A multiplicity result for double singularly perturbed elliptic systems

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To Professor Andrzej Granas

Abstract. We show that the number of low-energy solutions of a double singularly perturbed Schrödinger–Maxwell-type system on a smooth three-dimensional manifold (M, g) depends on the topological properties of the manifold. The result is obtained via the Lusternik–Schnirelmann category theory.

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1. Introduction

Given real numbers $q > 0$, $\omega > 0$ and $p > 4$, we consider the following system of Schrödinger–Maxwell type on a smooth manifold M endowed with a Riemannian metric g:

$$\begin{cases} -\varepsilon^2\Delta_g u + u + \omega uv = |u|^{p-2}u & \text{in } M, \\ -\varepsilon^2\Delta_g v + v = qu^2 & \text{in } M, \\ u > 0 & \text{in } M, \end{cases} \tag{1.1}$$

where Δ_g is the Laplace–Beltrami operator on M.

We want to prove that when the parameter ε is sufficiently small, there are many low-energy solutions of (1.1). In particular, the number of solutions of (1.1) is related to the topology of the manifold M. We suppose, without loss of generality, that the manifold M is isometrically embedded in $\mathbb{R}^n$ for some n.

Here there is a competition between the two equations, since both share the same singular perturbation of order ε^2. In [10, 11] we dealt with a similar system where only the first equation had a singular perturbation. In this case, the second equation disappears in the limit. In Section 2.1 we write the limit problem taking care of the competition, and we find the model solution for system (1.1).

A problem similar to (1.1), namely the Schrödinger–Newton system, has been studied from a dynamical point of view in [9]. Also in that paper the two equations have the ε^2 singular perturbation.

Recently, Schrödinger–Maxwell-type systems received considerable attention from the mathematical community, we refer the reader, e.g., to [1, 2, 5, 6, 7, 8, 15, 16]. A special case of Schrödinger–Maxwell-type systems, namely when the system is set in $\mathbb{R}^3$, takes the name of Schrödinger–Poisson–Slater equation and it arises in Slater approximation of the Hartree–Fock model. We want here to especially mention some result of the existence of solutions, i.e., [2, 7, 12, 15, 17], since the limit problem (1.1) is a Schrödinger–Poisson–Slater-type equation. (For a more exhaustive discussion on Schrödinger–Poisson–Slater equations and on the physical models that lead to this equation we refer the reader to [13, 14] and the references therein.)

Our main result is the following.

Theorem 1.1. *Let $4 < p < 6$. For ε small enough there exist at least* $\operatorname{cat}(M)$ *positive solutions of* (1.1).

Here we recall the definition of the Lusternik–Schnirelmann category of a set.

Definition 1.2. Let X be a topological space and consider a closed subset $A \subset X$. We say that A has category k relative to X ($\operatorname{cat}_X A = k$) if A is covered by k closed sets A_j, $j = 1, \dots, k$, which are contractible in X, and k is the minimum integer with this property. We simply denote $\operatorname{cat} X = \operatorname{cat}_X X$.

2. Preliminary results

We endow $H^1(M)$ and $L^p(M)$ with the following equivalent norms:

$$\|u\|_\varepsilon^2 = \frac{1}{\varepsilon^3}\int_M \varepsilon^2|\nabla u|^2 + u^2\, d\mu_g, \quad |u|_{\varepsilon,p}^p = \frac{1}{\varepsilon^3}\int_M |u|^p\, d\mu_g,$$
$$\|u\|_{H^1}^2 = \int_M |\nabla u|^2 + u^2\, d\mu_g, \qquad |u|_p^p = \int_M |u|^p\, d\mu_g$$

and we refer to H_ε (resp., L_ε^p) as the space $H^1(M)$ (resp., L_ε^p) endowed with the $\|\cdot\|_\varepsilon$ norm (resp., $|\cdot|_{\varepsilon,p}$ norm). Obviously, we refer to the scalar product on H_ε as

$$\langle u, v\rangle_\varepsilon = \frac{1}{\varepsilon^3}\int_M \varepsilon^2 \nabla u \nabla v + uv\, d\mu_g.$$

Following an idea by Benci and Fortunato [5], for any ε we introduce the map $\psi_\varepsilon : H^1(M) \to H^1(M)$ which is the solution of the equation

$$-\varepsilon^2 \Delta_g v + v = qu^2 \quad \text{in } M. \tag{2.1}$$

Lemma 2.1. *The map $\psi : H^1(M) \to H^1(M)$ is of class C^2 with derivatives $\psi'(u)$ and $\psi''(u)$ which satisfy*

$$-\varepsilon^2 \Delta_g \psi_\varepsilon'(u)[\varphi] + \psi_\varepsilon'(u)[\varphi] = 2qu\varphi, \tag{2.2}$$

$$-\varepsilon^2 \Delta_g \psi_\varepsilon''(u)[\varphi_1, \varphi_2] + \psi_\varepsilon''(u)[\varphi_1, \varphi_2] = 2q\varphi_1\varphi_2 \tag{2.3}$$

for any $\varphi, \varphi_1, \varphi_2 \in H^1(M)$. Moreover, $\psi_\varepsilon(u) \geq 0$.

Proof. The proof is standard. □

Remark 2.2. We observe that by simple computation, for any $t > 0$ we have

$$\psi_\varepsilon(tu) = t^2 \psi_\varepsilon(u).$$

In fact, if $\psi_\varepsilon(u)$ solves (2.1), multiplying both sides of (2.1) by t^2 we get the claim.

Lemma 2.3. *The map $T_\varepsilon : H_\varepsilon \to \mathbb{R}$ given by*

$$T_\varepsilon(u) = \int_M u^2 \psi_\varepsilon(u)\, d\mu_g$$

is a C^2 map and its first derivative is

$$T_\varepsilon'(u)[\varphi] = 4 \int_M \varphi u \psi_\varepsilon(u)\, d\mu_g.$$

Proof. The regularity is standard. The first derivative is

$$T_\varepsilon'(u)[\varphi] = 2 \int u\varphi\psi_\varepsilon(u) + \int u^2 \psi_\varepsilon'(u)[\varphi].$$

By (2.1) and (2.2) we have

$$\begin{aligned} 2 \int u\varphi\psi_\varepsilon(u) &= \frac{1}{q}\left(-\varepsilon^2 \int \Delta\big(\psi_\varepsilon'(u)[\varphi]\big)\psi_\varepsilon(u) + \int \psi_\varepsilon'(u)[\varphi]\psi_\varepsilon(u)\right) \\ &= \frac{1}{q}\left(-\varepsilon^2 \int \psi_\varepsilon'(u)[\varphi]\Delta\psi_\varepsilon(u) + \int \psi_\varepsilon'(u)[\varphi]\psi_\varepsilon(u)\right) \\ &= \int \psi_\varepsilon'(u)[\varphi]u^2, \end{aligned}$$

and the claim follows. □

At this point, we consider the following functional $I_\varepsilon \in C^2(H_\varepsilon, \mathbb{R})$:

$$I_\varepsilon(u) = \frac{1}{2}\|u\|_\varepsilon^2 + \frac{\omega}{4} G_\varepsilon(u) - \frac{1}{p}\left|u^+\right|_{\varepsilon,p}^p, \tag{2.4}$$

where

$$G_\varepsilon(u) = \frac{1}{\varepsilon^3}\int_\Omega u^2\psi_\varepsilon(u)\, dx = \frac{1}{\varepsilon^3} T_\varepsilon(u).$$

By Lemma 2.3 we have

$$\begin{aligned} I_\varepsilon'(u)[\varphi] &= \frac{1}{\varepsilon^3}\int_\Omega \varepsilon^2 \nabla u \nabla \varphi + u\varphi + \omega u \psi_\varepsilon(u)\varphi - \left(u^+\right)^{p-1}\varphi, \\ I_\varepsilon'(u)[u] &= \|u\|_\varepsilon^2 + \omega G_\varepsilon(u) - \left|u^+\right|_{\varepsilon,p}^p. \end{aligned}$$

Then if u is a critical point of the functional I_ε, the pair of positive functions $(u, \psi_\varepsilon(u))$ is a solution of (1.1).

We define the following Nehari set:

$$\mathcal{N}_\varepsilon = \left\{ u \in H^1(M) \setminus 0 : N_\varepsilon(u) := I'_\varepsilon(u)[u] = 0 \right\}.$$

The Nehari set has the following properties (for a complete proof see [11]).

Lemma 2.4. *If $p > 4$, $\mathcal{N}_\varepsilon$ is a C^2 manifold and $\inf_{\mathcal{N}_\varepsilon} \|u\|_\varepsilon > 0$.*

If $u \in \mathcal{N}_\varepsilon$, then

$$\begin{aligned} I_\varepsilon(u) &= \left(\frac{1}{2} - \frac{1}{p}\right) \|u\|_\varepsilon^2 + \omega \left(\frac{1}{4} - \frac{1}{p}\right) G_\varepsilon(u) \\ &= \left(\frac{1}{2} - \frac{1}{p}\right) \left|u^+\right|_{p,\varepsilon}^p - \frac{\omega}{4} G_\varepsilon(u) \\ &= \frac{1}{4} \|u\|_\varepsilon^2 + \left(\frac{1}{4} - \frac{1}{p}\right) \left|u^+\right|_{p,\varepsilon}^p, \end{aligned} \tag{2.5}$$

and the Palais–Smale condition holds for the functional I_ε on $\mathcal{N}_\varepsilon$.

Finally, for all $w \in H^1(M)$ such that $|w^+|_{\varepsilon,p} = 1$ there exists a unique positive number $t_\varepsilon = t_\varepsilon(w)$ such that $t_\varepsilon(w)w \in \mathcal{N}_\varepsilon$. The number t_ε is the critical point of the function

$$H(t) = I_\varepsilon(tw) = \frac{1}{2} t^2 \|w\|_\varepsilon^2 + \frac{t^4}{4} \omega \, G_\varepsilon(w) - \frac{t^p}{p}.$$

2.1. The limit problem

Consider the following problem in the whole space:

$$\begin{cases} -\Delta u + u + \omega u v = |u|^{p-2} u & \text{in } \mathbb{R}^3, \\ -\Delta v + v = q u^2 & \text{in } \mathbb{R}^3, \\ u > 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{2.6}$$

In an analogous way we define the function $\psi_\infty(u)$ as a solution of the second equation and, as before, we can define a functional

$$I_\infty(u) = \frac{1}{2} \|u\|_{H^1}^2 + \frac{\omega}{4} G(u) - \frac{1}{p} \left|u^+\right|_p^p,$$

where $G(u) = \int_{\mathbb{R}^3} u^2 \psi_\infty(u) \, dx$, and the Nehari manifold

$$\mathcal{N}_\infty = \left\{ u \in H^1(\mathbb{R}^3) \setminus 0 : I'_\infty(u)[u] = 0 \right\}.$$

It is easy to prove that (see [12]) the value

$$m_\infty = \inf_{\mathcal{N}_\infty} I_\infty$$

is attained at least by a function U which is a solution of problem (2.6).

We will refer to problem (2.6) as the limit problem. We set

$$U_\varepsilon(x) = U\left(\frac{x}{\varepsilon}\right)$$

and the function U_ε will be the model solution for a solution of problem (1.1).

3. Main ingredient of the proof

We sketch the proof of Theorem 1.1. First of all, it is easy to see that the functional $I_\varepsilon \in C^2$ is bounded from below and it satisfies the Palais–Smale condition on the complete C^2 manifold $\mathcal{N}_\varepsilon$. Then we have, by well-known results, that I_ε has at least $\operatorname{cat} I_\varepsilon^d$ critical points in the sublevel

$$I_\varepsilon^d = \left\{u \in H^1 : I_\varepsilon(u) \le d\right\}.$$

We prove that, for ε and δ small enough, it holds that

$$\operatorname{cat} M \le \operatorname{cat}\left(\mathcal{N}_\varepsilon \cap I_\varepsilon^{m_\infty+\delta}\right), \tag{3.1}$$

where m_∞ has been defined in the previous section.

To get estimate (3.1) we build two continuous operators:

$$\Phi_\varepsilon : M \to \mathcal{N}_\varepsilon \cap I_\varepsilon^{m_\infty+\delta},$$
$$\beta : \mathcal{N}_\varepsilon \cap I_\varepsilon^{m_\infty+\delta} \to M^+,$$

where

$$M^+ = \{x \in \mathbb{R}^n : d(x, M) < R\}$$

with R small enough so that $\operatorname{cat}(M^+) = \operatorname{cat}(M)$. Without loss of generality, we can suppose that $R = r$ is the injectivity radius of M, in order to simplify the notations.

Following an idea in [4], we build the operators Φ_ε and β such that

$$\beta \circ \Phi_\varepsilon : M \to M^+$$

is homotopic to the immersion $i : M \to M^+$. By a classical result on topology (which we summarize in Remark 3.1) we have

$$\operatorname{cat} M \le \operatorname{cat}\left(\mathcal{N}_\varepsilon \cap I_\varepsilon^{m_\infty+\delta}\right),$$

and the first claim of Theorem 1.1 is proved.

Remark 3.1. Let X_1, X_2 and X_3 be topological spaces with X_1 and X_3 homotopically identical. If $g_1 : X_1 \to X_2$ and $g_2 : X_2 \to X_3$ are continuous operators such that $g_2 \circ g_1$ is homotopic to the identity on X_1, then

$$\operatorname{cat} X_1 \le \operatorname{cat} X_2.$$

4. The map Φ_ε

For every $\xi \in M$ we define the function

$$W_{\xi,\varepsilon}(x) = U_\varepsilon\left(\exp_\xi^{-1} x\right) \chi\left(|\exp_\xi^{-1} x|\right), \tag{4.1}$$

where $\chi : \mathbb{R}^+ \to \mathbb{R}^+$ is a cutoff function, that is, $\chi \equiv 1$ for $t \in [0, r/2)$, $\chi \equiv 0$ for $t > r$ and $|\chi'(t)| \le 2/r$. Here $\exp_\xi$ are the normal coordinates centered at $\xi \in M$ and r is the injectivity radius of M. We recall the following well-known expansion of the metric g in normal coordinates:

$$g_{ij}(\varepsilon z) = \delta_{ij} + o(\varepsilon|z|), \qquad |g(\varepsilon z)|^{1/2} = 1 + o(\varepsilon|z|). \tag{4.2}$$

We can define a map

$$\Phi_\varepsilon : M \to \mathcal{N}_\varepsilon,$$
$$\Phi_\varepsilon(\xi) = t_\varepsilon(W_{\xi,\varepsilon})W_{\xi,\varepsilon}.$$

Remark 4.1. We have that $W_{\varepsilon,\xi} \in H^1(M)$ and the following limits hold uniformly with respect to $\xi \in M$:

$$\|W_{\varepsilon,\xi}\|_\varepsilon \to \|U\|_{H^1(\mathbb{R}^3)},$$
$$|W_{\varepsilon,\xi}|_{\varepsilon,t} \to \|U\|_{L^t(\mathbb{R}^3)} \quad \text{for all } 2 \le t \le 6.$$

Lemma 4.2. *We have that*

$$\lim_{\varepsilon\to 0} G_\varepsilon(W_{\varepsilon,\xi}) = G(U) = \int_{\mathbb{R}^3} qU^2\psi(U)\,dx$$

uniformly with respect to $\xi \in M$.

Proof. For the sake of simplicity, we set $\psi_\varepsilon(x) := \psi_\varepsilon(W_{\varepsilon,\xi})(x)$, and we define

$$\tilde\psi_\varepsilon(z) = \psi_\varepsilon\big(\exp_\xi(\varepsilon z)\big)\,\chi_r\left(|\varepsilon z|\right) \quad \text{for } z \in \mathbb{R}^3.$$

It is easy to see that $\|\tilde\psi_\varepsilon\|_{H^1(\mathbb{R}^3)} \le C\|\psi_\varepsilon\|_\varepsilon$. Moreover, by (2.1),

$$\|\psi_\varepsilon\|_\varepsilon^2 \le C\|W_{\varepsilon,\xi}\|^2_{12/5,\,\varepsilon}\|\psi_\varepsilon\|_\varepsilon \le C\|U\|^2_{12/5}\|\psi_\varepsilon\|_\varepsilon,$$

so $\tilde\psi_\varepsilon$ is bounded in $H^1(\mathbb{R}^3)$ and there exists $\bar\psi \in H^1(\mathbb{R}^3)$ such that, up to extracting a subsequence, $\tilde\psi_{\varepsilon_k} \rightharpoonup \bar\psi$ weakly in $H^1(\mathbb{R}^3)$.

First, we want to prove that $\bar\psi$ is a weak solution of

$$-\Delta v + v = qU^2,$$

that is, $\bar\psi = \psi_\infty(U)$. Given $f \in C_0^\infty(\mathbb{R}^3)$, we have that $\operatorname{spt} f \subset B(0,T)$ for some $T > 0$, so eventually $\operatorname{spt} f \subset B(0, r/\varepsilon_k)$, where $\operatorname{spt} f$ is the support of the function f. Thus we can define

$$f_k(x) := f\left(\frac{1}{\varepsilon_k}\exp_\xi^{-1}(x)\right)$$

and we have that $f_k(x)$ is compactly supported in $B_g(\xi, r)$. By definition of $\psi_\varepsilon(x)$ we have

$$\int_M \varepsilon_k^2\,\nabla_g\psi_{\varepsilon_k}\nabla_g f_k + \psi_{\varepsilon_k} f_k\,d\mu_g = q\int_M W^2_{\varepsilon_k,\xi} f_k\,d\mu_g. \tag{4.3}$$

By the change of variables, $x = \exp_\xi(\varepsilon_k z)$ and by (4.2) we get

$$\begin{aligned}
&\frac{1}{\varepsilon_k^3}\int_M \varepsilon_k^2\,\nabla_g\psi_{\varepsilon_k}\nabla_g f_k + \psi_{\varepsilon_k} f_k\,d\mu_g\\
&\quad= \int_{B(0,r/\varepsilon_k)} \Big[g_{ij}(\varepsilon_k z)\partial_i\tilde\psi_{\varepsilon_k}(z)\partial_j f(z) + \tilde\psi_{\varepsilon_k}(z)f(z)\Big]\,\big|g(\varepsilon_k z)\big|^{1/2}\,dz\\
&\quad= \int_{B(0,T)} \nabla\tilde\psi_{\varepsilon_k}(z)\nabla f(z) + \tilde\psi_{\varepsilon_k}(z)f(z)\,dz + o(\varepsilon_k),
\end{aligned}$$

thus, by the weak convergence of $\tilde{\psi}_\varepsilon$ we get

$$\frac{1}{\varepsilon_k^3}\int_M \varepsilon_k^2\,\nabla_g\psi_{\varepsilon_k}\nabla_g f_k+\psi_{\varepsilon_k}f_k\,d\mu_g\to\int_{\mathbb{R}^3}\nabla\bar{\psi}(z)\nabla f(z)+\bar{\psi}(z)f(z)\,dz \quad (4.4)$$

as $\varepsilon_k\to 0$. In the same way we get

$$\begin{aligned}\frac{q}{\varepsilon_k^3}\int_M W_{\varepsilon_k,\xi}^2 f_k\,d\mu_g &= q\int_{B(0,r/\varepsilon_k)} U^2(z)f(z)\left|g(\varepsilon_k z)\right|^{1/2}dz\\ &= q\int_{\mathbb{R}^3}U^2(z)f(z)\,dz+o(\varepsilon_k)\end{aligned}$$

and

$$\frac{q}{\varepsilon_k^3}\int_M W_{\varepsilon_k,\xi}^2 f_k\,d\mu_g\to q\int_{\mathbb{R}^3}U^2(z)f(z)\,dz. \quad (4.5)$$

By (4.3), (4.4) and (4.5) we get that, for any $f\in C_0^\infty(\mathbb{R}^3)$ it holds that

$$\int_{\mathbb{R}^3}\nabla\bar{\psi}\nabla f+\bar{\psi}f=q\int_{\mathbb{R}^3}U^2 f,$$

which proves that

$$\tilde{\psi}_{\varepsilon_k}\rightharpoonup\psi_\infty(U)\quad\text{weakly in } H^1(\mathbb{R}^3). \quad (4.6)$$

To conclude, again by change of variables we have

$$\begin{aligned}G_{\varepsilon_k}\left(W_{\varepsilon_k,\xi}\right) &= \frac{1}{\varepsilon_k^3}\int_{B_g(\xi,r)} W_{\varepsilon_k,\xi}^2\,\psi\left(W_{\varepsilon_k,\xi}\right)d\mu_g\\ &= \int_{\mathbb{R}^3}U^2(z)\,\chi^2\left(|\varepsilon_k z|\right)\tilde{\psi}_{\varepsilon_k}\left|g(\varepsilon_k z)\right|^{1/2}dz.\end{aligned}$$

Since $U^2\in L^{6/5}(\mathbb{R}^3)$, one has

$$U^2(z)\,\chi^2\left(|\varepsilon_k z|\right)\left|g(\varepsilon_k z)\right|^{1/2}\to U^2(z)\quad\text{strongly in } L^{6/5}(\mathbb{R}^3),$$

that, combined with (4.6), concludes the proof. □

Proposition 4.3. *For all $\varepsilon>0$ the map Φ_ε is continuous. Moreover, for any $\delta>0$ there exists $\varepsilon_0=\varepsilon_0(\delta)$ such that, if $\varepsilon<\varepsilon_0$, then $I_\varepsilon\left(\Phi_\varepsilon(\xi)\right)<m_\infty+\delta$.*

Proof. It is easy to see that Φ_ε is continuous because $t_\varepsilon(w)$ depends continuously on $w\in H_g^1(M)$.

At this point, we prove that $t_\varepsilon(W_{\varepsilon,\xi})\to 1$ uniformly with respect to $\xi\in M$. In fact, by Lemma 2.4, $t_\varepsilon(W_{\varepsilon,\xi})$ is the unique solution of

$$t^2\left\|W_{\varepsilon,\xi}\right\|_\varepsilon^2+\omega\,G_\varepsilon\left(tW_{\varepsilon,\xi}\right)-t^p\left|W_{\varepsilon,\xi}\right|_{\varepsilon,p}^p=0$$

which, in light of Remark 2.2, can by rewritten as

$$\left\|W_{\varepsilon,\xi}\right\|_\varepsilon^2+\omega t^2 G_\varepsilon\left(W_{\varepsilon,\xi}\right)-t^{p-2}\left|W_{\varepsilon,\xi}\right|_{\varepsilon,p}^p=0.$$

By Remark 4.1 and Lemma 4.2 we have the claim. In fact, we recall that, since U is a solution of (2.6), it holds that

$$\|U\|_{H^1(\mathbb{R}^3)}^2+\omega\,G(U)-|U|_{L^p(\mathbb{R}^3)}^p=0.$$

At this point, we have

$$I_\varepsilon\big(t_\varepsilon(W_{\varepsilon,\xi})W_{\varepsilon,\xi}\big)=\left(\frac{1}{2}-\frac{1}{p}\right)\|W_{\varepsilon,\xi}\|_\varepsilon^2 t_\varepsilon^2+\omega\left(\frac{1}{4}-\frac{1}{p}\right)t_\varepsilon^4 G_\varepsilon(W_{\varepsilon,\xi}).$$

Again, by Remark 4.1 and Lemma 4.2 and since $t_\varepsilon(W_{\varepsilon,\xi})\to 1$ we have

$$I_\varepsilon\big(t_\varepsilon(W_{\varepsilon,\xi})W_{\varepsilon,\xi}\big)\to\left(\frac{1}{2}-\frac{1}{p}\right)\|U\|_{H^1(\mathbb{R}^3)}^2+\omega\left(\frac{1}{4}-\frac{1}{p}\right)G(U)=m_\infty,$$

which concludes the proof. □

Remark 4.4. We set $m_\varepsilon=\inf_{\mathcal{N}_\varepsilon} I_\varepsilon$. By Proposition 4.3 we have

$$\limsup_{\varepsilon\to 0} m_\varepsilon\le m_\infty. \tag{4.7}$$

5. The map β

For any $u\in\mathcal{N}_\varepsilon$ we can define a point $\beta(u)\in\mathbb{R}^n$ by

$$\beta(u)=\frac{\int_M x\Gamma(u)\,d\mu_g}{\int_M \Gamma(u)\,d\mu_g},$$

where

$$\Gamma(u)=\left(\frac{1}{2}-\frac{1}{p}\right)\frac{1}{\varepsilon^3}\,|u^+|^p-\frac{\omega}{4}\,\frac{1}{\varepsilon^3}\,u^2\psi_\varepsilon(u).$$

Immediately, since $\int_M \Gamma(u)d\mu_g=I_\varepsilon(u)\ge m_\varepsilon$, the function β is well defined in $\mathcal{N}_\varepsilon$.

Lemma 5.1. *There exists $\alpha>0$ such that $m_\varepsilon\ge\alpha$ for all ε.*

Proof. Take w such that $|w^+|_{\varepsilon,p}=1$ and $t_\varepsilon=t_\varepsilon(w)$ such that $t_\varepsilon w\in\mathcal{N}_\varepsilon$. By (2.5) we have

$$I_\varepsilon(t_\varepsilon w)=\frac{t_\varepsilon^2}{4}\,\|w\|_\varepsilon^2+\left(\frac{1}{4}-\frac{1}{p}\right)t_\varepsilon^p\ge\left(\frac{1}{4}-\frac{1}{p}\right)t_\varepsilon^p.$$

Moreover, we have $\inf_{|w^+|_{\varepsilon,p}=1} t_\varepsilon(w)>0$. In fact, suppose that there exists a sequence w_n such that $|w^+|_{\varepsilon,p}=1$ and $t_\varepsilon(w_n)\to 0$. Since $t_\varepsilon(w_n)w_n\in\mathcal{N}_\varepsilon$ it holds that

$$1=|w_n^+|_{\varepsilon,p}=\frac{1}{t_\varepsilon(w_n)^{p-2}}\left(\|w_n\|_\varepsilon^2+\omega G_\varepsilon(t_\varepsilon(w_n))\right)\ge\frac{1}{t_\varepsilon(w_n)^{p-2}}\,\|w_n\|_\varepsilon^2.$$

Also, there exists a constant $C>0$ which does not depend on ε such that $|w_n^+|_{\varepsilon,p}\le|w_n|_{\varepsilon,p}\le C\|w_n\|_\varepsilon$, so

$$1\ge\frac{1}{Ct_\varepsilon(w_n)^{p-2}}\to+\infty$$

that is a contradiction. This proves that $m_\varepsilon\ge\alpha$ for some $\alpha>0$. □

Now we have to prove that, if $u \in \mathcal{N}_\varepsilon \cap I_\varepsilon^{m_\infty+\delta}$, then $\beta(u) \in M^+$.

Let us consider the following partitions of M. For a given $\varepsilon > 0$ we say that a finite partition

$$\mathcal{P}_\varepsilon = \left\{P_j^\varepsilon\right\}_{j\in\Lambda_\varepsilon}$$

of M is a "good" partition if

(1) for any $j \in \Lambda_\varepsilon$ the set P_j^ε is closed;
(2) $P_i^\varepsilon \cap P_j^\varepsilon \subset \partial P_i^\varepsilon \cap \partial P_j^\varepsilon$ for any $i \neq j$;
(3) there exist $r_1(\varepsilon), r_2(\varepsilon) > 0$ such that there are points $q_j^\varepsilon \in P_j^\varepsilon$ for which $B_g(q_j^\varepsilon, \varepsilon) \subset P_j^\varepsilon \subset B_g(q_j^\varepsilon, r_2(\varepsilon)) \subset B_g(q_j^\varepsilon, r_1(\varepsilon))$, with $r_1(\varepsilon) \geq r_2(\varepsilon) \geq C\varepsilon$ for some positive constant C;
(4) lastly, there exists a finite number $\nu(M) \in \mathbb{N}$ such that every $\xi \in M$ is contained in at most $\nu(M)$ balls $B_g(q_j^\varepsilon, r_1(\varepsilon))$, where $\nu(M)$ does not depend on ε.

Remark 5.2. We recall that there exists a constant $\gamma > 0$ such that, for any $\delta > 0$ and for any $\varepsilon < \varepsilon_0(\delta)$ as in Proposition 4.3, given any "good" partition $\mathcal{P}_\varepsilon = \left\{P_j^\varepsilon\right\}_j$ of the manifold M and for any function $u \in \mathcal{N}_\varepsilon \cap I_\varepsilon^{m_\infty+\delta}$ there exists, for an index $\bar{j}$, a set $P_{\bar{j}}^\varepsilon$ such that

$$\frac{1}{\varepsilon^3} \int_{P_{\bar{j}}^\varepsilon} |u^+|^p \, dx \geq \gamma. \tag{5.1}$$

Indeed we can proceed verbatim as in [10, Lemma 12], considering that, since $I'(u)[u] = 0$,

$$\begin{aligned}\|u\|_\varepsilon^2 &= \left|u^+\right|_{\varepsilon,p}^p - \frac{1}{\varepsilon^3}\int_M \omega u^2 \psi(u) \leq \left|u^+\right|_{\varepsilon,p}^p \\ &= \sum_j \left|u_j^+\right|_{\varepsilon,p}^p \leq \max_j \left\{\left|u_j^+\right|_{\varepsilon,p}^{p-2}\right\} \sum_j \left|u_j^+\right|_{\varepsilon,p}^2,\end{aligned}$$

where u_j^+ is the restriction of the function u^+ on the set P_j, and arguing as in [3, Lemma 5.3], we obtain that for some $C > 0$ it holds that

$$\sum_j \left|u_j^+\right|_{\varepsilon,p}^2 \leq C\nu \left\|u^+\right\|_\varepsilon^2,$$

and there the proof follows since

$$\max_j \left\{\left|u_j^+\right|_{\varepsilon,p}^{p-2}\right\} \geq \frac{1}{C\nu}.$$

Proposition 5.3. *For any $\eta \in (0,1)$ there exists $\delta_0 < m_\infty$ such that for any $\delta \in (0, \delta_0)$ and any $\varepsilon \in (0, \varepsilon_0(\delta))$ as in Proposition 4.3, for any function $u \in \mathcal{N}_\varepsilon \cap I_\varepsilon^{m_\infty+\delta}$ we can find a point $q = q(u) \in M$ such that*

$$\int_{B_g(q, r/2)} \Gamma(u) > (1-\eta)\, m_\infty.$$

Proof. First, we prove the proposition for $u \in \mathcal{N}_\varepsilon \cap I_\varepsilon^{m_\varepsilon + 2\delta}$.

By contradiction, we assume that there exists $\eta \in (0,1)$ such that we can find two sequences of vanishing real number δ_k and ε_k and a sequence of functions $\{u_k\}_k$ such that $u_k \in \mathcal{N}_{\varepsilon_k}$,

$$\begin{aligned} m_{\varepsilon_k} &\le I_{\varepsilon_k}(u_k) \\ &= \left(\frac{1}{2} - \frac{1}{p}\right) \|u_k\|^2_{\varepsilon_k} + \omega \left(\frac{1}{4} - \frac{1}{p}\right) G_{\varepsilon_k}(u_k) \\ &\le m_{\varepsilon_k} + 2\delta_k \le m_\infty + 3\delta_k \end{aligned} \tag{5.2}$$

for k large enough (see Remark 4.4), and for any $q \in M$,

$$\int_{B_g(q,r/2)} \Gamma(u_k) \le (1-\eta)\, m_\infty.$$

By the Ekeland principle and by definition of $\mathcal{N}_{\varepsilon_k}$ we can assume that

$$\left| I'_{\varepsilon_k}(u_k)[\varphi] \right| \le \sigma_k \|\varphi\|_{\varepsilon_k} \quad \text{as } \sigma_k \to 0. \tag{5.3}$$

By Remark 5.2 there exists a set $P_k^{\varepsilon_k} \in \mathcal{P}_{\varepsilon_k}$ such that

$$\frac{1}{\varepsilon_k^3} \int_{P_k^{\varepsilon_k}} |u_k^+|^p \, d\mu_g \ge \gamma,$$

so, we choose a point $q_k \in P_k^{\varepsilon_k}$ and we define, in analogy with the proof of Lemma 4.2,

$$w_k(z) := u_k\big(\exp_{q_k}(\varepsilon_k z)\big)\, \chi\left(\varepsilon_k |z|\right),$$

where $z \in B(0, r/\varepsilon_k) \subset \mathbb{R}^3$. Extending trivially w_k by zero to the whole $\mathbb{R}^3$, we have that $w_k \in H^1(\mathbb{R}^3)$ and, by (5.2),

$$\|w_k\|^2_{H^1(\mathbb{R}^3)} \le C \|u_k\|^2_{\varepsilon_k} \le C.$$

So there exists a $w \in H^1(\mathbb{R}^3)$ such that, up to subsequences, $w_k \to w$ weakly in $H^1(\mathbb{R}^3)$ and strongly in $L^t_{\text{loc}}(\mathbb{R}^3)$ for $2 \le t < 6$. Moreover, we set

$$\psi_k(x) := \psi_\varepsilon(u_k)(x) \quad \text{and} \quad \tilde{\psi}_k = \psi_k\big(\exp_{q_k}(\varepsilon_k z)\big)\, \chi\left(\varepsilon_k |z|\right).$$

Arguing as in Lemma 4.2, we get that $\tilde{\psi}_k \to \psi_\infty(w)$ weakly in $H^1(\mathbb{R}^3)$ and strongly in $L^t_{\text{loc}}(\mathbb{R}^3)$ for all $2 \le t < 6$.

Again, given $f \in C_0^\infty(\mathbb{R}^3)$, with $\operatorname{spt} f \subset B(0,T)$ for some $T > 0$, we can define

$$f_k(x) := f\left(\frac{1}{\varepsilon_k} \exp_\xi^{-1}(x)\right)$$

and, by (5.3), we have

$$\left|I'_{\varepsilon_k}(u_k)[f_k]\right| \to 0 \quad \text{as } k \to \infty.$$

Now, by change of variables we have

$$\begin{aligned}
&I'_{\varepsilon_k}(u_k)[f_k] \\
&\quad = \frac{1}{\varepsilon_k^3}\int_M \varepsilon_k^2 \nabla_g u_k \nabla_g f_k + u_k f_k + \omega q u_k \psi_k f_k - (u_k^+)^{p-1} f_k \, d\mu_g \\
&\quad = \int_{B(0,T)} \left[g_{ij}(\varepsilon_k)\partial_i w_k \partial_j f + w_k f + \omega q w_k \tilde{\psi}_k f - (w_k^+)^{p-1} f\right] \left|g(\varepsilon_k z)\right|^{1/2} dz \\
&\quad = \int_{\mathbb{R}^3} \nabla w_k \nabla f + w_k f + \omega q w_k \tilde{\psi}_k f - (w_k^+)^{p-1} f dz + o(\varepsilon_k) \\
&\quad \to \int_{\mathbb{R}^3} \nabla w \nabla f + w f + \omega q w \psi_\infty(w) f - (w_k^+)^{p-1} f \, dz = I'_\infty(w)[f]
\end{aligned}$$

and, by (5.3), we get that w is a weak solution of the limit problem (2.6) and that $w \in \mathcal{N}_\infty$. By Lemma 5.2 and by the choice of q_k we have that $w \neq 0$, so $w > 0$ and $I_\infty(w) \geq m_\infty$.

Now, consider the functions

$$h_k := \frac{1}{\varepsilon^3}\left|u_k^+\right|^{1/p}\left(\exp_{q_k}(\varepsilon_k z)\right)\left|g_{q_k}(\varepsilon_k z)\right|^{1/2p} \mathbb{I}_{B_g(q_k,r)},$$

where $\mathbb{I}_{B_g(q_k,r)}$ is the indicatrix function on $B_g(q_k, r)$. Since $|u_k|_{\varepsilon,p}$ is bounded, then h_k is bounded in $L^p(\mathbb{R}^3)$ so, it converges weakly to some $\bar{h} \in L^p(\mathbb{R}^3)$. We have that $h = |w^+|^{1/p}$. Take $f \in C_0^\infty(\mathbb{R}^3)$, with $\operatorname{spt} f \subset B(0,T)$ for some $T > 0$. Since, eventually $B(0,T) \subset B(0, r/2\varepsilon_k)$, $|u_k^+|^{1/p}(\exp_{q_k}(\varepsilon_k z)) = w_k^+$ on $B(0,T)$. Moreover, on $B(0,T)$ we have

$$\left|g_{q_k}(\varepsilon_k z)\right|^{1/2p} = 1 + o(\varepsilon_k).$$

Thus, since $w_k \rightharpoonup w$ in $L^p(\mathbb{R}^3)$ we get

$$\int_{\mathbb{R}^3} h_k f \, dz \to \int_{\mathbb{R}^3} |w^+|^{1/p} f \, dz$$

for any $f \in C_0^\infty(\mathbb{R}^3)$. In the same way we can consider the functions

$$\begin{aligned}
j_k &= \frac{1}{\varepsilon^3}\Big(g_{ij}(\varepsilon_k z)\partial_i u_k\left(\exp_{q_k}(\varepsilon_k z)\right) \\
&\qquad \times \partial_j u_k\left(\exp_{q_k}(\varepsilon_k z)\right)\left|g_{q_k}(\varepsilon_k z)\right|^{1/2}\Big)^{1/2} \mathbb{I}_{B_g(q_k,r)}, \\
l_k &:= \frac{1}{\varepsilon^3}\left|u_k\right|^{1/2}\left(\exp_{q_k}(\varepsilon_k z)\right)\left|g_{q_k}(\varepsilon_k z)\right|^{1/4} \mathbb{I}_{B_g(q_k,r)}.
\end{aligned}$$

We have that $j_k, l_k \in L^2(\mathbb{R}^3)$ and that $j_k \rightharpoonup |\nabla w|^{1/2}$, $l_k \rightharpoonup |w|^{1/2}$ in $L^2(\mathbb{R}^3)$.

At this point, since $w \in \mathcal{N}_\infty$ and by (5.2), we get

$$\begin{aligned} m_\infty \le I_\infty(w) &= \frac{1}{4}\|w\|^2_{H^1} + \left(\frac{1}{4} - \frac{1}{p}\right)\left|w^+\right|^p_p \\ &\le \liminf_{k\to\infty} \frac{1}{4}\|j_k\|^2_{L^2} + \frac{1}{4}\|i_k\|^2_{L^2} + \left(\frac{1}{4} - \frac{1}{p}\right)|h_k|^p_p \\ &\le \frac{1}{4}\|u_k\|^2_\varepsilon + \left(\frac{1}{4} - \frac{1}{p}\right)\left|u_k^+\right|^p_p \\ &\le m_\infty + 3\delta_k, \end{aligned}$$

so w is a ground state for the limit problem (2.6).

Given $T > 0$, by the definition of w_k we get, for k large enough,

$$\begin{aligned} \int_{B(0,T)} &\left[\left(\frac{1}{2} - \frac{1}{p}\right)(w_k^+)^p - \frac{\omega}{4} w_k^2 \tilde{\psi}_k\right] g(\varepsilon_k z)\,dz \\ &= \frac{1}{\varepsilon^3}\int_{B(q_k,\varepsilon_k T)} \left(\frac{1}{2} - \frac{1}{p}\right)(u_k^+)^p - \frac{\omega}{4} u_k^2 \psi_\varepsilon(u_k)\,d\mu_g \\ &= \int_{B(q_k,\varepsilon_k T)} \Gamma(u_k)\,dx \le \int_{B(q_k,r/2)} \Gamma(u_k)\,dx \\ &\le (1-\eta)\,m_\infty \end{aligned} \tag{5.4}$$

and, if we choose T sufficiently large, this leads to a contradiction since

$$w_k \to w \quad \text{and} \quad \tilde{\psi}_k \to \psi_\infty(w) \quad \text{in } L^t(B(0,T))$$

for any $T > 0$. Since

$$m_\infty = I_\infty(w) = \left(\frac{1}{2} - \frac{1}{p}\right)\left|w^+\right|^p - \frac{\omega}{4} G(w),$$

it is possible to choose T such that (5.4) is false, so the proposition is proved for $u \in \mathcal{N}_\varepsilon \cap I_\varepsilon^{m_\varepsilon + 2\delta}$.

The above arguments also prove that

$$\liminf_{k\to\infty} m_{\varepsilon_k} \ge \lim_{k\to\infty} I_{\varepsilon_k}(u_k) = m_\infty$$

and, in light of (4.7), this leads to

$$\lim_{\varepsilon\to 0} m_\varepsilon = m_\infty. \tag{5.5}$$

Hence, when ε and δ are small enough, $\mathcal{N}_\varepsilon \cap I_\varepsilon^{m_\infty+\delta} \subset \mathcal{N}_\varepsilon \cap I_\varepsilon^{m_\varepsilon+2\delta}$ and the general claim follows. $\square$

Proposition 5.4. *There exists $\delta_0 \in (0, m_\infty)$ such that for any $\delta \in (0,\delta_0)$ and any $\varepsilon \in (0, \varepsilon(\delta_0))$ (see Proposition 4.3), for every function $u \in \mathcal{N}_\varepsilon \cap I_\varepsilon^{m_\infty+\delta}$ it holds that $\beta(u) \in M^+$. Moreover, the composition*

$$\beta \circ \Phi_\varepsilon : M \to M^+$$

is homotopic to the immersion $i : M \to M^+$.

Proof. By Proposition 5.3, for any function $u \in \mathcal{N}_\varepsilon \cap I_\varepsilon^{m_\infty+\delta}$, for any $\eta \in (0,1)$ and for ε, δ small enough, we can find a point $q = q(u) \in M$ such that

$$\int_{B(q,r/2)} \Gamma(u) > (1-\eta)\, m_\infty.$$

Moreover, since $u \in \mathcal{N}_\varepsilon \cap I_\varepsilon^{m_\infty+\delta}$, we have

$$I_\varepsilon(u) = \int_M \Gamma(u) \le m_\infty + \delta.$$

Hence

$$\begin{aligned}
|\beta(u) - q| &\le \frac{\left|\int_M (x-q)\Gamma(u)\right|}{\int_M \Gamma(u)} \\
&\le \frac{\left|\left(1/\varepsilon^3\right)\int_{B(q,r/2)}(x-q)\Gamma(u)\right|}{\int_M \Gamma(u)} + \frac{\left|\left(1/\varepsilon^3\right)\int_{M\setminus B(q,r/2)}(x-q)\Gamma(u)\right|}{\int_M \Gamma(u)} \\
&\le \frac{r}{2} + 2\,\mathrm{diam}\,(M)\left(1 - \frac{1-\eta}{1+\delta/m_\infty}\right),
\end{aligned}$$

and the second term can be made arbitrarily small, choosing η, δ and ε sufficiently small. The second claim of the proposition is standard. □

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