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Twelve monotonicity conditions arising from algorithms for equilibrium problems

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In the last years many solution methods for equilibrium problems (EPs) have been developed. Several different monotonicity conditions have been exploited to prove convergence. The paper investigates all the relationships between them in the framework of the so-called abstract EP. The analysis is further detailed for variational inequalities and linear equilibrium problems, which include also Nash equilibrium problems with quadratic payoffs.

Keywords: equilibrium problems; monotonicity; variational inequalities; Nash equilibria.

AMS Subject Classification: 90C33; 47H05; 90C30

1. Introduction

Many mathematical models, including optimization, multiobjective optimization, variational inequalities, fixed point and complementarity problems, Nash equilibria in noncooperative games and inverse optimization, can be formulated in the same format, namely the *equilibrium problem*

$$\text{find} \quad x^* \in C \quad \text{s.t.} \quad f(x^*, y) \geq 0, \qquad \forall\, y \in C, \tag{EP}$$

where $C \subseteq \mathbb{R}^n$ is a nonempty, closed and convex set and $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a bifunction which satisfies $f(x, x) = 0$ for all $x \in C$ (see, for instance, [7, 8]). This general format clearly stems from variational inequalities, and solution methods for (EP) generally extend those originally designed for optimization or variational inequalities exploiting the underlying common structure.

Algorithms for (EP) can be roughly divided into a few classes (see the survey paper [7]): fixed point, extragradient, convex feasibility and descent methods, which generally require the solution of a sequence of (convex) optimization problems, and proximal point and Tikhonov-Browder regularization methods, which generally require the solution of a sequence of equilibrium problems with better properties than the given one. Almost all these methods share a common feature: convergence is guaranteed under suitable monotonicity assumptions on the bifunction f. This is not really surprising as the monotonicity of the operator is a crucial assumption in the algorithms for variational inequalities (see for instance [13]). Indeed, in equilibrium problems monotonicity plays a role which is similar to convexity in optimization. For instance, stationarity points of gap and D-gap

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functions for (EP) are actually global minima if appropriate monotonicity conditions hold (see [3–5, 31, 47]).

Several definitions of monotonicity for bifunctions have been introduced. Anyway, to the best of our knowledge, a thorough study of their relations has never been carried out. Therefore, the paper aims at analysing all the relationships between those monotonicity conditions which are relevant in the algorithms for (EP). Twelve different conditions have been identified and they are briefly recalled in Section 2, while Section 3 provides a full picture of the relationships between them. Finally, the analysis is further detailed in Section 4 for two particular cases: variational inequalities and the so-called linear equilibrium problems, which include also Nash equilibrium problems with quadratic payoffs.

Since the convexity of $f(x, \cdot)$ is required (for any $x \in C$) by all the algorithms for (EP) and some require also that f is continuously differentiable, both assumptions are taken for granted in the whole paper.

2. Monotonicity conditions

Monotonicity is a straightforward concept for real-valued functions of one single variable: for instance, $g : \mathbb{R} \to \mathbb{R}$ is monotone increasing on $C \subseteq \mathbb{R}$ if $g(y) \geq g(x)$ holds whenever $x, y \in C$ satisfy $y > x$. The statement

$$(g(y) - g(x))(y - x) \geq 0, \qquad \forall\, x, y \in C,$$

provides an elegant alternative definition, which can be easily extended to vector-valued mappings.

Definition 2.1 $F : \mathbb{R}^n \to \mathbb{R}^n$ is called *monotone on* $C \subseteq \mathbb{R}^n$ if

$$\langle F(y) - F(x), y - x \rangle \geq 0, \qquad \forall\, x, y \in C, \tag{1}$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^n$. *Strict monotonicity* means that (1) holds replacing $\geq$ by $>$ whenever $y \neq x$, while *strong monotonicity* requires the existence of some $\tau > 0$ such that the left-hand side of (1) is greater or equal to $\tau \|y - x\|^2$.

When (EP) is a variational inequality, i.e. $f(x, y) = \langle F(x), y - x \rangle$ for some operator $F : \mathbb{R}^n \to \mathbb{R}^n$, the left-hand side of (1) can be rewritten exploiting f:

$$\langle F(y) - F(x), y - x \rangle = -\big(f(x, y) + f(y, x)\big).$$

Therefore, monotonicity conditions for a bifunction $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ can be introduced relying on this relationship.

Definition 2.2 f is called *monotone on* C if

$$f(x, y) + f(y, x) \leq 0, \qquad \forall\, x, y \in C, \tag{2}$$

while it is called *strictly monotone on* C if (2) holds replacing $\leq$ by $<$ whenever $y \neq x$. f is called *strongly monotone on* C if there exists $\tau > 0$ such that

$$f(x, y) + f(y, x) \leq -\tau \|y - x\|^2, \qquad \forall\, x, y \in C. \tag{3}$$

Strong monotonicity is exploited in those algorithms which are based on the reformulation of (EP) as a fixed point problem [30, 36, 42], while no known algorithm relies directly on the strict monotonicity of f. Both strict and strong monotonicity are rather restrictive assumptions as each of them guarantees the uniqueness of the solution. Therefore, a few algorithms rely on monotonicity: they belong to the classes of extragradient [40], proximal point [15, 22, 32–35] and Tikhonov-Browder regularization methods [21, 23, 26]. Actually, many efforts have been devoted to further weaken the monotonicity requirements.

Definition 2.3 f is called *pseudomonotone on* C if

$$f(x,y) \geq 0 \quad \Longrightarrow \quad f(y,x) \leq 0, \qquad \forall\, x,y \in C. \tag{4}$$

f is called *weakly monotone on* C if there exists $\tau > 0$ such that

$$f(x,y) + f(y,x) \leq \tau\, \|y - x\|^2, \qquad \forall\, x,y \in C. \tag{5}$$

Pseudomonotonicity has been first used in the extragradient algorithms of [14] without naming it so, but exploiting (4) for the points x which solve (EP). Recently, other extragradient algorithms have been developed exploiting the above definition [1, 38, 41]. Also fixed point [37] and so-called combined relaxation methods [19, 28, 46], which exploit the standard fixed point iteration together with suitable projections, have been developed relying on the pseudomonotonicity of f, as well as convex feasibility [16, 43] and proximal point methods [2, 17, 20, 22, 29]. It is worth stressing that pseudomonotonicity is often exploited to get the existence of a solution of the following equilibrium problem

$$\text{find} \quad x^* \in C \quad \text{s.t.} \quad f(y,x^*) \leq 0, \qquad \forall\, y \in C.$$

Actually, the convergence of some algorithms is achieved supposing directly the solvability of this latter problem [20, 22, 38, 43]. Finally, weak monotonicity has been used in proximal point algorithms only [17, 22, 29].

Regularization methods involve a sequence of equilibrium problems, which have to be solved exploiting other techniques: for instance, descent methods have been exploited in [21, 23, 26] within a regularization framework. Descent methods rely on the reformulation of an equilibrium problem as an optimization problem. Since the objective function is generally nonconvex, appropriate monotonicity conditions are needed to solve them.

Definition 2.4 f is called ∇*-monotone on* C if

$$\langle \nabla_x f(x,y) + \nabla_y f(x,y), y - x \rangle \geq 0, \qquad \forall\, x,y \in C. \tag{6}$$

f is called *strictly* ∇*-monotone on* C if

$$\langle \nabla_x f(x,y) + \nabla_y f(x,y), y - x \rangle > 0, \qquad \forall\, x,y \in C,\ x \neq y. \tag{7}$$

f is called *strongly* ∇*-monotone on* C if there exists $\tau > 0$ such that

$$\langle \nabla_x f(x,y) + \nabla_y f(x,y), y - x \rangle \geq \tau\, \|y - x\|^2, \qquad \forall\, x,y \in C. \tag{8}$$

Both strict and strong ∇-monotonicity guarantee that all the stationary points of the reformulations as constrained optimization problems through gap functions are global minima and hence solutions of the equilibrium problem (see [3–5, 31]). Therefore, these conditions have been widely exploited to devise specific descent methods for (EP): algorithms for the latter case have been provided in [10, 24, 27, 31], while algorithms for the former in [5, 12, 31]. On the contrary, ∇-monotonicity does not guarantee the above "stationarity property" (see [3]), and no algorithm is yet available for this case. Anyway, when this property is not satisfied, some algorithms have been developed relying on the following condition [3, 4].

Definition 2.5 f is called *c-monotone on* C if

$$f(x,y) + \langle \nabla_x f(x,y), y - x \rangle \geq 0, \qquad \forall\, x, y \in C. \tag{9}$$

If $f(\cdot, y)$ is concave for all $y \in C$, then (9) holds. As a consequence, it is often referred to as a concavity-type condition.

Yet another kind of monotonicity comes into play when reformulations of (EP) as unconstrained optimization problems through D-gap functions are exploited to devise descent algorithms.

Definition 2.6 f is called ∇_{xy}*-monotone on* C if the mapping $\nabla_x f(x, \cdot)$ is monotone on C for any $x \in C$, that is

$$\langle \nabla_x f(x,y) - \nabla_x f(x,z), y - z \rangle \geq 0, \qquad \forall\, x, y, z \in C. \tag{10}$$

f is called *strictly* ∇_{xy}*-monotone on* C if the mapping $\nabla_x f(x, \cdot)$ is strictly monotone on C for any $x \in C$, that is

$$\langle \nabla_x f(x,y) - \nabla_x f(x,z), y - z \rangle > 0, \qquad \forall\, x, y, z \in C,\ y \neq z. \tag{11}$$

f is called *strongly* ∇_{xy}*-monotone on* C if the mappings $\nabla_x f(x, \cdot)$ are strongly monotone on C uniformly with respect to $x \in C$, that is, there exists $\tau > 0$ such that

$$\langle \nabla_x f(x,y) - \nabla_x f(x,z), y - z \rangle \geq \tau \, \|y - z\|^2, \qquad \forall\, x, y, z \in C. \tag{12}$$

Both strict and strong ∇_{xy}-monotonicity guarantee that all the stationary points of a D-gap function are global minima and hence solutions of the equilibrium problem [47]. Specific descent methods have been developed exploiting both the former [48, 49] and the latter condition [11, 25, 48], while ∇_{xy}-monotonicity, which does not guarantee the above property, has been exploited up to now only in [6].

3. Relations between monotonicity conditions

The twelve monotonicity conditions introduced in the previous section can be grouped into 3 families: Definitions 2.2 and 2.3 provide the most well-known conditions, Definitions 2.4 and 2.5 collect those conditions which are exploited by methods based on gap functions, while Definition 2.6 brings together those conditions which are exploited by methods based on D-gap functions. In the following they will be often referred to as the first, the second and the third family, respectively.

Inside each family some relationships are straightforward, thus they are not discussed explicitly. The following collection of results outlines some further non trivial relations between the twelve conditions.

Theorem 3.1 *Given any convex set* $C \subseteq \mathbb{R}^n$*, the following statements hold:*

a) if f is strongly ∇_{xy}-monotone on C, then f is strongly ∇-monotone on C;
b) if f is strongly ∇_{xy}-monotone on C, then f is strongly monotone on C;
c) if f is strictly ∇_{xy}-monotone on C, then f is strictly ∇-monotone on C;
d) if f is strictly ∇_{xy}-monotone on C, then f is strictly monotone on C;

e) if f is ∇_{xy}-monotone on C, then f is c-monotone on C;
f) if f is ∇_{xy}-monotone on C, then f is monotone on C;
g) if f is c-monotone on C, then f is ∇-monotone on C.

Proof.

a) Since $f(x,\cdot)$ is convex for any $x \in C$, the mapping $\nabla_y f(x,\cdot)$ is monotone on C. Therefore, the strong monotonicity of $\nabla_x f(x,\cdot)$ implies

$$\langle \nabla_x f(x,y) + \nabla_y f(x,y) - \nabla_x f(x,x) - \nabla_y f(x,x), y - x \rangle \geq \tau \, \|y - x\|^2$$

for all $x, y \in C$, i.e., the mapping $\nabla_x f(x,\cdot) + \nabla_y f(x,\cdot)$ is strongly monotone on C.
On the other hand, $\nabla_x f(x,x) + \nabla_y f(x,x) = 0$ since $f(x,x) = 0$ holds for all $x \in C$. Thus,

$$\langle \nabla_x f(x,y) + \nabla_y f(x,y), y - x \rangle \geq \tau \, \|y - x\|^2, \qquad \forall \, x, y \in C,$$

i.e., f is strongly ∇-monotone.

b) Given any $x, y \in C$, consider the function

$$g(t) := f(x + t(y-x), x) - f(x + t(y-x), y)$$

for $t \in [0,1]$. Since f is continuously differentiable, g has a derivative and

$$g'(t) = [\nabla_x f(x + t(y-x), x) - \nabla_x f(x + t(y-x), y)] \, (y - x) \leq -\tau \, \|y - x\|^2$$

for all $t \in [0,1]$, where the inequality follows form the strong ∇_{xy}-monotonicity of f. Therefore,

$$g(1) - g(0) = \int_0^1 g'(t) dt \leq -\tau \, \|y - x\|^2,$$

while also

$$g(1) - g(0) = f(y,x) - f(y,y) - f(x,x) + f(x,y) = f(x,y) + f(y,x).$$

Thus, f is strongly monotone.

c) The proof is analogous to a).

d) The proof is analogous to b).

e) Given any $x, y \in C$, the ∇_{xy}-monotonicity of f guarantees

$$\langle \nabla_x f(x,y) - \nabla_x f(x,x), y - x \rangle \geq 0.$$

Moreover, $\nabla_x f(x,x) + \nabla_y f(x,x) = 0$ since $f(x,x) = 0$ holds for any $x \in C$, while

$$f(x,y) \geq f(x,x) + \langle \nabla_y f(x,x), y - x \rangle = \langle \nabla_y f(x,x), y - x \rangle$$

follows from the convexity of the function $f(x,\cdot)$. As a consequence,

$$\begin{aligned} f(x,y) + \langle \nabla_x f(x,y), y - x \rangle &\geq f(x,y) + \langle \nabla_x f(x,x), y - x \rangle \\ &= f(x,y) - \langle \nabla_y f(x,x), y - x \rangle \geq 0, \end{aligned}$$

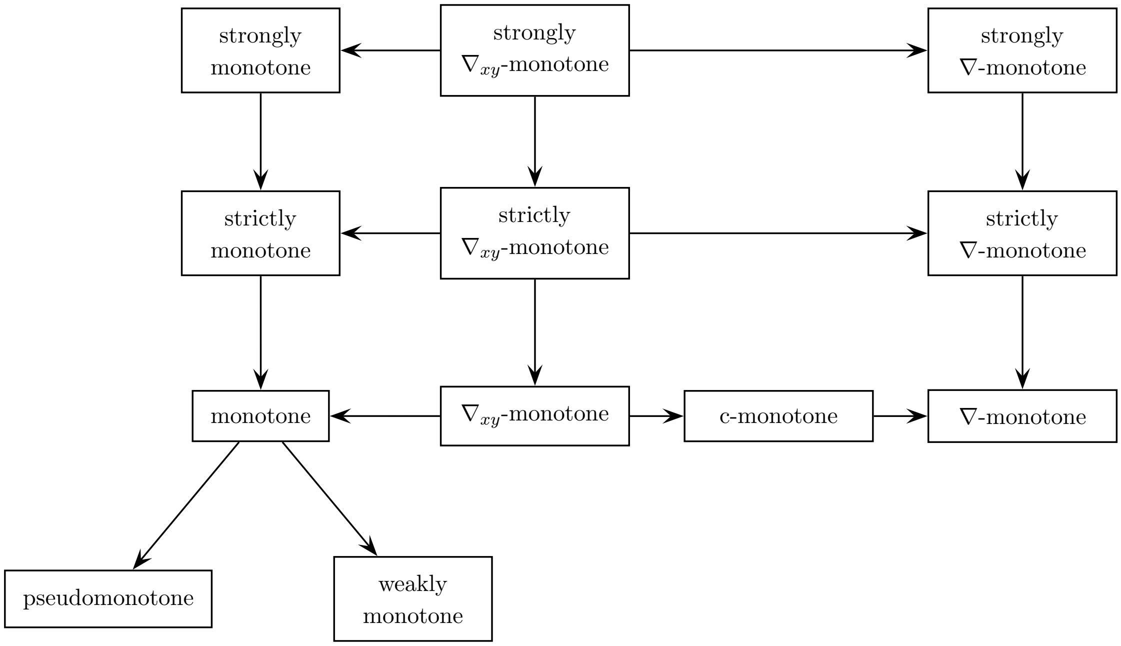

Figure 1. Relationships between monotonicity conditions: the general case.

and therefore f is c-monotone.

f) The proof is similar to b).

g) (see also [3, Theorem 3.1]). Since $f(x, \cdot)$ is convex, then

$$0 = f(x,x) \geq f(x,y) + \langle \nabla_y f(x,y), x-y \rangle$$

holds for any $x, y \in C$. The above inequality and the c-monotonicity of f guarantee

$$\begin{aligned} &\langle \nabla_x f(x,y) + \nabla_y f(x,y), y-x \rangle = \\ &= (f(x,y) + \langle \nabla_x f(x,y), y-x \rangle) - (f(x,y) + \langle \nabla_y f(x,y), x-y \rangle) \geq 0, \end{aligned}$$

and thus f is ∇–monotone.

■

Figure 1 depicts the relationships between the twelve monotonicity conditions. Note that the monotonicity conditions of the third family, that is the family related to D-gap functions, imply the corresponding conditions of the two other families.

The following counterexamples show that no further relation holds between the above conditions. To begin with, the next example shows that no condition belonging to the first family implies any condition of the other two families.

Example 3.2 Let $C = \mathbb{R}$ and $f(x,y) = e^{x^2}(y^2 - x^2) + x(y-x)$. Then, f is strongly monotone since

$$f(x,y) + f(y,x) = (e^{x^2} - e^{y^2})(y^2 - x^2) - (y-x)^2 \leq -(y-x)^2 \qquad \forall\ x,y \in C,$$

but it is not ∇-monotone since

$$\langle \nabla_x f(x,y) + \nabla_y f(x,y), y-x \rangle = (y-x)^2 [2e^{x^2}(x^2 + xy + 1) + 1],$$

which is negative for $x = 1$ and $y = -3$.

Hence, f is strongly monotone, strictly monotone, monotone, pseudomonotone and weakly monotone, but it does not satisfies any of the other seven conditions.

The following four examples show that only the obvious relationships hold between the conditions of the first family.

Example 3.3 Let $C = \mathbb{R}$ and $f(x, y) = x^3(y - x)$. Since

$$f(x, y) + f(y, x) = (x^3 - y^3)(y - x) < 0, \qquad \forall\, x, y \in \mathbb{R},\ x \neq y,$$

f is strictly monotone. On the other hand, it is not strongly monotone since

$$[f(x, y) + f(y, x)]/(y - x)^2 = -(x^2 + xy + y^2) \to 0$$

if $x \to 0$ and $y \to 0$.

Example 3.4 Let $C = \mathbb{R}$ and $f(x, y) = x - y$. Since

$$f(x, y) + f(y, x) = 0$$

holds for any $x, y \in \mathbb{R}$, f is monotone but not strictly monotone. Furthermore, f is ∇_{xy}-monotone since

$$\nabla_x f(x, y) - \nabla_x f(x, z) = 0$$

holds for any $x, y, z \in \mathbb{R}$. Hence, it is also c-monotone and ∇-monotone. On the other hand, it is not strictly ∇-monotone since

$$\nabla_x f(x, y) + \nabla_y f(x, y) = 0$$

holds for any $x, y \in \mathbb{R}$. Hence, f is ∇_{xy}-monotone, ∇-monotone, c-monotone, monotone, pseudomonotone and weakly monotone, but it does not satisfies any of the other six conditions.

Example 3.5 Let $C = \mathbb{R}$ and $f(x, y) = (x^2 + 1)(y - x)$. then, f is pseudomonotone since

$$f(x, y) \geq 0 \implies y \geq x \implies f(y, x) \leq 0,$$

but it is not weakly monotone since

$$[f(x, y) + f(y, x)]/(y - x)^2 = -(x + y) \to +\infty$$

if, for instance, $x \to -\infty$ and $y = 0$. As a consequence, it is not monotone, strictly and strongly monotone as well.

Example 3.6 Let $C = \mathbb{R}$ and $f(x, y) = -x^2 - xy + 2y^2$. Then, f is weakly monotone since

$$f(x, y) + f(y, x) = (y - x)^2,$$

but is not pseudomonotone since $f(2, -1) = 0$ and $f(-1, 2) = 9$. Furthermore, f is c-monotone since

$$f(x, y) + \langle \nabla_x f(x, y), y - x \rangle = -x^2 - xy + 2y^2 + (-2x - y)(y - x) = (x - y)^2 \geq 0,$$

and it is strongly ∇-monotone since

$$\langle \nabla_x f(x, y) + \nabla_y f(x, y), y - x\rangle = 3\,(y - x)^2.$$

Hence, f is strongly ∇-monotone, strictly ∇-monotone, ∇-monotone, c-monotone and weakly monotone, but it does not satisfies any of the other seven conditions.

The following two examples, together with Example 3.6, show that no further relation holds between the conditions of the second family and those of the other two families.

Example 3.7 Let $C = \mathbb{R}$ and $f(x, y) = (y-x)^4 + x(y-x)$. Then, f is strongly ∇-monotone since

$$\langle \nabla_x f(x, y) + \nabla_y f(x, y), y - x\rangle = (y - x)^2.$$

Thus, it is both strictly ∇-monotone and ∇-monotone as well. However, f is not weakly monotone since

$$[f(x, y) + f(y, x)]/(y - x)^2 = 2(y - x)^2 - 1 \to +\infty$$

if, for instance, $x = 0$ and $y \to +\infty$.

Example 3.8 Let $C = [0, +\infty)$ and $f(x, y) = (y - x)^4 + 3(y^4 - x^4)$. The concavity-type condition (9) reads

$$f(x, y) + \langle \nabla_x f(x, y), y - x\rangle = 6x\,(x^3 + 2y^3 - 3xy^2).$$

If $x = 0$ and $y \in C$ then the above right-hand side is 0. On the other hand, if $x > 0$ and $y \in C$, then there exists $\alpha \geq 0$ such that $y = \alpha x$ so that

$$f(x, y) + \langle \nabla_x f(x, y), y - x\rangle = 6x^4(2\alpha^3 - 3\alpha^2 + 1)$$

which is non-negative since $2\alpha^3 - 3\alpha^2 + 1 \geq 0$ for all $\alpha \geq 0$. Therefore, f is c-monotone.
However, f is not weakly monotone since

$$[f(x, y) + f(y, x)]/(y - x)^2 = 2(y - x)^2 \to +\infty$$

if, for instance, $x = 0$ and $y \to +\infty$.

The following counterexample, paired with Examples 3.4 and 3.8, shows that no further relation holds between the conditions of the second family.

Example 3.9 Let $C = \mathbb{R}$ and $f(x, y) = x^2 - 3xy + 2y^2$. Then, f is strongly ∇-monotone since

$$\langle \nabla_x f(x, y) + \nabla_y f(x, y), y - x\rangle = (y - x)^2.$$

Thus, it is both strictly ∇-monotone and ∇-monotone as well. However, it is not c-monotone since

$$f(x, y) + \langle \nabla_x f(x, y), y - x\rangle = x^2 - 3xy + 2y^2 + (2x - 3y)(y - x) = -(y - x)^2 < 0$$

for any pair $x, y \in C$ with $x \neq y$.

Note that Example 3.4 shows also that no further relation involving ∇_{xy}-monotonicity holds beyond those provided by Theorem 3.1. The following example provides an analogous result for strict ∇_{xy}-monotonicity.

Example 3.10 Let $C = \mathbb{R}$ and $f(x, y) = e^x(y - x)$. Since

$$\langle \nabla_x f(x, y) - \nabla_x f(x, z), y - z\rangle = e^x(y - z)^2,$$

f is strictly ∇_{xy}-monotone but not strongly ∇_{xy}-monotone. Furthermore, f is neither strongly ∇-monotone nor strongly monotone since

$$\langle \nabla_x f(x, y) + \nabla_y f(x, y), y - x\rangle/(y - x)^2 = e^x \to 0$$

if $x \to -\infty$, and

$$[f(x, y) + f(y, x)]/(y - x)^2 = (e^x - e^y)(y - x)/(y - x)^2 \to 0$$

if, for instance, $y = x + 1$ and $x \to -\infty$.

Table 1 summarizes all the relationships between the twelve monotonicity conditions.

4. Particular cases

In this section two particular cases of (EP) are studied: variational inequalities and linear equilibrium problems. In both cases some further relationships between monotonicity conditions hold beyond those already given by Theorem 3.1.

4.1 *Variational inequalities*

Consider (EP) with

$$f(x, y) = \langle F(x), y - x\rangle$$

for some continuously differentiable mapping $F : \mathbb{R}^n \to \mathbb{R}^n$. Since the first family of monotonicity conditions for f (Definitions 2.2 and 2.3) is built relying on the similarity with variational inequalities, they obviously collapse to the corresponding monotonicity conditions for F. Furthermore, the core formulas for the other two families read

$$\langle \nabla_x f(x, y) + \nabla_y f(x, y), y - x\rangle = f(x, y) + \langle \nabla_x f(x, y), y - x\rangle = \langle y - x, \nabla F(x)(y - x)\rangle,$$

and

$$\langle \nabla_x f(x, y) - \nabla_x f(x, z), y - z\rangle = \langle y - z, \nabla F(x)(y - z)\rangle.$$

Therefore, the monotonicity, c-monotonicity, ∇-monotonicity and ∇_{xy}-monotonicity of f coincide with the monotonicity of F (see [45, Proposition 12.3]). The analogous equivalences hold also for strong monotonicity conditions. On the contrary, the strict monotonicity conditions are all different: strict ∇_{xy}-monotonicity implies strict ∇-monotonicity, which in turn implies strict monotonicity, while both the reverse implications do not hold as shown by Example 3.3 for the latter and by the example below for the former.

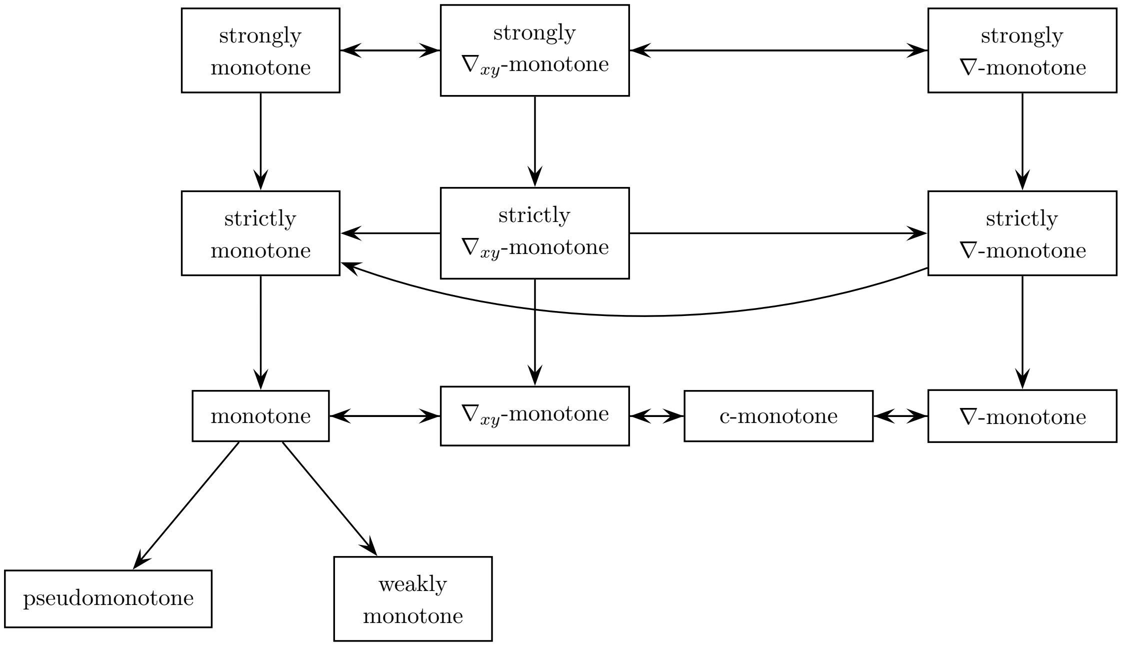

Figure 2. Relationships between monotonicity conditions: variational inequalities.

Example 4.1 Let $C = \mathbb{R}^2_+$ and $F(x_1, x_2) = (x_1 + x_2 + x_1^2,\ x_1 + x_2 + x_2^2)$. The matrix

$$\nabla F(x) = \begin{pmatrix} 1 + 2x_1 & 1 \\ 1 & 1 + 2x_2 \end{pmatrix}$$

is positive definite for any $x \neq 0$. Thus, if $x \neq 0$, then the inequality

$$\langle y - x, \nabla F(x)(y - x) \rangle > 0$$

holds for any $x, y \in C$ satisfying $y \neq x$. Moreover, $x = 0$ implies

$$\langle y - x, \nabla F(x)(y - x) \rangle = \langle y, \nabla F(0) y \rangle = (y_1 + y_2)^2 > 0$$

for any $y \in C$ with $y \neq x$. Therefore, f is strictly ∇-monotone. On the contrary, f is not strictly ∇_{xy}-monotone since $x = 0$, $y = (1, 0)$ and $z = (0, 1)$ provide

$$\langle y - z, \nabla F(x)(y - z) \rangle = 0.$$

Figure 2 shows how the twelve monotonicity conditions collapse to seven in the case of variational inequalities.

Furthermore, (EP) itself can be turned into a variational inequality (see [9, 18, 19, 44]). In fact, $x^* \in C$ solves (EP) if and only if it satisfies

$$\langle G(x^*), y - x^* \rangle \geq 0, \qquad \forall\ y \in C,$$

where $G(x) := \nabla_y f(x, x)$. The relations between the monotonicity conditions of f and G are reported in the following theorems.

Theorem 4.2 [18, Proposition 2.1.17] *Given any convex set $C \subseteq \mathbb{R}^n$, the following statements hold:*

a) if f is strongly monotone on C, then G is strongly monotone on C;
b) if f is strictly monotone on C, then G is strictly monotone on C;
c) if f is monotone on C, then G is monotone on C;
d) if f is weakly monotone on C, then G is weakly monotone on C;
e) if f is pseudomonotone on C, then G is pseudomonotone on C.

Proof.

a)–d) Since $f(z,\cdot)$ is convex for any $z \in C$, the inequalities

$$f(x,y) \geq f(x,x) + \langle \nabla_y f(x,x), y-x \rangle = \langle G(x), y-x \rangle, \tag{13}$$
$$f(y,x) \geq f(y,y) + \langle \nabla_y f(y,y), x-y \rangle = \langle G(y), x-y \rangle, \tag{14}$$

hold for any $x,y \in C$. Summing them,

$$\langle G(y) - G(x), y-x \rangle \geq -f(x,y) - f(y,x)$$

follows. Therefore, the monotonicity of f implies the monotonicity of G and the same implication works also for the strong, strict and weak cases.

e) If $\langle G(x), y-x \rangle \geq 0$, then (13) guarantees $f(x,y) \geq 0$, and therefore the pseudomonotonicity of f implies $f(y,x) \leq 0$, which together with (14) provides $\langle G(y), x-y \rangle \leq 0$.

■

None of the reverse implications holds: in Example 3.7 the bifunction f is neither weakly monotone nor pseudomonotone, while the mapping G is strongly monotone (indeed $G(x) = x$).

Theorem 4.3 *Suppose that f is twice continuously differentiable. Given any convex set $C \subseteq \mathbb{R}^n$, then*

a) if f is strongly ∇-monotone on C, then G is strongly monotone on C;
b) if f is ∇-monotone on C, then G is monotone on C.

Proof.

a) Given any $x,y \in C$, consider the function

$$h(t) := \langle \nabla_x f(x,x_t) + \nabla_y f(x,x_t), y-x \rangle,$$

where $x_t := x + t\,(y-x)$ and $t \in [0,1]$. Since $f(z,z) = 0$ for all $z \in C$, then $\nabla_x f(x,x) + \nabla_y f(x,x) = 0$, that is $h(0) = 0$. Furthermore, any $t \in (0,1]$ satisfies

$$\begin{aligned} [h(t) - h(0)]/t &= \langle \nabla_x f(x,x_t) + \nabla_y f(x,x_t), x_t - x \rangle / t^2 \\ &\geq \tau\, \|x_t - x\|^2 / t^2 \\ &= \tau\, \|y-x\|^2, \end{aligned}$$

where the inequality follows from the strong ∇-monotonicity of f. Taking the limit as $t \downarrow 0$, then $h'(0) \geq \tau\, \|y-x\|^2$. On the other hand,

$$h'(0) = \langle y-x, [\nabla^2_{xy} f(x,x) + \nabla^2_{yy} f(x,x)](y-x) \rangle = \langle y-x, \nabla G(x)(y-x) \rangle.$$

Therefore, G is strongly monotone.

$b)$ The proof is analogous to $a)$.

■

Both the reverse implications do not hold: in Example 3.2 the bifunction f is not ∇-monotone, while the mapping G is strongly monotone (indeed $G(x) = 2xe^{x^2} + x$).

Notice that the above proof does not work in the strictly monotone case since it involves a limit. Whether or not the strict ∇-monotonicity of f implies the strict monotonicity of G remains an open question.

4.2 *Linear equilibrium problems*

Consider the so-called linear equilibrium problem [42], that is (EP) with

$$f(x,y) = \langle Px + Qy + r, y - x\rangle \tag{15}$$

for some $r \in \mathbb{R}^n$ and some $P, Q \in \mathbb{R}^{n\times n}$, where Q is positive semidefinite (in order to guarantee that $f(x,\cdot)$ is convex). In this case the core formulas read

$$\langle \nabla_x f(x,y) - \nabla_x f(x,z), y - z\rangle = \langle y - z, (P^T - Q)(y - z)\rangle = \langle y - z, (P - Q)(y - z)\rangle,$$

$$f(x,y) + f(y,x) = -\langle y - x, (P - Q)(y - x)\rangle,$$

$$f(x,y) + \langle \nabla_x f(x,y), y - x\rangle = \langle y - x, P(y - x)\rangle,$$

and

$$\langle \nabla_x f(x,y) + \nabla_y f(x,y), y - x\rangle = \langle y - x, (P + Q)(y - x)\rangle.$$

Hence, f is monotone and ∇_{xy}-monotone if the matrix $P - Q$ is positive semidefinite (shortly, psd), it is c-monotone if P is psd and ∇-monotone if $P + Q$ is psd.

Note that monotonicity coincides with ∇_{xy}-monotonicity, but they are not equivalent to either c-monotonicity or ∇-monotonicity as it happens in the case of variational inequalities. Analogous relations hold also for strong and strict monotonicity conditions. Furthermore, strong ∇_{xy}-monotonicity (∇-monotonicity) is equivalent to strict ∇_{xy}-monotonicity (∇-monotonicity). Finally, weak monotonicity is always satisfied with $\tau = \|P - Q\|$, but it is not equivalent to pseudomonotonicity (see Example 3.6).

Figure 3 shows how the twelve monotonicity conditions collapse to seven in the case of linear equilibrium problems.

Nash equilibrium problems in noncooperative games with quadratic cost functions are an interesting particular case of linear equilibrium problems: each player i has a set of feasible strategies $K_i \subseteq \mathbb{R}^{n_i}$ and aims at minimizing a quadratic cost function which depends also on the strategies of the other players, namely

$$c_i(x) = \frac{1}{2}\langle x_i, A_{ii}x_i\rangle + \sum_{\substack{j=1\\ j\neq i}}^{N} \langle x_i, A_{ij}x_j\rangle + \langle b_i, x_i\rangle,$$

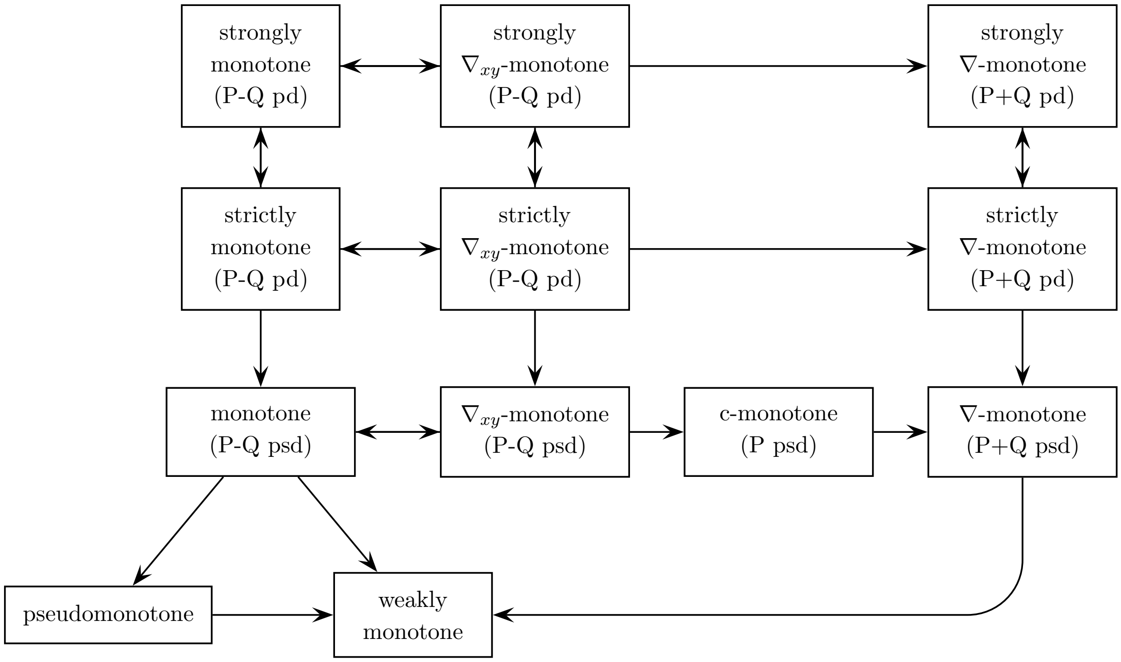

Figure 3. Relationships between monotonicity conditions: linear equilibria.

where N denotes the number of players, $x \in K_1 \times \cdots \times K_N$ and the squared matrices $A_{11}, \dots, A_{NN}$ are symmetric and positive semidefinite. Finding a Nash equilibrium amounts to solving (EP) with $C = K_1 \times \cdots \times K_N$ and the Nikaido-Isoda aggregate bifunction

$$f(x, y) = \sum_{i=1}^{N} [c_i(x) - c_i(x(y_i))]$$

where $x(y_i)$ denotes the vector obtained from x replacing x_i by y_i [39]. This bifunction can be written in the form (15) just setting

$$P = \begin{pmatrix} A_{11}/2 & A_{12} & \dots & A_{1N} \\ A_{21} & A_{22}/2 & \dots & A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N1} & A_{N2} & \dots & A_{NN}/2 \end{pmatrix}, \quad Q = \begin{pmatrix} A_{11}/2 & 0 & \dots & 0 \\ 0 & A_{22}/2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & A_{NN}/2 \end{pmatrix}, \tag{16}$$

and $r = (b_1, \dots, b_N)^T$. Hence, it is monotone if $A_{ij} = -A_{ji}^T$ for any $i, j = 1, \dots, N$; it is c-monotone if the matrix P in (16) is psd and ∇-monotone if the block matrix $A = (A_{ij})$ is psd.

5. Conclusions

Twelve monotonicity conditions, which are relevant in the algorithms for (EP), have been analysed. They include the most widespread definitions of monotonicity and some further conditions, which have been introduced in order to guarantee the convergence of descent methods based on gap or D-gap functions. All the relations between the twelve conditions have been identified.

Furthermore, two particular cases have been examined: variational inequalities and linear equilibrium problems. In both cases the twelve kinds of monotonicity collapse to seven different conditions. Anyway, the additional relationships are not all the same in the two cases: indeed, just two are common to both, while the other four of the first case are different from the other six of the second case.

Finally, it is worth stressing that almost all the algorithms for (EP) involve some further assumptions in addition to monotonicity. Thus, the relationships between monotonicity conditions are very useful to compare convergence results, but alone they are not enough to make full comparisons.

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Table 1. Relationships between monotonicity conditions.

$\Longrightarrow$	strongly monotone	strictly monotone	monotone	pseudo monotone	weakly monotone	strongly ∇-monotone	strictly ∇-monotone	∇-monotone	c-monotone	strongly ∇_{xy}-monotone	strictly ∇_{xy}-monotone	∇_{xy}-monotone
strongly monotone	–	yes	yes	yes	yes	no (Ex. 3.2)	no (Ex. 3.2)	no (Ex. 3.2)	no (Ex. 3.2)	no (Ex. 3.2)	no (Ex. 3.2)	no (Ex. 3.2)
strictly monotone	no (Ex. 3.3)	–	yes	yes	yes	no (Ex. 3.2)	no (Ex. 3.2)	no (Ex. 3.2)	no (Ex. 3.2)	no (Ex. 3.2)	no (Ex. 3.2)	no (Ex. 3.2)
monotone	no (Ex. 3.4)	no (Ex. 3.4)	–	yes	yes	no (Ex. 3.2)	no (Ex. 3.2)	no (Ex. 3.2)	no (Ex. 3.2)	no (Ex. 3.2)	no (Ex. 3.2)	no (Ex. 3.2)
pseudomonotone	no (Ex. 3.5)	no (Ex. 3.5)	no (Ex. 3.5)	–	no (Ex. 3.5)	no (Ex. 3.2)	no (Ex. 3.2)	no (Ex. 3.2)	no (Ex. 3.2)	no (Ex. 3.2)	no (Ex. 3.2)	no (Ex. 3.2)
weakly monotone	no (Ex. 3.6)	no (Ex. 3.6)	no (Ex. 3.6)	no (Ex. 3.6)	–	no (Ex. 3.2)	no (Ex. 3.2)	no (Ex. 3.2)	no (Ex. 3.2)	no (Ex. 3.2)	no (Ex. 3.2)	no (Ex. 3.2)
strongly ∇-monotone	no (Ex. 3.6)	no (Ex. 3.6)	no (Ex. 3.6)	no (Ex. 3.6)	no (Ex. 3.7)	–	yes	yes	no (Ex. 3.9)	no (Ex. 3.6)	no (Ex. 3.6)	no (Ex. 3.6)
strictly ∇-monotone	no (Ex. 3.6)	no (Ex. 3.6)	no (Ex. 3.6)	no (Ex. 3.6)	no (Ex. 3.7)	no (Ex. 3.8)	–	yes	no (Ex. 3.9)	no (Ex. 3.6)	no (Ex. 3.6)	no (Ex. 3.6)
∇-monotone	no (Ex. 3.6)	no (Ex. 3.6)	no (Ex. 3.6)	no (Ex. 3.6)	no (Ex. 3.7)	no (Ex. 3.8)	no (Ex. 3.4)	–	no (Ex. 3.9)	no (Ex. 3.6)	no (Ex. 3.6)	no (Ex. 3.6)
c-monotone	no (Ex. 3.6)	no (Ex. 3.6)	no (Ex. 3.6)	no (Ex. 3.6)	no (Ex. 3.8)	no (Ex. 3.8)	no (Ex. 3.4)	yes	–	no (Ex. 3.6)	no (Ex. 3.6)	no (Ex. 3.6)
strongly ∇_{xy}-monotone	yes	yes	yes	yes	yes	yes	yes	yes	yes	–	yes	yes
strictly ∇_{xy}-monotone	no (Ex 3.10)	yes	yes	yes	yes	no (Ex. 3.10)	yes	yes	yes	no (Ex. 3.10)	–	yes
∇_{xy}-monotone	no (Ex 3.4)	no (Ex 3.4)	yes	yes	yes	no (Ex. 3.4)	no (Ex. 3.4)	yes	yes	no (Ex. 3.4)	no (Ex. 3.4)	–