# Global regularity for a slightly supercritical hyperdissipative Navier-Stokes system 

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#### Abstract

We prove global existence of smooth solutions for a slightly supercritical hyperdissipative Navier-Stokes under the optimal condition on the correction to the dissipation. This proves a conjecture formulated by Tao [Tao09].


## 1 Introduction

Let $d \geq 3$ and consider the generalized Navier-Stokes system

$$
\left\{\begin{array}{l}
\frac{\partial u}{\partial t}+(u \cdot \nabla) u+\nabla p+D_{0}^{2} u=0  \tag{1.1}\\
\nabla \cdot u=0 \\
\int_{[0,2 \pi]^{d}} u(t, x) d x=0
\end{array}\right.
$$

on $[0,2 \pi]^{d}$ with periodic boundary conditions, where $D_{0}$ is a Fourier multiplier with non-negative symbol $m$. The Navier-Stokes system is recovered when $m(k)=|k|$. If

$$
\begin{equation*}
m(k) \geq c \frac{|k|^{\frac{d+2}{4}}}{G(|k|)} \tag{1.2}
\end{equation*}
$$

where $G:[0, \infty) \rightarrow[0, \infty)$ is a non-decreasing function such that

$$
\begin{gather*}
\int_{1}^{\infty} \frac{d s}{s G(s)^{4}}=\infty  \tag{1.3}\\
\frac{G(x)}{|x|^{\frac{d+2}{4}}} \text { eventually non-increasing, } \tag{1.4}
\end{gather*}
$$

then in [Tao09] it is proved ${ }^{1}$ that (1.1) has a global smooth solution for every smooth initial condition. The result has been later extended to the two dimensional case by [KT12].

A heuristic argument developed in [Tao09] and based on the comparison between the speed of propagation of a (possible) blow-up and the rate of dissipation suggests that regularity should still hold under the weaker condition

$$
\begin{equation*}
\int_{1}^{\infty} \frac{d s}{s G(s)^{2}}=\infty \tag{1.5}
\end{equation*}
$$

The main result of this paper, contained in the following theorem, is a complete proof of this conjecture.

Theorem 1.1. Let $d \geq 2$ and assume (1.2), (1.4) and (1.5) for a non-decreasing function $G:[0, \infty) \rightarrow[0, \infty)$. Then (1.1) has a global smooth solution for every smooth initial condition.

A simple version of this conjecture, when reformulated on a toy model, has been proved for the dyadic model in [BMR14]. Actually, for that model one could prove regularity in full supercritical regime, with $m(k)=|k|$, as was done in [BMR11], but it was natural to develop there some of the main ideas on which also this paper is based. In fact here we prove that the equations for the velocity can be reduced to a suitable dyadic-like model, with infinitely many interactions though. A more sophisticated version of the arguments of [BMR14] ensures regularity of this dyadic model and, in turns, of the solution of the problem (1.1) above.

Our technique for proving Theorem 1.1 is flexible enough to include an additional critical parameter. Consider the following generalized Leray $\alpha$-model,

$$
\left\{\begin{array}{l}
\frac{\partial v}{\partial t}+(u \cdot \nabla) v+\nabla p+D_{1} v=0  \tag{1.6}\\
v=D_{2} u \\
\nabla \cdot v=0 \\
\int_{[0,2 \pi]^{d}} v(t, x) d x=\int_{[0,2 \pi]^{d}} u(t, x) d x=0
\end{array}\right.
$$

where $D_{1}$ and $D_{2}$ are Fourier multipliers with non-negative symbols $m_{1}$ and $m_{2}$.
Theorem 1.2. Let $d \geq 2, \alpha, \beta \geq 0$, and assume

$$
m_{1}(k) \geq c \frac{|k|^{\alpha}}{g(|k|)}, \quad m_{2}(k) \geq c|k|^{\beta}, \quad \alpha+\beta \geq \frac{d+2}{2}
$$

[^0]where $g:[0, \infty) \rightarrow[0, \infty)$ is a non-decreasing function such that $x^{-\alpha} g(x)$ is eventually non-increasing, and
\[

$$
\begin{equation*}
\int_{1}^{\infty} \frac{d s}{s g(s)}=\infty \tag{1.7}
\end{equation*}
$$

\]

Then (1.6) has a global smooth solution for every smooth initial condition.
Under the assumptions of Theorem 1.1, if $\beta=0, \alpha=\frac{d+2}{2}, g(x)=G(x)^{2}$, $m_{2}(k)=1$, and $m_{1}(k)=m(k)^{2}$, then the assumptions of Theorem 1.2 are met. Therefore Theorem 1.1 follows immediately from Theorem 1.2, and it is sufficient to prove only the second result.

Our results hold as well when the problems are considered in $\mathbb{R}^{d}$, since in our method large scales play no significant role (see Remark 2.9).

The model (1.6) with $g \equiv 1$ was introduced by Olson and Titi in [OT07]. They proposed the idea that a weaker non-linearity and a stronger viscous dissipation could work together to yield regularity. Their statement uses though a stronger hypothesis $\alpha+\frac{\beta}{2} \geq \frac{d+2}{2}$ and this result was later logarithmically improved in [Yam12] with condition (1.3).

Our results are also relevant in view of the analysis in [Tao14] (see Remark 5.2 therein), since they confirm that the condition (1.7) is optimal, when general non-linear terms with the same scaling are considered.

The proof of the above theorem is based on two crucial ideas. The first idea is that smoothness of (1.6) can be reduced to the smoothness of a suitable shell model, obtained by averaging the energy of a solution of (1.6) over dyadic shells in Fourier space. We believe that this reduction may be interesting beyond the scope of this paper. The second idea is that the overall contribution of energy and dissipation over large shells satisfies a recursive inequality. Under condition (1.7) dissipation dumps significantly the flow of energy towards small scales and ensures smoothness. This is a more sophisticated version of the result obtained in [BMR14], due to the larger number of interactions between shells.

The paper is organized as follows. In Section 2 we derive the shell approximation of a solution of (1.6). The recursive formula is obtained in Section 3. In Section 4 we deduce exponential decays of shell modes by the recursive formula. The appendix A contains, for the sake of completeness, a standard existence and uniqueness result.

## 2 From the generalized Fourier Navier-Stokes to the dyadic equation

This section contains one of the crucial steps in our approach. We show that the proof of Theorem 1.2 can be reduced to a proof of decay of solutions of a suitable
shell model. For simplicity and without loss of generality from now on we assume that

$$
m_{1}(k)=\frac{|k|^{\alpha}}{g(|k|)^{\prime}}, \quad m_{2}(k) \geq|k|^{\beta} .
$$

### 2.1 The shell approximation

The dynamics of our generalized version of Navier-Stokes equation in Fourier decomposition reads

$$
\left\{\begin{array}{l}
v_{k}^{\prime}=-\frac{|k|^{\alpha}}{g(|k|)} v_{k}-i \sum_{h \in \mathbb{Z}^{d} \backslash\{0\}} \frac{\left\langle v_{h}, k\right\rangle}{|h|^{\beta}} P_{k}\left(v_{k-h}\right),  \tag{2.1}\\
\left\langle v_{k}, k\right\rangle=0, \\
v_{-k}=\overline{v_{k}},
\end{array} \quad k \in \mathbb{Z}^{d} \backslash\{0\}\right.
$$

where $P_{k}(w):=w-\frac{\langle w, k\rangle}{|k|^{2}} k$ and $v_{0}=0$. A solution is a family $\left(v_{k}\right)_{k \in \mathbb{Z}^{d} \backslash\{0\}}$ where each $v_{k}=v_{k}(t)$ is a differentiable map from $[0, \infty)$ to $\mathbb{C}^{d}$ satisfying (2.1) for all times.

As is common in Littlewood-Paley theory, let $\Phi:[0, \infty) \rightarrow[0,1]$ be a smooth function such that $\Phi \equiv 1$ on $[0,1], \Phi \equiv 0$ on $[2, \infty)$ and $\Phi$ is strictly decreasing on $[1,2]$. For $x \geq 0$, let $\psi(x):=\Phi(x)-\Phi(2 x)$, so that $\psi$ is a smooth bump function supported on $\left(\frac{1}{2}, 2\right)$ satisfying

$$
\sum_{n=0}^{\infty} \psi\left(x / 2^{n}\right)=1-\Phi(2 x) \equiv 1, \quad x \geq 1
$$

Notice that it is elementary to show that $\sqrt{\psi}$ is Lipschitz continuous.
Let $\mathbb{N}_{0}$ denote the set of non-negative integers. For all $n \in \mathbb{N}_{0}$ we introduce the radial maps $\psi_{n}: \mathbb{R}^{d} \rightarrow[0,1]$ defined by $\psi_{n}(x)=\psi\left(2^{-n}|x|\right)$. Notice that

$$
\sum_{n \in \mathbb{N}_{0}} \psi_{n}(x) \equiv 1, \quad x \in \mathbb{Z}^{d} \backslash\{0\}
$$

In Littlewood-Paley theory one typically defines $\psi_{n}$ for all $n \in \mathbb{Z}$, introduces objects like

$$
P_{n}(x):=\sum_{k \in \mathbb{Z}^{d}} \psi_{n}(k) v_{k} e^{i\langle k, x\rangle}
$$

and then proves that $u=\sum_{n} P_{n}$. Since these $P_{n}$ are not orthogonal ${ }^{2}$ this does not give a nice decomposition of energy, as

$$
\sum_{n \in \mathbb{Z}}\left\|P_{n}\right\|_{L^{2}}^{2} \neq \sum_{k \in \mathbb{Z}^{d}}\left|v_{k}\right|^{2}=\|u\|_{L^{2}}^{2} .
$$

[^1]Thus instead of $P_{n}(x)$ we introduce a sort of square-averaged Littlewood-Paley decomposition. Let

$$
\begin{equation*}
X_{n}(t):=\left(\sum_{k \in \mathbb{Z}^{d}} \psi_{n}(k)\left|v_{k}(t)\right|^{2}\right)^{1 / 2}, \quad n \in \mathbb{N}_{0}, \quad t \geq 0 \tag{2.2}
\end{equation*}
$$

Then clearly

$$
\sum_{n \in \mathbb{N}_{0}} X_{n}^{2}=\sum_{k \in \mathbb{Z}^{d}}\left|v_{k}\right|^{2}=\|u\|_{L^{2}}^{2} .
$$

Remark 2.1. One major difference with respect to the usual Littlewood-Paley theory is that it is impossible to recover $v$ from these $X_{n}$ (as it was with the components $P_{n}(x)$ ), since they are averaged both in the physical space and over one shell of the frequency space.

We will denote by $H^{\gamma}$ the Hilbert-Sobolev space of periodic functions with differentiation index $\gamma$, namely

$$
\begin{equation*}
H^{\gamma}=\left\{v=\left(v_{k}\right)_{k \in \mathbb{Z}^{d}}: \sum\left(1+|k|^{2}\right)^{\gamma}\left|v_{k}\right|^{2}<\infty\right\} . \tag{2.3}
\end{equation*}
$$

Definition 2.2. If (2.2) holds, we say that $X=\left(X_{n}(t)\right)_{n \in \mathbb{N}_{0}, t \geq 0}$ is the shell approximation of $v$.

If $v \in H^{\gamma}$ and $X$ is its shell approximation, then

$$
\sum_{n} 2^{2 \gamma n} X_{n}^{2}=\sum_{k}\left(\sum_{n} 2^{2 \gamma n} \psi_{n}(k)\right)\left|v_{k}\right|^{2} \approx \sum_{k}|k|^{2 \gamma}\left|v_{k}\right|^{2}=\|v\|_{H \gamma}^{2}
$$

Hence, $v(t) \in C^{\infty}$ if and only if $\sup _{n} 2^{\gamma n} X_{n}<\infty$ for every $\gamma>0$. In view of Theorem A. 1 (see page 22), Theorem 1.2 follows if we can prove the following result.

Theorem 2.3. Under the same assumptions of Theorem 1.2, let $v(0)$ be smooth and periodic, and let $m \geq 2+\frac{d}{2}$. If $v$ is a solution of (1.6) in $H^{m}$ on its maximal interval of existence $\left[0, T_{\star}\right), X$ is its shell approximation and

$$
\sup _{\left[0, T_{\star}\right)} \sum 2^{2 m n} X_{n}^{2}<\infty
$$

then $T_{\star}=\infty$.

### 2.2 The shell solution

We want to write a system of equation for the shell approximation of a solution of (1.6). We give a more formal connection between (1.6) and its shell equation because we believe the notion will result useful beyond the scopes of the present work.

Define the set $I$ as follows,

$$
I:=\left\{(l, m, n) \in \mathbb{N}_{0}^{3}: \begin{array}{l}
\text { the difference between the two largest }  \tag{2.4}\\
\text { integers among } l, m \text { and } n \text { is at most } 2
\end{array}\right\} .
$$

We are now ready to introduce the shell model ODE for the energy of each shell, equation (2.5).

Definition 2.4 (shell solution). Let $X=\left(X_{n}\right)_{n \in \mathbb{N}_{0}}$ be a sequence of real valued maps, $X_{n}:[0, \infty) \rightarrow \mathbb{R}$. We say that $X$ is a shell solution if there are two families of real valued maps $\chi=\left(\chi_{n}\right)_{n \in \mathbb{N}_{0}}$ and $\phi=\left(\phi_{(l, m, n)}\right)_{(l, m, n) \in I}$ such that

$$
\begin{equation*}
\frac{d}{d t} X_{n}^{2}(t)=-\chi_{n}(t) X_{n}^{2}(t)+\sum_{\substack{l, m \in \mathbb{N}_{0} \\(l, m, n) \in I}} \phi_{(l, m, n)}(t) X_{l}(t) X_{m}(t) X_{n}(t), \tag{2.5}
\end{equation*}
$$

for all $n \in \mathbb{N}_{0}$ and $t>0$ where the sum above is understood as absolutely convergent, and $\eta, \phi$ satisfies the following properties,

1. the family $\phi$ is antisymmetric, in the sense that

$$
\phi_{(l, m, n)}(t)=-\phi_{(l, n, m)}(t), \quad(l, m, n) \in I, t \geq 0
$$

2. there exist two positive constants $c_{1}$ and $c_{2}$ for which,

$$
\begin{equation*}
\chi_{n}(t) \geq c_{1} \frac{2^{\alpha n}}{g\left(2^{n+1}\right)} \quad \text { and } \quad\left|\phi_{(l, m, n)}(t)\right| \leq c_{2} 2^{\left(\frac{d}{2}+1-\beta\right) \min \{l, m, n\}} \tag{2.6}
\end{equation*}
$$

for all $(l, m, n) \in I$ and $t \geq 0$.
Remark 2.5. We will prove below that the shell approximation of a solution of (1.6) is a shell solution. It is easy to check that the dissipation term is local as expected, due to the way the shell components of a solution interact in the model's dynamics. As for the nonlinear term, it turns out that the set $I$ of the triples of indices $(l, m, n)$ for which there may be interaction between the shell components $l, m$ and $n$ is quite small. This is basically because in the Fourier space, three components may interact only if they are the sides of a triangle and by triangle inequality their lengths cannot be in three shells far away from each other.

Remark 2.6. To ensure that the sum in (2.5) is absolutely convergent, it is sufficient to assume that the sequence $\left(X_{n}(t)\right)_{n \in \mathbb{N}_{0}}$ is square summable (this will be a consequence of the energy inequality, see Definition 3.1). Indeed, if $n$ is not the smallest index, then the sum is extended to a finite number if indices. Otherwise, $\phi_{(l, m, n)}$ is constant with respect to $l, m$.
Remark 2.7. The antisymmetric property is what makes the non-linearity of (2.5) formally conservative. In fact using antisymmetry, a change of variable ( $m^{\prime}=n$ and $n^{\prime}=m$ ) and the fact that $\left(l, m^{\prime}, n^{\prime}\right) \in I$ if and only if $\left(l, n^{\prime}, m^{\prime}\right) \in I$, one could formally write,

$$
\begin{aligned}
&-\sum_{\substack{l, m, n \in \mathbb{N}_{0} \\
(l, m, n) \in I}} \phi_{(l, m, n)} X_{l} X_{m} X_{n}=\sum_{\substack{l, m, n \in \mathbb{N}_{0} \\
(l, m, n) \in I}} \phi_{(l, n, m)} X_{l} X_{m} X_{n} \\
&=\sum_{\substack{l, m^{\prime}, \prime^{\prime} \in \in \mathbb{N} \\
\left(l, n^{\prime}, m^{\prime}\right) \in I}} \phi_{\left(l, m^{\prime}, n^{\prime}\right)} X_{l} X_{m^{\prime}} X_{n^{\prime}}=\sum_{\substack{l, m^{\prime}, n^{\prime} \in \mathbb{N}_{0} \\
\left(l, m^{\prime}, n^{\prime}\right) \in I}} \phi_{\left(l, m^{\prime}, n^{\prime}\right) \in I} X_{l} X_{m^{\prime}} X_{n^{\prime}}
\end{aligned}
$$

If these sums are absolutely convergent, this would prove indeed that the expression itself is equal to zero.

Since these are infinite sums, these computations are not rigorous unless we know, for instance, that $\sum_{n} 2^{2 \gamma n} X_{n}^{2}<\infty$, with $\gamma \geq \frac{1}{3}\left(\frac{d}{2}+1-\beta\right)$, as it can be verified by an elementary computation.

### 2.3 The shell model as a shell approximation

The bounds on the coefficients given in Definition 2.4 are in the correct direction to prove regularity results (and hence Theorem 2.3). The following theorem, which is the main result of this section shows that they capture the natural scaling of the shell interactions for the physical solutions.

Theorem 2.8. Ifv is a solution of (1.6) on $[0, T]$ and $X$ is its shell approximation, then $X$ is a shell solution.

Remark 2.9. At this stage it is easy to realize that our main results hold also in $\mathbb{R}^{d}$ with minimal changes. Indeed when passing to the shell approximation, all large frequencies are considered together in the first element of the shell model.

The proof of Theorem 2.8 can be found at the end of this section. It is based on Propositions 2.10-2.11 below, which give the actual definitions of $\chi$ and $\phi$ and prove their properties.

Proposition 2.10. Let $X$ be the shell approximation of a solution $v$. Define $\chi_{n}(t)$ for all $n \in \mathbb{N}_{0}$ and $t \geq 0$ as follows

$$
\chi_{n}(t):= \begin{cases}\frac{2}{X_{n}^{2}(t)} \sum_{k \in \mathbb{Z}^{d} \backslash\{0\}} \psi_{n}(k) \frac{|k|^{\alpha}}{g(|k|)}\left|v_{k}(t)\right|^{2}, & \text { if } X_{n}(t) \neq 0  \tag{2.7}\\ \frac{2^{\alpha n-\alpha+1}}{g\left(2^{n+1}\right)}, & \text { if } X_{n}(t)=0\end{cases}
$$

Then

$$
\chi_{n}(t) \geq \frac{2^{\alpha n-\alpha+1}}{g\left(2^{n+1}\right)}, \quad n \in \mathbb{N}_{0}, t \geq 0
$$

Proof. Fix $n \in \mathbb{N}_{0}$ and $t \geq 0$. The map $\psi_{n}$ is supported on $\left\{x \in \mathbb{Z}^{d}: 2^{n-1}<|x|<\right.$ $\left.2^{n+1}\right\}$ and $g$ is non-decreasing, so

$$
\sum_{k \in \mathbb{Z}^{d} \backslash\{0\}} \psi_{n}(k) \frac{|k|^{\alpha}}{g(|k|)}\left|v_{k}(t)\right|^{2} \geq \sum_{k \in \mathbb{Z}^{d} \backslash\{0\}} \psi_{n}(k) \frac{2^{(n-1) \alpha}}{g\left(2^{n+1}\right)}\left|v_{k}(t)\right|^{2}=\frac{2^{(n-1) \alpha}}{g\left(2^{n+1}\right)} X_{n}^{2}(t)
$$

where we used (2.2). By (2.7) we get the thesis.
We finally turn our attention to the antisymmetry property and an upper bound for $\phi_{(l, m, n)}(t)$. The statement is as follows.

Proposition 2.11. Let $X$ be the shell approximation of a solution $v$. Define $\phi_{(l, m, n)}(t)$ for all $l, m, n \in \mathbb{N}_{0}$ and $t \geq 0$ as

$$
\begin{align*}
\phi_{(l, m, n)}(t):= & \frac{2}{X_{l}(t) X_{m}(t) X_{n}(t)} . \\
& \cdot \sum_{\substack{h, k \in \mathbb{Z}^{d} \\
h \neq 0}} \psi_{l}(h) \psi_{m}(k-h) \psi_{n}(k) \frac{\operatorname{Im}\left\{\left\langle v_{h}(t), k\right\rangle\left\langle v_{k-h}(t), v_{k}(t)\right\rangle\right\}}{|h|^{\beta}}, \tag{2.8}
\end{align*}
$$

(unless $X_{l}(t) X_{m}(t) X_{n}(t)=0$, in which case $\left.\phi_{(l, m, n)}(t):=0\right)$.
Then:

1. $\phi_{(l, m, n)}(t)=0$ for all $(l, m, n) \notin I$ and all $t \geq 0$.
2. $\phi_{(l, m, n)}(t)=-\phi_{(l, n, m)}(t)$ for all $l, m, n \in \mathbb{N}_{0}$ and all $t \geq 0$.
3. For any $\beta \geq 0$ there exists a constant $c_{3}>0$ depending only on $d, \beta$ and $\psi$ such that

$$
\begin{equation*}
\left|\phi_{(l, m, n)}(t)\right| \leq c_{3} 2^{\left(\frac{d}{2}+1-\beta\right) \min \{l, m, n\}}, \quad(l, m, n) \in I, t \geq 0 \tag{2.9}
\end{equation*}
$$

For the proof we need a couple of lemmas.
Lemma 2.12. Suppose $v=\left(v_{k}\right)_{k \in \mathbb{Z}^{d}}$ is a complex field over $\mathbb{Z}^{d}$ such that for all $k \in \mathbb{Z}^{d},\left\langle k, v_{k}\right\rangle=0$ and $\overline{v_{k}}=v_{-k}$. Then for all $h \in \mathbb{Z}^{d}$,

$$
\begin{aligned}
\sum_{k \in \mathbb{Z}^{d}} \psi_{m}(k-h) \psi_{n}(k) \operatorname{Im}\left\{\left\langle v_{h}, k\right\rangle\right. & \left.\left\langle v_{k-h}, v_{k}\right\rangle\right\} \\
& =-\sum_{k \in \mathbb{Z}^{d}} \psi_{m}(k) \psi_{n}(k-h) \operatorname{Im}\left\{\left\langle v_{h}, k\right\rangle\left\langle v_{k-h}, v_{k}\right\rangle\right\}
\end{aligned}
$$

Proof. Consider the left-hand side. By performing the change of variable $k^{\prime}=h-k$ we obtain

$$
\begin{gathered}
\psi_{m}(k-h)=\psi_{m}\left(-k^{\prime}\right)=\psi_{m}\left(k^{\prime}\right), \\
\psi_{n}(k)=\psi_{n}\left(h-k^{\prime}\right)=\psi_{n}\left(k^{\prime}-h\right), \\
\left\langle v_{h}, k\right\rangle=\left\langle v_{h}, h-k^{\prime}\right\rangle=-\left\langle v_{h}, k^{\prime}\right\rangle \\
\left\langle v_{k-h}, v_{k}\right\rangle=\left\langle v_{-k^{\prime}}, v_{h-k^{\prime}}\right\rangle=\left\langle\overline{v_{k^{\prime}}}, \overline{v_{k^{\prime}}-h}\right\rangle=\left\langle v_{k^{\prime}-h}, v_{k^{\prime}}\right\rangle .
\end{gathered}
$$

The sum for $k \in \mathbb{Z}^{d}$ is equivalent to the sum for $k^{\prime} \in \mathbb{Z}^{d}$ and this concludes the proof.

Lemma 2.13. Let $v$ be a solution and $X$ its shell approximation. Then for all $a, b, c \in \mathbb{N}_{0}$ and for all $t \geq 0$,

$$
\begin{aligned}
\sum_{h \in \mathbb{Z}^{d}} \psi_{a}(h)\left|v_{h}(t)\right| \sum_{k \in \mathbb{Z}^{d}} \sqrt{\psi_{b}(k) \psi_{c}(k-h)}\left|v_{k}(t)\right|\left|v_{k-h}(t)\right| & \leq \\
& \leq 2^{\frac{d}{2}(a+3)} X_{a}(t) X_{b}(t) X_{c}(t) .
\end{aligned}
$$

Proof. By Cauchy-Schwarz inequality and formula (2.2) we have that for all $h \in$ $\mathbb{Z}^{d}$,

$$
\sum_{k \in \mathbb{Z}^{d}} \sqrt{\psi_{b}(k) \psi_{c}(k-h)}\left|v_{k}(t)\right|\left|v_{k-h}(t)\right| \leq X_{b}(t) X_{c}(t)
$$

Then, let $S_{a}$ denote the intersection between $\mathbb{Z}^{d}$ and the support of $\psi_{a}$. By inscribing $S_{a}$ in a cube we can bound its cardinality with $\left|S_{a}\right| \leq\left(2^{a+2}+1\right)^{d} \leq 2^{(a+3) d}$, so

$$
\sum_{k \in \mathbb{Z}^{d}} \psi_{a}(k)\left|v_{k}(t)\right| \leq\left(\left|S_{a}\right| \sum_{k \in S_{a}} \psi_{a}^{2}(k) v_{k}^{2}(t)\right)^{1 / 2} \leq\left(2^{(a+3) d}\right)^{1 / 2} X_{a}(t)
$$

where we used the fact that $\psi_{a}(k) \leq 1$.

Proof of Proposition 2.11. Consider the definition of $\phi_{(l, m, n)}$, equation (2.8). By applying Lemma 2.12, for fixed $t$, we immediately conclude that

$$
\phi_{(l, n, m)}=-\phi_{(l, m, n)} \quad l, m, n \in \mathbb{N}_{0}
$$

and in particular that $\phi_{(l, m, m)}=0$.
Moreover, for all choices of $h$ and $k$, the arguments of $\psi_{l}, \psi_{m}$ and $\psi_{n}$ are the sides of a triangle in $\mathbb{R}^{d}$, so by the triangle inequality the size of the largest (wlog $k)$ is at most twice the size of the second largest $(\operatorname{wlog} h)$. On the other hand for all $j \in \mathbb{N}_{0}$ the support of $\psi_{j}$ is $\left\{x \in \mathbb{R}^{d}: 2^{j-1}<|x|<2^{j+1}\right\}$. Thus whenever $\psi_{l}(h) \psi_{n}(k) \neq 0$, necessarily $n \leq l+2$ since

$$
2^{n-1}<|k| \leq 2|h|<2^{l+2} .
$$

This proves that $\phi_{(l, m, n)}=0$ outside $I$ as defined in equation (2.4).
Finally we prove inequality (2.9) for $(l, m, n) \in I$ with $m<n$. We will consider separately the two cases $n-m>2$ and $n-m \in\{1,2\}$, starting from the former.

Case 1. Since $m<n-2$ and $(l, m, n) \in I$, then $m=\min \{l, m, n\}$ and $\mid l-$ $n \mid \leq 2$. This means in particular that for all the non-zero terms of the sum in equation (2.8), tipically $|k-h|<|k|$, so it is convenient to substitute $\left\langle v_{h}, k\right\rangle=$ $\left\langle v_{h}, k-h\right\rangle$ in the equation to obtain the following bound

$$
\left|\phi_{(l, m, n)}\right| \leq \frac{2}{X_{l} X_{m} X_{n}} \sum_{\substack{h, k \in \mathbb{Z}^{d} \\ h \neq 0}} \psi_{l}(h) \psi_{m}(k-h) \psi_{n}(k) \frac{\left|v_{h}\right||k-h|\left|v_{k-h}\right|\left|v_{k}\right|}{|h|^{\beta}} .
$$

By the definition of $\psi_{l}$, either $\psi_{l}(h)=0$ or $|h| \geq 2^{l-1} \geq 2^{m}$. Applying this and the change of variable $k^{\prime}=k-h$ one gets,

$$
\left|\phi_{(l, m, n)}\right| \leq \frac{2^{1-\beta m}}{X_{l} X_{m} X_{n}} \sum_{k^{\prime} \in \mathbb{Z}^{d}} \psi_{m}\left(k^{\prime}\right)\left|k^{\prime}\right|\left|v_{k^{\prime}}\right| \sum_{h \in \mathbb{Z}^{d}} \psi_{l}(h) \psi_{n}\left(k^{\prime}+h\right)\left|v_{h}\right|\left|v_{k^{\prime}+h}\right| .
$$

In the same way we can substitute $\left|k^{\prime}\right| \leq 2^{m+1}$ and apply Lemma 2.13 (recall that $\psi \leq 1$, so $\psi \leq \sqrt{\psi}$ ) to get

$$
\left|\phi_{(l, m, n)}\right| \leq 2^{1-\beta m+m+1+\frac{d}{2}(m+3)} .
$$

Since in the present case $\min \{l, m, n\}=m$, this proves inequality (2.9) with $c_{3}=2^{2+3 d / 2}$.

Case 2. Suppose now that $n-m \in\{1,2\}$ and $(l, m, n) \in I$, then $l \leq n+2$ and $\min \{l, m, n\} \geq l-4$. In this case it is $l$ that can be small with respect to $m$ and
$n$, so we take the terms in $l$ and $h$ outside the internal sum,

$$
\left|\phi_{(l, m, n)}\right| \leq \frac{2}{X_{l} X_{m} X_{n}} \sum_{h \in \mathbb{Z}^{d} \backslash\{0\}} \frac{\psi_{l}(h)}{|h|^{\beta}}\left|\sum_{k \in \mathbb{Z}^{d}} \psi_{m}(k-h) \psi_{n}(k) \operatorname{Im}\left\{\left\langle v_{h}, k\right\rangle\left\langle v_{k-h}, v_{k}\right\rangle\right\}\right| .
$$

The idea is to exploit the cancellations in the sum over $k$ that happen when $k-h$ and $k$ are switched. By Lemma 2.12 and the bound $|k| \leq 2^{n+1}$ for $k$ in the support of $\psi_{m}$ or $\psi_{n}$,

$$
\begin{aligned}
& \left|\phi_{(l, m, n)}\right| \leq \frac{2}{X_{l} X_{m} X_{n}} \sum_{h \in \mathbb{Z}^{d} \backslash\{0\}} \frac{\psi_{l}(h)}{|h|^{\beta}} \\
& \quad \cdot \frac{1}{2}\left|\sum_{k \in \mathbb{Z}^{d}}\left(\psi_{m}(k-h) \psi_{n}(k)-\psi_{m}(k) \psi_{n}(k-h)\right) \operatorname{Im}\left\{\left\langle v_{h}, k\right\rangle\left\langle v_{k-h}, v_{k}\right\rangle\right\}\right| \\
& \leq \frac{2^{n+1}}{X_{l} X_{m} X_{n}} \sum_{h \in \mathbb{Z}^{d} \backslash\{0\}} \frac{\psi_{l}(h)\left|v_{h}\right|}{|h|^{\beta}} \sum_{k \in \mathbb{Z}^{d}}\left|\psi_{m}(k-h) \psi_{n}(k)-\psi_{m}(k) \psi_{n}(k-h)\right|\left|v_{k-h}\right|\left|v_{k}\right| .
\end{aligned}
$$

We turn our attention to the term $\psi_{m}(k-h) \psi_{n}(k)-\psi_{m}(k) \psi_{n}(k-h)$ and show that it is small. Let $L$ denote the Lipschitz constant of the function $\psi^{1 / 2}$. Then for all $h, k \in \mathbb{Z}^{d}$ and all $m, n \in \mathbb{N}$ such that $m \geq n-2$,

$$
\begin{aligned}
& \left|\sqrt{\psi_{m}(k-h) \psi_{n}(k)}-\sqrt{\psi_{m}(k) \psi_{n}(k-h)}\right| \\
& =\left|\sqrt{\psi_{m}(k-h) \psi_{n}(k)}-\sqrt{\psi_{m}(k) \psi_{n}(k)}+\sqrt{\psi_{m}(k) \psi_{n}(k)}-\sqrt{\psi_{m}(k) \psi_{n}(k-h)}\right| \\
& \quad \leq L \frac{|h|}{2^{m}} \sqrt{\psi_{n}(k)}+L \frac{|h|}{2^{n}} \sqrt{\psi_{m}(k)} \leq L \frac{|h|}{2^{n-3}} .
\end{aligned}
$$

Moreover by simmetry with respect to $m$ and $n$,

$$
\begin{aligned}
& \sum_{k \in \mathbb{Z}^{d}}\left(\sqrt{\psi_{m}(k-h) \psi_{n}(k)}+\sqrt{\psi_{m}(k) \psi_{n}(k-h)}\right)\left|v_{k-h}\right|\left|v_{k}\right| \\
&=2 \sum_{k \in \mathbb{Z}^{d}} \sqrt{\psi_{m}(k-h) \psi_{n}(k)}\left|v_{k-h}\right|\left|v_{k}\right|
\end{aligned}
$$

so that

$$
\left|\phi_{(l, m, n)}\right| \leq \frac{2^{5} L}{X_{l} X_{m} X_{n}} \sum_{h \in \mathbb{Z}^{d} \backslash\{0\}}|h|^{1-\beta} \psi_{l}(h)\left|v_{h}\right| \sum_{k \in \mathbb{Z}^{d}} \sqrt{\psi_{m}(k-h) \psi_{n}(k)}\left|v_{k-h}\right|\left|v_{k}\right| .
$$

By the usual bound $2^{l-1} \leq|h| \leq 2^{l+1}$, since $\beta \geq 0$, we see that $|h|^{1-\beta} \leq 2^{l(1-\beta)+1+\beta}$, so by Lemma 2.13,

$$
\left|\phi_{(l, m, n)}\right| \leq 2^{5} 2^{(1-\beta) l+1+\beta} 2^{(l+3) \frac{d}{2}} L \leq 2^{\left(\frac{d}{2}+1-\beta\right)(l-4)+9-3 \beta+\frac{11}{2} d} L
$$

Since in the present case $\min \{l, m, n\} \geq l-4$, this proves inequality (2.9) with $c_{3}=2^{9+\frac{11}{2} d-3 \beta} L$.

Finally we have all the ingredients to prove the main theorem of this section. Proof of Theorem 2.8. A direct computation using (2.2) and (2.1) shows that

$$
\begin{aligned}
& \frac{1}{2} \frac{d}{d t} X_{n}^{2}=\operatorname{Re} \sum_{k \in \mathbb{Z}^{d}} \psi_{n}(k)\left\langle v_{k}^{\prime}, v_{k}\right\rangle \\
& =-\sum_{k \in \mathbb{Z}^{d} \backslash\{0\}} \psi_{n}(k) \frac{|k|^{\alpha}}{g(|k|)}\left|v_{k}\right|^{2}+\operatorname{Im} \sum_{k \in \mathbb{Z}^{d}} \sum_{h \in \mathbb{Z}^{d} \backslash\{0\}} \psi_{n}(k) \frac{\left\langle v_{h}, k\right\rangle}{|h|^{\beta}}\left\langle P_{k}\left(v_{k-h}\right), v_{k}\right\rangle \\
& \quad=-\sum_{k \in \mathbb{Z}^{d} \backslash\{0\}} \psi_{n}(k) \frac{|k|^{\alpha}}{g(|k|)}\left|v_{k}\right|^{2}+\sum_{\substack{h, k \in \mathbb{Z}^{d} \\
h \neq 0}} \psi_{n}(k) \frac{\operatorname{Im}\left\{\left\langle v_{h}, k\right\rangle\left\langle v_{k-h}, v_{k}\right\rangle\right\}}{|h|^{\beta}} .
\end{aligned}
$$

To deal with the first sum, define $\chi$ as in Proposition 2.10. By applying (2.7) for $X_{n}(t) \neq 0$ and (2.2) for $X_{n}(t)=0$ we see that in both cases,

$$
2 \sum_{k \in \mathbb{Z}^{d} \backslash\{0\}} \psi_{n}(k) \frac{|k|^{\alpha}}{g(|k|)}\left|v_{k}\right|^{2}=\chi_{n}(t) X_{n}^{2}(t)
$$

Now consider the second sum. Since the terms with $h=k$ give no contribution, we can apply

$$
\sum_{l \in \mathbb{N}_{0}} \psi_{l}(h)=\sum_{m \in \mathbb{N}_{0}} \psi_{m}(k-h)=1, \quad h, k \in \mathbb{Z}^{d}, \quad 0 \neq h \neq k
$$

to get

$$
\begin{aligned}
& \sum_{\substack{h, k \in \mathbb{Z}^{d} \\
h \neq 0}} \psi_{n}(k) \frac{\operatorname{Im}\left\{\left\langle v_{h}, k\right\rangle\left\langle v_{k-h}, v_{k}\right\rangle\right\}}{|h|^{\beta}} \\
&=\sum_{\substack{h, k \in \mathbb{Z}^{d} \\
h \neq 0}} \sum_{l, m \in \mathbb{N}_{0}} \psi_{l}(h) \psi_{m}(k-h) \psi_{n}(k) \frac{\operatorname{Im}\left\{\left\langle v_{h}, k\right\rangle\left\langle v_{k-h}, v_{k}\right\rangle\right\}}{|h|^{\beta}} \\
&=\sum_{\substack{l, m \in \mathbb{N}_{0}}} \sum_{\substack{h, k \in \mathbb{Z}^{d} \\
h \neq 0}} \psi_{l}(h) \psi_{m}(k-h) \psi_{n}(k) \frac{\operatorname{Im}\left\{\left\langle v_{h}, k\right\rangle\left\langle v_{k-h}, v_{k}\right\rangle\right\}}{|h|^{\beta}},
\end{aligned}
$$

where it was possible to exchange the order of summation because the middle expression is clearly absolutely convergent.

Now define $\phi$ as in Proposition 2.11. By applying (2.8) or (2.2) depending on $X_{l}(t) X_{m}(t) X_{n}(t)$ being positive or zero, we see that for all $l, m, n \in \mathbb{N}_{0}$ and $t \geq 0$,

$$
2 \sum_{\substack{h, k \in \mathbb{Z}^{d} \\ h \neq 0}} \psi_{l}(h) \psi_{m}(k-h) \psi_{n}(k) \frac{\operatorname{Im}\left\{\left\langle v_{h}, k\right\rangle\left\langle v_{k-h}, v_{k}\right\rangle\right\}}{|h|^{\beta}}=
$$

$$
=\phi_{(l, m, n)}(t) X_{l}(t) X_{m}(t) X_{n}(t) .
$$

Putting all together we get

$$
\frac{d}{d t} X_{n}^{2}(t)=-\chi_{n}(t) X_{n}^{2}(t)+\sum_{l, m \in \mathbb{N}_{0}} \phi_{(l, m, n)}(t) X_{l}(t) X_{m}(t) X_{n}(t) \quad n \in \mathbb{N}_{0}, t \geq 0
$$

Finally recalling by Proposition 2.11 that $\phi \equiv 0$ outside $I$, we may restrict the scope of the sum and obtain equation (2.5). The required properties of the coefficients $\chi$ and $\psi$ follow again from Propositions 2.10-2.11.

## 3 From the dyadic equation to the recursive inequality

In view of the results of the previous section, we can now concentrate on shell solutions and forget equation (1.6). In this section we proceed as in [BMR14] and we deduce a recursive inequality between the tails of energy and dissipation. Clearly here, due to the more complex non-linear interaction, the relation is less trivial than in [BMR14].

Definition 3.1. A shell solution $X$ satisfies the energy inequality on $[0, T]$ if the sum $\sum_{n} X_{n}^{2}(0)$ is finite and

$$
\begin{equation*}
\sum_{n \in \mathbb{N}_{0}} X_{n}^{2}(t)+\int_{0}^{t} \sum_{n \in \mathbb{N}_{0}} \chi_{n}(s) X_{n}^{2}(s) d s \leq \sum_{n \in \mathbb{N}_{0}} X_{n}^{2}(0), \quad t \in[0, T] . \tag{3.1}
\end{equation*}
$$

Definition 3.2. Let $X$ be a shell solution and define the sequences of real valued maps $\left(F_{n}\right)_{n \in \mathbb{N}_{0}}$ and $\left(d_{n}\right)_{n \in \mathbb{N}_{0}}$ for $t \geq 0$ by

$$
\begin{gathered}
F_{n}(t):=\sum_{k \geq n} X_{k}^{2}(t), \\
d_{n}(t):=\left(F_{n}(t)+\sum_{h \geq n} \int_{0}^{t} \chi_{h}(s) X_{h}^{2}(s) d s\right)^{\frac{1}{2}} .
\end{gathered}
$$

We will call $\left(F_{n}\right)_{n \in \mathbb{N}_{0}}$ the tail of $X$ and $\left(d_{n}\right)_{n \in \mathbb{N}_{0}}$ the energy bound of $X$.

The recursive inequality between the tails and the energy bound is given in the next result.

Proposition 3.3. Let $X$ be a shell solution that satisfies the energy inequality on a time interval $[0, t]$, let $\left(d_{n}\right)_{n \in \mathbb{N}_{0}}$ be its sequence of energy bounds, and set $\lambda=2^{\alpha}$.

Then there is a positive constant $c_{4}>0$ such that for all $n \in \mathbb{N}_{0}$,

$$
\begin{equation*}
d_{n}^{2}(t) \leq F_{n}(0)+c_{4} \sum_{l=0}^{n-1} \frac{\bar{d}_{l}}{\lambda^{n-l}} \sum_{m \geq n-2} \frac{g\left(2^{m+1}\right)}{\lambda^{m-n}}\left(d_{m}^{2}(t)-d_{m+1}^{2}(t)\right), \tag{3.2}
\end{equation*}
$$

where $\bar{d}_{l}:=\max _{s \in[0, t]} d_{l}(s)$.
Proof. Fix $n \in \mathbb{N}_{0}$. Differentiate $\sum_{h=0}^{n-1} X_{h}^{2}$ using equation (2.5),

$$
\frac{d}{d t} \sum_{h=0}^{n-1} X_{h}^{2}=-\sum_{h=0}^{n-1} \chi_{h} X_{h}^{2}+\sum_{\substack{l, m, h \in \mathbb{N}_{0} \\(l, m, h \in I \\ h \leq n-1}} \phi_{(l, m, h)} X_{l} X_{m} X_{h}
$$

Apply Lemma 3.4 below to the second sum and integrate on $[0, t]$ to obtain

$$
\sum_{h=0}^{n-1} X_{h}^{2}(t)-\sum_{h=0}^{n-1} X_{h}^{2}(0)=-\int_{0}^{t} \sum_{h=0}^{n-1} \chi_{h} X_{h}^{2} d s-\int_{0}^{t} \sum_{\substack{(l, m, h) \in I \\ m<n \leq h}} \phi_{(l, m, h)} X_{l} X_{m} X_{h} d s
$$

so that by the energy inequality (3.1),

$$
F_{n}(t)+\int_{0}^{t} \sum_{h \geq n} \chi_{h}(s) X_{h}^{2}(s) d s \leq F_{n}(0)+\int_{0}^{t} \sum_{\substack{(l, m, h) \in I \\ m<n \leq h}} \phi_{(l, m, h)} X_{l}(s) X_{m}(s) X_{h}(s) d s,
$$

where the $F_{n}$ are the tails of $X$ and $F_{n}(0)<\infty$ by hypothesis. Thus by (3.2),

$$
d_{n}^{2}(t) \leq F_{n}(0)+\int_{0}^{t} \sum_{\substack{(l, m, h) \in I \\ m<n \leq h}} \phi_{(l, m, h)} X_{l}(s) X_{m}(s) X_{h}(s) d s
$$

Recall that $\alpha+\beta \geq \frac{d}{2}+1$, hence the bound (2.6) for $\phi$ yields $\phi_{(l, m, h)} \leq c_{2} \lambda^{\min \{l, m, h\}}$. Therefore

$$
d_{n}^{2}(t) \leq F_{n}(0)+\int_{0}^{t} \sum_{\substack{(l, m, h) \in I \\ m<n \leq h}} c_{2} \lambda^{\min \{l, m\}}\left|X_{l}(s) X_{m}(s) X_{h}(s)\right| d s
$$

It is convenient to split the set over which the sum is done into $\{l<m\}$ and $\{m \leq l\}$,

$$
\begin{gathered}
\sum_{\substack{(l, m, h) \in I \\
m<n \leq h}} \lambda^{\min \{l, m\}}\left|X_{l} X_{m} X_{h}\right| \leq \sum_{\substack{(l, m, h) \in I \\
l<m<n \leq h}} \lambda^{l}\left|X_{l} X_{m} X_{h}\right|+\sum_{\substack{(l, m, h) \in I \\
m<n \leq h \\
m \leq l}} \lambda^{m}\left|X_{l} X_{m} X_{h}\right| \\
\leq \sum_{\substack{(l, m, h) \in I \\
l<m<n \leq h}} \lambda^{l}\left|X_{l} X_{m} X_{h}\right|+\sum_{\substack{(l, m, h) \in I \\
l n \leq h \\
l \leq m}} \lambda^{l}\left|X_{l} X_{m} X_{h}\right| \\
\quad \leq 2 \sum_{\substack{(l, m, h) \in I \\
l<n \leq h \\
l \leq m}} \lambda^{l}\left|X_{l} X_{m} X_{h}\right| \leq 2 \sum_{l=0}^{n-1} \lambda^{l} \bar{d}_{l} \sum_{h \geq n} \sum_{m=h-2}^{h+2}\left|X_{m} X_{h}\right| .
\end{gathered}
$$

Apply the Cauchy-Schwarz inequality to get

$$
2 \sum_{h \geq n} \sum_{m=h-2}^{h+2}\left|X_{h} X_{m}\right| \leq \sum_{h \geq n} \sum_{m=h-2}^{h+2}\left(X_{h}^{2}+X_{m}^{2}\right) \leq 10 \sum_{m \geq n-2} X_{m}^{2} .
$$

Then by the bound on $\chi$ in (2.6), on all $[0, t]$,

$$
\sum_{m \geq n-2} X_{m}^{2} \leq c_{1}^{-1} \sum_{m \geq n-2} \frac{g\left(2^{m+1}\right)}{\lambda^{m}} \chi_{m} X_{m}^{2}
$$

Finally the integral of $\chi_{m} X_{m}^{2}$ can be bounded using (3.2),

$$
d_{m}^{2}(t)-d_{m+1}^{2}(t)=F_{m}(t)-F_{m+1}(t)+\int_{0}^{t} \chi_{m}(s) X_{m}^{2}(s) d s \geq \int_{0}^{t} \chi_{m}(s) X_{m}^{2}(s) d s
$$

Putting all together we obtain

$$
d_{n}^{2}(t) \leq F_{n}(0)+10 \frac{c_{2}}{c_{1}} \sum_{l=0}^{n-1} \frac{\bar{d}_{l}}{\lambda^{-l}} \sum_{m \geq n-2} \frac{g\left(2^{m+1}\right)}{\lambda^{m}}\left(d_{m}^{2}(t)-d_{m+1}^{2}(t)\right),
$$

thus proving equation (3.2) with $c_{4}=10 \frac{c_{2}}{c_{1}}$.
Lemma 3.4. Let $X$ be a shell solution, then for all $n \in \mathbb{N}_{0} \backslash\{0\}$ and $s \in[0, t]$,

$$
\begin{equation*}
\sum_{\substack{(l, m, h) \in I \\ h \leq n-1}} \phi_{(l, m, h)} X_{l} X_{m} X_{h}=-\sum_{\substack{(l, m, h) \in I \\ m \leq n-1<h}} \phi_{(l, m, h)} X_{l} X_{m} X_{h} . \tag{3.3}
\end{equation*}
$$

Proof. By using (2.6), noticing that $\min (l, m, h) \leq n-1$, we see that by definition of shell solution (Definition 2.4) the left-hand side of (3.3) is an absolutely convergent sum. Therefore we can exploit the cancellations due to the antisymmetry of $\phi$, as in Remark 2.7. Indeed

$$
\begin{equation*}
\sum_{\substack{(l, m, h) \in I \\ h \leq n-1}} \phi_{(l, m, h)} X_{l} X_{m} X_{h}=\sum_{\substack{(l, m, h) \in I \\ m<h \leq n-1}} \phi_{(l, m, h)} X_{l} X_{m} X_{h}+\sum_{\substack{(l, m, h) \in I \\ h \leq n-1 \\ m>h}} \phi_{(l, m, h)} X_{l} X_{m} X_{h}, \tag{3.4}
\end{equation*}
$$

and

$$
\begin{align*}
& \sum_{\substack{(l, m, h) \in I \\
h \leq n-1 \\
m>h}} \phi_{(l, m, h)} X_{l} X_{m} X_{h}=-\sum_{\substack{(l, m, h) \in I \\
h \leq n-1 \\
m>h}} \phi_{(l, h, m)} X_{l} X_{m} X_{h}= \\
& =-\sum_{\substack{\left(l, h^{\prime}, m^{\prime}\right) \in I \\
m^{\prime} \leq n-1}}^{h^{\prime}>m^{\prime}} ⿺
\end{align*} \phi_{\left(l, m^{\prime}, h^{\prime}\right)} X_{l} X_{m^{\prime}} X_{h^{\prime}}=-\sum_{\substack{\left(l, m^{\prime}, h^{\prime}\right) \in I  \tag{3.5}\\
m^{\prime} \backslash-1 \\
m^{\prime}<h^{\prime}}} \phi_{\left(l, m^{\prime}, h^{\prime}\right)} X_{l} X_{m^{\prime}} X_{h^{\prime}} .
$$

By using (3.5) into (3.4) the conclusion follows.

## 4 Solving the recursion

In this section we complete the proof of our main result. In the previous section we have shown a recursive inequality involving the energy bounds of a shell solution. The following theorem shows that shell solutions are smooth. By Theorem 2.8 the shell approximation of a solution of (1.6) is a shell solution, hence Theorem 2.3 holds, and in turns Theorem 1.2 holds as well.

Theorem 4.1. Let $X$ be a shell solution satisfying the energy inequality on $[0, t)$. If $\sup _{n} 2^{m n}\left|X_{n}(0)\right|<\infty$ for every $m \geq 1$, then

$$
\sup _{s \in[0, t]} \sup _{n} 2^{m n}\left|X_{n}(s)\right|<\infty
$$

for every $m \geq 1$.
Let $b_{n}=g\left(2^{n+1}\right)^{-1}, n \geq 0$, then the assumptions of Theorem 1.2 for $g$ read in terms of the sequence $b$ as

- $\left(b_{n}\right)_{n \in \mathbb{N}}$ non-increasing,
- $\left(\lambda^{n} b_{n}\right)_{n \in \mathbb{N}}$ non-decreasing,
- $\sum_{n} b_{n}=\infty$.

Let $X$ be a shell solution as in the statement of Theorem 4.1, denote by $\left(d_{n}\right)_{n \in \mathbb{N}}$ and $\left(F_{n}\right)_{n \in \mathbb{N}}$ the energy bound and the tail of $X$ (see Definition 3.2), and set $\bar{d}_{n}=\sup _{[0, t]} d_{n}(t)$ for every $n$. Set

$$
Q_{n}=\sum_{j=0}^{n-1} \frac{\bar{d}_{j}}{\lambda^{n-j}}
$$

and

$$
R_{n}(t)=\sum_{j \geq n} \frac{d_{j}(t)^{2}-d_{j+1}(t)^{2}}{\lambda^{j-n} b_{j}}
$$

where $\lambda=2^{\alpha}$ as in the previous section. We recall that, by Proposition 3.3, the following inequality holds,

$$
\begin{equation*}
d_{n}(t)^{2} \leq F_{n}(0)+c_{4} Q_{n} R_{n-2}(t) \tag{4.1}
\end{equation*}
$$

In the following lemma we collect some properties of the quantities $R_{n}, Q_{n}, \bar{d}_{n}$ that will be crucial in the proof of Theorem 4.1 above.

Lemma 4.2. The following properties hold.

1. For every $1 \leq m_{1} \leq m_{2}$ and $t>0$,

$$
\begin{equation*}
\min \left\{R_{m_{1}}(t), R_{m_{1}+1}(t) \ldots, R_{m_{2}}(t)\right\} \leq \frac{\lambda}{\lambda-1} \frac{d_{m_{1}}(t)^{2}}{\sum_{n=m_{1}}^{m_{2}} b_{n}} \tag{4.2}
\end{equation*}
$$

2. For every $t>0, \liminf _{n} R_{n}(t)=0$.
3. $\bar{d}_{n} \downarrow 0$ as $n \rightarrow \infty$.
4. $Q_{n} \rightarrow 0$ as $n \rightarrow \infty$.
5. $\left(Q_{n}\right)_{n \geq 1}$ is eventually non-increasing.

Proof. Since $\lambda^{n} b_{n}$ is non-decreasing, we know that $b_{n}-\lambda^{-1} b_{n-1} \geq 0$. Hence by exchanging the sums,

$$
\begin{aligned}
& \sum_{n=m_{1}}^{\infty}\left(b_{n}-\lambda^{-1} b_{n-1}\right) R_{n}(t)= \\
& =\sum_{k=m_{1}}^{\infty} \frac{d_{k}(t)^{2}-d_{k+1}(t)^{2}}{\lambda^{k} b_{k}} \sum_{n=m_{1}}^{k}\left(\lambda^{n} b_{n}-\lambda^{n-1} b_{n-1}\right) \leq \\
& \quad \leq \sum_{k=m_{1}}^{\infty}\left(d_{k}(t)^{2}-d_{k+1}(t)^{2}\right) \leq d_{m_{1}}(t)^{2}
\end{aligned}
$$

If $m_{2} \geq m_{1}$, since $\left(b_{n}\right)_{n \geq 1}$ is non-increasing,

$$
\begin{aligned}
\sum_{n=m_{1}}^{m_{2}}\left(b_{n}-\lambda^{-1} b_{n-1}\right) R_{n}(t) & \geq \min \left\{R_{m_{1}}(t), \ldots, R_{m_{2}}(t)\right\} \sum_{n=m_{1}}^{m_{2}}\left(b_{n}-\lambda^{-1} b_{n-1}\right) \\
& \geq \frac{\lambda-1}{\lambda}\left(\sum_{n=m_{1}}^{m_{2}} b_{n}\right) \min \left\{R_{m_{1}}(t), \ldots, R_{m_{2}}(t)\right\}
\end{aligned}
$$

The claim $\lim \inf _{n} R_{n}(t)=0$ follows from (4.2), since $d_{n}(t) \leq d_{1}(t)$ for every $n$, and since, by the assumptions on $\left(b_{n}\right)_{n \geq 1}$, we can find a sequence $\left(m_{k}\right)_{k \geq 1}$ such that $\sum_{n=m_{k}}^{m_{k+1}-1} b_{n} \uparrow \infty$.

To prove that $\bar{d}_{n} \downarrow 0$, we notice that the sequence $\left(m_{k}\right)_{k \geq 1}$ mentioned above does not depend on $t$, hence using the monotonicity of $\left(d_{n}(t)\right)_{n \geq 1}$ and formula (4.2), we can prove that $\lim \inf _{n} \bar{d}_{n}=0$, and hence $\bar{d}_{n} \downarrow 0$ by monotonicity. Once we know that $\bar{d}_{n} \downarrow 0$, an easy and standard argument proves that $Q_{n} \rightarrow 0$.

To prove that $\left(Q_{n}\right)_{n \geq 1}$ is eventually non-increasing, we notice that, since $\left(\bar{d}_{n}\right)_{n \geq 1}$ is non-increasing,

$$
\left(Q_{n+1}-Q_{n}\right)=\frac{1}{\lambda}\left(Q_{n}-Q_{n-1}\right)+\frac{1}{\lambda}\left(\bar{d}_{n}-\bar{d}_{n-1}\right) \leq \frac{1}{\lambda}\left(Q_{n}-Q_{n-1}\right) .
$$

In view of the above inequality, it is sufficient to show that for some $m$ the increment $Q_{m}-Q_{m-1} \leq 0$. This is true because otherwise the sequence $\left(Q_{n}\right)_{n \geq 1}$ would be non-decreasing, in contradiction with $Q_{n} \rightarrow 0$ and $Q_{n} \geq 0$.

Given $\theta>0$ and $n_{0} \geq 1$, define by recursion the sequence

$$
\begin{equation*}
n_{k+1}=2+\min \left\{n \geq n_{k}-1: \sum_{j=n_{k}-1}^{n} b_{j} \geq \theta \lambda^{-\frac{k}{4}}\right\} . \tag{4.3}
\end{equation*}
$$

The definition of $Q_{n}$ and the fact that the sequence $\left(\bar{d}_{n}\right)_{n \geq 1}$ is non-increasing yield the following recursive formula for $Q_{n_{k}}$,

$$
\begin{equation*}
Q_{n_{k+1}}=\frac{1}{\lambda^{n_{k+1}-n_{k}}} Q_{n_{k}}+\sum_{j=n_{k}}^{n_{k+1}-1} \frac{\bar{d}_{j}}{\lambda^{n_{k+1}-j}} \leq \frac{1}{\lambda} Q_{n_{k}}+c \bar{d}_{n_{k}} \tag{4.4}
\end{equation*}
$$

for a constant $c>0$ depending only from $\lambda$. Moreover, if we choose $n_{0}$ large enough that $\left(Q_{n}\right)_{n \geq 0}$ is non-increasing,

$$
d_{n_{k+1}}(t)^{2} \leq d_{n}(t)^{2} \leq F_{n}(0)+c_{4} Q_{n} R_{n-2}(t) \leq F_{n_{k}}(0)+c_{4} Q_{n_{k}} R_{n-2}(t)
$$

for each $n \in\left\{n_{k}+1, \ldots, n_{k+1}\right\}$, hence by formula (4.2) and the definition of the sequence $\left(n_{k}\right)_{k \geq 1}$,

$$
\begin{aligned}
d_{n_{k+1}}(t)^{2} \leq F_{n_{k}}(0)+ & c_{4} Q_{n_{k}} \min \left\{R_{n_{k}-1}, \ldots, R_{n_{k+1}-2}\right\} \leq \\
& \leq F_{n_{k}}(0)+c Q_{n_{k}} \frac{d_{n_{k}-1}(t)^{2}}{\sum_{n_{k}-1}^{n_{k+1}-2} b_{j}} \leq F_{n_{k}}(0)+c \frac{\lambda^{\frac{k}{4}}}{\theta} Q_{n_{k}} d_{n_{k}-1}(t)^{2}
\end{aligned}
$$

and in conclusion,

$$
\begin{equation*}
\bar{d}_{n_{k+1}}^{2} \leq F_{n_{k}}(0)+c \frac{\lambda^{\frac{k}{4}}}{\theta} Q_{n_{k}} \bar{d}_{n_{k}-1}^{2} \tag{4.5}
\end{equation*}
$$

Lemma 4.3 (initial step of the cascade). Given $M>0$, there are $n_{0} \geq 1$ and $\theta>0$ such that

$$
\begin{gathered}
Q_{n_{k}} \leq \lambda^{-\frac{k}{2}} \\
\bar{d}_{n_{k}}^{2} \leq \lambda^{-M k}
\end{gathered}
$$

for all $k \geq 0$.
Proof. Without loss of generality we can choose $M$ large (depending only on the value of $\lambda$, see below at the end of the proof). Choose $n_{0}$ large enough that $\left(Q_{n}\right)_{n \geq n_{0}}$ is non-increasing and

$$
Q_{n_{0}-i} \leq \epsilon, \quad \bar{d}_{n_{0}-i} \leq \epsilon, \quad i=0,1, \quad \text { and } \quad \lambda^{M n} F_{n}(0) \leq \epsilon, \quad n \geq n_{0}
$$

for a number $\epsilon \in(0,1)$ suitably chosen below. We will prove by induction that

$$
\begin{equation*}
Q_{n_{k}-i} \leq \lambda^{-\frac{1}{2}(k-i)}, \quad \bar{d}_{n_{k-i}}^{2} \leq \lambda^{-M(k-i)}, \quad i=0,1, \quad k \geq 1 . \tag{4.6}
\end{equation*}
$$

For the initial step of the induction $(k=1)$, we notice that by (4.4) and (4.5),

$$
\begin{gathered}
Q_{n_{1}} \leq \frac{1}{\lambda} Q_{n_{0}}+c \bar{d}_{n_{0}} \leq \frac{\epsilon}{\lambda}+c \epsilon \leq \frac{1}{\lambda^{1 / 2}} \\
\bar{d}_{n_{1}}^{2} \leq F_{n_{0}}(0)+\frac{c}{\theta} Q_{n_{0}} \bar{d}_{n_{0}-1}^{2} \leq \epsilon+\frac{c}{\theta} \epsilon^{3} \leq \lambda^{-M},
\end{gathered}
$$

if we choose $\epsilon$ small enough, depending from the values of $\lambda, M$, and $\theta$.
Assume now that (4.6) holds for some $k \geq 1$, and let us prove that the same holds for $k+1$. To this end it is sufficient to give the estimate for $Q_{n_{k+1}}$ and $\bar{d}_{n_{k+1}}^{2}$. Again by (4.4), (4.5) and the induction hypothesis, and since $\left(n_{k}\right)_{k \geq 0}$ is increasing by definition,

$$
\begin{gathered}
Q_{n_{k+1}} \leq \frac{1}{\lambda} Q_{n_{k}}+c \bar{d}_{n_{k}} \leq \lambda^{-\frac{k}{2}-1}+c \lambda^{-\frac{M}{2} k} \leq \lambda^{-\frac{1}{2}(k+1)} \\
\bar{d}_{n_{k+1}}^{2} \leq F_{n_{k}}(0)+c \frac{\lambda^{\frac{k}{4}}}{\theta} Q_{n_{k}} \bar{d}_{n_{k}-1}^{2} \leq \epsilon \lambda^{-M k}+\frac{c}{\theta} \lambda^{-\frac{k}{4}} \lambda^{-M(k-1)} \leq \lambda^{-M(k+1)}
\end{gathered}
$$

if $M$ is large (depending on $\lambda$ ), and $\epsilon$ is small and $\theta$ is large (depending only on $M, \lambda)$.

Before giving the last step of the proof of Theorem 4.1, we show a property of the sequence $\left(n_{k}\right)_{k \geq 0}$. The proof is the same as [BMR14, Lemma 11], we detail it for completeness.

Lemma 4.4. Given $n_{0} \geq 1$ and $\theta>0$, consider the sequence defined in (4.3). For infinitely many $k, n_{k+1}=n_{k}+1$. In particular $b_{n_{k}-1} \geq \theta \lambda^{-k / 4}$ for all such $k$.

Proof. Assume by contradiction that there is $r$ such that $n_{k+1} \geq n_{k}+2$ for $k \geq r$. On the one hand

$$
\sum_{j=n_{k}-1}^{n_{k+1}-3} b_{j} \leq \theta \lambda^{-\frac{k}{4}},
$$

and summing up in $k \geq r$ yields

$$
\sum_{k \geq r} \sum_{j=n_{k}-1}^{n_{k+1}-3} b_{j}<\infty \quad \rightsquigarrow \quad \sum_{k} b_{n_{k}-2}=\infty .
$$

On the other hand, $b_{n_{k}-2} \leq b_{n_{k}-3} \leq \theta \lambda^{-\frac{1}{4}(k-1)}$ and the series $\sum_{k} b_{n_{k}-2}$ converges.

Lemma 4.5 (cascade recursion). For every $M>0$ there is $c_{M}>0$ such that

$$
\bar{d}_{n}^{2} \leq c_{M} \lambda^{-M n}, \quad Q_{n} \leq c_{M} \lambda^{-n}
$$

Proof. There is no loss of generality if we assume $M$ is large. Let $n_{0}, \theta$ be the values provided by Lemma 4.3. By Lemma 4.3 and Lemma 4.4 there are infinitely many $k \geq 1$ such that

$$
\begin{equation*}
b_{n_{k}-1} \geq \theta \lambda^{-\frac{k}{4}}, \quad Q_{n_{k}} \leq \lambda^{-\frac{k}{2}}, \quad \bar{d}_{n_{k}}^{2} \leq \lambda^{-M k} \tag{4.7}
\end{equation*}
$$

Let $k_{0}$ be one of such indices, large enough (the size of $k_{0}$ will be chosen at the end of the proof). We will prove by induction that

$$
\begin{equation*}
\bar{d}_{n_{k_{0}}+m}^{2} \leq c \lambda^{-M m}, \quad Q_{n_{k_{0}}+m} \leq c^{\prime} \lambda^{-m}, \quad b_{n_{k_{0}}-1+m} \geq \theta \lambda^{-\frac{k_{0}}{4}-m} \tag{4.8}
\end{equation*}
$$

for a suitable choice of the constants $c>0, c^{\prime}>0$. We first notice that there is nothing to prove concerning $b_{n_{k_{0}-1+m}}$, since this is a straightforward consequence of the choice of $k_{0}$ and the monotonicity of $\left(\lambda^{n} b_{n}\right)_{n \geq 1}$.

The initial step $m=0$ holds, since inequalities (4.7) hold true for the index $k_{0}$. For $m=1$,

$$
\begin{gathered}
\bar{d}_{n_{k_{0}}+1}^{2} \leq \bar{d}_{n_{k_{0}}}^{2} \leq c \lambda^{-M}, \\
Q_{n_{k_{0}}+1}=\frac{1}{\lambda} Q_{n_{k_{0}}}+\frac{1}{\lambda} \bar{d}_{n_{k_{0}}} \leq \frac{1}{\lambda}\left(\lambda^{-\frac{k_{0}}{2}}+\lambda^{-\frac{M}{2} k_{0}}\right) \leq \frac{c^{\prime}}{\lambda},
\end{gathered}
$$

if $c=\lambda^{-M\left(k_{0}-1\right)}$ and $c^{\prime} \geq \lambda^{-k_{0} / 2}+\lambda^{-M k_{0} / 2}$.
Assume that (4.8) holds for $1, \ldots, m$, for some $m \geq 1$. By its definition,

$$
\begin{aligned}
Q_{n_{k_{0}}+m+1} & =Q_{n_{k_{0}}} \lambda^{-(m+1)}+\sum_{j=n_{k_{0}}}^{n_{k_{0}}+m} \frac{\bar{d}_{j}}{\lambda^{n_{k_{0}}+m+1-j}} \\
& \leq \lambda^{-\frac{k_{0}}{2}-(m+1)}+\sqrt{c} \lambda^{-(m+1)} \sum_{j=0}^{m} \lambda^{-\left(\frac{M}{2}-1\right) j} \\
& \leq\left(\lambda^{-\frac{k_{0}}{2}}+\frac{\lambda}{\lambda-1} \sqrt{c}\right) \lambda^{-(m+1)} \\
& \leq c^{\prime} \lambda^{-(m+1)}
\end{aligned}
$$

if $c^{\prime}=\lambda^{-\frac{k_{0}}{2}}+\lambda(\lambda-1)^{-1} \sqrt{c}$ (the previous constraint on $c^{\prime}$ is met by this choice).
By (4.1) and (4.2) we have that for every $n \geq 2$,

$$
d_{n+1}(t)^{2} \leq F_{n+1}(0)+04 Q_{n+1} R_{n-1}(t) \leq F_{n+1}(0)+c_{4} Q_{n+1} \frac{\bar{d}_{n-1}^{2}}{b_{n-1}}
$$

hence, using the inequality for $Q_{n_{k_{0}}+m+1}$ already proved and the induction hypothesis,

$$
\begin{aligned}
\bar{d}_{n_{k_{0}}+m+1}^{2} & \leq F_{n_{k_{0}}+m+1}(0)+c_{4} Q_{n_{k_{0}+m+1}} \frac{\bar{d}_{n_{k_{0}}+m-1}^{2}}{b_{n_{k_{0}}+m-1}} \\
& \leq c \lambda^{-M(m+1)}\left(\lambda^{M\left(n_{k_{0}}+m+1\right)} F_{n_{k_{0}}+m+1}(0)+\frac{c_{4}}{\theta} c^{\prime} \lambda^{2 M+\frac{k_{0}}{4}}\right) \\
& \leq c 2^{-M(m+1)}
\end{aligned}
$$

where the last inequality follows if $k_{0}$ is large enough, since $\lambda^{n} F_{n}(0) \rightarrow 0$ by assumption, and by our choice of $c, c^{\prime}$, we have that $\lambda^{k_{0} / 4} c^{\prime} \rightarrow 0$ as $k_{0} \rightarrow \infty$.

## A Local existence and uniqueness

Consider the generalised system (1.6), under the same assumptions of Theorem 1.2. Assume ${ }^{3}$, for simplicity, that $m_{1}(k)=\frac{|k|^{\alpha}}{g(|k|)}$. Denote by $V_{m}$ the subspace of $H^{m}$ (see (2.3)) of divergence free vector fields with mean zero. Our main theorem on local existence and uniqueness for (1.6) is as follows.

[^2]Theorem A.1. Let $m \geq 2+\frac{d}{2}$ and $v_{0} \in V_{m}$. Then there are $T>0$ and a unique solution $v$ of (1.6) on $[0, T]$ with initial condition $v_{0}$ such that

$$
\begin{align*}
v \in L^{\infty}\left([0, T] ; V_{m}\right) & \cap \operatorname{Lip}\left([0, T] ; V_{m-\alpha}\right) \cap C\left([0, T] ; V_{m}^{\text {weak }}\right), \\
& \int_{0}^{T}\left\|D_{1}^{\frac{1}{2}} v\right\|_{m}^{2} d t<\infty, \tag{A.1}
\end{align*}
$$

where $V_{m}^{\text {weak }}$ is the space $V_{m}$ with the weak topology. Moreover, $v$ is right-continuous with values in $V_{m}$ for the strong topology.

If $T_{\star}$ is the maximal time of existence of the solution started from $v_{0}$, then either $T_{\star}=\infty$ or

$$
\limsup _{t \uparrow T_{\star}}\|v(t)\|_{m}=\infty
$$

The proof of the theorem is based on a proof of existence of a local unique solution for the Euler equation taken from [MB02, Section 3.2]. The idea is that we cannot use the $D_{1}$ operator as a replacement for the Laplacian, since in general $D_{1}$ may not have smoothing properties (indeed, it is easy to adapt the counterexample in [BMR14, Remark 15] to $D_{1}$ on $\mathbb{R}^{d}$ or on the $d$-dimensional torus). Likewise we do not use any smoothing properties of $D_{2}$, so that our proof includes the case $\beta=0$. The result is by no means optimal, but fits the needs of our paper.

We work on the torus $[0,2 \pi]^{d}$, although the proof, essentially unchanged, works in $\mathbb{R}^{d}$. Denote by $H$ the projection of $L^{2}\left([0,2 \pi]^{d}\right)$ onto divergence free vector fields, and for every $s>0$, by $V_{s}$ the projection of the Sobolev space $H^{s}\left([0,2 \pi]^{d}\right)$ onto divergence free vector fields. We will denote by $\|\cdot\|_{H}$ and by $\langle\cdot, \cdot\rangle_{H}$ the norm and the scalar product in $H$, and by $\|\cdot\|_{s}$ and by $\langle\cdot, \cdot\rangle_{s}$ the norm and the scalar product in $V_{s}$.

We denote by $\hat{B}\left(v_{1}, v_{2}\right)$ the (Leray) projection of the non-linearity, namely

$$
\hat{B}\left(v_{1}, v_{2}\right)=\Pi_{\text {Leray }}\left[\left(\left(D_{2}^{-1} v_{1} \cdot \nabla\right) v_{2}\right] .\right.
$$

Since $\beta \geq 0,\left\|D_{2}^{-1} v\right\|_{s} \leq\|v\|_{s}$ for every $s \in \mathbb{R}$. Hence, (see for instance [Kat72], or [CF88]), for every $m \geq 1+\left[\frac{d}{2}\right]$, there exists $c_{m}>0$ such that

$$
\begin{aligned}
& \left\|\hat{B}\left(v_{1}, v_{2}\right)\right\|_{m} \leq c_{m}\left\|v_{1}\right\|_{m}\left\|v_{2}\right\|_{m+1}, \\
& \left\langle\hat{B}\left(v_{1}, v_{2}\right), v_{2}\right\rangle_{m} \leq c_{m}\left\|v_{1}\right\|_{m}\left\|v_{2}\right\|_{m}^{2}
\end{aligned}
$$

In the rest of the section we briefly outline the proof of Theorem A.1, following [MB02, Section 3.2]. The proof of the following result is a slight modification of the arguments to prove [MB02, Theorem 3.4].

Proposition A.2. Given an integer $m \geq 2+\frac{d}{2}$, there exists a number $c_{\star}>0$ such that for every $v_{0} \in V_{m}$, if $T<c_{\star} /\left\|v_{0}\right\|_{m}$, there is a unique solution of (1.6)
with initial condition $v_{0}$. Moreover $v_{\epsilon} \rightarrow v$ in $C\left([0, T] ; V_{m^{\prime}}\right)$, for $m^{\prime}<m$, and in $C\left([0, T] ; V_{m}^{\text {weak }}\right)$, (A.1) hold for $v$, and for every $\epsilon>0$,

$$
\begin{equation*}
\sup _{[0, T]}\left\|v_{\epsilon}\right\|_{m} \leq \frac{\left\|v_{0}\right\|_{m}}{1-c_{\star} T\left\|v_{0}\right\|_{m}} . \tag{A.2}
\end{equation*}
$$

Unfortunately, at this stage, we cannot prove the analogous of Theorem 3.5 of [MB02] for our $v$, namely that $v$ is continuous in time for the strong topology of $V_{m}$. The reason is that their proof uses either the reversibility of the Euler equation (that we do not have due to the presence of $D_{1}$ ), or the smoothing of the Laplace operator, that we do not have here either (as already mentioned). On the other hand we can prove right-continuity.

Lemma A.3. The solution $v$ from Proposition A. 2 is right-continuous with values in $V_{m}$ for the strong topology, and $\frac{d}{d t} v$ is right continuous with values in $V_{m-\alpha}$.

Proof. Given $t \in[0, T]$, the same computations leading to (A.2) yield

$$
\sup _{[0, t]}\|v(s)\|_{m} \leq\left\|v_{0}\right\|_{m}+\frac{c_{\star} t\left\|v_{0}\right\|_{m}^{2}}{1-c_{\star} t\left\|v_{0}\right\|_{m}},
$$

therefore $\lim \sup _{t \downarrow 0}\|v(t)\|_{m} \leq\left\|v_{0}\right\|_{m}$. On the other hand, by weak continuity, $\left\|v_{0}\right\|_{m} \leq \liminf _{t \downarrow 0}\|v(t)\|_{m}$ and $v$ is right continuous in 0 . Uniqueness for (1.6) and the same argument applied to $t \in(0, T]$ yield right-continuity in $t$.

Nevertheless, we can still define a maximal solution and a maximal time of existence. Given $v_{0} \in V_{m}$, let $T_{\star}$ be the maximal time of existence of the solution starting from $v_{0}$, that is the supremum over all $T>0$ such that there exists a solution $v$ of (1.6) on $[0, T]$ with $v(0)=u_{0}, v$ is right-continuous with values in $V_{m}$, continuous with values in $V_{m}^{\text {weak }}$ and with $\frac{d}{d t} v$ right continuous with values in $V_{m-\alpha}$. Due to uniqueness, any two such solutions coincide on the common interval of definition.

Proposition A.4. Given $v_{0} \in V_{m}$, if $T_{\star}$ is the maximal time of existence of the solution started from $v_{0}$, then either $T_{\star}=\infty$ or

$$
\underset{t \uparrow T_{*}}{\limsup }\|v(t)\|_{m}=\infty
$$

Proof. Assume by contradiction that $T_{\star}<\infty$ and that $M:=\sup _{t<T_{\star}}\|v(t)\|_{m}<\infty$. Let $T_{0}=T_{\star}-c_{\star} /(4 M)$, and start a solution with initial condition $v\left(T_{0}\right)$ at time $T_{0}$. By Proposition A. 2 there is a solution of (1.6) on a time span of length at least $c_{\star} /\left(2\left\|v\left(T_{0}\right)\right\|_{m}\right) \geq c_{\star} /(2 M)$, hence at least up to time $T_{0}+c_{\star} /(2 M)>T_{\star}$. By uniqueness, this solution is equal to $v$ up to time $T_{\star}$.

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[^0]:    ${ }^{1}$ The proof of the result of [Tao09] is given in $\mathbb{R}^{d}$, but it can be easily extended to the periodic setting, see [Tao09, Remark 2.1].

[^1]:    ${ }^{2}$ They are in fact almost orthogonal in the sense that $\left\langle P_{n}, P_{m}\right\rangle_{L^{2}}=0$ whenever $|m-n| \geq 2$.

[^2]:    ${ }^{3}$ Existence and uniqueness can be proved also in the general case $m_{1}(k) \geq|k|^{\alpha} g(|k|)^{-1}$. A simple assumption that keeps our proof almost unchanged is a control from above, say $m(k) \leq$ $|k|^{\beta}$, for some $\beta \geq \alpha$.

