary 4.4). It follows *a posteriori* that Ghys' definition of semi-conjugacy is an equivalence relation for fixed point free circle actions, a highly non-obvious fact.

Our amended definition of semi-conjugacy will be stated in Definition 2.5 below. For this definition, Theorem 1.2 will be established in Section 4. We will then recall in Subsection 4.E the argument of Ghys which shows that Theorem 1.2 implies Theorem 1.1. Our proof of Theorem 1.2 is based on Ghys' original proof, but involves an additional argument to deal with potential fixed points of the circle actions under considerations. For a sketch of an alternative proof, due originally to Thurston [Thu], see [Cal07]. (The definition of semi-conjugacy should be modified accordingly.)

Besides the proofs of Theorem 1.2 and Theorem 1.1 we also discuss a number of related issues of independent interest. In Subsection 2.A we discuss characterisations of non-decreasing degree one maps. In Section 3 we discuss thoroughly three well-known characterisations of the bounded Euler class on $\mathrm{Homeo}^+(S^1)$ and establish carefully their mutual equivalence. A fourth characterization, which will not be needed in the proof of Ghys' theorem, but generalizes nicely to higher dimensions, is discussed in an appendix to this note. In Subsection 5.A we explain how Poincaré's classification of $\mathbb{Z}$-actions on the circle can be derived from Ghys' theorem. We also show that every action of an amenable group on the circle is semi-conjugate to an action by rotations, a result originally due to Hirsch-Thurston. The final two sections, Section 5.C and Section 5.D, discuss respectively a characterization of circle actions with vanishing real bounded Euler classes and regularity questions concerning non-decreasing degree one maps.

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2. On the definition of semi-conjugacy

2.A. **Non-decreasing degree one maps.** Throughout this article we consider the circle $S^1 = \mathbb{R}/\mathbb{Z}$ as a quotient of the real line. A pre-image $\widetilde{x}$ of a point $x \in S^1$ under the canonical projection $\mathbb{R} \to S^1$ will be called a *lift* of x and we write $[\widetilde{x}] := x$.

Definition 2.1. An ordered k-tuple $(x_1, \dots, x_k) \in (S^1)^k$, for $k \in \mathbb{N}$ is said to be

- *weakly positively oriented* if there exists $a \in \mathbb{R}$ and lifts $\widetilde{x_i} \in \mathbb{R}$ of the x_i's such that
$$a \leq \widetilde{x_1} \leq \cdots \leq \widetilde{x_q} \leq a + 1,$$
- *positively oriented* if furthermore
$$a \leq \widetilde{x_1} < \cdots < \widetilde{x_q} < a + 1.$$

Replacing $\leq, <$ and $a, a+1$ respectively by $\geq, >$ and $a+1, a$ we obtain the corresponding notion of (weakly) negatively oriented k-tuples.

Note that if $k \leq 2$ then a k-tuple is both weakly positively oriented and weakly negatively oriented. Furthermore, the property of being (weakly) positively oriented is obviously invariant under cyclic permutations.

Definition 2.2. A (not necessarily continuous) map $\varphi\colon S^1 \to S^1$ is a *non-decreasing degree one map* if the following condition holds for all $k \in \mathbb{N}$: If $(x_1, \dots, x_k) \in (S^1)^k$ is weakly positively oriented, then $(\varphi(x_1), \dots, \varphi(x_k))$ is weakly positively oriented.

As we will see in Lemma 2.4 below it is actually enough to check the condition for $k = 4$. Observe that non-decreasing degree one maps are closed under composition and that every constant map is a non-decreasing degree one map.

Definition 2.3. Let $\varphi\colon S^1 \to S^1$ be any map. A set-theoretical lift $\widetilde{\varphi}\colon \mathbb{R} \to \mathbb{R}$ of φ is called a *good lift* of φ if $\widetilde{\varphi}(x+1) = \widetilde{\varphi}(x) + 1$ for every $x \in \mathbb{R}$ and $\widetilde{\varphi}$ is non-decreasing, i.e. $\widetilde{\varphi}(x) \leq \widetilde{\varphi}(y)$ whenever $x \leq y$.

Being a non-decreasing degree one map is equivalent to admitting a good lift (see Lemma 2.4) and the latter property is often taken as the definition of a non-decreasing degree one map, since it is maybe easier to state. Of course, one may obtain infintely many good lifts of a non-decreasing degree one map just by taking one such lift and composing it with an integral translation. But a given non-decreasing degree one map may also have good lifts which are not obtained one from the other by the composing with an integral translation. For example, for every $\alpha \in \mathbb{R}$ the maps $x \mapsto \lfloor x + \alpha \rfloor$ and $x \mapsto \lceil x + \alpha \rceil$ are good lifts of the constant map $\varphi\colon S^1 \to S^1$ mapping every point to $[0]$.

Lemma 2.4. *Let $\varphi\colon S^1 \to S^1$ be any map. Then the following conditions are equivalent:*

(i) The map φ is a non-decreasing degree one map.
(ii) If $(x_1, \dots, x_4) \in (S^1)^4$ is weakly positively oriented, then $(\varphi(x_1), \dots, \varphi(x_4))$ is weakly positively oriented;
(iii) There exists a good lift of φ.

Proof. The implication (i) $\Rightarrow$ (ii) holds by definition.

(ii) $\Rightarrow$ (iii): Let $x_0 \in S^1$ be a base point and $y_0 = \varphi(x_0) \in S^1$ be its image. Choose lifts $\widetilde{x_0}, \widetilde{y_0} \in \mathbb{R}$ of x_0, y_0 respectively and define $\widetilde{\varphi}$ on $[\widetilde{x_0}, \widetilde{x_0}+1)$ as follows: for $\widetilde{x_0} \leq \widetilde{x} < \widetilde{x_0} + 1$, let $\widetilde{\varphi}(x)$ be the unique lift of $\varphi([\widetilde{x}])$ lying in $[\widetilde{y_0}, \widetilde{y_0}+1)$. Now extend $\widetilde{\varphi}$ to $\mathbb{R}$ in the unique possible way such that it commutes with integral translations.

In order to see that $\widetilde{\varphi}$ is non-decreasing it suffices to show that it is non-decreasing on $[\widetilde{x_0}, \widetilde{x_0}+1)$. Thus let $\widetilde{x_0} \leq \widetilde{x} < \widetilde{y} < \widetilde{x_0}+1$. Then the quadruple $(x_0, [\widetilde{x}], [\widetilde{y}], x_0+1)$ is weakly positively oriented, and thus also the quadruple $(\varphi(x_0), \varphi([\widetilde{x}]), \varphi([\widetilde{y}]), \varphi(x_0)) = (y_0, [\widetilde{\varphi}(\widetilde{x})],\ [\widetilde{\varphi}(\widetilde{y})], y_0)$ is weakly positively-oriented by (ii). By definition this means that there exists a real number a and integers $n, n_x, n_y, m \in \mathbb{Z}$ such that

$$a \leq \widetilde{y_0} + n \leq \widetilde{\varphi}(\widetilde{x}) + n_x \leq \widetilde{\varphi}(\widetilde{y}) + n_y \leq \widetilde{y_0} + m \leq a + 1\ .$$

It follows from the first and last inequality, that $m \in \{n, n+1\}$. In both cases, substracting n we obtain

$$\widetilde{y_0} \leq \widetilde{\varphi}(\widetilde{x}) + (n_x - n) \leq \widetilde{\varphi}(\widetilde{y}) + (n_y - n) \leq \widetilde{y_0} + 1.$$

Since $\widetilde{\varphi}(\widetilde{x})$ and $\widetilde{\varphi}(\widetilde{y})$ both belong to the interval $[\widetilde{y_0}, \widetilde{y_0}+1)$ it follows that $n_x = n = n_y$ and $\widetilde{\varphi}(\widetilde{x}) \leq \widetilde{\varphi}(\widetilde{y})$, which finishes the proof of this implication.

(iii) $\Rightarrow$ (i): Let $x_0, \dots, x_k$ be weakly positively oriented. By definition this means that there exists $a \in \mathbb{R}$ and lifts $\widetilde{x_i} \in \mathbb{R}$ of the x_i's such that

$$a \leq \widetilde{x_1} \leq \cdots \leq \widetilde{x_q} \leq a + 1.$$

Applying the non-decreasing map $\widetilde{\varphi}$ to the above inequalities gives

$$\widetilde{\varphi}(a) \leq \widetilde{\varphi}(\widetilde{x_1}) \leq \cdots \leq \widetilde{\varphi}(\widetilde{x_q}) \leq \widetilde{\varphi}(a+1) = \widetilde{\varphi}(a) + 1,$$

where the last equality uses the fact that $\widetilde{\varphi}$ commutes with integral translations. Since the $\widetilde{\varphi}(x_i)$'s are lifts of $\varphi(x_i)$, this by definition implies that the k-tuple $(\varphi(x_1), \ldots, \varphi(x_k))$ is weakly positively oriented. □

It is clear from the proof that we cannot replace the statement in (ii) with the corresponding statement for triples. To give an explicit counterexample, consider the function $\varphi : S^1 \to S^1$ given by

$$\varphi([t]) = \begin{cases} [0], & \lfloor t \rfloor \in [0, 1/4) \cup [1/2, 3/4), \\ [1/2], & \lfloor t \rfloor \in [1/4, 1/2) \cup [3/4, 1). \end{cases}$$

This function φ takes any triple into a weakly positively oriented one, but the quadruple $([0], [1/4], [1/2], [3/4])$ is taken by φ to $([0], [1/2], [0], [1/2])$, which is not weakly positively oriented.

2.B. **Semi-conjugacy.** The following is the key definition of this note.

Definition 2.5. Let $\rho_j : \Gamma \to \mathrm{Homeo}^+(S^1)$ be circle actions, $j = 1, 2$. We say that ρ_1 is *left-semi-conjugate* to ρ_2 (and ρ_2 is right-semi-conjugate to ρ_1) if there exists a non-decreasing degree one map φ such that

$$\rho_1(g)\varphi = \varphi\rho_2(g)$$

for every $g \in \Gamma$. In this case, φ is called a *left-semi-conjugacy* from ρ_1 to ρ_2 and we say that ρ_1 is left-semi-conjugate to ρ_2 *via* φ.

The circle action ρ_1 is called *semi-conjugate* to ρ_2 if it is both left- and right-semi-conjugate to ρ_2.

We recall some standard terminology for group actions: A circle action $\rho : \Gamma \to \mathrm{Homeo}^+(S^1)$ is said to have a *global fixed point* if there exists $x \in S^1$ such that $\rho(\gamma)(x) = x$ for every $\gamma \in \Gamma$. An action is *fixed point free* if it does not admit a global fixed point.

Proposition 2.6. *(i) Semi-conjugacy is an equivalence relation.*
(ii) Every circle action is right-semi-conjugate to the trivial action.
(iii) A circle action is left-semi-conjugate to the trivial action if and only if it has a global fixed point.

Proof. (i) Reflexivity and symmetry are obvious, while transitivity readily follows from the fact that non-decreasing degree one maps are closed under composition. (ii) Choose φ to be an arbitrary constant map. (iii) If ρ is left-semi-conjugate to the trivial action, then there exists φ such that for all $g \in \Gamma$ and $x \in S^1$

$$\rho(g)(\varphi(x)) = \varphi(x)$$

whence the image of φ consists of fixed points of $\rho(\Gamma)$. On the other hand, if x_0 is fixed by $\rho(\Gamma)$ and $\varphi(x) = x_0$ for every $x \in S^1$, then ρ is left-semi-conjugate to the trivial action by φ. □

Remark 2.7. The definition of semi-conjugacy given in [Ghy87] coincides with our definition of left-semi-conjugacy. As it obviously follows from Proposition 2.6 (ii)-(iii) that left-semi-conjugacy is not even an equivalence relation, it cannot be the correct notion. However, for *fixed point free* circle actions it does indeed coincide with our notion of semi-conjugacy, see Corollary 4.4.

The definition of semi-conjugation is changed in [Ghy01] to be a left semi-conjugation where one further requires the map $\varphi : S^1 \to S^1$ to be continuous. This is still not symmetric (see Example 5.5 below) but allows for a correct reformulation of Theorem 1.2.

In some sense, semi-conjugacy in the sense of Definition 2.5 is the most obvious way to turn left-semi-conjugacy into an equivalence relation. However, contrary to what is sometimes claimed, it is *not* the equivalence relation generated by left-semi-conjugacy. Namely, by Proposition 2.6 the equivalence relation generated by left-semi-conjugacy is the all relation, in which any two circle actions are related.

By definition, conjugate circle actions are semi-conjugate. We will see in Proposition 4.7 below that for minimal circle actions the converse holds. However, in general the notion of semi-conjugacy is much weaker than the notion of conjugacy. For example Proposition 2.6 shows that every circle action admitting a fixed point is semi-conjugate to the trivial circle action (but of course not conjugate to the trivial circle action unless it is trivial itself).

3. Three characterisations of the bounded Euler class

The goal of this section is to introduce the bounded Euler class and provide three different characterizations: as a bounded obstruction class (Subsection 3.B), via the translation number (Subsection 3.C) and as a bounded geometric class on the circle (Subsection 3.D). Yet another description of the bounded Euler class, which generalizes readily to higher dimensions, will be discussed in the appendix. In order to keep this note self-contained we collect in the next subsection various basic facts concerning (bounded) group cohomology. The expert can skip that subsection without loss of continuity.

3.A. **Preliminaries on (bounded) group cohomology.** Given a group H acting on a space X we define a cocomplex $(\mathcal{C}^n(H, X; \mathbb{Z}), \delta)$ by setting $\mathcal{C}^n(H, X; \mathbb{Z}) := \mathrm{Map}(X^{n+1}; \mathbb{Z})^H$, where the superscript ^{H} denotes H-invariants under the diagonal H-action, and defining the *homogeneous differential* δ by

$$\delta f(g_0, \ldots, g_n) = \sum_{i=0}^{n} (-1)^i f(g_0, \ldots, \widehat{g_i}, \ldots, g_n).$$

We then denote by $H^\bullet(H, X; \mathbb{Z})$ the cohomology of this cocomplex. For $X = H$ with the left-H-action this cohomology is precisely the classical group cohomology $H^\bullet(H;\mathbb{Z})$ with $\mathbb{Z}$-coefficients. Given a cocycle $c \in \mathcal{C}^n(H,X;\mathbb{Z})$ and a basepoint $x_0 \in X$ we obtain a cocycle $c_{x_0} \in \mathcal{C}^n(H,H;\mathbb{Z})$ by

$$c_{x_0}(h_0,\dots,h_n) = c(h_0\cdot x_0,\dots,h_n\cdot x_0).$$

The class of c_{x_0} is independent of the choice of basepoint x_0. We thus obtain a map $\iota_X : H^\bullet(H,X;\mathbb{Z}) \to H^\bullet(H;\mathbb{Z})$ and we say that a class $\alpha \in H^\bullet(H;\mathbb{Z})$ is *represented over* X if it is in the image of this map.

There is a more efficient representation for classes in $H^\bullet(H;\mathbb{Z})$ based on the fact that we can identify $\mathcal{C}(H,H;\mathbb{Z})$ with $C^n(H;\mathbb{Z}) := \mathrm{Map}(H^n;\mathbb{Z})$ via the isomorphism

$$\iota : C^n(H;\mathbb{Z}) \to \mathcal{C}^n(H,H;\mathbb{Z})$$

given by

$$\begin{aligned}
\iota(f)(h_0,\dots,h_n) &:= f(h_0^{-1}h_1, h_1^{-1}h_2,\dots,h_{n-1}^{-1}h_n)\\
\iota^{-1}(g)(h_1,\dots,h_n) &:= g(e,h_1,h_1h_2,\dots,h_1h_2\cdot\ldots\cdot h_n).
\end{aligned}$$

Thus $H^\bullet(H;\mathbb{Z}) = H^\bullet(C^\bullet(H;\mathbb{Z}),d)$, where the differential $d = \iota^{-1}\circ\delta\circ\iota$ is given by

$$\begin{aligned}
df(g_1,\dots,g_{n+1}) &= f(g_2,\dots,g_{n+1}) + \sum_{i=1}^{n}(-1)^i f(g_1,\dots,g_ig_{i+1},\dots,g_{n+1})\\
&\quad +(-1)^{n+1}f(g_1,\dots,g_n).
\end{aligned}$$

Cocycles in this model are called *inhomogeneous cocycles*, and are particularly useful to compute low degree cohomologies. We will be specifically interested in cohomology of degree 2; we thus recall briefly the relation between $H^2(H;\mathbb{Z})$ and central extensions. Given a central extension of groups of the form

$$\xi = \left(0 \longrightarrow \mathbb{Z} \xrightarrow{\ i\ } \widetilde{H} \xrightarrow{\ p\ } H \longrightarrow \{e\}\right)$$

and a set theoretic section $\sigma : H \to \widetilde{H}$ of p we define a function $c_\sigma : H^2 \to \widetilde{H}$ by

$$c_\sigma(g,h) = \sigma(h)\sigma(gh)^{-1}\sigma(g).$$

Since $p(c_\sigma(g,h)) = e$ we can consider c_σ as a function into $i(\mathbb{Z})$. We will often tacitly identify $\mathbb{Z}$ with its image in $\widetilde{H}$ and thus consider c_σ as a function $c_\sigma : H^2 \to \mathbb{Z}$. It is straightforward to check that c_σ satisfies the cocycle identity

$$dc_\sigma(g,h,k) = c_\sigma(h,k) - c_\sigma(gh,k) + c_\sigma(g,hk) - c_\sigma(g,h) = 0,$$

whence we refer to it as the *obstruction cocycle* associated with the extension ξ and the section σ. It turns out that the class $e(\xi) := [c_\sigma] \in H^2(H;\mathbb{Z})$ is independent of the choice of section. This independence can easily be proved directly, but it also follows from the following universal property of the class $[c_\sigma]$:

Lemma 3.1 (Lifting obstruction). *If $\rho : G \to H$ is a homomorphism, then there exists a lift*

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "i"] & \widetilde{H} \arrow[r, "p"] & H \arrow[r] & \{e\} \\ & & & G \arrow[u, "\rho"'] \arrow[ul, dotted, "\widetilde{\rho}"] & \end{tikzcd}$$

if and only if $\rho^*[c_\sigma] = 0 \in H^2(G;\mathbb{Z})$.

In the sequel we will need the following explicit version of (one direction of) the lemma:

Proposition 3.2 (Lifting formula). *Let $\rho : G \to H$ be a homomorphism. Assume that $\rho^* c_\sigma = du$ for some $u : G \to \mathbb{Z}$. Then a homomorphic lift $\widetilde{\rho} : G \to \widetilde{H}$ is given by the formula*

$$\widetilde{\rho}(g) = \sigma(\rho(g)) \cdot i(-u(g)).$$

Proof. Since this formula is at the heart of our argument we carry out the straightforward computation. Since $i(\mathbb{Z})$ is central in $\widetilde{H}$ we obtain

$$\begin{aligned} \sigma(\rho(g_1g_2)) &= \sigma(\rho(g_1)) \left(\sigma(\rho(g_1))^{-1}\sigma(\rho(g_1)\rho(g_2))\sigma(\rho(g_2))^{-1}\right) \sigma(\rho(g_2)) \\ &= \sigma(\rho(g_1)) \cdot c_\sigma(g_1, g_2)^{-1} \cdot \sigma(\rho(g_2)) \\ &= \sigma(\rho(g_1)) \cdot i(-du(g_1, g_2)) \cdot \sigma(\rho(g_2)) \\ &= \sigma(\rho(g_1))i(-u(g_1)) \cdot \sigma(\rho(g_2))i(-u(g_2)) \cdot i(-u(g_1g_2))^{-1} \end{aligned}$$

for all $g_1, g_2 \in G$. Multiplying both sides by $i(-u(g_1g_2))$ now yields $\widetilde{\rho}(g_1g_2) = \widetilde{\rho}(g_1)\widetilde{\rho}(g_2)$ and finishes the proof. □

Conversely, a class $e \in H^2(\Gamma;\mathbb{Z})$ determines a central extension, which is unique up to a suitable notion of isomorphism between extensions. We refer the reader to [Bro82, Chapter IV] for the details.

The subcomplex $\mathcal{C}_b^n(H, X;\mathbb{Z}) \subset \mathcal{C}^n(H, X;\mathbb{Z})$ of bounded functions is invariant under δ, and its cohomology is called the *(integral) bounded cohomology* of the H-action on X and denoted $H_b^\bullet(H, X;\mathbb{Z})$. In particular, $H_b^\bullet(H;\mathbb{Z}) := H_b^\bullet(H, H;\mathbb{Z})$ is the bounded group cohomology of H in the sense of [Gro82]. Note that the isomorphism $\iota : C^n(H;\mathbb{Z}) \to \mathcal{C}^n(H, H;\mathbb{Z})$ identifies $\mathcal{C}_b^n(H, H;\mathbb{Z})$ with the subspace $C_b^n(H;\mathbb{Z}) < C^n(H;Z)$ of bounded functions, whence $H_b^\bullet(H;\mathbb{Z})$ can also be computed from bounded inhomgeneous cocycles.

The inclusion of complexes $(C_b^n(H;\mathbb{Z}), \delta) \hookrightarrow (C^n(H;Z), \delta)$ induces on the level of cohomology a *comparison map* $H_b^\bullet(H;\mathbb{Z}) \to H^\bullet(H;\mathbb{Z})$, whose kernel is classically (and somewhat unfortunately) denoted by $EH_b^\bullet(H;\mathbb{Z})$. Note that an inhomogeneous bounded cocycle representing a class in $EH_b^2(H;\mathbb{Z})$ is of the form dT for some $T : H \to \mathbb{Z}$ with the property that $|T(gh) - T(g) - T(h)| = |dT(g, h)|$ is uniformly bounded. Such a function T is called an integral *quasimorphism* and the number $D(T) := \|dT\|_\infty$ is called its *defect*.

Given two quasimorphisms T_1, T_2 we have $[dT_1] = [dT_2] \in EH_b^2(H;\mathbb{Z})$ if and only if $T_1 - T_2 \in \mathrm{Hom}(H;\mathbb{Z}) \oplus \mathrm{Map}_b(H;\mathbb{Z})$. In particular, changing T by a bounded amount does not change the bounded cohomology class of $[dT]$.

Bounded group cohomology can also be defined with real coefficients. In this case, bounded inhomogeneous cocycles in $EH_b^2(H;\mathbb{R})$ are of the form dT where T is a real-valued quasimorphism. Every real-valued quasimorphism (and in particular every integral one) is at bounded distance from a unique homogeneous real-valued quasimorphism called its *homogeneization*. Here a real-valued function f is called *homogeneous* provided $f(h^n) = n\cdot f(h)$ for all $n \in \mathbb{N}$. Homogeneous quasimorphisms have the additional properties of being conjugation-invariant and linear on abelian subgroups. They also satisfy $f(h^n) = n\cdot f(h)$ for all $n \in \mathbb{Z}$, positive or not. Note that two quasimorphisms are at bounded distance if and only if their homogeneizations coincide. The following lemma illustrates how bounded cohomology with real coefficients can be used to obtain results concerning integral bounded cohomology; we will apply this in our second characterization of the bounded Euler class below.

Lemma 3.3. *If $q : \widetilde{H} \to H$ is a surjective homomorphism with amenable (e.g. abelian) kernel, then $q^* : H_b^2(H;\mathbb{Z}) \to H_b^2(\widetilde{H};\mathbb{Z})$ is injective.*

Proof. The short exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$ of coefficients induces a natural long exact sequence in bounded cohomology, called the Gersten sequence (see [Mon01, Prop. 8.2.12]), and the corresponding ladder associated with the homomorphism q starts from

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathrm{Hom}(H;\mathbb{R}/\mathbb{Z}) \arrow[r] \arrow[d, "q^*"'] & H_b^2(H;\mathbb{Z}) \arrow[r] \arrow[d, "q^*"'] & H_b^2(H;\mathbb{R}) \arrow[d, "q^*"'] \\
0 \arrow[r] & \mathrm{Hom}(\widetilde{H};\mathbb{R}/\mathbb{Z}) \arrow[r] & H_b^2(\widetilde{H};\mathbb{Z}) \arrow[r] & H_b^2(\widetilde{H};\mathbb{R})
\end{tikzcd}$$

Now surjectivity of q implies that the pullback map $q^* : \mathrm{Hom}(H;\mathbb{R}/\mathbb{Z}) \to \mathrm{Hom}(\widetilde{H};\mathbb{R}/\mathbb{Z})$ is injective, and the map $q^* : H_b^2(H;\mathbb{R}) \to H_b^2(\widetilde{H};\mathbb{R})$ is an isomorphism by [Mon01, Cor. 8.5.3], whence the lemma follows from the 4-lemma. $\square$

3.B. **The bounded Euler class as a bounded lifting obstruction.** From now on we reserve the letter H to denote the group $H := \mathrm{Homeo}^+(S^1)$ of orientation-preserving homeomorphisms of the circle $S^1 = \mathbb{R}/\mathbb{Z}$ and abbreviate by

$$\widetilde{H} := \{f \in \mathrm{Homeo}^+(\mathbb{R}) \mid \forall x \in \mathbb{R} : \; f(x+1) = f(x) + 1\}$$

its universal covering group (with respect to the compact-open topology). We then have a central extension

$$\xi = \left(0 \longrightarrow \mathbb{Z} \xrightarrow{\;i\;} \widetilde{H} \xrightarrow{\;p\;} H \longrightarrow \{e\} \right),$$

where $i(n)(x) := x + n$ and $p(\widetilde{f})([x]) = [\widetilde{f}(x)]$.

A section $\sigma : H \to \widetilde{H}$ is provided by specifying $\sigma(f)(0)$ for each $f \in H$; the section is called *bounded* provided $E_\sigma := \{\sigma(f)(0) \mid f \in H\}$ is bounded. In this case the obstruction cocycle $c_\sigma : H^2 \to \mathbb{Z}$ is bounded and thus defines also a class in the *bounded* second cohomology $H_b^2(H; \mathbb{Z})$. Again it is easy to see that this class is independent of the choice of bounded section. We then obtain two classes $\mathrm{eu} := [c_\sigma] \in H^2(H; \mathbb{Z})$ and $\mathrm{eu}_b := [c_\sigma] \in H_b^2(H; \mathbb{Z})$.

Definition 3.4. The classes eu and eu_b are called the *Euler class*, respectively *bounded Euler class.*

One special section σ is obtained by taking $E_\sigma = [0, 1)$. Let us give an explicit formula for the cocycle c_σ. For all $g, h \in G$ we have $\sigma(h)\sigma(gh)^{-1}\sigma(g) = i(c_\sigma(g, h))$. Since $i(\mathbb{Z}) < \widetilde{H}$ is central this can be written as $\sigma(g)\sigma(h) = \sigma(gh)i(c_\sigma(g, h))$. Evaluating at 0 we obtain

$$\sigma(g)\sigma(h)(0) = \sigma(gh)(0) + c_\sigma(g, h).$$

Observe that $\sigma(gh)(0)$ and $\sigma(h)(0)$ are contained in $[0, 1)$. The latter implies that $\sigma(g)\sigma(h)(0) \in [0, 2)$. Thus

$$c_\sigma(g, h) = \begin{cases} 1 & \text{if } \sigma(g)\sigma(h)(0) \in [1, 2), \\ 0 & \text{if } \sigma(g)\sigma(h)(0) \in [0, 1). \end{cases} \tag{3.1}$$

Another equivalent description can be given as follows: Observe that $\sigma(g)(1) = \sigma(g)(0) + 1 \in [1, 2)$ and that $\sigma(h)(0) < 1$ implies $\sigma(g)\sigma(h)(0) < \sigma(g)(1)$, and similarly $0 \le \sigma(h)(0)$ implies $\sigma(g)(0) \le \sigma(g)\sigma(h)(0)$. We may thus rewrite (3.1) as

$$c_\sigma(g, h) = \begin{cases} 1 & \text{if } 1 \le \sigma(g)\sigma(h)(0) < \sigma(g)(1) < 2, \\ 0 & \text{if } 0 \le \sigma(g)(0) \le \sigma(g)\sigma(h)(0) < 1. \end{cases} \tag{3.2}$$

Both formulas will be used below.

3.C. **The bounded Euler class and the translation number.** The *Poincaré translation number* $T : \widetilde{H} \to \mathbb{R}$ is the homogeneous quasimorphism on $\widetilde{H}$ given by

$$T(g) = \lim_{n\to\infty} \frac{g^n x - x}{n} \quad (x \in \mathbb{R}),$$

which by a classical theorem of Poincaré is independent of the choice of basepoint $x \in \mathbb{R}$. Let $T_\mathbb{Z} : \widetilde{H} \to \mathbb{Z}$ be any function at bounded distance from T. Then the cocyle $dT_\mathbb{Z}$ is bounded and thus defines a class $[dT_\mathbb{Z}] \in H_b^2(\widetilde{H}; \mathbb{Z})$, which is independent of the concrete choice of function $T_\mathbb{Z}$. We can now state the second characterization of the bounded Euler class. We recall that $p : \widetilde{H} \to H$ denotes the canonical projection.

Proposition 3.5. *The Euler class* eu_b *is the unique class in* $H_b^2(H; \mathbb{Z})$ *such that* $p^*\mathrm{eu}_b = -[dT_\mathbb{Z}] \in H_b^2(\widetilde{H}; \mathbb{Z})$.

Proof. Let c_σ denote the cocycle given by (3.1) and let $\bar{g}, \bar{h} \in \widetilde{H}$. We abbreviate $g := p(\bar{g})$, $h := p(\bar{h})$. Given a real number $r \in \mathbb{R}$ we denote by $r = \lfloor r \rfloor + \{r\}$ the unique decomposition of r with $\lfloor r \rfloor \in \mathbb{Z}$ and $\{r\} \in [0,1)$. Since $\bar{g}$ and $\sigma(g)$ have the same projection, we see that

$$\bar{g}(x) - \sigma(g)(x) = \bar{g}(0) - \sigma(g)(0) = \lfloor \bar{g}(0) \rfloor,$$

i.e. $\sigma(g)(x) = \bar{g}(x) - \lfloor \bar{g}(0) \rfloor$ and similarly $\sigma(h)(x) = \bar{h}(x) - \lfloor \bar{h}(0) \rfloor$. We deduce that

$$\begin{aligned} \sigma(g)\sigma(h)(0) &= \sigma(g)(\bar{h}(0) - \lfloor \bar{h}(0) \rfloor) = \sigma(g)(\bar{h}(0)) - \lfloor \bar{h}(0) \rfloor \\ &= \bar{g}\bar{h}(0) - \lfloor \bar{g}(0) \rfloor - \lfloor \bar{h}(0) \rfloor \\ &= \lfloor \bar{g}\bar{h}(0) \rfloor - \lfloor \bar{g}(0) \rfloor - \lfloor \bar{h}(0) \rfloor + \{\bar{g}\bar{h}(0)\} \end{aligned}$$

Since the last term is contained in $[0,1)$, this expression is in $[1,2)$ respectively $[0,1)$ if the sum of the first three terms is equal to 1 and 0 respectively. We thus obtain

$$p^* c_\sigma(\bar{g}, \bar{h}) = \lfloor \bar{g}\bar{h}(0) \rfloor - \lfloor \bar{g}(0) \rfloor - \lfloor \bar{h}(0) \rfloor.$$

Now the function $T_{\mathbb{Z}} : \widetilde{H} \to \mathbb{Z}$ given by $\bar{g} \mapsto \lfloor \bar{g}(0) \rfloor$ is at bounded distance from the translation number T and the last identity can be written as $p^* c_\sigma = -dT_{\mathbb{Z}}$. We thus deduce that $p^* \mathrm{eu}_b = -[dT_{\mathbb{Z}}]$ and uniqueness follows from Lemma 3.3. □

3.D. **The bounded Euler class realized over the circle.** In this subsection we are going to show that the Euler class and the bounded Euler class are representable over the circle, i.e. that they are in the respective images of the maps $H^2(H, S^1; \mathbb{Z}) \to H^2(H; \mathbb{Z})$ and $H^2_b(H, S^1; \mathbb{Z}) \to H^2_b(H; \mathbb{Z})$. Recall that throughout we think of S^1 as the quotient space $\mathbb{R}/\mathbb{Z}$. In order to describe cocycles in $\mathcal{C}^n(H, S^1; \mathbb{Z})$ we need to understand H-orbits in $(S^1)^{n+1}$. For $n \leq 2$ the classification of orbits is as follows:

Orbits of H acting on $(S^1)^{n+1}$.

(n=0) The action of H on one factor S^1 clearly has exactly one orbit.

(n=1) The action of H on two factors $(S^1)^2$ has two orbits: one degenerate orbit $\{(x,x) \mid x \in S^1\}$ and one non degenerate orbit $\{(x,y) \mid x \neq y \in S^1\}$.

(n=2) The action of H on three factors $(S^1)^3$ has six orbits. Choose distinct points $x, y, z \in S^1$ and suppose that (x,y,z) is a positively oriented triple. Then the orbits are given as follows:
degenerate: $H\cdot(x,x,x)$, $H\cdot(x,x,y)$, $H\cdot(x,y,x)$, $H\cdot(y,x,x)$.
nondegenerate: $H \cdot (x,y,z)$, $H \cdot (y,x,z)$.

For general n there are still only finitely many H-orbits. This implies $\mathcal{C}^n_b(H, S^1; \mathbb{Z}) = \mathcal{C}^n(H, S^1; \mathbb{Z})$ and thus the comparison map $H^n_b(H, S^1; \mathbb{Z}) \cong H^n(H, S^1; \mathbb{Z})$ is an isomorphism. In particular, if an element of $H^n(H; \mathbb{Z})$ is representable over S^1, then it is bounded. In degree 2 we can actually parametrize all possible cocycles and coboundaries using the above description of orbits. For coboundaries the result is as follows.

Lemma 3.6. *Let $b : (S^1)^2 \to \mathbb{R}$ be an H-invariant 2-cochain determined by its two values on (x,x) and (x,y) (where $x \neq y$), say $b(x,x) = \alpha$ and $b(x,y) = \beta$. If $z \notin \{x,y\}$, then*

$$\begin{aligned} \delta b(x,x,x) = \delta b(x,x,y) = \delta b(y,x,x) &= \alpha, \\ \delta b(x,y,x) &= 2\beta - \alpha, \\ \delta b(x,y,z) = \delta b(y,x,z) &= \beta. \end{aligned}$$

One very familiar H-invariant 2-cocycle on S^1 is the orientation cocycle Or, which assigns the value $+1$, respectively -1, to positively oriented, resp. negatively oriented non-degenerate triples, and 0 to all degenerate triples. By the previous lemma it is not a coboundary, since the value on positively and negatively oriented triples is not the same. We now describe general 2-cocycles:

Lemma 3.7. *Let $f : (S^1)^3 \to \mathbb{R}$ be an H-invariant cochain. Then f is a cocycle if and only if*

$$\begin{aligned} &f(x,x,x) = f(y,x,x) = f(x,x,y), \\ &f(y,x,z) - f(x,y,x) = f(x,x,z) - f(x,y,x) \end{aligned}$$

for every triple (x,y,z) of distint points.

Proof. That the three linear conditions given in the lemma are necessary follow from the cocycle relations

$$\delta f(y,x,x,x) = \delta f(x,x,x,y) = \delta f(x,y,x,z) = 0.$$

There cannot be any further relation since the space of 2-coboundaries is 2-dimensional by Lemma 3.6 and the quotient of the space of 2-cocycles by the space of 2-coboundaries is nontrivial as it contains $0 \neq [\mathrm{Or}] \in H^2_b(H, S^1; \mathbb{Z})$. □

Definition 3.8. Let (x,y,z) be a positively oriented triple in S^1. Then the unique H-invariant function $c : (S^1)^3 \to \mathbb{Z}$ given by

$$c(x,x,x) = c(y,x,x) = c(x,x,y) = 0, \quad c(x,y,x) = 1$$
$$c(x,y,z) = 0, \quad c(y,x,z) = 1$$

is called the *Euler cocycle.*

By Lemma 3.7 the Euler cocycle is indeed a cocycle in $\mathcal{C}^2_b(H, S^1; \mathbb{Z})$. Now a small computation shows (see also [Ioz02, Lemma 2.1]):

Lemma 3.9. *If $c \in \mathcal{C}^2_b(H, S^1; \mathbb{Z})$ is the Euler cocycle and $c_\sigma \in C^2_b(H)$ denotes the obstruction cocycle associated with the special section $\sigma : H \to \widetilde{H}$ with $E_\sigma = [0,1)$, then*

$$c_\sigma(g,h) = c([0], g \cdot [0], gh \cdot [0]).$$

Moreover, if Or *defines the orientation cocycle and* b *is the* 2*-cochain defined by* $b(x,x) = 0$ *and* $b(x,y) = 1$ *then* $\mathrm{Or} = 2c + \delta b$.

Proof. It follows from the explicit definition of c that

$$c([0], g \cdot [0], gh \cdot [0]) = \begin{cases} 1, & 1 \leq \sigma(g)\sigma(h)(0) < \sigma(g)(1) < 2 \\ 0, & 0 \leq \sigma(g)(0) \leq \sigma(g)\sigma(h)(0) < 1 \end{cases}$$

In view of (3.2) this implies $c_\sigma(g,h) = c([0], g \cdot [0], gh \cdot [0])$. The relation $\mathrm{Or} = 2c + \delta b$ is straightforward. □

From this computation we draw the following conclusions.

Corollary 3.10. *The bounded Euler class* eu_b *is representable over the circle. In fact it is represented by the Euler cocycle* $c : (S^1)^3 \to \mathbb{Z}$*. Similarly, the class* $2 \cdot \mathrm{eu}_b$ *is represented over the circle by the orientation cocycle* Or.

Note that, in particular, for every $x \in S^1$ the homogeneous cocycle $c_x : H^3 \to \mathbb{Z}$ given by

$$(g_0, g_1, g_2) \mapsto c_x(g_0, g_1, g_2) = c(g_0 x, g_1 x, g_2 x)$$

represents the bounded Euler class.

4. Ghys' theorem

4.A. **Circle actions with vanishing bounded Euler class.** Before we turn to the proof of Ghys' theorem in the general case we provide a characterization of circle actions with vanishing bounded Euler class. This characterization can be seen as a special case of Ghys' theorem, but it is also of independent interest and has a particularly simple proof. Parts of this special case will also be used in the proof of the general theorem.

Recall that the Euler class eu was defined as an obstruction class. It thus follows from Lemma 3.1 that if $\rho : \Gamma \to H$ is a circle action, then

$$\rho^*\mathrm{eu} = 0 \Leftrightarrow \text{ the action lifts to an action on the real line.}$$

The following result shows that the vanishing of the *bounded* Euler class has much more drastical consequences:

Proposition 4.1. *Let* $\rho : \Gamma \to H$ *be a circle action with* $\rho^*\mathrm{eu}_b = 0$*. Then the action lifts to an action of the real line which moreover has a fixed point.*

Proof. By assumption there exists a bounded function $u : \Gamma \to \mathbb{Z}$ with $\rho^* c_\sigma = du$. By Proposition 3.2 we have a homomorphism

$$\widetilde{\rho} : \Gamma \to \widetilde{H}, \quad \widetilde{\rho}(g) = \sigma(\rho(g)) \cdot i(-u(g)).$$

In particular,

$$\widetilde{\rho}(g)(0) = \sigma(\rho(g))(0) - u(g).$$

Now, since σ is a bounded section and u is bounded, also $\widetilde{\rho}(g)(0)$ is bounded. It follows that

$$F^+(\widetilde{\rho}) := \sup_{g\in G} \widetilde{\rho}(g)(0);$$

is well-defined, and it is clearly a fixed point for $\widetilde{\rho}(G)$. □

Using the second characterization of the bounded Euler class via the translation number we obtain a converse to this result, leading to the following characterization:

Corollary 4.2 (Circle actions with vanishing bounded Euler class)**.** *Let $\rho : \Gamma \to \mathrm{Homeo}^+(S^1)$ be a circle action. Then the following are equivalent:*

(i) $\rho^*\mathrm{eu}_b = 0$*.*
(ii) The circle action ρ lifts to an action on the real line which moreover has a fixed point.
(iii) $\rho(\Gamma)$ fixes a point in S^1.
(iv) ρ is semi-conjugate to the trivial circle action.

Proof. We have already seen that (i)⇒(ii). Conversely, if (ii) holds for a lift $\widetilde{\rho} : \Gamma \to \widetilde{H}$ with fixed point x_0, then

$$\rho^*\mathrm{eu}_b = -\widetilde{\rho}^*[dT_{\mathbb{Z}}] = -[d\widetilde{\rho}^*T_Z].$$

However we have for every $g \in \Gamma$,

$$\widetilde{\rho}^*T(g) = \lim_{n\to\infty} \frac{\widetilde{\rho}(g)^n(x_0) - x_0}{n} = 0,$$

whence $\widetilde{\rho}^*T_{\mathbb{Z}}$ is bounded and thus (i) holds. The implication (ii)⇒(iii) is obvious, since the projection of a fixed point of a lift is a fixed point for the original action. Conversely, if $\rho(\Gamma)$ fixes $[x_0] \in S^1$, then it acts on $S^1 \setminus \{[x_0]\}$ and this action can be lifted to an action on $(x_0, x_0 + 1)$ and periodically to an action on $\mathbb{R}$ fixing all points in $x_0 + \mathbb{Z}$. This shows (ii)⇔(iii) and the equivalence (iii) ⇔ (iv) follows from Proposition 2.6. □

Although Corollary 4.2 is only a very simple special case of Ghys' theorem, it is sufficient for many applications. E.g. most of the applications of Ghys' theorem in higher Teichmüller theory depend only on Corollary 4.2 (see e.g. [BIW10, BSBH$^+$13]). We therefore find it important to point out the above simple proof. Note that a slightly stronger version of Corollary 4.2 is established in the appendix.

4.B. **A refined statement of Ghys' theorem.** We will now prove Ghys' Theorem 1.2 (with our corrected definition of semi-conjugation), thus establishing that the bounded Euler class is a complete invariant of semi-conjugacy. We will actually prove the following more precise version:

Theorem 4.3. *Let ρ_1, ρ_2 be circle actions of Γ.*

(i) If $\rho_1^*\mathrm{eu}_b = \rho_2^*\mathrm{eu}_b$, then ρ_1 and ρ_2 are semi-conjugate.
(ii) If ρ_1 and ρ_2 are semi-conjugate and either of them has a fixed point, then both have a fixed point and $\rho_1^*\mathrm{eu}_b = \rho_2^*\mathrm{eu}_b = 0$.

(iii) If ρ_1 is fixed point free and left-semi-conjugate to ρ_2, then $\rho_1^*\mathrm{eu}_b = \rho_2^*\mathrm{eu}_b \neq 0$.

Note that in the situation of (iii), ρ_1 and ρ_2 are actually semi-conjugate by (i). This proves the following result alluded to in the introduction.

Corollary 4.4. *If a fixed point free circle action ρ_1 is left-semi-conjugate to a circle action ρ_2, then they are semi-conjugate. In particular, left-semi-conjugacy defines an equivalence relation on the set of all fixed point free circle actions.*

Part (ii) of Theorem 4.3 follows directly from Corollary 4.2: If, say, ρ_1 has a fixed point, then it is semi-conjugate to the trivial circle action by the implication (iii) $\Rightarrow$ (iv), whence also ρ_2 is semi-conjugate to the trivial circle action and thus has a fixed point by the implication (iv) $\Rightarrow$ (iii). Then, by the implication (iii) $\Rightarrow$ (i) we have $\rho_1^*\mathrm{eu}_b = \rho_2^*\mathrm{eu}_b = 0$. Thus it remains to show only (i) and (iii), which we will do in the next two subsections.

4.C. **Same bounded Euler class implies semi-conjugacy.** In this subsection we are going to establish Part (i) of Theorem 4.3. Our proof is a slight variation of Ghys' original proof, which emphasizes the similarity to the proof of Proposition 4.1.

To fix notation, let ρ_1, ρ_2 be circle actions with the same bounded Euler class $\rho_1^*\mathrm{eu}_b = \rho_2^*\mathrm{eu}_b$. We claim that ρ_1 and ρ_2 are semi-conjugate. By symmetry it suffices to show that ρ_1 is left-semi-conjugate to ρ_2.

Let $\widetilde{\Gamma}$ be the central extension of Γ which corresponds to $\rho_1^*\mathrm{eu} = \rho_2^*\mathrm{eu}$. Then we can choose lifts $\widetilde{\rho_1}, \widetilde{\rho_2}$ making the diagrams

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "i"] & \widetilde{H} \arrow[r, "p"] & H \arrow[r] & 1 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, "i"] \arrow[u, "\widetilde{\rho_i}"'] & \widetilde{\Gamma} \arrow[r] \arrow[u, "\widetilde{\rho_i}"'] & \Gamma \arrow[r] \arrow[u, "\rho_i"'] & 1 \end{tikzcd}$$

commute. Since $\rho_1^*\mathrm{eu}_b = \rho_2^*\mathrm{eu}_b$ and the diagrams commute we have

$$[d\widetilde{\rho_1}^*T_{\mathbb{Z}}] = \widetilde{\rho_1}^*[dT_{\mathbb{Z}}] = -\widetilde{\rho_1}^*(p^*\mathrm{eu}_b) = -\widetilde{\rho_2}^*(p^*\mathrm{eu}_b) = \widetilde{\rho_2}^*[dT_{\mathbb{Z}}] = [d\widetilde{\rho_2}^*T_{\mathbb{Z}}].$$

This implies that there exist a homomorphism $u : \widetilde{\Gamma} \to \mathbb{Z}$ and a bounded function $b : \widetilde{\Gamma} \to \mathbb{Z}$ such that $\widetilde{\rho_1}^*T_{\mathbb{Z}} - \widetilde{\rho_2}^*T_{\mathbb{Z}} = u + b$. It follows that $\widetilde{\rho_1}^*T - \widetilde{\rho_2}^*T - u$ is a bounded homogeneous function, hence 0. Thus,

$$\widetilde{\rho_1}^*T - \widetilde{\rho_2}^*T = u.$$

Replacing the lift $\widetilde{\rho_2}$ by $\widetilde{\rho_2} + i(u)$ we can ensure that $u = 0$. Assume that $\widetilde{\rho_2}$ is chosen in that way. Then for every $g \in \widetilde{H}$,

$$|T(\widetilde{\rho_1}(g)^{-1}\widetilde{\rho_2}(g))| \leq |-T(\widetilde{\rho_1}(g)) + T(\widetilde{\rho_2}(g))| + D(T) = D(T),$$

where $D(T)$ is the defect of the quasimorphism T. In particular, $\widetilde{\rho_1}(g)^{-1}\widetilde{\rho_2}(g)$ has uniformly bounded translation number and thus

$$\widetilde{\varphi}(x) := \sup_{g\in\widetilde{\Gamma}}(\widetilde{\rho_1}(g)^{-1}\widetilde{\rho_2}(g)(x))$$

is well-defined. By definition we have for every $g_0 \in \widetilde{\Gamma}$,

$$\begin{aligned}\widetilde{\varphi}(\widetilde{\rho}_2(g_0)(x)) &= \sup_{g \in \widetilde{\Gamma}} \widetilde{\rho}_1(g)^{-1}(\widetilde{\rho}_2(g)(\widetilde{\rho}_2(g_0)(x))) \\ &= \sup_{g \in \widetilde{\Gamma}} \widetilde{\rho}_1(g\, g_0^{-1})^{-1}(\widetilde{\rho}_2(g)(x)) \\ &= \widetilde{\rho}_1(g_0)\left(\sup_{g \in \widetilde{\Gamma}} \widetilde{\rho}_1(g)^{-1}(\widetilde{\rho}_2(g)(x))\right) \\ &= \widetilde{\rho}_1(g_0)(\widetilde{\varphi}(x)).\end{aligned}$$

Moreover, being the supremum of increasing maps which commute with integral translations, the map $\widetilde{\varphi}\colon \mathbb{R} \to \mathbb{R}$ is non-decreasing and commutes with integral translations, so it is a good lift of a non-decreasing degree one map $\varphi\colon S^1 \to S^1$. It follows that φ realizes the desired left-semi-conjugation from ρ_1 to ρ_2. This finishes the proof of Part (i) of Theorem 4.3.

4.D. **Semi-conjugacy implies same bounded Euler class.** In this subsection we establish the remaining Part (iii) of Theorem 4.3 thereby finishing the proof of the theorem. Here we will finally make use of the third (geometric) characterization of the bounded Euler class.

Instead of Theorem 4.3.(iii) we will actually prove a slightly stronger statement. To state this result we introduce the following notation. Throughout this section we will fix two circle actions ρ_1, ρ_2 of Γ and a semi-conjugacy φ from ρ_1 to ρ_2. We will *not* assume a priori that ρ_1 is fixed point free. For each $\gamma \in \Gamma$ we fix lifts $\widetilde{\rho_1}(\gamma)$ and $\widetilde{\rho_2}(\gamma)$ of $\rho_1(\gamma)$ respectively $\rho_2(\gamma)$ with the additional property that

$$\widetilde{\rho_j}(\gamma)^{-1} = \widetilde{\rho_j}(\gamma^{-1}) \quad (j = 1, 2). \tag{4.1}$$

Suppose now that $\widetilde{\varphi}$ is some good lift of φ. Since $\widetilde{\rho_1}(\gamma)\widetilde{\varphi}$ and $\widetilde{\varphi}\widetilde{\rho_2}(\gamma)$ are lifts of the same map and are invariant under integral translations, there exists a map $n_\gamma : \mathbb{R} \to \mathbb{Z}$ (dependent on $\widetilde{\varphi}$), invariant under integral translations, such that for all $x \in \mathbb{R}$,

$$\widetilde{\rho_1}(\gamma)\widetilde{\varphi}(x) = \widetilde{\varphi}(\widetilde{\rho_2}(\gamma)(x)) + n_\gamma(x). \tag{4.2}$$

Note that our symmetry assumption (4.1) implies that $n_{\gamma^{-1}} = -n_\gamma \circ \widetilde{\rho_2}(\gamma^{-1})$.

Proposition 4.5. *Let ρ_1, ρ_2 be circle-actions of Γ and let φ be a semi-conjugacy from ρ_1 to ρ_2. Let a good lift $\widetilde{\varphi}$ of φ be fixed and let $n_\gamma : \mathbb{R} \to \mathbb{Z}$ be defined by (4.2). Consider the following statements:*

(1) $\rho_1(\Gamma)$ does not have a global fixed point in S^1.
(2) φ is not the constant map.
(3) There exists a good lift $\widetilde{\varphi}$ of φ such that for each $\gamma \in \Gamma$ the map n_γ given by (4.2) is constant.
(4) There exists a good lift $\widetilde{\varphi}$ of φ such that $\widetilde{\rho_1}(\gamma)\widetilde{\varphi}(x) = \widetilde{\varphi}(\widetilde{\rho_2}(\gamma)(x))$ for all $\gamma \in \Gamma$ and $x \in \mathbb{R}$.

(5) There exists a non-empty $\rho_2(\Gamma)$-invariant subset $K_2 \subset S^1$ such that $\varphi|_{K_2}$ is injective.

(6) $\rho_1^*\mathrm{eu}_b = \rho_2^*\mathrm{eu}_b$.

Then the implications (1)⇒(2)⇒(3)⇒(4)⇒(5)⇒(6) hold.

Note that the implication (1)⇒(6) gives Part (iii) of Theorem 4.3.

Proof of Proposition 4.5. The implication (1)⇒(2) is obvious, so we turn directly to the proofs of the implications (2)⇒(3)⇒(4)⇒(5)⇒(6).

Assume that (2) holds and fix $\gamma \in \Gamma$. Let $\widetilde{\varphi}$ be a good lift of φ. Since φ is non-constant we find distinct elements $a_0, b_0 \in \mathbb{R}$ with $b_0 - a_0 \in (0,1)$ and $\widetilde{\varphi}(b_0) - \widetilde{\varphi}(a_0) \in (0,1)$. Since $\widetilde{\rho_1}(\gamma)$ is strictly increasing and commutes with integral translations, this implies at once that

$$0 < \widetilde{\rho_1}(\gamma)(\widetilde{\varphi}(b_0)) - \widetilde{\rho_1}(\gamma)(\widetilde{\varphi}(a_0)) < 1. \tag{4.3}$$

On the other hand, since $\widetilde{\varphi} \circ \widetilde{\rho_2}(\gamma)$ is non-decreasing and commutes with integral translations, we also have $0 \leq \widetilde{\varphi}(\widetilde{\rho_2}(\gamma)(b_0)) - \widetilde{\varphi}(\widetilde{\rho_2}(\gamma)(a_0)) \leq 1$. However, these inequalities must both be strict, because otherwise we would have

$$\rho_1(\gamma)(\varphi([b_0])) = \varphi(\rho_2(\gamma)([b_0])) = \varphi(\rho_2(\gamma)([a_0])) = \rho_1(\gamma)(\varphi([a_0])),$$

which contradicts (4.3). We have thus shown that

$$0 < \widetilde{\rho_1}(\gamma)(\widetilde{\varphi}(b_0)) - \widetilde{\rho_1}(\gamma)(\widetilde{\varphi}(a_0)) < 1\,, \quad 0 < \widetilde{\varphi}(\widetilde{\rho_2}(\gamma)(b_0)) - \widetilde{\varphi}(\widetilde{\rho_2}(\gamma)(a_0)) < 1\,.$$

Subtracting the second inequality from the first we deduce that $n_\gamma(b) - n_\gamma(a) \in [0,1) - [0,1) = (-1,1)$. Since both are integers we deduce that $n_\gamma(b) = n_\gamma(a)$, which implies that n_γ is constant on $E := (a_0 + \mathbb{Z}) \cup (b_0 + \mathbb{Z})$.

Now let $x \in \mathbb{R} \setminus E$. Then the interval $(x-1, x+1)$ contains one translate of a_0 and one translate of b_0, and these take different values under $\widetilde{\varphi}$. We thus find $e \in E$ with $|x - e| < 1$ and $\widetilde{\varphi}(x) \neq \widetilde{\varphi}(e)$, whence either $\{x - e, \widetilde{\varphi}(x) - \widetilde{\varphi}(e)\} \subset [0,1)$ or $\{e - x, \widetilde{\varphi}(e) - \widetilde{\varphi}(x)\} \subset [0,1)$. In both cases we have $n_\gamma(x) - n_\gamma(e) \in (-1,1)$, so $n_\gamma(x) = n_\gamma(e)$. This finishes the proof of the implication (2)⇒(3).

Now assume that (3) holds, i.e. for every $\gamma \in \Gamma$ we have $n_\gamma(x) = n_\gamma$ for some constant n_γ. We can then replace the lift $\widetilde{\rho_1}(\gamma)$ by the lift $\widetilde{\rho_1}(\gamma) - n_\gamma$ and thereby achieve that for all $x \in \mathbb{R}$,

$$\widetilde{\rho_1}(\gamma)\widetilde{\varphi}(x) = \widetilde{\varphi}(\widetilde{\rho_2}(\gamma)(x)), \tag{4.4}$$

which is (4). We emphasise that (4.1) still holds after this modification. Indeed, by the remark following (4.2) we have $n_\gamma = -n_\gamma^{-1}$ and thus

$$(\widetilde{\rho_1}(\gamma) - n_\gamma)^{-1} = \widetilde{\rho_1}(\gamma)^{-1} + n_\gamma = \widetilde{\rho_1}(\gamma^{-1}) - n_{\gamma^{-1}}.$$

We now deduce (5) from (4), where according to the previous remark we may also assume (4.1). Given $x \in \mathbb{R}$ we define a subset S_x by

$$S_x = \{y \in \mathbb{R} \mid \widetilde{\varphi}(y) = \widetilde{\varphi}(x)\}.$$

Since $\widetilde{\varphi}$ is non-decreasing and commutes with integral translations we have $S_x \subset (x-1, x]$. In particular, $\inf S_x$ is a well-defined real number. This shows that the subset $\widetilde{K_2} \subset \mathbb{R}$ given by

$$\widetilde{K_2} := \{x \in \mathbb{R} \mid x = \inf S_x\}$$

is non-empty.

We claim that $\widetilde{\varphi}|_{K_2}$ is injective. Indeed, assume $x_1, x_2 \in \widetilde{K_2}$ and

$$\widetilde{\varphi}(x_1) = \widetilde{\varphi}(x_2).$$

Assume without loss of generality that $x_1 \leq x_2$. Then $x_1 \in S_{x_2}$ and hence $x_2 = \inf S_{x_2}$ implies $x_2 \leq x_1$ whence $x_1 = x_2$, finishing the proof of injectivity.

Now we claim that $\widetilde{K_2}$ is invariant under $\widetilde{\rho_2}(\gamma)$ for every $\gamma \in \Gamma$. Thus let $x \in \widetilde{K_2}$ and $y \in \mathbb{R}$ with $y < \widetilde{\rho_2}(\gamma)x$. Then (4.1) yields $\widetilde{\rho_2}(\gamma^{-1})y = \widetilde{\rho_2}(\gamma)^{-1}y < x$, and since $x = \inf S_x$ we have

$$\widetilde{\varphi}(\widetilde{\rho_2}(\gamma^{-1})y) < \widetilde{\varphi}(x).$$

Combining this with (4.4) we obtain

$$\widetilde{\varphi}(x) > \widetilde{\varphi}(\widetilde{\rho_2}(\gamma^{-1})(y)) = \widetilde{\rho_1}(\gamma)^{-1}\widetilde{\varphi}(y).$$

Multiplying both sides by $\widetilde{\rho_1}(\gamma)$ and using (4.4) again we find

$$\widetilde{\varphi}(y) < \widetilde{\varphi}(\widetilde{\rho_2}(\gamma)(x)).$$

This shows that $\widetilde{\rho_2}(\gamma)x \in \widetilde{K_2}$. It then follows that the image K_2 of $\widetilde{K_2}$ in S^1 is non-empty and $\rho_2(\Gamma)$-invariant, and that $\varphi|_{K_2}$ is injective. This finishes the proof of the implication (4)⇒(5).

Finally, we establish the implication (5)⇒(6): Let K_2 be as in (4) and let $x \in K_2$. By Lemma 3.7 the cohomology class $\rho_2^*\mathrm{eu}_b$ is represented by the cocycle

$$\rho_2^*c_x(g_0, g_1, g_2) = c(\rho_2(g_0)x, \rho_2(g_1)x, \rho_2(g_2)x).$$

Note that for $j = 0, 1, 2$ the points $\rho_j(g_0)x$ are all contained in K_2, since K_2 is $\rho_2(\Gamma)$-invariant. It thus follows from injectivity of φ on K_2 that they are pairwise distinct if and only if their images under φ are pairwise distinct. Since φ also preserves their weak orientation, we deduce that the triples $(\rho_2(g_0)x, \rho_2(g_1)x, \rho_3(g_2)x)$ and $(\widetilde{\varphi}(\rho_2(g_0)x), \widetilde{\varphi}(\rho_2(g_1)x), \widetilde{\varphi}(\rho_2(g_2)x))$ are in the same H-orbit. Indeed, this follows from the classification of H-orbits on $(S^1)^3$ in Subsection 3.D. Since c is H-invariant we obtain

$$\begin{aligned}
\rho_2^*c_x(g_0, g_1, g_2) &= c(\varphi(\rho_2(g_0)x), \varphi(\rho_2(g_1)x), \varphi(\rho_2(g_2)x)) \\
&= c(\rho_1(g_0)\varphi(x), \rho_1(g_1)\varphi(x), \rho_1(g_2)\varphi(x)) \\
&= \rho_1^*c_{\varphi(x)}(g_0, g_1, g_2).
\end{aligned}$$

Since the cocycle $\rho_1^*c_{\varphi(x)}$ represents $\rho_1^*\mathrm{eu}_b$, we deduce that $\rho_1^*\mathrm{eu}_b = \rho_2^*\mathrm{eu}_b$. This finishes the proof. □

At this point we have finished the proof of Theorem 4.3 and thereby of Theorem 1.2.

Remark 4.6. In [Ghy87, Proof of Proposition 5.2] Ghys claims that the map n_γ is constant independently of whether φ is constant or not. (Note that our n_γ is denoted $u(\gamma)$ in [Ghy87, Equation (1), Proof of Proposition 5.2]). The following example shows that this claim is wrong. Let ρ_1 be the trivial circle action of $\mathbb{Z}$ and ρ_2 be the circle action sending 1 to the rotation by 1/2. Then ρ_1 is left semi-conjugate to ρ_2 by Proposition 2.6 (ii). The left semi-conjugation can be given by the constant map $\varphi(x) \equiv 0$ which lifts to $\widetilde{\varphi} : x \mapsto \lfloor x \rfloor$. A lift of $\rho_1(1)$ is the identity and a lift of $\rho_2(1)$ is the translation $T_{1/2}$ by 1/2. Then $\rho_1(1)\varphi = \varphi\rho_2(1)$ on the circle but the translation $x \mapsto \widetilde{\varphi}(x) - \widetilde{\varphi}(T_{1/2})(x) = \lfloor x \rfloor - \lfloor x + 1/2 \rfloor$ depends on x since it is 0 for $x \in [0, 1/2) + \mathbb{Z}$ and -1 for $x \in [1/2, 1) + \mathbb{Z}$. More generally, neither of the statements (2)–(5) is correct without the assumption that ρ_1 is fixed point free. For example, if ρ_1 has a fixed point then we can alway choose φ to be constant. In that case, every set $K_2 \subset S^1$ on which φ is injective constructed is a singleton. If $\rho_2(\Gamma)$ is fixed point free, then such a set cannot be invariant. The reader may check that in this case our set K_2 constructed in the proof is indeed a singleton, and that the proof of invariance breaks down in the absence of (3), e.g. in the situation of the example above.

4.E. **The minimal case: Semi-conjugacy equals conjugacy.** Recall that a circle action $\rho : \Gamma \to \mathrm{Homeo}^+(S^1)$ is *minimal* if every $\rho(\Gamma)$-orbit is dense in S^1. The following proposition shows that for minimal circle actions, the notions of conjugacy and semi-conjugacy coincide. This implies in particular that Theorem 1.1 follows from Theorem 1.2.

Proposition 4.7 (Ghys)**.** *Let $\rho_1, \rho_2 \colon \Gamma \to \mathrm{Homeo}^+(S^1)$ be minimal circle actions. Then the following are equivalent:*

(i) ρ_1 is left-semi-conjugate to ρ_2.
(ii) ρ_1 and ρ_2 are semi-conjugate.
(iii) ρ_1 and ρ_2 are conjugate.

Proof. Since minimal actions are fixed point free, the equivalence (i)$\Leftrightarrow$(ii) follows from Corollary 4.4. Moreover, the implication (iii)$\Rightarrow$(i) holds trivially. Concerning the implication (i)$\Rightarrow$(iii) assume that ρ_1 is left-semi-conjugate to ρ_2 via φ. Then the image of φ is $\rho_1(\Gamma)$-invariant, whence dense in S^1 by minimality. This in turn implies that the image of $\widetilde{\varphi}$ is dense in $\mathbb{R}$. So the map $\widetilde{\varphi}$, being non-decreasing and commuting with integral translations, is continuous and surjective. Therefore, the same is true for φ, and we are left to show that φ is also injective.

Suppose by contradiction that there exist distinct points $x, y \in S^1$ such that $\varphi(x) = \varphi(y)$, and choose lifts $\widetilde{x}, \widetilde{y}$ of x, y in $\mathbb{R}$ such that $\widetilde{x} < \widetilde{y} < \widetilde{x} + 1$. Since $\widetilde{\varphi}$ is non-decreasing and commutes with integral translations, we have either $\widetilde{\varphi}(\widetilde{y}) = \widetilde{\varphi}(\widetilde{x})$ or $\widetilde{\varphi}(\widetilde{y}) = \widetilde{\varphi}(\widetilde{x} + 1)$. In any case, $\widetilde{\varphi}$ is constant on a non-trivial interval, so there exists an open subset $U \subseteq S^1$ such that $\varphi|_U$ is constant. Let now x be an arbitrary point of S^1. By minimality of ρ_2 there exists $g \in \Gamma$ such that $\rho_2(g)^{-1}(x) \in U$, and consequently $V := \rho_2(g)(U)$ is an open neighborhood of x. Now

$$\varphi|_V = (\varphi\rho_2(g))|_U \circ \rho_2(g)^{-1}|_V = (\rho_1(g)\varphi)|_U \circ \rho_2(g)^{-1}|_V,$$

whence φ is locally constant. It follows that φ is constant, and this contradicts the fact that φ is surjective. □

We have now established both of the theorems mentioned in the introduction.

5. Variations and examples

5.A. **Classical examples.** Let us spell out a few special immediate consequences of Ghys' theorem. We start with the case where $\Gamma = \mathbb{Z}$. In this case a circle action $\rho : \Gamma \to \mathrm{Homeo}^+(S^1)$ is given by a single invertible transformation $\rho(1) \in \mathrm{Homeo}^+(S^1)$. The action lifts to $\widetilde{\rho} : \mathbb{Z} \to \widetilde{H}$ and following Poincaré we define its rotation number as

$$R(\rho) := [\widetilde{\rho}^* T(1)] \in \mathbb{R}/\mathbb{Z}.$$

The fact that any $\mathbb{Z}$ action lifts is illustrated by $\rho^*(\mathrm{eu}) = 0 \in H^2(\mathbb{Z};\mathbb{Z})$. Thus, the unbounded Euler class cannot give any information for $\mathbb{Z}$-actions. The case of the bounded Euler class is much more interesting:

Corollary 5.1 (Poincaré)**.** *For circle actions $\rho_1, \rho_2 : \mathbb{Z} \to \mathrm{Homeo}^+(S^1)$ the following are equivalent:*

(i) ρ_1 and ρ_2 are semi-conjugate.
(ii) $\rho_1^*\mathrm{eu}_b = \rho_2^*\mathrm{eu}_b$.
(iii) $R(\rho_1) = R(\rho_2)$.

In particular, Poincaré's rotation number is a complete semi-conjugacy invariant for circle actions of $\mathbb{Z}$ (and a complete conjugacy invariant for minimal $\mathbb{Z}$-actions).

Proof. The equivalence (i)⇔(ii) is a special case of Theorem 1.2. For $j = 1, 2$ we have

$$\rho_j^*\mathrm{eu}_b = \widetilde{\rho_j}^* p^*\mathrm{eu}_b = -\widetilde{\rho_j}^*[dT_{\mathbb{Z}}] = -[d\widetilde{\rho_j}^* T_{\mathbb{Z}}],$$

whence (ii) is equivalent to $[d(\widetilde{\rho_1}^* T_{\mathbb{Z}} - \widetilde{\rho_2}^* T_{\mathbb{Z}})] = 0$. This in turn means that there exists a homomorphism $f \in \mathrm{Hom}(\mathbb{Z},\mathbb{Z})$ such that the quasimorphism $\widetilde{\rho_1}^* T_{\mathbb{Z}} - \widetilde{\rho_2}^* T_{\mathbb{Z}} - f$ is bounded. Now using the fact that a homogeneous quasimorphism is bounded if and only if its homogeneization is trivial we see that the latter condition is equivalent to

$$\widetilde{\rho_1}^* T - \widetilde{\rho_2}^* T = f \in \mathrm{Hom}(\mathbb{Z},\mathbb{Z}).$$

Since two homogeneous functions on $\mathbb{Z}$ agree iff they agree on 1 we see that this condition is equivalent to

$$\widetilde{\rho_1}^* T(1) - \widetilde{\rho_2}^* T(1) \in \mathbb{Z},$$

i.e. $R(\rho_1) = R(\rho_2)$. □

Given $\alpha \in \mathbb{R}/\mathbb{Z}$ we denote by $R_\alpha \in \mathrm{Homeo}^+(S^1)$ the rotation by α. We denote by $\mathrm{Rot}(S^1) \cong \mathbb{R}/\mathbb{Z}$ the subgroup of $\mathrm{Homeo}^+(S^1)$ given by rotations. Note that the $\mathbb{Z}$-action ρ with $\rho(1) = R_\alpha$ has rotation number α. In particular, every $\mathbb{Z}$-action is semi-conjugate to a rotation action. This is more generally true in the following context (see [HT75, Cal07]):

Corollary 5.2 (Hirsch-Thurston)**.** *Every circle action $\rho : \Gamma \to \mathrm{Homeo}^+(S^1)$ of an amenable group is semi-conjugate to an action by rotations, i.e. a homomorphism $\rho' : \Gamma \to \mathrm{Rot}(S^1) < \mathrm{Homeo}^+(S^1)$.*

Proof. By a classical result of Trauber (see [Mon01, Cor. 7.5.11]) the bounded cohomology of Γ with *real* coefficient vanishes. Thus the connecting homomorphism

$$\delta : H^1(\Gamma; \mathbb{R}/\mathbb{Z}) \to H^2_b(\Gamma; \mathbb{Z})$$

of the Gersten exact sequence (see [Mon01, Prop. 8.2.12]) is an isomorphism. Let $\alpha := \rho^*\mathrm{eu}_b \in H^2_b(\Gamma; \mathbb{Z})$ and $\beta := \delta^{-1}(\alpha)$. Then under the isomorphism $H^1(\Gamma; \mathbb{R}/\mathbb{Z}) \cong \mathrm{Hom}(\Gamma, \mathbb{R}/\mathbb{Z}) = \mathrm{Hom}(\Gamma, \mathrm{Rot}(S^1))$ the class β corresponds to a homomorphism $\rho' : \Gamma \to \mathrm{Rot}(S^1)$. Now a standard diagram chase shows that $(\rho')^*\mathrm{eu}_b = \delta(\beta) = \rho^*\mathrm{eu}_b$, whence ρ and ρ' are semi-conjugate. $\square$

5.B. **Alternative characterizations of semi-conjugacy.** In this subsection we present two alternative characterisations of semi-conjugacy, which are respectively due to Maxime Wolff [Wol] and the first author [Buc08]. Both of these characterisations have the advantage that they only require one semi-conjugacy with certain additional properties rather than two semi-conjugacies as in Definition 2.5; on the downside, it is not obvious a priori that either of these definitions actually yields an equivalence relation.

Corollary 5.3. *For circle actions ρ_1 and ρ_2 the following are equivalent:*

(i) There exists a left-semi-conjugacy from ρ_1 to ρ_2 which satisfies Property (4) of Proposition 4.5.
(ii) There exists a left-semi-conjugacy from ρ_1 to ρ_2 which satisfies Property (5) of Proposition 4.5.
(iii) ρ_1 and ρ_2 are semi-conjugate.

Characterization (i) was pointed out to us by Maxime Wolff, and characterization (ii) is taken from [Buc08].

Proof. The implications (i)⇒(ii)⇒(iii) of the corollary follow from the implications (4)⇒(5)⇒(6) in Proposition 4.5 and Part (i) of Theorem 4.3. Conversely assume that (iii) holds and that ρ_1 is left-semi-conjugate to ρ_2 via φ. If φ is non-constant then (i) and (ii) hold by the implications (2)⇒(4)⇒(5) of Proposition 4.5. Now assume, on the other hand, that φ is constant. Then the image of φ is a fixed point $[x_1]$ for ρ_1. According to Part (ii) of Theorem 4.3 there is also a fixed point $[x_2]$ of ρ_2. Let $x_1, x_2 \in \mathbb{R}$ be lifts of x_1 and x_2 respectively. Then there exists a unique good lift $\widetilde{\varphi}$ of φ such that $\widetilde{\varphi}([x_2, x_2 + 1)) = \{x_1\}$, and this lift clearly satisfies Property (4) of Proposition 4.5. This shows that (iii) implies (i) and finishes the proof. $\square$

5.C. **Real bounded Euler class.** In many applications, computations in integral bounded cohomology are difficult, and thus one relies on real bounded cohomology. The image of eu_b in $H^2_b(H;\mathbb{R})$ under the change of coefficient map $H^2_b(H;\mathbb{Z}) \to H^2_b(H;\mathbb{R})$ is called the *real bounded Euler class* and denoted $\mathrm{eu}_b^{\mathbb{R}}$. Corollary 4.1 has the following real counterpart:

Corollary 5.4. *Let $\rho : \Gamma \to \mathrm{Homeo}^+(S^1)$ be a circle action with $\rho^*\mathrm{eu}_b^{\mathbb{R}} = 0$. Then $\rho([\Gamma,\Gamma])$ fixes a point on S^1.*

Proof. Since $\rho^*\mathrm{eu}_b^{\mathbb{R}} = 0$ we can argue as in the proof of Corollary 5.2 and prove that ρ is semi-conjugate to an action $\rho' : \Gamma \to \mathrm{Rot}(S^1) < \mathrm{Homeo}^+(S^1)$. In particular, $\rho|_{[\Gamma,\Gamma]}$ is semi-conjugate to $\rho'|_{[\Gamma,\Gamma]}$. Now since $\mathrm{Rot}(S^1)$ is abelian, ρ' vanishes on $[\Gamma,\Gamma]$. It follows that $(\rho|_{[\Gamma,\Gamma]})^*\mathrm{eu}_b = (\rho'|_{[\Gamma,\Gamma]})^*\mathrm{eu}_b = 0$, whence $\rho([\Gamma,\Gamma])$ fixes a point on S^1 by Corollary 4.1. □

5.D. **Regularity of semi-conjugacies.** The following example shows that semi-conjugacies may not be chosen to be continuous in general.

Example 5.5. Let ρ_1 be the action of $\mathbb{Z}$ given by sending the generator 1 to the rotation by π. Let ρ_2 be an action of $\mathbb{Z}$ with rotation number $\frac{1}{2}$ for which $\rho(2)$ has precisely two fixed points. For example, the generator could be sent to fixed point free lift of a parabolic isometry to the double cover of $S^1 = \partial\mathbb{H}^2$. Both actions have rotation number $1/2$, so that they are semi-conjugate, say, ρ_1 is right-semi-conjugate to ρ_2 via $\varphi : S^1 \to S^1$. By definition, φ sends orbits for the ρ_1 action to orbits for the ρ_2 action. Now all ρ_1 orbits have precisely two points, while only one ρ_2 orbit has two points (and the other orbits have infinite order). It follows that the image of φ is equal to the unique ρ_2 orbit consisting of two points, hence the map φ cannot be continuous. Even worse, the semi-conjugacy $\varphi' : S^1 \to S^1$ in the opposite direction, i.e. from ρ_1 to ρ_2 cannot be chosen continuous either. Indeed, let $\{x_1, x_2\}$ be the unique ρ_2-orbit containing two points. Then φ' has to send x_1 and x_2 to a pair of antipodal points $y, \overline{y}$. Now restrict to the index two subgroup $2\mathbb{Z} < \mathbb{Z}$ and look at the restricted orbits: The restricted ρ_1-action is trivial, so orbits for the restricted ρ_2 action have to be sent to points. But x_1 and x_2 are accumulation points of the same restricted ρ_2-orbit, which is all mapped to a point z. Then z cannot be both equal to y and $\overline{y}$, so that φ' is not continuous.

Things get better if we replace continuity with the following less demanding notion:

Definition 5.6. Let $\varphi\colon S^1 \to S^1$ be an non-decreasing degree one map. Then φ is called *upper semicontinuous* if it admits an upper semicontinuous good lift $\widetilde{\varphi}\colon \mathbb{R} \to \mathbb{R}$.

Indeed we can show:

Lemma 5.7. *If a circle action $\rho_1 : \Gamma \to H$ is left-semi-conjugate to a circle action $\rho_2 : \Gamma \to H$, then it is left-semi-conjugate to ρ_2 via an upper semicontinuous map $\varphi'\colon S^1 \to S^1$.*

Proof. Let φ be an arbitrary left-semi-conjugacy from ρ_1 to ρ_2. If φ is constant then there is nothing to show, hence we may assume that φ is non-constant. We then define $\widetilde{\varphi}$, $\widetilde{\rho_1}$, $\widetilde{\rho_2}$ and n_γ as in the beginning of Subsection 4.D and also define a new function $\widetilde{\varphi}' : \mathbb{R} \to \mathbb{R}$ by

$$\widetilde{\varphi}'(x); = \sup\{\widetilde{\varphi}(y) \,|\, y < x\}.$$

Since $\widetilde{\varphi}'$ is non-decreasing and commutes with integral translations, it is the good lift of a non-decreasing degree one map $\varphi' \colon S^1 \to S^1$. We claim that φ' is a left-semi-conjugacy from ρ_1 to ρ_2.

In order to prove our claim we fix $\gamma \in \Gamma$ and abbreviate $g_j := \widetilde{\rho}_j(\gamma) \in \widetilde{H}$ for $j = 1, 2$. By (4.2) we have for every $y \in \mathbb{R}$,

$$g_1\widetilde{\varphi}(y) = \widetilde{\varphi}(g_2(y)) + n_\gamma(y). \tag{5.1}$$

By the implication (2)⇒(3) in Proposition 4.5, there exists an integer $m \in \mathbb{N}$ such that $n_\gamma \equiv m$. Now for every $x \in \mathbb{R}$ we have

$$\begin{aligned} g_1\widetilde{\varphi}'(x) &= g_1(\sup\{\widetilde{\varphi}(y) \mid y < x\}) = \sup\{g_1\widetilde{\varphi}(y) \mid y < x\} \\ &= \sup\{\widetilde{\varphi}(g_2(y)) + m \mid y < x\} = \sup\{\widetilde{\varphi}(y) \mid y < g_2(x)\} + m \\ &= \widetilde{\varphi}'(g_2(x)) + m, \end{aligned}$$

which implies that $\rho_1(\gamma)\varphi' = \varphi'\rho_2(\gamma)$, and concludes the proof. □

Thus in the definition of semi-conjugacy we could have added the requirement that the semi-conjugacies in question are upper semi-continuous.

Appendix A. The action of the double cover of H on the circle

Let $\overline{H}$ be the unique double cover of $H := \mathrm{Homeo}^+(S^1)$ and $p : \overline{H} \to H$ denote the canonical projection. The double cover $\overline{H}$ acts on the circle by preserving antipodal points. The aim of this appendix is twofold: On the one hand, we describe all cocycles obtained as $\overline{H}$-invariant functions $(S^1)^3 \to \mathbb{Z}$ and relate them to the cohomology class $p^*(\mathrm{eu}_b) \in H^2_b(\overline{H}, \mathbb{Z})$. On the other hand, we establish a fixed point theorem (Theorem A.5) which is stronger then its analogue for H (Corollary 4.2) since in this case a fixed point is not only equivalent to the vanishing of the pullback of the Euler class, but further to the vanishing of the pullback of a particular cocycle.

Non-degenerate orbits of $\overline{H}$ acting on $(S^1)^{n+1}$. For every point $x \in S^1$, we denote by $\overline{x}$ its antipodal point.

(n=0) The action of $\overline{H}$ on one factor S^1 clearly has exactly one orbit.

(n=1) The action of $\overline{H}$ on two factors $(S^1)^2$ has two non degenerate orbits: If $x, y \in S^1$ are such that $y \neq x, \overline{x}$, then the two non degenerate orbits are

$$\overline{H} \cdot (x, y) \quad \text{and} \quad \overline{H} \cdot (y, x).$$

(There are further two degenerate orbits: $H \cdot (x, \overline{x})$ and $H \cdot (x, x)$.)

(n=2) The action of $\overline{H}$ on three factors $(S^1)^3$ has eight non-degenerate orbits. Choose distinct points $x_0, x_1, x_2 \in S^1$ and suppose that $(x_0, x_1, x_2, \overline{x_0})$ is a positively oriented triple. Then the orbits are given as follows, where the first six orbits are parametrized by the symmetric group $\mathrm{Sym}(3)$:

$$\overline{H} \cdot (x_{\sigma(0)}, x_{\sigma(1)}, x_{\sigma(2)}), \quad \text{for every } \sigma \in \mathrm{Sym}(3),$$
$$\overline{H} \cdot (x_0, x_2, \overline{x_1}) \quad \text{and} \quad \overline{H} \cdot (x_0, \overline{x_1}, x_2).$$

Cocycles on S^1. Given an $\overline{H}$-invariant n-cochain f defined only on non-degenerate orbits and with values in $\mathbb{Z}$, there is a simple trick (see [BM12]) to obtain a cochain defined on all orbits and taking the same values as f. Indeed, define $f(x_0, \ldots, x_n)$ as follows: Intuitively, we want to move $x_n, \ldots, x_0$ (in this order) a very small amount in the positive direction to make the $(n+1)$-tuple non-degenerate. More precisely, if x_n is equal to x_i or $\overline{x_i}$ for $i \neq n$, replace x_n by a point x_n^+ such that $(x_n, x_n^+, \overline{x_n})$ is positively oriented and no x_i or $\overline{x_i}$, for $i \neq n$, lies in the positive direction between x_n and x_n^+. Continue inductively for all x_i's and set $f(x_0, \ldots, x_n) := f(x_0^+, \ldots, x_n^+)$.

We keep now to the described procedure of extending a cochain defined only on non-degenerate orbits to all orbits. In particular, the cocycle and coboundary conditions need only to be verified on non-degenerate orbits. A straightforward computation gives:

Lemma A.1. *Let $b : (S^1)^2 \to \mathbb{Z}$ be a 1-cochain given by its two values $b(x, y) = w_+$ and $b(y, x) = w_-$, where $x, y \in S^1$ are chosen so that $(x, y, \overline{x})$*

is a positively oriented triple. Let $z \in S^1$ such that $(x, y, z, \overline{x})$ is a positively oriented 4-tuple. Then

$$\begin{aligned} \delta b(x,y,z) &= w_+, \\ \delta b(x,z,y) &= w_-, \\ \delta b(x,z,\overline{y}) &= 2w_+ - w_-, \\ \delta b(x,\overline{y},z) &= 2w_- - w_+. \end{aligned}$$

Given an $\overline{H}$-invariant 2-cochain $f : (S^1)^3 \to \mathbb{R}$ and $\sigma \in S_3$ we denote by a_σ, b^+ and b_- the values it takes on the corresponding non-degenerate orbits as listed above. Observe that again, the classical orientation cocycle Or attributing the value $+1$ on positively oriented triples, and -1 on negatively oriented triples, thus with

$$b_+ = 1, b_- = -1, \quad \text{and} \quad a_\sigma = \operatorname{sign}(\sigma)$$

is indeed a cocycle. (However with our convention of perturbing degenerate configuration of points, it will not agree with the orientation cocycle defined previously on degenerate orbits.) According to Lemma A.1, the orientation cocycle gives rise to a nontrivial cohomology class in $H^2(\overline{H}, S^1; \mathbb{Z})$.

Lemma A.2. *Let as above a_σ, b^+ and b_- denote the values on the non-degenerate $\overline{H}$-orbits of an $\overline{H}$-invariant cochain $f : (S^1)^3 \to \mathbb{R}$. Then f is a cocycle if and only if*

$$\begin{aligned} a_{\mathrm{Id}} = a_{(0\ 1\ 2)} &= a_{(0\ 2\ 1)}, \\ a_{(0\ 1)} = a_{(0\ 2)} &= a_{(1\ 2)}, \\ a_{\mathrm{Id}} + a_{(0\ 1)} &= b_+ + b_-. \end{aligned}$$

Proof. Elementary case by case consideration of configurations of four points on the circle shows that the five linearly independent conditions are necessary. Since by Lemma A.1, the space of coboundaries is 2-dimensional, it follows that $H^2(\overline{H}, S^1; \mathbb{Z})$ is at most $8-5-2 = 1$-dimensional. Since further $H^2(\overline{H}, S^1; \mathbb{Z})$ is nontrivial, we have found all relations. □

Let us simplify the notation and write $a_+ = a_{\mathrm{Id}}$ and $a_- = a_{(0\ 1)}$. Thus an $\overline{H}$-invariant cocycle is given by the four values a_+, a_-, b_+, b_- subject to the relation $a_+ + a_- = b_+ + b_-$. Various cocycles appear in the literature, among them also the cocycle $p_2^*(c)$, where c is the H-invariant cocycle on S^1 defined in Section 3.D and $p_2 : S^1 \to S^1$ is the double cover given by identifying antipodal points. We summarize their relations in the following table:

	a_+	a_-	b_+	b_-	$H^2(\overline{H}, S^1; \mathbb{Z})$
δb	w_+	w_-	$2w_+ - w_-$	$2w_- - w_+$	0
Or	1	-1	1	-1	$-2[E_{Sull}]$
$p_2^*(\mathrm{Or})$	1	-1	-1	1	$-4\,[E_{Sull}]$
$p_2^*(c)$	1	0	0	1	$-2\,[E_{Sull}]$
E_{Sull}	0	0	1	-1	$[E_{Sull}]$

The Sullivan cocycle can be described geometrically as follows: it is nonzero on a non degenerate triple (x, y, z) if and only if the triple contains 0 in (the interior of) its convex hull and in that case it is $+1$ or -1 depending on the orientation of the triple. This cocycle was found by Sullivan as an explicit representative for the Euler class of flat oriented $\mathbb{R}^2$-vector bundles. Observe that the Sullivan cocycle is not H-invariant, but only $\overline{H}$-invariant.

The Sullivan cocycle. One quite attractive aspect of the Sullivan cocycle and its higher-dimensional analoga is that they detect small subsets of spheres. Here a subset of a sphere is called *small* if its spherical convex hull is not the whole sphere. In the case of S^1 a set $X \subset S^1$ is small if and only if it is contained in a half-open half-circle.

Proposition A.3. *Let $X \subset S^1$ be any subset. Then E_{Sull} vanishes on X^3 if and only if X is small.*

Proof. If $X \subset S^1$ is a small subset then no three points in X ever contain 0 in their convex hull, so that E_{Sull} vanishes on X^3.

Conversely, suppose that E_{Sull} vanishes on X^3. View X as a subset of $\mathbb{R}^2$ and consider its convex hull in $\mathbb{R}^2$. By Caratheodory's Theorem, if 0 is contained in the convex hull of X, then there exists $x_0, x_1, x_2 \in X$ such that 0 belongs to the convex hull of x_0, x_1, x_2 and hence $E_{Sull}(x_0, x_1, x_2) \neq 0$, which is impossible. If 0 is not on the boundary of the convex hull, then by Hahn-Banach there exists a hyperplane separating 0 and the convex hull of X, so X is in particular contained in the intersection of S^1 with the (appropriate) half plane delimited by the hyperplane. If 0 is in the boundary of the convex hull, then by the supporting hyperplane theorem, there exists a hyperplane through 0 so that the convex hull of X is contained in one half of one closed half space delimited by that hyperplane. We are almost done, except that we need to exclude the case that X is contained in one closed half-circle, but is not contained in a half-open half-circle. Suppose that x and $\overline{x}$ belong to X. Then $E_{Sull}(x, \overline{x}, x) = E_{Sull}(x, \overline{x}^+, x^+) = 1$, where the points $\overline{x}^+, x^+ \in S^1$ are very small perturbations of $\overline{x}, x$ in the positive direction. $\square$

Note that the same proof holds also for the higher dimensional generalization of the Sullivan cocycle.

The cohomology class $[E_{Sull}]$. Given a basepoint $x \in S^1$ we obtain a cocycle $E^x_{Sull} : \overline{H}^3 \to \mathbb{Z}$ by pullback along the corresponding orbit map, i.e.

$$E^x_{Sull}(g_0, g_1, g_2) := E_{Sull}(g_0.x, g_1.x, g_2.x).$$

This cocycle determines a class in the group cohomology $H^2(\overline{H}; \mathbb{Z})$; since E^x_{Sull} is bounded, it also determines a class in the bounded group cohomology $H^2_b(\overline{H}; \mathbb{Z})$. In order to provide an interpretation of these classes, we consider the following diagram of central extensions

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "i"] \arrow[d, "\cdot 2"] & \widetilde{H} \arrow[r, "p"] \arrow[d, "\mathrm{Id}"] & \overline{H} \arrow[r] \arrow[d, "p"] & \{e\} \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, "i"] & \widetilde{H} \arrow[r, "p"] & H \arrow[r] & \{e\} \end{tikzcd}$$

where $\widetilde{H}$ denotes, as before, the common universal cover of H and $\overline{H}$ which consists of homeomorphisms of the real line commuting with integral translations. The cohomology class in $H^2(\overline{H}, \mathbb{Z})$ corresponding to the central extension in the top row is twice the pullback by p of the class $\mathrm{eu} \in H^2(H, \mathbb{Z})$ hence can be represented by $-E_{Sull}$.

By Lemma 3.1 this yields the following interpretation of $[E_{Sull}]$ as an obstruction class: Given a group Γ, the S^1-action associated with a homomorphism $\rho : \Gamma \to \overline{H}$ lifts to an action of Γ on the real line if and only if $\rho^*[E_{Sull}] = 0 \in H^2(\Gamma, \mathbb{Z})$.

We now turn to an interpretation of the bounded class defined by E_{Sull}. It turns out that the case of the bounded Sullivan cocycle in degree 2 is very particular since the vanishing of the cohomology class is equivalent to the vanishing of the cocycle:

Proposition A.4. *Let Γ be a group and $\rho : \Gamma \to \overline{H}$ be any homomorphism, then $\rho^*[E^x_{Sull}]_b = 0 \in H^2_b(\Gamma, \mathbb{Z})$ if and only if $\rho^*(E^x_{Sull}) = 0$ for any base point $x \in S^1$.*

Proof. One direction is trivial. For the other direction, suppose that $\rho^* E^x_{\mathrm{Sull}} = \delta b$ for some $x \in S^1$ and a Γ-invariant bounded cochain $b : \Gamma^2 \to \mathbb{Z}$. We will show that $b \equiv 0$. Writing out the cocycle equation in a special case yields for all $\gamma \in \Gamma$,

$$\rho^* E^x_{\mathrm{Sull}}(e, \gamma, \gamma^2) = 2b(e, \gamma) - b(e, \gamma^2).$$

This implies in particular $|2b(e, \gamma) - b(e, \gamma^2)| \leq 1$, hence inductively

$$|2^k b(e, \gamma) - b(e, \gamma^{2^k})| \leq 2^k - 1. \tag{A.1}$$

Since b is bounded, we can choose k sufficiently big so that $|b(e, \gamma^{2^k})| \leq 2^{k-1}$. Dividing (A.1) by 2^k we obtain

$$|b(e, \gamma)| \leq \frac{1}{2^k} |b(e, \gamma)^{2^k}| + 1 - \frac{1}{2^k} \leq 1 + \frac{1}{2} - \frac{1}{2^k} < 2.$$

Since b takes integral values, it follows that it takes values in $\{-1, 0, 1\}$. Assume that $b(e, \gamma) = 1$. Then (A.1) yields

$$|2^k - b(e, \gamma^{2^k})| \leq 2^k - 1,$$

hence $b(e, \gamma^{2^k}) = 1$. A similar argument in the negative case shows that for every $\gamma \in \Gamma$, either $b(e, \gamma) = 0$ or $0 \neq b(e, \gamma) = b(e, \gamma^{2^k})$ for every $k > 0$. Thus if $b(e, \gamma) \neq 0$ for some γ, then

$$\begin{aligned} E_{\text{Sull}}(x, \rho(\gamma)x, \rho(\gamma)^2 x) &= 2b(e, \gamma) - b(e, \gamma^2) = b(e, \gamma) = b(e, \gamma^2) \\ &= E_{\text{Sull}}(x, \rho(\gamma)^2 x, \rho(\gamma)^4 x). \end{aligned}$$

This means that there exist $w, x, y, z \in S^1$ such that

$$E_{\text{Sull}}(x, y, z) = E_{\text{Sull}}(x, z, w) \neq 0.$$

By our extension of the Sullivan cocycle to degenerate orbits, we can without loss of generality suppose that both triples (x, y, z) and (x, z, w) are non degenerate. Since their evaluations on the Sullivan cocycle agree both triples contain 0 in the interior of their convex hull and have the same orientation. This is impossible. $\square$

For the Sullivan cocycle we now obtain the following stronger version of Corollary 4.2:

Theorem A.5. *Let Γ be a group, $\rho : \Gamma \to \overline{H}$ a homomorphism. Then the following are equivalent:*

(1) $\rho^*[E^x_{Sull}]_b = 0 \in H^2_b(\Gamma; \mathbb{Z})$;
(2) ρ lifts to a homomorphism $\widetilde{\rho} : \Gamma \to \widetilde{H}$ and $\widetilde{\rho}(\Gamma)$ has a fixed point in $\mathbb{R}$.
(3) $\rho(\Gamma)$ fixes a point in S^1.
(4) Every $\rho(\Gamma)$-orbit on S^1 is small.
(5) There exists a small $\rho(\Gamma)$-orbit in S^1.
(6) For every $x \in S^1$, $\rho^* E^x_{Sull} = 0$.
(7) There exists $x \in S^1$ such that $\rho^* E^x_{Sull} = 0$.

Proof. We summarize the shown implications in the following diagram:

$$\begin{tikzcd}
(4) \arrow[r, Leftrightarrow, "\text{Prop} A.3"] \arrow[d, Rightarrow, "\text{trivial}"'] & (6) \arrow[r, Leftrightarrow, "\text{Prop} A.4"] \arrow[d, Rightarrow, "\text{trivial}"'] & (1) \\
(5) \arrow[r, Leftrightarrow, "\text{Prop} A.3"'] & (7) \arrow[ur, Rightarrow, "\text{trivial}"'] & (3). \arrow[l, Rightarrow, "\text{trivial}"]
\end{tikzcd}$$

The remaining equivalence between (1), (2) and (3) admits the same proof as the equivalence between (i), (ii) and (iii) in Corollary 4.2. $\square$

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