arXiv:1302.4734v1 [math-ph] 19 Feb 2013

THE ROLE OF THE SCALAR CURVATURE IN SOME SINGULARLY PERTURBED COUPLED ELLIPTIC SYSTEMS ON RIEMANNIAN MANIFOLDS

MARCO GHIMENTI, ANNA MARIA MICHELETTI, AND ANGELA PISTOIA

ABSTRACT. Given a 3-dimensional Riemannian manifold (M, g), we investigate the existence of positive solutions of the Klein-Gordon-Maxwell system

$$\begin{cases} -\varepsilon^2 \Delta_g u + au = u^{p-1} + \omega^2(qv-1)^2 u & \text{in } M \\ -\Delta_g v + (1 + q^2 u^2)v = qu^2 & \text{in } M \end{cases}$$

and Schrödinger-Maxwell system

$$\begin{cases} -\varepsilon^2 \Delta_g u + u + \omega uv = u^{p-1} & \text{in } M \\ -\Delta_g v + v = qu^2 & \text{in } M \end{cases}$$

when $p \in (2, 6)$. We prove that if ε is small enough, any *stable* critical point ξ_0 of the scalar curvature of g generates a positive solution $(u_\varepsilon, v_\varepsilon)$ to both the systems such that u_ε concentrates at ξ_0 as ε goes to zero.

1. INTRODUCTION

Let (M, g) be a smooth compact, boundaryless $3-$dimensional Riemannian manifold.

Given real numbers $\varepsilon > 0$, $a > 0$, $q > 0$, $\omega \in (-\sqrt{a}, \sqrt{a})$ and $2 < p < 6$, we consider the following singularly perturbed electrostatic Klein-Gordon-Maxwell system

$$(1) \qquad \begin{cases} -\varepsilon^2 \Delta_g u + au = u^{p-1} + \omega^2(qv-1)^2 u & \text{in } M \\ -\Delta_g v + (1 + q^2 u^2)v = qu^2 & \text{in } M \\ u, v > 0 \end{cases}$$

and the Schrödinger-Maxwell system

$$(2) \qquad \begin{cases} -\varepsilon^2 \Delta_g u + u + \omega uv = u^{p-1} & \text{in } M \\ -\Delta_g v + v = qu^2 & \text{in } M \\ u, v > 0 \end{cases}$$

KGM systems and SM systems provide a model for the description of the interaction between a charged particle of matter u constrained to move on M and its own electrostatic field v.

The Schrödinger-Maxwell and the Klein-Gordon-Maxwell systems have been object of interest for many authors.

In the pioneering paper [6] Benci-Fortunato studied the following Schrödinger-Maxwell system

$$\begin{cases} -\Delta u + u + \omega uv = 0 & \text{in } \Omega \subset \mathbb{R}^3 \text{ or in } \mathbb{R}^3 \\ -\Delta v = \gamma u^2 & u = v = 0 \text{ on } \partial\Omega \end{cases}$$

Regarding the system in a semiclassical regime

$$(3) \qquad \begin{cases} -\varepsilon^2 \Delta u + u + \omega uv = f(u) & \text{in } \Omega \subset \mathbb{R}^3 \text{ or in } \mathbb{R}^3 \\ -\Delta v = \gamma u^2 & u = v = 0 \text{ on } \partial\Omega, \end{cases}$$

(here ε is a positive parameter small enough) Ruiz [34] and D'Aprile-Wei [13] showed the existence of a family of radially symmetric solutions respectively for $\Omega = \mathbb{R}^3$ or a ball. D'Aprile-Wei [14] also proved the existence of clustered solutions in the case of a bounded domain Ω in $\mathbb{R}^3$. Ghimenti-Micheletti [21] give an estimate on the number of solutions of (3).

Moreover, when $\varepsilon = 1$ we have results of existence and nonexistence of solutions for pure power nonlinearities $f(v) = |v|^{p-2}v$, $2 < p < 6$ or in presence of a more general nonlinearity (see [1, 2, 3, 5, 11, 24, 25, 33, 36]). In particular, Siciliano [35] proves an estimate on the number of solution for a pure power nonlinearity when p is close to the critical exponent.

Klein-Gordon-Maxwell systems are widely studied in physics and in mathematical physics (see for example [10, 18, 26, 28, 29]). In this setting, there are results of existence and non existence of solutions for subcritical nonlinear terms in a bounded domain Ω (see [4, 7, 8, 9, 12, 15, 16, 17, 32]).

As far as we know, the first result concerning the Klein-Gordon systems on manifold is due to Druet-Hebey [19]. They prove uniform bounds and the existence of a solution for the system (1) when $\varepsilon = 1$, a is positive function and the exponent p is either subcritical or critical, i.e. $p \in (2, 6]$. In particular, the existence of a solution in the critical case, i.e. $p = 6$, is obtained provided the function a is suitable small with respect to the scalar curvature of the metric g. Recently, Ghimenti-Micheletti [20] give an estimate on the number of low energy solution for the system (1) in terms of the topology of the manifold.

In this paper, we show that the existence and the multiplicity of solutions of both systems (1) and (2) in the subcritical case when ε is small enough is strictly related to the geometry of the manifold (M, g). More precisely, we prove that the number of solutions to (1) or (2) is affected by the number of *stable* critical points of the scalar curvature S_g of the metric g. Indeed, our result reads as follows.

Theorem 1. *Assume K is a C^1-stable critical set of S_g. Then there exists $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon})$ the KGM system (1) and the SM system (2) have a solution $(u_\varepsilon, v_\varepsilon)$ such that u_ε concentrates at a point $\xi_0 \in K$ as ε goes to 0. More precisely, there exists a point $\xi_\varepsilon \in M$ such that if ε goes to zero $\xi_\varepsilon \to \xi_0 \in K$*

$$u_\varepsilon - W_{\varepsilon, \xi_\varepsilon} \to 0 \text{ in } H_g^1(M), \ v_\varepsilon \to 0 \text{ in } H_g^1(M)$$

where the function $W_{\varepsilon, \xi_\varepsilon}$ is defined in (17).

We recall the the definition of C^1-stable critical set.

Definition 2. Let $f \in C^1(M, \mathbb{R})$. We say that $K \subset M$ is a C^1-stable critical set of f if $K \subset \{x \in M \ : \ \nabla_g f(x) = 0\}$ and for any $\mu > 0$ there exists $\delta > 0$ such that,

if $h \in C^1(M, \mathbb{R})$ with

$$\max_{d_g(x,K) \leq \mu} |f(x) - h(x)| + |\nabla_g f(x) - \nabla_g h(x)| \leq \delta,$$

then h has a critical point ξ with $d_g(\xi, K) \leq \mu$. Here d_g denotes the geodesic distance associated to the Riemannian metric g.

It is easy to see that if K is the set of the strict local minimum (or maximum) points of f, then K is a C^1-stable critical set of f. Moreover, if K consists of nondegenerate critical points, then K is a C^1-stable critical set of f.

By Theorem 1 we deduce that multiplicity of solutions of (1) and (2) is strictly related to stable critical points of the scalar curvature. At this aim, it is useful to point out that Micheletti-Pistoia [31] proved that, generically with respect to the metric g, the scalar curvature S_g is a Morse function on the manifold M. More precisely, they proved

Theorem 3. *Let $\mathscr{M}^k$ be the set of all C^k Riemannian metrics on M with $k \geq 3$. The set*

$$\mathscr{A} = \left\{ g \in \mathscr{M}^k \; : \; \text{all the critical points of } S_g \text{ are non degenerate} \right\}$$

is a open dense subset of $\mathscr{M}^k$.

Then generically with respect to the metric g, the critical points of the scalar curvature S_g are nondegenerate, in a finite number and at least $P_1(M)$ where $P_t(M)$ is the Poincaré polynomial of M in the t variable. Therefore, we can conclude as follows.

Corollary 4. *Generically with respect to the metric g, if ε is small enough the KGM system (1) and the SM system (2) have at least $P_1(M)$ positive solutions $(u_\varepsilon, v_\varepsilon)$ such that u_ε concentrates at one nondegenerate critical point of the scalar curvature S_g as ε goes to 0.*

The proof of our results relies on a very well known Ljapunov-Schmidt reduction. In Section 2 we recall some known results, we write the approximate solution, we sketch the proof of the Ljapunov Schmidt procedure and we prove Theorem 1. In Section 3 we reduce the problem to a finite dimensional one, while in Section 4 we study the reduced problem. In Appendix A we give some important estimates. All the proofs are given for the system (1), but it is clear that up some minor modifications they also hold true for the system (2).

2. Preliminaries and scheme of the proof of Theorem 1

2.1. **The function Ψ.** First of all, we reduce the system to a single equation. In order to overcome the problems given by the competition between u and v, using an idea of Benci and Fortunato [7], we introduce the map $\Psi : H^1_g(M) \to H^1_g(M)$ defined by the equation

$$(4) \qquad\qquad -\Delta_g \Psi(u) + \Psi(u) + q^2 u^2 \Psi(u) = q u^2.$$

It follows from standard variational arguments that Ψ is well-defined in $H^1_g(M)$ as soon as $\lambda := a - \omega^2 > 0$, i.e. $\omega \in]-\sqrt{a}, +\sqrt{a}[$.

By the maximum principle and by regularity theory is not difficult to prove that

$$(5) \qquad\qquad 0 < \Psi(u) < 1/q \text{ for all } u \text{ in } H^1_g(M).$$

Moreover, it holds true that

Lemma 5. *The map* $\Psi : H^1_g(M) \to H^1_g(M)$ *is* C^1 *and its differential* $\Psi'(u)[h] = V_u[h]$ *at* u *is the map defined by*

(6) $- \Delta_g V_u[h] + V_h[h] + q^2 u^2 V_u[h] = 2qu(1 - q\Psi(u))h$ *for all* $h \in H^1_g(M).$

Also, we have

$$0 \leq \Psi'(u)[u] \leq \frac{2}{q}$$

Lemma 6. *The map* $\Theta : H^1_g(M) \to \mathbb{R}$ *given by*

$$\Theta(u) = \frac{1}{2} \int_M (1 - q\Psi(u))u^2 d\mu_g$$

is C^1 *and*

$$\Theta'(u)[h] = \int_M (1 - q\Psi(u))^2 uh d\mu_g \text{ for any } u, h \in H^1_g(M)$$

For the proofs of these results we refer to [19].
Now, we introduce the functionals $I_\varepsilon, J_\varepsilon, G_\varepsilon : H^1_g(M) \to \mathbb{R}$

(7) $I_\varepsilon(u) = J_\varepsilon(u) + \frac{\omega^2}{2} G_\varepsilon(u),$

where

(8) $J_\varepsilon(u) := \frac{1}{\varepsilon^3} \int_M \left[\frac{1}{2}\varepsilon^2 |\nabla_g u|^2 + \frac{\lambda}{2} u^2 - F(u) \right] d\mu_g$

and

(9) $G_\varepsilon(u) := \frac{1}{\varepsilon^3} q \int_M \Psi(u) u^2 d\mu_g.$

Here $F(u) := \frac{1}{p} (u^+)^p$, so that $F'(u) = f(u) := (u^+)^{p-1}$. By Lemma 6 we deduce that

$$\frac{1}{2}G'_\varepsilon(u)[\varphi] = \frac{1}{\varepsilon^3} \int_M [q^2\Psi^2(u) - 2q\Psi(u)]u\varphi d\mu_g,$$

so

$$I'_\varepsilon(u)\varphi = \frac{1}{\varepsilon^3} \int_M \varepsilon^2 \nabla_g u \nabla_g \varphi + au\varphi - (u^+)^{p-1}\varphi - \omega^2(1 - q\Psi(u))^2 u\varphi d\mu_g.$$

Therefore, if u is a critical point of the functional I_ε we have

(10) $- \varepsilon^2 \Delta_g u + (a - \omega^2)u + \omega^2 q\Psi(u)(2 - q\Psi(u))u = (u^+)^{p-1} \quad \text{in } M.$

In particular, if $u \neq 0$ by the maximum principle and by the regularity theory we have that $u > 0$. Thus the pair $(u, \Psi(u))$ is a solution of Problem (1). Finally, the problem is reduced to find a solution to the single equation (10).

2.2. **Setting of the problem.** In the following we denote by $B_g(\xi, r)$ the geodesic ball in M centered in ξ with radius r and by $B(x, r)$ the ball in $\mathbb{R}^3$ centered in x with radius r.

It is possible to define a system of coordinates on M called normal coordinates. We denote by g_ξ the Riemannian metric read in $B(0, r) \subset \mathbb{R}^3$ through the normal coordinates defined by the exponential map $\exp_\xi$ at ξ. We denote $|g_\xi(z)| := \det(g_{ij}(z))$ and $\left(g_\xi^{ij}(z)\right)$ is the inverse matrix of $g_\xi(z)$. In particular, it holds

$$(11) \qquad g_\xi^{ij}(0) = \delta_{ij} \text{ and } \frac{\partial g_\xi^{ij}}{\partial z_k}(0) = 0 \text{ for any } i, j, k.$$

Here δ_{ij} denotes the Kronecker symbol.

We denote by

$$\|u\|_g^2 := \int_M \left(|\nabla_g u|^2 + u^2\right) d\mu_g \text{ and } |u|_g^q := \int_M |u|^q d\mu_g$$

the standard norms in the spaces $H_g^1(M)$ and $L^q(M)$.

Let H_ε be the Hilbert space $\mathrm{H}_g^1(M)$ equipped with the inner product

$$\langle u, v \rangle_\varepsilon := \frac{1}{\varepsilon^3} \left(\varepsilon^2 \int_M \nabla_g u \nabla_g v d\mu_g + \lambda \int_M uv d\mu_g\right),$$

which induces the norm

$$\|u\|_\varepsilon^2 := \frac{1}{\varepsilon^3} \left(\varepsilon^2 \int_M |\nabla_g|^2 d\mu_g + \lambda \int_M u^2 d\mu_g\right).$$

Let L_ε^q be the Banach space $\mathrm{L}_g^q(M)$ equipped the norm

$$|u|_{q,\varepsilon} := \left(\frac{1}{\varepsilon^3} \int_M |u|^q d\mu_g\right)^{1/q}.$$

It is clear that for any $q \in [2, 6)$ the embedding $H_\varepsilon \hookrightarrow L_\varepsilon^q$ is a continuous map. It is not difficult to check that

$$(12) \qquad |u|_{q,\varepsilon} \le c \|u\|_\varepsilon, \text{ for any } u \in H_\varepsilon,$$

where the constant c does not depend on ε.

In particular, the embedding $i_\varepsilon : H_\varepsilon \hookrightarrow L_\varepsilon^p$ is a compact continuous map. The adjoint operator $i_\varepsilon^* : L_\varepsilon^{p'} \to H_\varepsilon$, $p' := \frac{p}{p-1}$, is a continuous map such that

$$u = i_\varepsilon^*(v) \iff \langle i_\varepsilon^*(v), \varphi \rangle_\varepsilon = \frac{1}{\varepsilon^3} \int_M v\varphi, \ \varphi \in H_\varepsilon \iff -\varepsilon^2 \Delta_g u + u = v \text{ on } M, \ u \in \mathrm{H}_g^1(M).$$

Moreover

$$\|i_\varepsilon^*(v)\|_\varepsilon \le c|v|_{p',\varepsilon}, \text{ for any } v \in L_\varepsilon^{p'},$$

where the constant c does not depend on ε.

We can rewrite problem (10) in the equivalent way

$$(13) \qquad u = i_\varepsilon^* \left[f(u) + \omega^2 g(u)\right], \ u \in H_\varepsilon,$$

where we set

$$(14) \qquad g(u) := \left(q^2 \Psi^2(u) - 2q\Psi(u) \right) u.$$

2.3. **An approximation for the solution.** It is well known (see [22, 27]) that there exists a unique positive spherically symmetric function $U \in \mathrm{H}^1(\mathbb{R}^N)$ such that

$$(15) \qquad -\Delta U + \lambda U = U^{p-1} \text{ in } \mathbb{R}^N.$$

Moreover, the function U and its derivatives are exponentially decaying at infinity, namely

$$(16) \qquad \lim_{|x|\to\infty} U(|x|)|x|^{\frac{N-1}{2}} e^{|x|} = c > 0, \quad \lim_{|x|\to\infty} U'(|x|)|x|^{\frac{N-1}{2}} e^{|x|} = -c.$$

Let χ_r be a smooth cut-off function such that $\chi_r(z) = 1$ if $z \in B(0, r/2)$, $\chi_r(z) = 0$ if $z \in \mathbb{R}^3 \setminus B(0, r)$, $|\nabla \chi_r(z)| \le 2/r$ and $|\nabla^2 \chi_r(z)| \le 2/r^2$ where r is the injectivity radius of M. Fixed a point $\xi \in M$ and $\varepsilon > 0$ let us define on M the function
$$(17)$$
$$W_{\varepsilon,\xi}(x) := U_\varepsilon \left(\exp_\xi^{-1}(x) \right) \chi_r \left(\exp_\xi^{-1}(x) \right) \text{ if } x \in B_g(\xi, r), \ W_{\varepsilon,\xi}(x) := 0 \text{ if } x \in M \setminus B_g(\xi, r),$$

where we set $U_\varepsilon(z) := U\left(\frac{z}{\varepsilon}\right)$.

We will look for a solution to (13) or equivalently to (10) as $u_\varepsilon := W_{\varepsilon,\xi} + \phi$, where the rest term ϕ belongs to a suitable space which will be introduced in the following.

It is well known that every solution to the linear equation

$$-\Delta \psi + \lambda \psi = (p-1)U^{p-2}\psi \text{ in } \mathbb{R}^3$$

is a linear combination of the functions

$$\psi^i(z) := \frac{\partial U}{\partial z_i}(z), \ i = 1, 2, 3.$$

Let us define on M the functions
$$(18)$$
$$Z_{\varepsilon,\xi}^i(x) := \psi_\varepsilon^i \left(\exp_\xi^{-1}(x) \right) \chi_r \left(\exp_\xi^{-1}(x) \right) \text{ if } x \in B_g(\xi, r), \ Z_{\varepsilon,\xi}^i(x) := 0 \text{ if } x \in M \setminus B_g(\xi, r),$$

where we set $\psi_\varepsilon^i(z) := \psi^i\left(\frac{z}{\varepsilon}\right)$. Let us introduce the spaces $K_{\varepsilon,\xi} := \text{span}\left\{ Z_{\varepsilon,\xi}^1, Z_{\varepsilon,\xi}^2, Z_{\varepsilon,\xi}^3 \right\}$ and
$K_{\varepsilon,\xi}^\perp := \left\{ \phi \in H_\varepsilon : \left\langle \phi, Z_{\varepsilon,\xi}^i \right\rangle_\varepsilon = 0, \ i = 1, 2, 3 \right\}$. Finally, let $\Pi_{\varepsilon,\xi} : H_\varepsilon \to K_{\varepsilon,\xi}$ and
$\Pi_{\varepsilon,\xi}^\perp : H_\varepsilon \to K_{\varepsilon,\xi}^\perp$ be the orthogonal projections.

In order to solve problem (13) we will solve the couple of equations

$$(19) \qquad \Pi_{\varepsilon,\xi}^\perp \left\{ W_{\varepsilon,\xi} + \phi - i_\varepsilon^* \left[f\left(W_{\varepsilon,\xi} + \phi \right) + \omega^2 g\left(W_{\varepsilon,\xi} + \phi \right) \right] \right\} = 0$$

$$(20) \qquad \Pi_{\varepsilon,\xi} \left\{ W_{\varepsilon,\xi} + \phi - i_\varepsilon^* \left[f\left(W_{\varepsilon,\xi} + \phi \right) + \omega^2 g\left(W_{\varepsilon,\xi} + \phi \right) \right] \right\} = 0.$$

2.4. **Scheme of the proof of Theorem 1.** The first step is to solve equation (19). More precisely, if ε is small enough for any fixed $\xi \in M$, we will find a function $\phi \in \Pi_{\varepsilon,\xi}^{\perp}$ such that (19) holds.

First of all, we define the linear operator $L_{\varepsilon,\xi} : K_{\varepsilon,\xi}^{\perp} \to K_{\varepsilon,\xi}^{\perp}$ by

$$L_{\varepsilon,\xi}(\phi) := \Pi_{\varepsilon,\xi}^{\perp} \left\{ \phi - i_{\varepsilon}^{*} \left[f'\left(W_{\varepsilon,\xi}\right) \phi \right] \right\}.$$

In Proposition 3.1 of [30] we proved the invertibility of $L_{\varepsilon,\xi}$.

Proposition 7. *There exists $\varepsilon_0 > 0$ and $c > 0$ such that for any $\xi \in M$ and for any $\varepsilon \in (0, \varepsilon_0)$*

$$\|L_{\varepsilon,\xi}(\phi)\|_{\varepsilon} \geq c\|\phi\|_{\varepsilon} \quad \text{for any } \phi \in K_{\varepsilon,\xi}^{\perp}.$$

Secondly, in Lemma 3.3 of [30] we estimated the error term $R_{\varepsilon,\xi}$ defined by

(21) $$R_{\varepsilon,\xi} := \Pi_{\varepsilon,\xi}^{\perp} \left\{ i_{\varepsilon}^{*} \left[f\left(W_{\varepsilon,\xi}\right) \right] - W_{\varepsilon,\xi} \right\}.$$

Proposition 8. *There exists $\varepsilon_0 > 0$ and $c > 0$ such that for any $\xi \in M$ and for any $\varepsilon \in (0, \varepsilon_0)$ it holds*

$$\|R_{\varepsilon,\xi}\|_{\varepsilon} \leq c\varepsilon^2.$$

Finally, we use a contraction mapping argument to solve equation (19). This is done in Section 3

Proposition 9. *There exists $\varepsilon_0 > 0$ and $c > 0$ such that for any $\xi \in M$ and for any $\varepsilon \in (0, \varepsilon_0)$ there exists a unique $\phi_{\varepsilon,\xi} = \phi(\varepsilon, \xi)$ which solves equation (19). Moreover*

(22) $$\|\phi_{\varepsilon,\xi}\|_{\varepsilon} \leq c\varepsilon^2.$$

Finally, $\xi \to \phi_{\varepsilon,\xi}$ is a C^1-map.

The second step is to solve equation (20). More precisely, for ε small enough we will find the point ξ in M such that equation (20) is satisfied.

Let us introduce the reduced energy $\widetilde{I}_{\varepsilon} : M \to \mathbb{R}$ defined by

$$\widetilde{I}_{\varepsilon}(\xi) := I_{\varepsilon}\left(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}\right),$$

where the energy I_{ε} whose critical points are solution to problem (10) is defined in (7).

First of all, arguing exactly as in Lemma 4.1 of [30] we get

Proposition 10. *ξ_{ε} is a critical point of $\widetilde{I}_{\varepsilon}$ if and only if the function $u_{\varepsilon} = W_{\varepsilon,\xi_{\varepsilon}} + \phi_{\varepsilon,\xi_{\varepsilon}}$ is a solution to problem (10).*

Thus, the problem is reduced to search for critical points of $\widetilde{I}_{\varepsilon}$ whose asymptotic expansion is given in Section 4 and reads as follows.

Proposition 11. *It holds true that*

(23) $$\widetilde{I}_{\varepsilon}(\xi) = c_1 + c_2\varepsilon^2 - c_3 S_g(\xi)\varepsilon^2 + o\left(\varepsilon^2\right),$$

C^1-uniformly with respect to ξ as ε goes to zero. Here $S_g(\xi)$ is the scalar curvature of M at ξ and c_i's are constants.

Finally, we can prove Theorem 1 by showing that $\widetilde{I}_{\varepsilon}$ has a critical point in M.

Proof of Theorem 1. If K is a C^1-stable critical set of the scalar curvature of M (see Definition 2), by Proposition 11, we deduce that if ε is small enough the function $\widetilde{I}_{\varepsilon}$ has a critical point ξ_{ε} such that $\xi_{\varepsilon} \to \xi_0$ as ε goes to zero. The claim follows by Proposition 10. $\qquad\square$

3. The finite dimensional reduction

This section is devoted to the proof of Proposition 9.

First, we remark that equation (19) is equivalent to

$$(24) \qquad L_{\varepsilon,\xi}(\phi) = N_{\varepsilon,\xi}(\phi) + S_{\varepsilon,\xi}(\phi) + R_{\varepsilon,\xi},$$

where

$$(25) \qquad N_{\varepsilon,\xi}(\phi) := \Pi_{\varepsilon,\xi}^{\perp} \left\{ i_{\varepsilon}^{*} \left[f\left(W_{\varepsilon,\xi} + \phi\right) - f\left(W_{\varepsilon,\xi}\right) - f'\left(W_{\varepsilon,\xi}\right)\phi \right] \right\},$$

$$(26) \qquad S_{\varepsilon,\xi}(\phi) := \omega^{2} \Pi_{\varepsilon,\xi}^{\perp} \left\{ i_{\varepsilon}^{*} \left[q^{2} \Psi^{2}\left(W_{\varepsilon,\xi} + \phi\right) - 2q\Psi\left(W_{\varepsilon,\xi} + \phi\right)\left(W_{\varepsilon,\xi} + \phi\right) \right] \right\}$$

and $R_{\varepsilon,\xi}$ is defined in (21). In order to solve equation (24), we need to find a fixed point for the operator $T_{\varepsilon,\xi} : K_{\varepsilon,\xi}^{\perp} \to K_{\varepsilon,\xi}^{\perp}$ defined by

$$T_{\varepsilon,\xi}(\phi) := L_{\varepsilon,\xi}^{-1}\left(N_{\varepsilon,\xi}(\phi) + S_{\varepsilon,\xi}(\phi) + R_{\varepsilon,\xi}\right).$$

We are going to prove that $T_{\varepsilon,\xi}$ is a contraction map on suitable ball of H_{ε}.

In Proposition 8 we estimate the error term $R_{\varepsilon,\xi}$, while in Proposition 3.5 of [30], we estimated the higher order term $N_{\varepsilon,\xi}(\phi)$.

Lemma 12. *There exists $\varepsilon_0 > 0$, $c > 0$ such that for any $\xi \in M$, $\varepsilon \in (0, \varepsilon_0)$ and $r > 0$ it holds true that*

$$\|N_{\varepsilon,\xi}(\phi)\|_{\varepsilon} \le c\|\phi\|_{\varepsilon}^{2} \text{ if } p \ge 3, \quad \|N_{\varepsilon,\xi}(\phi)\|_{\varepsilon} \le c\|\phi\|_{\varepsilon}^{p-1} \text{ if } 2 < p < 3$$

and

$$\|N_{\varepsilon,\xi}(\phi_1) - N_{\varepsilon,\xi}(\phi_2)\|_{\varepsilon} \le \ell_{\varepsilon}\|\phi_1 - \phi_2\|_{\varepsilon}$$

provided $\phi, \phi_1, \phi_2 \in \left\{\phi \in H_{\varepsilon} \ : \ \|\phi\|_{\varepsilon} \le r\varepsilon^2\right\}$. Here $\ell_{\varepsilon} \to 0$ while $\varepsilon \to 0$

It only remains to estimate the term $S_{\varepsilon,\xi}(\phi)$.

Lemma 13. *There exists $\varepsilon_0 > 0$, $c > 0$ such that for any $\xi \in M$, $\varepsilon \in (0, \varepsilon_0)$ and $r > 0$ it holds true that*

$$(27) \qquad \|S_{\varepsilon,\xi}(\phi)\|_{\varepsilon} \le c\varepsilon^2$$

and

$$(28) \qquad \|S_{\varepsilon,\xi}(\phi_1) - S_{\varepsilon,\xi}(\phi_2)\|_{\varepsilon} \le \ell_{\varepsilon}\|\phi_1 - \phi_2\|_{\varepsilon}$$

provided $\phi, \phi_1, \phi_2 \in \left\{\phi \in H_{\varepsilon} \ : \ \|\phi\|_{\varepsilon} \le r\varepsilon^2\right\}$. Here $\ell_{\varepsilon} \to 0$ while $\varepsilon \to 0$

Proof. Let us prove (27). By Remark 2.2 in [30] it follows that

$$\|S_{\varepsilon,\xi}(\phi)\|_{\varepsilon} \le c \left|\Psi^2\left(W_{\varepsilon,\xi} + \phi\right)\left(W_{\varepsilon,\xi} + \phi\right)\right|_{\varepsilon,p'} + \left|\Psi\left(W_{\varepsilon,\xi} + \phi\right)\left(W_{\varepsilon,\xi} + \phi\right)\right|_{\varepsilon,p'}.$$

By Lemma 19 we have

$$|\Psi \left(W_{\varepsilon,\xi} + \phi\right) \left(W_{\varepsilon,\xi} + \phi\right)|_{\varepsilon,p'}$$

$$= \left(\frac{1}{\varepsilon^3} \int_M |\Psi \left(W_{\varepsilon,\xi} + \phi\right)|^{p'} |W_{\varepsilon,\xi} + \phi|^{p'}\right)^{1/p'}$$

$$\leq c \frac{1}{\varepsilon^{3/p'}} \left(\int_M |\Psi \left(W_{\varepsilon,\xi} + \phi\right)|^p\right)^{1/p} \left(|W_{\varepsilon,\xi} + \phi|^{\frac{p}{p-2}}\right)^{\frac{p-2}{p}}$$

$$\leq c \frac{1}{\varepsilon^{3/p'}} \|\Psi \left(W_{\varepsilon,\xi} + \phi\right)\|_{H_g^1} |W_{\varepsilon,\xi} + \phi|_{\frac{p}{p-2},g}$$

$$\leq c \frac{1}{\varepsilon^{3/p'}} \left(\varepsilon^{\frac{5}{2}} + \|\phi\|_{H_g^1}^2\right) \left(\varepsilon^{3\frac{p}{p-2}} + \|\phi\|_{H_g^1}\right) \ \left(\text{because } \|u\|_{H_g^1} \leq \sqrt{\varepsilon}\|u\|_\varepsilon\right)$$

$$\leq c \frac{1}{\varepsilon^{3/p'}} \left(\varepsilon^{\frac{5}{2}} + \varepsilon\|\phi\|_\varepsilon^2\right) \left(\varepsilon^{3\frac{p}{p-2}} + \sqrt{\varepsilon}\|\phi\|_\varepsilon\right) \ \left(\text{because } \|\phi\|_\varepsilon \leq r\varepsilon^2\right)$$

$$\leq c \frac{\varepsilon^5}{\varepsilon^{3/p'}}.$$

By Lemma 19 and the previous estimate we deduce the following

$$\left|\Psi^2 \left(W_{\varepsilon,\xi} + \phi\right) \left(W_{\varepsilon,\xi} + \phi\right)\right|_{\varepsilon,p'} \leq c\varepsilon^2,$$

and then (27) follows.

Let us prove (28). By Remark 2.2 in [30] it follows that

$$\|S_{\varepsilon,\xi}(\phi_1) - S_{\varepsilon,\xi}(\phi_2)\|_\varepsilon$$
$$\leq c \left|[\Psi \left(W_{\varepsilon,\xi} + \phi_1\right) - \Psi \left(W_{\varepsilon,\xi} + \phi_2\right)] W_{\varepsilon,\xi}\right|_{\varepsilon,p'} + c \left|\left[\Psi^2 \left(W_{\varepsilon,\xi} + \phi_1\right) - \Psi^2 \left(W_{\varepsilon,\xi} + \phi_2\right)\right] W_{\varepsilon,\xi}\right|_{\varepsilon,p'}$$
$$+ c \left|\Psi \left(W_{\varepsilon,\xi} + \phi_1\right) \phi_1 - \Psi \left(W_{\varepsilon,\xi} + \phi_2\right) \phi_2\right|_{\varepsilon,p'} + c \left|\Psi \left(W_{\varepsilon,\xi} + \phi_1\right) \phi_1 - \Psi \left(W_{\varepsilon,\xi} + \phi_2\right) \phi_2\right|_{\varepsilon,p'}$$
$$=: I_1 + I_2 + I_3 + I_4.$$

By Remark 18 and Lemma 20 we have for some $\theta \in (0,1)$:

$$I_1^{p'} = \frac{1}{\varepsilon^3} \int_M |\Psi' \left(W_{\varepsilon,\xi} + \theta\phi_1 + (1-\theta)\phi_2\right) \left(\phi_1 - \phi_2\right)|^{p'} |W_{\varepsilon,\xi}|^{p'}$$

$$\leq \frac{1}{\varepsilon^3} \left(\int_M |\Psi' \left(W_{\varepsilon,\xi} + \theta\phi_1 + (1-\theta)\phi_2\right) \left(\phi_1 - \phi_2\right)|^p\right)^{p'/p} \left(\int_M |W_{\varepsilon,\xi}|^{\frac{p'p}{p-p'}}\right)^{\frac{p-p'}{p}}$$

$$\leq c \frac{\varepsilon^{3\frac{p-p'}{p}}}{\varepsilon^3} \left(\varepsilon^2 + \|\phi_1\|_{H_g^1} + \|\phi_2\|_{H_g^1}\right)^{p'} \|\phi_1 - \phi_2\|_{H_g^1}^{p'}$$

$$\left(\text{because } \|\phi_i\|_{H_g^1} \leq \sqrt{\varepsilon}\|\phi_i\|_\varepsilon \leq c\varepsilon^{5/2}\right)$$

$$\leq c\varepsilon^{\left(2-\frac{3}{p}\right)p'} \left(\sqrt{\varepsilon}\right)^{p'} \|\phi_1 - \phi_2\|_\varepsilon^{p'}.$$

By Lemma 19 and the estimate of I_1 we have

$$I_2^{p'} = \frac{1}{\varepsilon^3} \int_M |\Psi(W_{\varepsilon,\xi} + \theta\phi_1) + \Psi(W_{\varepsilon,\xi} + \theta\phi_2)|^{p'} |\Psi(W_{\varepsilon,\xi} + \theta\phi_1) - \Psi(W_{\varepsilon,\xi} + \theta\phi_2)|^{p'} |W_{\varepsilon,\xi}|^{p'}$$

$$\leq c \left(\varepsilon^{3/2} + \|\phi_1\|_{H_g^1}^2 + \|\phi_2\|_{H_g^1}^2\right)^{p'} \varepsilon^{\left(2 - \frac{3}{p}\right)p'} \left(\sqrt{\varepsilon}\right)^{p'} \|\phi_1 - \phi_2\|_\varepsilon^{p'}$$

$$\left(\text{because } \|\phi_i\|_{H_g^1} \leq \sqrt{\varepsilon}\|\phi_i\|_\varepsilon \leq c\varepsilon^{5/2}\right)$$

$$\leq c\varepsilon^{\frac{3}{2}p'} \varepsilon^{\left(2-\frac{3}{p}\right)p'} \left(\sqrt{\varepsilon}\right)^{p'} \|\phi_1 - \phi_2\|_\varepsilon^{p'},$$

By Lemma 19 and Lemma 20 we have

$$I_3^{p'} \leq \frac{1}{\varepsilon^3} \int_M |\Psi'(W_{\varepsilon,\xi} + \theta\phi_1 + (1-\theta)\phi_2)(\phi_1 - \phi_2)|^{p'} |\phi_1|^{p'} + \frac{1}{\varepsilon^3} \int_M |\Psi(W_{\varepsilon,\xi} + \theta\phi_2)|^{p'} |\phi_1 - \phi_2|^{p'}$$

$$\leq c\frac{1}{\varepsilon^3} \left(\int_M |\Psi'(W_{\varepsilon,\xi} + \theta\phi_1 + (1-\theta)\phi_2)(\phi_1 - \phi_2)|^p\right)^{p'/p} \left(\int_M |\phi_1|^{\frac{p'p}{p-p'}}\right)^{\frac{p-p'}{p}}$$

$$+ c\frac{1}{\varepsilon^3} \left(\int_M |\phi_1 - \phi_2|^p\right)^{p'/p} \left(\int_M |\Psi(W_{\varepsilon,\xi} + \phi_2)|^{\frac{p'p}{p-p'}}\right)^{\frac{p-p'}{p}}$$

$$\leq c\frac{1}{\varepsilon^3} \left(\varepsilon^2 + \|\phi_1\|_{H_g^1} + \|\phi_2\|_{H_g^1}\right)^{p'} \|\phi_1 - \phi_2\|_{H_g^1}^{p'} \|\phi_1\|_{H_g^1}^{p'} + c\frac{1}{\varepsilon^3} \left(\varepsilon^{5/2} + \|\phi_2\|_{H_g^1}^2\right)^{p'} \|\phi_1 - \phi_2\|_{H_g^1}^{p'}$$

$$\left(\text{because } \|\phi_i\|_{H_g^1} \leq \sqrt{\varepsilon}\|\phi_i\|_\varepsilon \leq c\varepsilon^{5/2}\right)$$

$$\leq c\varepsilon^{\frac{5}{2}p'-3}\|\phi_1 - \phi_2\|_\varepsilon$$

and

$$I_4^{p'} \leq \frac{1}{\varepsilon^3} \int_M |\Psi^2(W_{\varepsilon,\xi} + \phi_1)|^{p'} |\phi_1 - \phi_2|^{p'}$$

$$+ \frac{1}{\varepsilon^3} \int_M |\Psi(W_{\varepsilon,\xi} + \phi_1) + \Psi(W_{\varepsilon,\xi} + \phi_2)|^{p'} |\Psi'(W_{\varepsilon,\xi} + \theta\phi_1 + (1-\theta)\phi_2)(\phi_1 - \phi_2)|^{p'} |\phi_2|^{p'}$$

$$\leq c\frac{1}{\varepsilon^3} \left(\varepsilon^{3/2} + \|\phi_1\|_{H_g^1}^2\right)^{p'} \left(\varepsilon^{5/2} + \|\phi_1\|_{H_g^1}^2\right)^{p'} \|\phi_1 - \phi_2\|_{H_g^1}^{p'}$$

$$+ c\frac{1}{\varepsilon^3} \left(\varepsilon^{3/2} + \|\phi_1\|_{H_g^1}^2 + \|\phi_2\|_{H_g^1}^2\right)^{p'} \left(\varepsilon^2 + \|\phi_1\|_{H_g^1} + \|\phi_2\|_{H_g^1}\right)^{p'} \|\phi_2\|_{H_g^1}^{p'} \|\phi_1 - \phi_2\|_{H_g^1}^{p'}$$

$$\left(\text{because } \|\phi_i\|_{H_g^1} \leq \sqrt{\varepsilon}\|\phi_i\|_\varepsilon \leq c\varepsilon^{5/2}\right)$$

$$\leq c\varepsilon^{4p'-3}\|\phi_1 - \phi_2\|_\varepsilon$$

Collecting the estimates of I_i's we get (28).

$\square$

Proof of Proposition 9 (completed). By Proposition 7, we deduce

$$\|T_{\varepsilon,\xi}(\phi)\|_\varepsilon \leq c \left(\|N_{\varepsilon,\xi}(\phi)\|_\varepsilon + \|S_{\varepsilon,\xi}(\phi)\|_\varepsilon + \|R_{\varepsilon,\xi}\|_\varepsilon\right)$$

and
$$\left\|T_{\varepsilon,\xi}(\phi_1) - T_{\varepsilon,\xi}(\phi_2)\right\|_{\varepsilon} \leq c \left\|N_{\varepsilon,\xi}(\phi_1) - N_{\varepsilon,\xi}(\phi_2)\right\|_{\varepsilon} + c \left\|S_{\varepsilon,\xi}(\phi_1) - S_{\varepsilon,\xi}(\phi_2)\right\|_{\varepsilon}.$$

By Lemma 12 and Lemma 13 together with Proposition 8, we immediately deduce that $T_{\varepsilon,\xi}$ is a contraction in the ball centered at 0 with radius $c\varepsilon^2$ in $K_{\varepsilon,\xi}^{\perp}$ for a suitable constant c. Then $T_{\varepsilon,\xi}$ has a unique fixed point.

In order to prove that the map $\xi \rightarrow \phi_{\varepsilon,\xi}$ is a C^1-map, we apply the Implicit Function Theorem to the C^1-function $G : M \times H_\varepsilon \rightarrow H_\varepsilon$ defined by

$$G(\xi, u) := \Pi_{\varepsilon,\xi}^{\perp} \left\{ W_{\varepsilon,\xi} + \Pi_{\varepsilon,\xi}^{\perp} u - i_\varepsilon^* \left[f \left(W_{\varepsilon,\xi} + \Pi_{\varepsilon,\xi}^{\perp} u \right) + \omega^2 g \left(W_{\varepsilon,\xi} + \Pi_{\varepsilon,\xi}^{\perp} u \right) \right] \right\} + \Pi_{\varepsilon,\xi} u.$$

Indeed, $G\left(\xi, \phi_{\varepsilon,\xi}\right) = 0$ and the linearized operator $\frac{\partial G}{\partial u}\left(\xi, \phi_{\varepsilon,\xi}\right) : H_\varepsilon \rightarrow H_\varepsilon$ defined by

$$\frac{\partial G}{\partial u}\left(\xi, \phi_{\varepsilon,\xi}\right)(u) = \Pi_{\varepsilon,\xi}^{\perp} \left\{ \Pi_{\varepsilon,\xi}^{\perp}(u) - i_\varepsilon^* \left[f' \left(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi} \right) \Pi_{\varepsilon,\xi}^{\perp}(u) + \omega^2 g' \left(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi} \right) \Pi_{\varepsilon,\xi}^{\perp}(u) \right] \right\} + \Pi_{\varepsilon,\xi}(u)$$

is invertible, provided ε is small enough. For any ϕ with $\|\phi\|_{\varepsilon} \leq c\varepsilon^2$ it holds true that

$$\left\| \frac{\partial G}{\partial u}\left(\xi, \phi_{\varepsilon,\xi}\right)(u) \right\|_{\varepsilon}$$
$$\geq c \left\|\Pi_{\varepsilon,\xi}(u)\right\|_{\varepsilon} + c \left\|\Pi_{\varepsilon,\xi}^{\perp} \left\{ \Pi_{\varepsilon,\xi}^{\perp}(u) - i_\varepsilon^* \left[f' \left(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi} \right) \Pi_{\varepsilon,\xi}^{\perp}(u) + \omega^2 g' \left(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi} \right) \Pi_{\varepsilon,\xi}^{\perp}(u) \right] \right\} \right\|_{\varepsilon}$$
$$\geq c \left\|\Pi_{\varepsilon,\xi}(u)\right\|_{\varepsilon} + c \left\|L_{\varepsilon,\xi} \left(\Pi_{\varepsilon,\xi}^{\perp}(u) \right) \right\|_{\varepsilon}$$
$$-c \left\|\Pi_{\varepsilon,\xi}^{\perp} \left\{ i_\varepsilon^* \left[\left(f' \left(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi} \right) - f' \left(W_{\varepsilon,\xi} \right) \right) \Pi_{\varepsilon,\xi}^{\perp}(u) \right] \right\} \right\|_{\varepsilon}$$
$$-c \left\|\Pi_{\varepsilon,\xi}^{\perp} \left\{ i_\varepsilon^* \left[\omega^2 g' \left(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi} \right) \Pi_{\varepsilon,\xi}^{\perp}(u) \right] \right\} \right\|_{\varepsilon}$$
$$\geq c \left\|\Pi_{\varepsilon,\xi}(u)\right\|_{\varepsilon} + c \left\|\Pi_{\varepsilon,\xi}^{\perp}(u)\right\|_{\varepsilon} - c\varepsilon^{2\min\{p-2,1\}} \left\|\Pi_{\varepsilon,\xi}^{\perp}(u)\right\|_{\varepsilon} - c\varepsilon^{3-\frac{3}{p'}} \left\|\Pi_{\varepsilon,\xi}^{\perp}(u)\right\|_{\varepsilon}$$
$$\geq c \left\|u\right\|_{\varepsilon}.$$

Indeed, at page 246 of [30] we proved that

$$\left\|\Pi_{\varepsilon,\xi}^{\perp} \left\{ i_\varepsilon^* \left[\left(f' \left(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi} \right) - f' \left(W_{\varepsilon,\xi} \right) \right) \Pi_{\varepsilon,\xi}^{\perp}(u) \right] \right\} \right\|_{\varepsilon}$$
$$\leq c \left(\|\phi\|_{\varepsilon}^{p-2} + \|\phi\|_{\varepsilon} \right) \left\|\Pi_{\varepsilon,\xi}^{\perp}(u)\right\|_{\varepsilon}.$$

Moreover we have

$$\left\|\Pi_{\varepsilon,\xi}^{\perp} \left\{ i_\varepsilon^* \left[\omega^2 g' \left(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi} \right) \Pi_{\varepsilon,\xi}^{\perp}(u) \right] \right\} \right\|_{\varepsilon}$$
$$\leq c \left| \left(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi} \right) \left(2q - 2q^2 \Psi \left(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi} \right) \right) \Psi' \left(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi} \right) \left[\Pi_{\varepsilon,\xi}^{\perp}(u) \right] \right|_{\varepsilon,p'}$$
$$+c \left| \left[2q\Psi \left(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi} \right) - q^2 \Psi^2 \left(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi} \right) \right] \Pi_{\varepsilon,\xi}^{\perp}(u) \right|_{\varepsilon,p'}$$
$$= I_1 + I_2,$$

by Lemma 20 we get

$$I_1 \leq \frac{1}{\varepsilon^{\frac{3}{p'}}} \left| W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi} \right|_{g,6} \left| \Psi' \left(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi} \right) \Pi_{\varepsilon,\xi}^{\perp}(u) \right|_{g,6} \left| 2q - 2q^2 \Psi \left(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi} \right) \right|_{g,\frac{3-p'}{3p'}}$$
$$\leq c \frac{1}{\varepsilon^{\frac{3}{p'}}} \left(\sqrt{\varepsilon} \left(\frac{1}{\varepsilon^3} \int_M W_{\varepsilon,\xi}^6 \right)^{\frac{1}{6}} + \|\phi\|_{H_g^1} \right) \left(\varepsilon^2 + \|\phi\|_{H_g^1} \right) \left\|\Pi_{\varepsilon,\xi}^{\perp} u\right\|_{H_g^1}$$
$$\leq c \frac{1}{\varepsilon^{\frac{3}{p'}}} \varepsilon^3 \left\|\Pi_{\varepsilon,\xi}^{\perp} u\right\|_{\varepsilon}$$

and by Lemma 19 we get

$$I_2 \leq \frac{1}{\varepsilon^{\frac{3}{p'}}} \left| \Pi_{\varepsilon,\xi}^{\perp} u \right|_{g,6} \left| \Psi \left(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi} \right) \right|_{g,6} \left| 2q - q^2 \Psi \left(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi} \right) \right|_{g,\frac{3-p'}{3p'}}$$

$$\leq c \frac{1}{\varepsilon^{\frac{3}{p'}}} \left(\varepsilon^{\frac{5}{2}} + \|\phi\|_{\varepsilon}^2 \right) \left\| \Pi_{\varepsilon,\xi}^{\perp} u \right\|_{H_g^1} \leq c \frac{1}{\varepsilon^{\frac{3}{p'}}} \varepsilon^{\frac{5}{2}} \sqrt{\varepsilon} \left\| \Pi_{\varepsilon,\xi}^{\perp} u \right\|_{\varepsilon}.$$

That concludes the proof. $\square$

4. The reduced energy

This section is devoted to the proof of Proposition 11.
The first important result is the following one.

Lemma 14. *It holds true that*
(29)

$$\widetilde{I}_{\varepsilon}(\xi) = I_{\varepsilon} \left(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi} \right) = I_{\varepsilon} \left(W_{\varepsilon,\xi} \right) + o \left(\varepsilon^2 \right) = J_{\varepsilon} \left(W_{\varepsilon,\xi} \right) + \frac{\omega^2}{2} G_{\varepsilon} \left(W_{\varepsilon,\xi} \right) + o \left(\varepsilon^2 \right)$$

uniformly with respect to ξ as ε goes to zero.

Moreover, setting $\xi(y) = \exp_{\xi}(y)$, $y \in B(0,r)$ it holds true that

$$\left(\frac{\partial}{\partial y_h} \widetilde{I}_{\varepsilon}(\xi(y)) \right)_{|y=0} = \left(\frac{\partial}{\partial y_h} I_{\varepsilon} \left(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)} \right) \right)_{|y=0}$$

$$= \left(\frac{\partial}{\partial y_h} I_{\varepsilon} \left(W_{\varepsilon,\xi(y)} \right) \right)_{|y=0} + o \left(\varepsilon^2 \right)$$

(30) $$= \left(\frac{\partial}{\partial y_h} J_{\varepsilon} \left(W_{\varepsilon,\xi(y)} \right) \right)_{|y=0} + \frac{\omega^2}{2} \left(\frac{\partial}{\partial y_h} G_{\varepsilon} \left(W_{\varepsilon,\xi(y)} \right) \right)_{|y=0} + o \left(\varepsilon^2 \right)$$

uniformly with respect to ξ as ε goes to zero.

Proof. We argue exactly as in Lemma 5.1 of [30], once we prove the the following
estimates:

(31) $$G_{\varepsilon} \left(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi} \right) - G_{\varepsilon} \left(W_{\varepsilon,\xi} \right) = o(\varepsilon^2)$$

(32) $$[G'_{\varepsilon} \left(W_{\varepsilon,\xi_0} + \phi_{\varepsilon,\xi_0} \right) - G'_{\varepsilon} \left(W_{\varepsilon,\xi_0} \right)] \left[\left(\frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right)_{|y=0} \right] = o(\varepsilon^2)$$

and

(33) $$G'_{\varepsilon} \left(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)} \right) \left[\frac{\partial}{\partial y_h} \phi_{\varepsilon,\xi(y)} \right] = o(\varepsilon^2)$$

Let us prove (31). We have (for some $\theta \in [0,1]$)

$$G_{\varepsilon} \left(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi} \right) - G_{\varepsilon} \left(W_{\varepsilon,\xi} \right)$$

$$= \frac{1}{\varepsilon^3} \int_M \left[\Psi \left(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi} \right) \left(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi} \right)^2 - \Psi \left(W_{\varepsilon,\xi} \right) \left(W_{\varepsilon,\xi} \right)^2 \right]$$

$$= \frac{1}{\varepsilon^3} \int_M \Psi' \left(W_{\varepsilon,\xi} + \theta\phi_{\varepsilon,\xi} \right) [\phi_{\varepsilon,\xi}] \left(W_{\varepsilon,\xi} \right)^2 + \frac{1}{\varepsilon^3} \int_M \Psi \left(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi} \right) \left[2\phi_{\varepsilon,\xi} W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}^2 \right]$$

$$=: I_1 + I_2$$

with

$$I_1 \leq \frac{1}{\varepsilon^3} \left(\int_M \left[\Psi' \left(W_{\varepsilon,\xi} + \theta \phi_{\varepsilon,\xi} \right) \right]^2 \right)^{1/2} \left(\frac{1}{\varepsilon^3} \int_M W_{\varepsilon,\xi}^4 \right)^{1/2} \varepsilon^{3/2}$$

$$\leq c \frac{1}{\varepsilon^{3/2}} \left\| \Psi' \left(W_{\varepsilon,\xi} + \theta \phi_{\varepsilon,\xi} \right) \left[\phi_{\varepsilon,\xi} \right] \right\|_{H_g^1} \quad \text{(because of Lemma 20)}$$

$$\leq c \frac{1}{\varepsilon^{3/2}} \left(\varepsilon^2 \|\phi\|_{H_g^1} + \|\phi\|_{H_g^1}^2 \right) \quad \left(\text{because } \|\phi_{\varepsilon,\xi}\|_{H_g^1} \leq \sqrt{\varepsilon} \|\phi_{\varepsilon,\xi}\|_\varepsilon \leq c\varepsilon^{5/2} \right)$$

$$= o(\varepsilon^2)$$

and

$$I_2 \leq \frac{1}{\varepsilon^3} \left(\int_M \left(\Psi \left(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi} \right) \left[\phi_{\varepsilon,\xi} \right] \right)^2 \right)^{1/2} \left(\int_M \phi_{\varepsilon,\xi}^4 \right)^{1/2}$$

$$+ \frac{1}{\varepsilon^3} \left(\int_M \left[\Psi \left(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi} \right) \right]^3 \right)^{1/3} \left(\int_M \phi_{\varepsilon,\xi}^3 \right)^{1/3} \left(\frac{1}{\varepsilon^3} \int_M W_{\varepsilon,\xi}^3 \right)^{1/3} \varepsilon$$

$$\leq c \frac{1}{\varepsilon^3} \left\| \Psi \left(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi} \right) \right\|_{H_g^1} \left\| \phi_{\varepsilon,\xi} \right\|_{H_g^1}^2 + c \frac{1}{\varepsilon^2} \left\| \Psi \left(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi} \right) \right\|_{H_g^1} \left\| \phi_{\varepsilon,\xi} \right\|_{H_g^1}$$

$$\left(\text{because of Lemma 19 and the fact that } \|\phi_{\varepsilon,\xi}\|_{H_g^1} \leq \sqrt{\varepsilon} \|\phi_{\varepsilon,\xi}\|_\varepsilon \leq c\varepsilon^{5/2} \right)$$

$$\leq c\varepsilon^{5+\frac{5}{2}-3} + c\varepsilon^{\frac{5}{2}+\frac{5}{2}-2} = o(\varepsilon^2).$$

Then (31) follows.

Let us prove (32). We have (for some $\theta \in [0, 1]$)

$$[G'_\varepsilon (W_{\varepsilon,\xi_0} + \phi_{\varepsilon,\xi_0}) - G'_\varepsilon (W_{\varepsilon,\xi_0})] \left[\left(\frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right)_{|y=0} \right]$$

$$\leq \left| \frac{q}{2\varepsilon^3} \int_M \{ [2\Psi (W_{\varepsilon,\xi_0} + \phi_{\varepsilon,\xi_0}) - \Psi (W_{\varepsilon,\xi_0})] - [q\Psi^2 (W_{\varepsilon,\xi_0} + \phi_{\varepsilon,\xi_0}) - q\Psi^2 (W_{\varepsilon,\xi_0})] \} W_{\varepsilon,\xi_0} \left(\frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right)_{|y=0} \right|$$

$$+ \left| \frac{q}{2\varepsilon^3} \int_M [2\Psi (W_{\varepsilon,\xi_0} + \phi_{\varepsilon,\xi_0}) - q\Psi^2 (W_{\varepsilon,\xi_0} + \phi_{\varepsilon,\xi_0})] \phi_{\varepsilon,\xi_0} \left(\frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right)_{|y=0} \right|$$

$$\leq \left| \frac{q}{2\varepsilon^3} \int_M 2\Psi' (W_{\varepsilon,\xi_0} + \theta\phi_{\varepsilon,\xi_0}) (\phi_{\varepsilon,\xi_0}) W_{\varepsilon,\xi_0} \left(\frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right)_{|y=0} \right|$$

$$+ \left| \frac{q^2}{\varepsilon^3} \int_M \Psi (W_{\varepsilon,\xi_0} + \theta\phi_{\varepsilon,\xi_0}) \Psi' (W_{\varepsilon,\xi_0} + \theta\phi_{\varepsilon,\xi_0}) (\phi_{\varepsilon,\xi_0}) W_{\varepsilon,\xi_0} \left(\frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right)_{|y=0} \right|$$

$$+ \left| \frac{q}{\varepsilon^3} \int_M \Psi (W_{\varepsilon,\xi_0}) \phi_{\varepsilon,\xi_0} \left(\frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right)_{|y=0} \right|$$

$$+ \left| \frac{q}{2\varepsilon^3} \int_M \Psi' (W_{\varepsilon,\xi_0}) (\phi_{\varepsilon,\xi_0}) \phi_{\varepsilon,\xi_0} \left(\frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right)_{|y=0} \right|$$

$$+ \left| \frac{q^2}{2\varepsilon^3} \int_M \Psi^2 (W_{\varepsilon,\xi_0}) \phi_{\varepsilon,\xi_0} \left(\frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right)_{|y=0} \right|$$

$$=: I_1 + I_2 + I_3 + I_4 + I_5.$$

By Lemma 20 and the facts that $\left\| \phi_{\varepsilon,\xi(y)} \right\|_{H^1_g} = O \left(\varepsilon^{5/2} \right)$ and Remark 18 we get

$$I_1 \leq c \frac{1}{\varepsilon} \left(\int_M [\Psi' (W_{\varepsilon,\xi_0} + \theta\phi_{\varepsilon,\xi_0}) (\phi_{\varepsilon,\xi_0})]^3 \right)^{1/3} \left(\frac{1}{\varepsilon^3} \int_M W^3_{\varepsilon,\xi_0} \right)^{1/3} \left(\frac{1}{\varepsilon^3} \int_M \left[\left(\frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right)_{|y=0} \right]^3 \right)^{1/3}$$

(we use (6.3) of [30])

$$\leq c\varepsilon^{7/2} \left(\int_M \left[\sum_{k=1}^3 \left| \frac{1}{\varepsilon} \frac{\partial U(z)}{\partial z_k} \chi(\varepsilon z) + \frac{\partial \chi(\varepsilon z)}{\partial z_k} U(z) \right| \left| \delta_{hk} + \varepsilon^2 |z|^2 \right| \right]^3 \right)^{1/3} = O \left(\varepsilon^{5/2} \right) = o(\varepsilon^2).$$

By the estimate of I_1 we get $I_2 = o(\varepsilon^2)$, because of Lemma 19 and we also get $I_4 = o(\varepsilon^2)$, because $\|\phi_{\varepsilon,\xi_0}\|_{H^1_g} = O \left(\varepsilon^{5/2} \right)$. Let us estimate I_3. We use the definition

of $\widetilde{v}_\varepsilon$ given in Lemma 21 and we get

$$
I_3 = \left| \frac{1}{\varepsilon^3} \int_{B_g(\xi_0,R)} \Psi\left(W_{\varepsilon,\xi_0}\right)(\phi_{\varepsilon,\xi_0})\left(\frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)}\right)_{|y=0} \right|
$$

$$
\leq c \int_{B(0,R/\varepsilon)} |\widetilde{v}_\varepsilon(z)|\left|\widetilde{\phi}_{\varepsilon,\xi_0}(z)\right| \left| \sum_{k=1}^{3} \left| \frac{1}{\varepsilon}\frac{\partial U(z)}{\partial z_k}\chi(\varepsilon z) + \frac{\partial \chi(\varepsilon z)}{\partial z_k}U(z) \right| \left| \delta_{hk} + \varepsilon^2|z|^2 \right| dz
$$

$$
\leq c \left(\int_{B(0,R/\varepsilon)} \widetilde{\phi}_{\varepsilon,\xi_0}^2(z) \right)^{1/2} \left(\varepsilon^4 \int_{B(0,R/\varepsilon)} \frac{\widetilde{v}_\varepsilon^2(z)}{\varepsilon^4} \left[\sum_{k=1}^{3} \left| \frac{1}{\varepsilon}\frac{\partial U(z)}{\partial z_k}\chi(\varepsilon z) + \frac{\partial \chi(\varepsilon z)}{\partial z_k}U(z) \right| \left| \delta_{hk} + \varepsilon^2|z|^2 \right| \right]^2 dz \right)^{1/2}
$$

$$
\left(\text{because } \frac{\widetilde{v}_\varepsilon^2(z)}{\varepsilon^4} \to \gamma \text{ weakly in } L^6 \right)
$$

$$
= o(\varepsilon^2).
$$

Here we used the fact that (see (6.3) of [30]) the function $\widetilde{\phi}_{\varepsilon,\xi_0}(z) := \phi_{\varepsilon,\xi_0}\left(\exp_{\xi_0}(\varepsilon z)\right) = \phi_{\varepsilon,\xi_0}(x)$ can be estimated as

$$
\|\phi_{\varepsilon,\xi_0}\|_\varepsilon^2 \geq \frac{1}{\varepsilon^3} \int_{B_g(\xi_0,R)} \left(\varepsilon^2|\nabla_g \phi_{\varepsilon,\xi_0}|^2 + \phi_{\varepsilon,\xi_0}^2 \right) d\mu_g =
$$

$$
= \frac{1}{\varepsilon^3} \int_{B(0,R/\varepsilon)} \left(\sum_{i,j=1}^{3} \varepsilon^2 g^{ij}(\varepsilon z)\frac{\partial \widetilde{\phi}_{\varepsilon,\xi_0}}{\partial z_i}\frac{\partial \widetilde{\phi}_{\varepsilon,\xi_0}}{\partial z_j} + \widetilde{\phi}_{\varepsilon,\xi_0}^2 \right) |g(\varepsilon z)|^{1/2} dz
$$

$$
\geq c \int_{B(0,R/\varepsilon)} \widetilde{\phi}_{\varepsilon,\xi_0}^2(z) dz
$$

which implies

$$
\int_{B(0,R/\varepsilon)} \widetilde{\phi}_{\varepsilon,\xi_0}^2(z) dz = O\left(\|\phi_{\varepsilon,\xi_0}\|_\varepsilon^2\right) = O\left(\varepsilon^4\right).
$$

By the estimate of I_3 we get $I_5 = o(\varepsilon^2)$, because of Lemma 19.

Let us prove (33). We have (for some $\theta \in [0,1]$) Arguing as in the proof of (5.10) of [30], the proof of (33) reduces to the proof of the following estimate

$$
(34) \qquad \frac{1}{\varepsilon^3} \int_M g\left(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}\right)\left(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}\right) Z_{\varepsilon,\xi(y)}^l = o(\varepsilon^2),
$$

where the functions $Z_{\varepsilon,\xi(y)}^l$ are defined in (18). First of all we point out that

$$
(35) \qquad \frac{1}{\varepsilon^3} \int_M \Psi\left(W_{\varepsilon,\xi(y)}\right) W_{\varepsilon,\xi(y)} Z_{\varepsilon,\xi(y)}^l = o(\varepsilon^2).
$$

By Lemma 21 we have that $\left\{\frac{1}{\varepsilon^2}\tilde{v}_{\varepsilon,\xi}\right\}_n$ converges to γ weakly in $L^6(\mathbb{R}^3)$. So, arguing as in the proof of (39) and using Lemma 21 we get

$$
\frac{1}{\varepsilon^3}\int_M \Psi\left(W_{\varepsilon,\xi(y)}\right)W_{\varepsilon,\xi(y)}Z^l_{\varepsilon,\xi(y)} = \frac{1}{\varepsilon^3}\int_{B_g(\xi,R)} \Psi\left(W_{\varepsilon,\xi(y)}\right)W_{\varepsilon,\xi(y)}Z^l_{\varepsilon,\xi(y)}
$$

$$
= \int_{B(0,R/\varepsilon)} \tilde{v}(z)\chi^2(\varepsilon z)U(z)\frac{\partial U}{\partial z_l}(z)dz = \frac{1}{2}\varepsilon^2 \int_{B(0,R/\varepsilon)} \frac{\tilde{v}(z)}{\varepsilon^2}\chi^2(\varepsilon z)\frac{\partial U^2}{\partial z_l}dz =
$$

$$
= \frac{1}{\varepsilon^2}\int_{\mathbb{R}^3}\gamma\frac{\partial U^2}{\partial z_l}dz + o(\varepsilon^2) = o(\varepsilon^2).
$$

because both U and γ are radially symmetric.

Therefore, by (35), using the definition of g in (14) we are lead to estimate (for some $\theta \in [0,1]$)

$$
\frac{1}{\varepsilon^3}\left|\int_M g\left(W_{\varepsilon,\xi(y)}+\phi_{\varepsilon,\xi(y)}\right)\left(W_{\varepsilon,\xi(y)}+\phi_{\varepsilon,\xi(y)}\right)Z^l_{\varepsilon,\xi(y)} + 2q\int_M \Psi\left(W_{\varepsilon,\xi(y)}\right)W_{\varepsilon,\xi(y)}Z^l_{\varepsilon,\xi(y)}\right|
$$

$$
= \frac{1}{\varepsilon^3}\left|\int_M \left[q^2\Psi^2\left(W_{\varepsilon,\xi(y)}+\phi_{\varepsilon,\xi(y)}\right) - 2q\Psi\left(W_{\varepsilon,\xi(y)}+\phi_{\varepsilon,\xi(y)}\right)\right]\left(W_{\varepsilon,\xi(y)}+\phi_{\varepsilon,\xi(y)}\right)Z^l_{\varepsilon,\xi(y)}\right.
$$

$$
\left. +2q\int_M \Psi\left(W_{\varepsilon,\xi(y)}\right)W_{\varepsilon,\xi(y)}Z^l_{\varepsilon,\xi(y)}\right|
$$

$$
\leq c\frac{1}{\varepsilon^3}\int_M\left|\Psi^2\left(W_{\varepsilon,\xi(y)}+\phi_{\varepsilon,\xi(y)}\right)\left(W_{\varepsilon,\xi(y)}+\phi_{\varepsilon,\xi(y)}\right)Z^l_{\varepsilon,\xi(y)}\right|
$$

$$
+c\frac{1}{\varepsilon^3}\int_M\left|\Psi\left(W_{\varepsilon,\xi(y)}\right)\phi_{\varepsilon,\xi(y)}Z^l_{\varepsilon,\xi(y)}\right|
$$

$$
+c\frac{1}{\varepsilon^3}\int_M\left|\Psi'\left(W_{\varepsilon,\xi(y)}+\theta\phi_{\varepsilon,\xi(y)}\right)\left(\phi_{\varepsilon,\xi(y)}\right)\left(W_{\varepsilon,\xi(y)}+\phi_{\varepsilon,\xi(y)}\right)Z^l_{\varepsilon,\xi(y)}\right| = I_1 + I_2 + I_3,
$$

with

$$
I_1 \leq \frac{1}{\varepsilon^3}\left|\Psi\left(W_{\varepsilon,\xi(y)}+\phi_{\varepsilon,\xi(y)}\right)\right|^2_{g,3}\left(\left|W_{\varepsilon,\xi(y)}\right|_{g,3}+\left|\phi_{\varepsilon,\xi(y)}\right|_{g,3}\right)\left|Z^l_{\varepsilon,\xi(y)}\right|_{g,3}
$$

(we use Lemma 19)

$$
\leq \frac{1}{\varepsilon^3}\left\|\Psi\left(W_{\varepsilon,\xi(y)}+\phi_{\varepsilon,\xi(y)}\right)\right\|^2_{H^1_g}\left(\left|W_{\varepsilon,\xi(y)}\right|_{g,3}+\left\|\phi_{\varepsilon,\xi(y)}\right\|_{H^1_g}\right)\left|Z^l_{\varepsilon,\xi(y)}\right|_{g,3}
$$

$$
\left(\text{we use that }\|\phi\|_{H^1_g}\leq\sqrt{\varepsilon}\|\phi\|_\varepsilon\leq c\varepsilon^{5/2},\ \left|W_{\varepsilon,\xi(y)}\right|_{g,3}=O(\varepsilon)\text{ and }\left|Z^l_{\varepsilon,\xi(y)}\right|_{g,3}=O(\varepsilon)\right)
$$

$$
= o(\varepsilon^2),
$$

$$I_2 \leq \frac{1}{\varepsilon^3} \left\| \Psi\left(W_{\varepsilon,\xi(y)}\right) \right\|_{H_g^1} \left\| \phi_{\varepsilon,\xi(y)} \right\|_{H_g^1} \left| Z_{\varepsilon,\xi(y)}^l \right|_{g,3}$$

$$\left(\text{we use Lemma 19 and also that } \|\phi\|_{H_g^1} \leq \sqrt{\varepsilon}\|\phi\|_\varepsilon \leq c\varepsilon^{5/2} \text{ and } \left| Z_{\varepsilon,\xi(y)}^l \right|_{g,3} = O(\varepsilon) \right)$$

$$= o(\varepsilon^2)$$

and

$$I_3 \leq \frac{1}{\varepsilon^3} \left| \Psi'\left(W_{\varepsilon,\xi(y)} + \theta\phi_{\varepsilon,\xi(y)}\right)\left(\phi_{\varepsilon,\xi(y)}\right) \right|_{g,3} \left| W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)} \right|_{g,3} \left| Z_{\varepsilon,\xi(y)}^l \right|_{g,3}$$

$$\leq \frac{1}{\varepsilon^3} \left\| \Psi'\left(W_{\varepsilon,\xi(y)} + \theta\phi_{\varepsilon,\xi(y)}\right)\left(\phi_{\varepsilon,\xi(y)}\right) \right\|_{H_g^1} \left(\left| W_{\varepsilon,\xi(y)} \right|_{g,3} + \left\| \phi_{\varepsilon,\xi(y)} \right\|_{H_g^1} \right) \left| Z_{\varepsilon,\xi(y)}^l \right|_{g,3}$$

(because of Lemma 20)

$$\leq \frac{1}{\varepsilon^3} \left(\varepsilon^2 + \left\| \phi_{\varepsilon,\xi(y)} \right\|_{H_g^1} \right) \left\| \phi_{\varepsilon,\xi(y)} \right\|_{H_g^1} \left(\left| W_{\varepsilon,\xi(y)} \right|_{g,3} + \left\| \phi_{\varepsilon,\xi(y)} \right\|_{H_g^1} \right) \left| Z_{\varepsilon,\xi(y)}^l \right|_{g,3}$$

$$\left(\text{we use that } \|\phi\|_{H_g^1} \leq \sqrt{\varepsilon}\|\phi\|_\varepsilon \leq c\varepsilon^{5/2}, \left| W_{\varepsilon,\xi(y)} \right|_{g,3} = O(\varepsilon) \text{ and } \left| Z_{\varepsilon,\xi(y)}^l \right|_{g,3} = O(\varepsilon) \right)$$

$$= o(\varepsilon^2).$$

$\square$

Lemma 15. *It holds true that*

(36) $$J_\varepsilon\left(W_{\varepsilon,\xi}\right) = C - \alpha\frac{\varepsilon^2}{6}S_g(\xi) + o\left(\varepsilon^2\right),$$

C^1-*uniformly with respect to* $\xi \in M$ *as* ε *goes to zero. Here*

$$C := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla U|^2 dz - \frac{\lambda}{2}\int_{\mathbb{R}^3} U^2 dz - \frac{1}{p}\int_{\mathbb{R}^3} U^p dz$$

and

$$\alpha := \int_{\mathbb{R}^3} \left(\frac{U'(|z|)}{|z|} \right)^2 z_1^4 dz.$$

Proof. See Lemma (4.2) of [30]. $\square$

Lemma 16. *It holds true that*

$$G_\varepsilon(W_{\varepsilon,\xi}) := \frac{1}{\varepsilon^3}\int_{B_g(\xi,r)} \Psi(W_{\varepsilon,\xi})W_{\varepsilon,\xi}^2 d\mu_g = \beta\varepsilon^2 + o(\varepsilon^2)$$

C^1-*uniformly with respect to* $\xi \in M$ *as* ε *goes to zero. Here*

$$\beta = \int_{\mathbb{R}^3} \gamma(z)U^2(z)dz = \frac{1}{q}\int_{\mathbb{R}^3} |\nabla\gamma(z)|^2 dz$$

with $\gamma \in D^{1,2}(\mathbb{R}^3)$ *such that* $-\Delta\gamma = qU^2$.

Proof. Step 1: the C^0-*estimate.*

By the weak convergence of $\left\{ \frac{1}{\varepsilon_n^2}v_{\varepsilon_n,\xi} \right\}_n$ in $L^6(\mathbb{R}^3)$ we infer

$$\frac{G_{\varepsilon_n}(W_{\varepsilon_n,\xi})}{\varepsilon_n^2} = \int_{\mathbb{R}^3} \frac{\tilde{v}_{\varepsilon_n,\xi}(z)}{\varepsilon_n^2}\chi_r^2(\varepsilon_n|z|)U^2(z)|g_\xi(\varepsilon_n z)|^{1/2}$$

$$= \int_{\mathbb{R}^3} \frac{v_{\varepsilon_n,\xi}(z)}{\varepsilon_n^2}\chi_r(\varepsilon_n|z|)U^2(z)|g_\xi(\varepsilon_n z)|^{1/2} \to \int_{\mathbb{R}^3} \gamma(z)U^2(z)dz.$$

We have to prove that the convergence is uniform with respect to $\xi \in M$.

By the expansions of $|g_\xi(\varepsilon z)|^{1/2}$ and $\chi(\varepsilon|z|)$, and by (47) we have

$$
\begin{aligned}
\frac{G_\varepsilon(W_{\varepsilon,\xi})}{\varepsilon^2} &= \int_{\mathbb{R}^3} \frac{\tilde{v}_{\varepsilon,\xi}(z)}{\varepsilon^2} \chi_r^2(\varepsilon|z|) U^2(z) |g_\xi(\varepsilon z)|^{1/2} dz \\
&= \frac{1}{q} \int_{\mathbb{R}^3} \frac{\tilde{v}_{\varepsilon,\xi}(z)}{\varepsilon^2} \chi_r^2(\varepsilon|z|) q U^2(z) dz + O(\varepsilon^2) \\
(37) \qquad &= -\frac{1}{q} \int_{\mathbb{R}^3} \frac{\tilde{v}_{\varepsilon,\xi}(z)}{\varepsilon^2} \chi_r^2(\varepsilon|z|) \Delta\gamma dz + O(\varepsilon^2) \\
&= -\frac{1}{q} \int_{\mathbb{R}^3} \left(\Delta \frac{\tilde{v}_{\varepsilon,\xi}(z)}{\varepsilon^2} \right) \chi_r^2(\varepsilon|z|) \gamma dz + O(\varepsilon) \\
&= -\frac{1}{q} \int_{\mathbb{R}^3} -\sum_{ij} \partial_j \left(|g_\xi(\varepsilon z)|^{1/2} g_\xi^{ij}(\varepsilon z) \partial_i \frac{\tilde{v}_{\varepsilon,\xi}(z)}{\varepsilon^2} \right) \chi_r^2(\varepsilon|z|) \gamma dz + O(\varepsilon)
\end{aligned}
$$

uniformly with respect to ξ as ε goes to zero.

By (48) and by the expansions of $|g_\xi(\varepsilon z)|^{1/2}$ and $\chi(\varepsilon|z|)$ we have

$$
\begin{aligned}
(38) \\
\left| \frac{G_\varepsilon(W_{\varepsilon,\xi})}{\varepsilon^2} - \int_{\mathbb{R}^3} U^2 \gamma dz \right| &\leq O(\varepsilon) + \left| \int_{\mathbb{R}^3} U^2(z)\gamma(z)[|g_\xi(\varepsilon z)|^{1/2}\chi_r^2(\varepsilon|z|) - 1] dz \right| \leq \\
+ \frac{1}{q} \left| \int_{\mathbb{R}^3} |g_\xi(\varepsilon z)|^{1/2} \left[\chi_r^2(\varepsilon|z|) + q^2 U^2(z)\chi_r^4(\varepsilon|z|) \right] \tilde{v}_{\varepsilon,\xi}(z)\gamma(z) dz \right| &\leq \\
&\leq O(\varepsilon) + \frac{1}{q} \left| \int_{\mathbb{R}^3} \tilde{v}_{\varepsilon,\xi}(z)\gamma(z) \right| \leq O(\varepsilon)
\end{aligned}
$$

uniformly with respect to ξ as ε goes to zero.

Step 2: the C^1-estimate.

More precisely, if $\xi(y) = \exp_\xi(y)$ for $y \in B(0,r)$,. we are going to prove that

$$
(39) \qquad \frac{\partial}{\partial y_h} G_\varepsilon(W_{\varepsilon,\xi(h)}) \bigg|_{y=0} = o(\varepsilon^2) \text{ uniformly with respect to } \xi \text{ as } \varepsilon \text{ goes to } 0.
$$

We have that

$$
\begin{aligned}
\frac{\partial}{\partial y_h} G_\varepsilon(W_{\varepsilon,\xi}) \bigg|_{y=0} &= \frac{\partial}{\partial y_h} \frac{1}{\varepsilon^3} \int_M \Psi(W_{\varepsilon,\xi(y)}) W_{\varepsilon,\xi(y)}^2 \bigg|_{y=0} d\mu_g \\
&= \frac{1}{\varepsilon^3} \int_M \Psi(W_{\varepsilon,\xi(y)}) 2 W_{\varepsilon,\xi(h)} \left(\frac{\partial}{\partial y_h} W_{\varepsilon,\xi(h)} \right) \bigg|_{y=0} d\mu_g \\
&\quad + \frac{1}{\varepsilon^3} \int_M W_{\varepsilon,\xi(h)}^2 \Psi'(W_{\varepsilon,\xi(y)}) \left[\frac{\partial}{\partial y_h} W_{\varepsilon,\xi(h)} \bigg|_{y=0} \right] d\mu_g
\end{aligned}
$$

We call $I_1(\varepsilon,\xi)$ and $I_2(\varepsilon,\xi)$ respectively the first and the second addendum of the above equation.

We recall that (see Section 6 of [30]) that

$$(40) \quad \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(h)}\bigg|_{y=0} = \sum_{k=1}^{3} \left[\frac{1}{\varepsilon} \frac{\partial U(z)}{\partial z_k} \chi_r(\varepsilon|z|) + U(z) \frac{\partial \chi_r(\varepsilon|z|)}{\partial z_k} \right] \frac{\partial}{\partial y_h} \mathcal{E}_k(0, \exp_\xi(\varepsilon z))$$

$$(41) \quad \frac{\partial}{\partial y_h} \mathcal{E}_k(0, \exp_\xi(\varepsilon z)) = \delta_{hk} + O(\varepsilon^2 |z|^2)$$

$$(42) \quad |g_\xi(\varepsilon z)|^{1/2} = 1 - \frac{\varepsilon^2}{4} \sum_{i,s,k=1}^{3} \frac{\partial^2 g_\xi^{ij}(0)}{\partial z_s \partial z_k} z_s z_k + O(\varepsilon^3 |z|^3)$$

Using the normal coordinates and the previous estimates we get

$$\frac{1}{\varepsilon^2} I_1(\varepsilon,\xi) = \int_{\mathbb{R}^3} \frac{\tilde{v}_{\varepsilon,\xi}(z)}{\varepsilon^2} 2U(z)\chi_r(\varepsilon|z|)|g_\xi(\varepsilon z)|^{1/2} \times$$

$$\times \left\{ \sum_{k=1}^{3} \left[\frac{1}{\varepsilon} \frac{\partial U(z)}{\partial z_k} \chi_r(\varepsilon|z|) + U(z) \frac{\partial \chi_r(\varepsilon|z|)}{\partial z_k} \right] \frac{\partial}{\partial y_h} \mathcal{E}_k(0, \exp_\xi(\varepsilon z)) \right\} dz.$$

By Lemma 21 we have that $\left\{ \frac{1}{\varepsilon^2} \tilde{v}_{\varepsilon,\xi} \right\}_n$ converges to γ weakly in $L^6(\mathbb{R}^3)$, so we have

$$I_1(\varepsilon,\xi) = 2\varepsilon \int_{\mathbb{R}^3} \gamma U(z) U'(z) \frac{z_h}{|z|} dz + o(\varepsilon^2).$$

Finally, we have that $\int_{\mathbb{R}^3} \gamma(z) U(z) U'(z) \frac{z_h}{|z|} dz = 0$ because both γ and U are radially symmetric on z.

At this point we have to prove the uniform convergence of $I_1(\varepsilon, \xi)$ with respect to $\xi \in M$. We remark that, by (47) we have, for all $k = 1, 2, 3$, $-\Delta \frac{\partial}{\partial z_k} \gamma(z) = \frac{\partial}{\partial z_k} U^2(z)$. Thus, by (40), (41), (42), we get

$$\begin{aligned}
\frac{1}{\varepsilon^2} I_1(\varepsilon,\xi) &= \frac{1}{q} \int_{\mathbb{R}^3} \frac{\tilde{v}_{\varepsilon,\xi}(z)}{\varepsilon^2} \chi_r(\varepsilon|z|) \frac{q}{\varepsilon} \frac{\partial U^2(z)}{\partial z_k} dz + O(\varepsilon) \\
&= -\frac{1}{\varepsilon q} \int_{\mathbb{R}^3} \frac{\tilde{v}_{\varepsilon,\xi}(z)}{\varepsilon^2} \chi_r(\varepsilon|z|) \Delta \left(\frac{\partial \gamma(z)}{\partial z_k} \right) dz + O(\varepsilon) \\
&= -\frac{1}{\varepsilon q} \int_{\mathbb{R}^3} \Delta \left(\frac{\tilde{v}_{\varepsilon,\xi}(z)}{\varepsilon^2} \right) \chi_r(\varepsilon|z|) \frac{\partial \gamma(z)}{\partial z_k} dz \\
&\quad + \frac{1}{q} \int_{\mathbb{R}^3} \nabla \left(\frac{\tilde{v}_{\varepsilon,\xi}(z)}{\varepsilon^2} \right) \chi_r'(\varepsilon|z|) \frac{z}{|z|} \frac{\partial \gamma(z)}{\partial z_k} dz + O(\varepsilon)
\end{aligned}$$

Now we have that

$$\frac{1}{q} \left| \int_{\mathbb{R}^3} \nabla \left(\frac{\tilde{v}_{\varepsilon,\xi}(z)}{\varepsilon^2} \right) \chi_r'(\varepsilon|z|) \frac{z}{|z|} \frac{\partial \gamma(z)}{\partial z_k} dz \right| \leq$$

$$\left\| \frac{\tilde{v}_{\varepsilon,\xi}(z)}{\varepsilon^2} \right\|_{D^{1,2}(B(0,r/\varepsilon))} \left\| \frac{\partial \gamma(z)}{\partial z_k} dz \right\|_{D^{1,2}(\mathbb{R}^3 \smallsetminus B(0,r/2\varepsilon))}$$

and the last term vanish uniformly in ξ when ε goes to zero because $\frac{\partial \gamma(z)}{\partial z_k}$ decays exponentially with respect to $|z|$.

Moreover, arguing as in (37) and in (38) we obtain

$$-\frac{1}{\varepsilon q}\int_{\mathbb{R}^3}\Delta\left(\frac{\tilde{v}_{\varepsilon,\xi}(z)}{\varepsilon^2}\right)\chi_r(\varepsilon|z|)\frac{\partial\gamma(z)}{\partial z_k}dz$$

$$=\frac{1}{\varepsilon}\int_{\mathbb{R}^3}U^2(z)\frac{\partial\gamma(z)}{\partial z_k}dz+O(\varepsilon)=-\frac{1}{\varepsilon}\int_{\mathbb{R}^3}\frac{\partial}{\partial z_k}\left[U^2(z)\right]\gamma(z)dz+O(\varepsilon)$$

and the last integral is zero because both U,γ and χ_r are radially symmetric.

By Equation (4), Lemma 5, and by (40), (41), (42), we have

$$
\begin{aligned}
I_2(\varepsilon,\xi) &= \frac{1}{q\varepsilon^3}\int_M\left\{-\Delta_g\Psi(W_{\varepsilon,\xi})+(1+qW_{\varepsilon,\xi}^2)\Psi(W_{\varepsilon,\xi})\right\}\Psi'(W_{\varepsilon,\xi(y)})\left[\left.\frac{\partial}{\partial y_h}W_{\varepsilon,\xi(h)}\right|_{y=0}\right]d\mu_g \\
&= \frac{1}{q\varepsilon^3}\int_M\Psi(W_{\varepsilon,\xi})\left\{-\Delta_g\Psi'(W_{\varepsilon,\xi})[\cdot]+(1+qW_{\varepsilon,\xi}^2)\Psi'(W_{\varepsilon,\xi(y)})[\cdot]\right\}d\mu_g \\
&= \frac{1}{q\varepsilon^3}\int_M\Psi(W_{\varepsilon,\xi})\left\{2qW_{\varepsilon,\xi}(1-q\Psi(W_{\varepsilon,\xi}))\left[\left.\frac{\partial}{\partial y_h}W_{\varepsilon,\xi(h)}\right|_{y=0}\right]\right\}d\mu_g = \\
&= 2\int_{\mathbb{R}^3}|g(\varepsilon z)|^{1/2}\tilde{v}_{\varepsilon,\xi}(z)U(z)\chi_r(\varepsilon|z|)(1-q\tilde{v}_{\varepsilon,\xi}(z))\times \\
&\quad \times\left\{\sum_{k=1}^3\left[\frac{1}{\varepsilon}\frac{\partial U(z)}{\partial z_k}\chi_r(\varepsilon|z|)+U(z)\frac{\partial\chi_r(\varepsilon|z|)}{\partial z_k}\right]\frac{\partial}{\partial y_h}\mathcal{E}_k(0,\exp_\xi(\varepsilon z))\right\}dz.
\end{aligned}
$$

Again, by Lemma 21 and by (40), (41), (42), we have that

$$I_2(\varepsilon,\xi)=2\varepsilon\int_{\mathbb{R}^3}\gamma U(z)U'(z)\frac{z_h}{|z|}dz+o(\varepsilon^2)=o(\varepsilon^2).$$

At this point we have to prove the uniform convergence of $I_2(\varepsilon,\xi)$ with respect to $\xi\in M$. By (40), (41) and (42) we have that

$$\frac{1}{\varepsilon^2}I_2(\varepsilon,\xi)=2\int_{\mathbb{R}^3}\frac{\tilde{v}_{\varepsilon,\xi}(z)}{\varepsilon^2}U(z)\chi_r(\varepsilon|z|)(1-q\tilde{v}_{\varepsilon,\xi}(z))\frac{1}{\varepsilon}\frac{\partial U(z)}{\partial z_k}dz+O(\varepsilon)$$

$$=\frac{1}{\varepsilon}\int_{\mathbb{R}^3}\frac{\tilde{v}_{\varepsilon,\xi}(z)}{\varepsilon^2}\chi_r(\varepsilon|z|)\frac{\partial U^2(z)}{\partial z_k}dz-\frac{q}{\varepsilon}\int_{\mathbb{R}^3}\frac{\tilde{v}_{\varepsilon,\xi}^2(z)}{\varepsilon^2}\chi_r(\varepsilon|z|)\frac{\partial U^2(z)}{\partial z_k}dz+O(\varepsilon)$$

and the last integral vanishes when ε goes to zero, because $\|\tilde{v}_{\varepsilon,\xi}\|_{L^6}\le\varepsilon^2$ uniformly with respect to ξ. Thus

$$\frac{1}{\varepsilon^2}I_2(\varepsilon,\xi)=\frac{1}{\varepsilon}\int_{\mathbb{R}^3}\frac{\tilde{v}_{\varepsilon,\xi}(z)}{\varepsilon^2}\chi_r(\varepsilon|z|)\frac{\partial U^2(z)}{\partial z_k}dz+o(1)$$

and we can conclude exactly as for $I_1(\varepsilon,\xi)$. $\qquad\square$

Remark 17. For every $\varphi\in H_g^1$ we have

$$
\begin{aligned}
\|\varphi\|_{H_g^1}^2 &= \int_M|\nabla_g\varphi|^2+\varphi^2d\mu_g\le\int_M|\nabla_g\varphi|^2+\frac{1}{\varepsilon}\varphi^2d\mu_g = \\
&= \varepsilon\int_M\frac{1}{\varepsilon}|\nabla_g\varphi|^2+\frac{1}{\varepsilon^2}\varphi^2d\mu_g=\varepsilon\|\varphi\|_\varepsilon^2
\end{aligned}
$$

Proof of Proposition 11. It follows from Lemma 14, Lemma 15 and Lemma 16. $\quad\square$

Appendix A. Some key estimates

Remark 18. The following limits hold uniformly with respect to $q \in M$.

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^3} |W_{\varepsilon,q}|_{p,g}^p = |U|_p^p, \quad 2 \le p \le 2^*$$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} |\nabla_g W_{\varepsilon,q}|_{2,g}^2 = |\nabla U|_2^2$$

Lemma 19. *For any $\varphi \in H_g^1(M)$ and for all $\xi \in M$ it holds*

$$\|\Psi(W_{\varepsilon,\xi} + \varphi)\|_{H_g^1} \le c_1 \varepsilon^{5/2} (1 + \|\varphi\|_\varepsilon^2)$$

$$\|\Psi(W_{\varepsilon,\xi} + \varphi)\|_{H_g^1} \le c_1 (\varepsilon^{5/2} + \|\varphi\|_{H_g^1}^2)$$

$$\|\Psi(W_{\varepsilon,\xi} + \varphi)\|_{L^\infty} \le c_2 (\varepsilon^{3/2} + \|\varphi\|_{H_g^1}^2)$$

where c_1 and c_2 are constants non depending on ξ and ε.

Proof. To simplify the notations we set $v = \Psi(W_{\varepsilon,\xi} + \varphi)$. By (4) we have

$$\|v\|_{H_q^1}^2 \le \int_M |\nabla_g v|^2 + v^2 + q^2 (W_{\varepsilon,\xi} + \varphi)^2 v^2 = q \int (W_{\varepsilon,\xi} + \varphi)^2 v \le$$

$$\le \left(\int_M v^6\right)^{1/6} \left(\int_M (W_{\varepsilon,\xi} + \varphi)^{12/5}\right)^{5/6} \le c\|v\|_{H_g^1} \|W_{\varepsilon,\xi} + \varphi\|_{L_g^{12/5}}^2 \le$$

$$\le c\|v\|_{H_g^1} \left(\|W_{\varepsilon,\xi}\|_{L_g^{12/5}}^2 + \|\varphi\|_{L_g^{12/5}}^2\right)$$

We recall (see Remark 18) that

$$(43) \qquad \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^3} |W_{\varepsilon,\xi}|_t^t = |U|_t^t \text{ uniformly w.r.t. } \xi \in M.$$

Then we have

$$(44) \qquad \|v\|_{H_g^1} \le c(\varepsilon^{5/2} + |\varphi|_{12/5,g}^2) \le c(\varepsilon^{5/2} + \|\varphi\|_{H_g^1}^2).$$

Also,

$$(45) \qquad \|v\|_{H_g^1} \le c\varepsilon^{5/2} (1 + |\varphi|_{12/5,\varepsilon}^2) \le c\varepsilon^{5/2} \left(1 + \|\varphi\|_\varepsilon^2\right).$$

By (4) and by standard regularity theory (see [23, Th. 8.8]), we have that $v \in H_g^2$, and that $\|v\|_{H_g^2} \le \|v\|_{H_g^1} + \|(W_{\varepsilon,\xi} + \varphi)^2 (1 - qv)\|_{L_g^2}$. By Sobolev embedding and by (5) and (45) we get

$$\|v\|_{L^\infty} \le c\|v\|_{H_g^2} \le c\left\{\|v\|_{H_g^1} + \|(W_{\varepsilon,\xi} + \varphi)^2 (1 - qv)\|_{L^2}\right\} \le$$

$$\le c\left\{\|v\|_{H_g^1} + \|W_{\varepsilon,\xi}\|_{L^4}^2 + \|\varphi\|_{L^4}^2\right\} \le$$

$$(46) \qquad \le c\left\{\varepsilon^{3/2} + \|\varphi\|_{H_g^1}^2\right\}$$

$\square$

Lemma 20. *For any $\xi \in M$ and $h, k \in H_g^1$ it holds*

$$\|\Psi'(W_{\varepsilon,\xi} + k)[h]\|_{H_g^1} \le c\left\{\varepsilon^2 \|h\|_{H_g^1} + \|h\|_{H_g^1} \|k\|_{H_g^1}\right\}$$

where the constant c does not depend on ξ and ε.

Proof. We have, by (5) and by (6)

$$
\begin{aligned}
\|\Psi'(W_{\varepsilon,\xi}+k)[h]\|_{H_g^1}^2 &= 2q\int_M (W_{\varepsilon,\xi}+k)(1-q\Psi(W_{\varepsilon,\xi}+k))h\Psi'(W_{\varepsilon,\xi}+k)[h] - \\
&\quad -q^2\int_M (W_{\varepsilon,\xi}+k)^2(\Psi'(W_{\varepsilon,\xi}+k)[h])^2 \leq \\
&\leq \int_M W_{\varepsilon,\xi}|h|\,|\Psi'(W_{\varepsilon,\xi}+k)[h]| + \int_M |k||h|\,|\Psi'(W_{\varepsilon,\xi}+k)[h]|
\end{aligned}
$$

We call each integral term respectively I_1, I_2, and we estimate each term separately. We have

$$
I_1 \leq \|\Psi'(W_{\varepsilon,\xi}+k)[h]\|_{L_g^6}\|h\|_{L_g^6}\|W_{\varepsilon,\xi}\|_{L_g^{3/2}} \leq \varepsilon^2\|\Psi'\|_{H_g^1}\|h\|_{H_g^1}
$$

$$
I_2 \leq \|k\|_{L_g^3}\|h\|_{L_g^3}\|\Psi'(W_{\varepsilon,\xi}+k)[h]\|_{L_g^3} \leq \|k\|_{H_g^1}\|h\|_{H_g^1}\|\Psi'\|_{H_g^1}
$$

that is our claim.   □

Lemma 21. *Let us consider the functions*

$$
\tilde{v}_{\varepsilon,\xi}(z) = \begin{cases} \Psi(W_{\varepsilon,\xi})\left(\exp_\xi(\varepsilon z)\right) & for\ z\in B(0,r/\varepsilon) \\[2mm] 0 & for\ z\in\mathbb{R}^3\smallsetminus B(0,r/\varepsilon) \end{cases}
$$

Then there exists a constant $c>0$ such that

$$
\|\tilde{v}_{\varepsilon,\xi}(z)\|_{L^6(\mathbb{R}^3)} \leq c\varepsilon^2.
$$

Furthermore, up to subsequences, $\left\{\dfrac{1}{\varepsilon^2}\tilde{v}_{\varepsilon,\xi}\right\}_\varepsilon$ converges weakly in $L^6(\mathbb{R}^3)$ as ε goes to 0 to a function $\gamma\in D^{1,2}(\mathbb{R}^3)$. The function γ solves, in a weak sense, the equation

$$
-\Delta\gamma = qU^2\ in\ \mathbb{R}^3 \tag{47}
$$

Proof. By definition of $\tilde{v}_{\varepsilon,\xi}(z)$ and by (4) we have, for all $z\in B(0,r/\varepsilon)$,

$$
-\sum_{ij}\partial_j\left(|g_\xi(\varepsilon z)|^{1/2}g_\xi^{ij}(\varepsilon z)\partial_i\tilde{v}_{\varepsilon,\xi}(z)\right) = \tag{48}
$$

$$
= \varepsilon^2|g_\xi(\varepsilon z)|^{1/2}\left\{qU^2(z)\chi_r^2(\varepsilon|z|) - \left[1+q^2U^2(z)\chi_r^2(\varepsilon|z|)\right]\tilde{v}_{\varepsilon,\xi}(z)\right\}
$$

By (48), and remarking that $\tilde{v}_{\varepsilon,\xi}(z)\geq 0$ we have

$$
\|\tilde{v}_{\varepsilon,\xi}(z)\|_{D^{1,2}(B(0,r/\varepsilon))}^2 \leq C\int_{B(0,r/\varepsilon)}|g_\xi(\varepsilon z)|^{1/2}g_\xi^{ij}(\varepsilon z)\partial_i\tilde{v}_{\varepsilon,\xi}(z)\partial_j\tilde{v}_{\varepsilon,\xi}(z)dz = \tag{49}
$$

$$
= C\varepsilon^2\int_{B(0,r/\varepsilon)}|g_\xi(\varepsilon z)|^{1/2}\left\{qU^2(z)\chi_r^2(\varepsilon|z|)\tilde{v}_{\varepsilon,\xi}(z) - \left[1+q^2U^2(z)\chi_r^2(\varepsilon|z|)\right]\tilde{v}_{\varepsilon,\xi}^2(z)\right\}dz \leq
$$

$$
\leq C\varepsilon^2\int_{B(0,r/\varepsilon)}|g_\xi(\varepsilon z)|^{1/2}qU^2(z)\chi_r^2(\varepsilon|z|)\tilde{v}_{\varepsilon,\xi}(z)dz \leq
$$

$$
\leq C\varepsilon^2 q\|\tilde{v}_{\varepsilon,\xi}(z)\|_{L^6(B(0,r/\varepsilon))}\|U\|_{L^{12/5}}^2 \leq C\varepsilon^2\|\tilde{v}_{\varepsilon,\xi}(z)\|_{D^{1,2}(B(0,r/\varepsilon))}
$$

Thus we have

$$
\|\tilde{v}_{\varepsilon,\xi}(z)\|_{D^{1,2}(B(0,r/\varepsilon))} \leq C\varepsilon^2\ \text{and}\ \|\tilde{v}_{\varepsilon,\xi}(z)\|_{L^6(\mathbb{R}^3)} \leq C\varepsilon^2. \tag{50}
$$

By (50), if ε_n is a sequence which goes to zero, the sequence $\left\{\frac{1}{\varepsilon_n^2}\tilde{v}_{\varepsilon_n,\xi}\right\}_n$ is bounded in $L^6(\mathbb{R}^3)$. Then, up to subsequence, $\left\{\frac{1}{\varepsilon_n^2}\tilde{v}_{\varepsilon_n,\xi}\right\}_n$ converges to some $\tilde{\gamma} \in L^6(\mathbb{R}^3)$ weakly in $L^6(\mathbb{R}^3)$. We have also that $\|\tilde{v}_{\varepsilon,\xi}\|_{L^2(\mathbb{R}^3)} \leq C\varepsilon$. In fact, by Holder inequality

$$\int_{B(0,r/\varepsilon)}\tilde{v}_{\varepsilon,\xi}^2 \leq \left(\int_{B(0,r/\varepsilon)}\tilde{v}_{\varepsilon,\xi}^6\right)^{1/3}\left(\int_{B(0,r/\varepsilon)}1\right)^{2/3} \leq C\varepsilon^4\left(\frac{r^3}{\varepsilon^3}\right)^{2/3} \leq C\varepsilon^2.$$

Moreover, by (48), for any $\varphi \in C_0^\infty(\mathbb{R}^3)$, it holds

(51) $$\int_{\text{supp }\varphi}\sum_{ij}|g_\xi(\varepsilon z)|^{1/2}g_\xi^{ij}(\varepsilon z)\partial_i\frac{\tilde{v}_{\varepsilon,\xi}(z)}{\varepsilon_n^2}\partial_j\varphi(z)dz =$$

$$\int_{\text{supp }\varphi}\left\{qU^2(z)\chi_r^2(\varepsilon|z|) - \left[1 + q^2U^2(z)\chi_r^2(\varepsilon|z|)\right]\tilde{v}_{\varepsilon,\xi}(z)\right\}|g_\xi(\varepsilon z)|^{1/2}\varphi(z)dz.$$

Consider now the functions

$$v_{\varepsilon,\xi}(z) := \Psi(W_{\varepsilon,\xi})\left(\exp_\xi(\varepsilon z)\right)\chi_r(\varepsilon|z|) = \tilde{v}_{\varepsilon,\xi}(z)\chi_r(\varepsilon|z|) \text{ for } z \in \mathbb{R}^3.$$

We have that

$$\|v_{\varepsilon,\xi}(z)\|_{D^{1,2}(\mathbb{R}^3)}^2 = \int|\nabla v_{\varepsilon,\xi}|^2dz \leq 2\int\chi_r^2(\varepsilon|z|)|\nabla\tilde{v}_{\varepsilon,\xi}(z)|^2 + \varepsilon^2|\chi_r'(\varepsilon|z|)|^2\tilde{v}_{\varepsilon,\xi}(z)^2dz \leq$$

$$\leq c\left(\|\tilde{v}_{\varepsilon,\xi}(z)\|_{D^{1,2}(B(0,r/\varepsilon))}^2 + \varepsilon^2\|\tilde{v}_{\varepsilon,\xi}(z)\|_{L^2(\mathbb{R}^3)}^2\right) \leq c\varepsilon^4.$$

Thus the sequence $\left\{\frac{1}{\varepsilon_n^2}v_{\varepsilon_n,\xi}\right\}_n$ converges to some $\gamma \in D^{1,2}(\mathbb{R}^3)$ weakly in $D^{1,2}(\mathbb{R}^3)$ and in $L^6(\mathbb{R}^3)$.

For any compact set $K \subset \mathbb{R}^3$ eventually $v_{\varepsilon_n,\xi} \equiv \tilde{v}_{\varepsilon_n,\xi}$ on K. So it is easy to see that $\tilde{\gamma} = \gamma$.

We recall the Taylor expansions

(52) $$|g_\xi(\varepsilon z)|^{1/2} = 1 + O(\varepsilon^2|z|^2), \quad \text{and} \quad g_\xi^{ij}(\varepsilon z) = \delta_{ij} + O(\varepsilon^2|z|^2),$$

so, by (52), and by the weak convergence of $\left\{\frac{1}{\varepsilon_n^2}v_{\varepsilon_n,\xi}\right\}_n$ in $D^{1,2}(\mathbb{R}^3)$, for any $\varphi \in C_0^\infty(\mathbb{R}^3)$ we get

(53) $$\int_{\text{supp }\varphi}\sum_{ij}|g_\xi(\varepsilon_n z)|^{1/2}g_\xi^{ij}(\varepsilon_n z)\partial_i\frac{\tilde{v}_{\varepsilon_n,\xi}(z)}{\varepsilon_n^2}\partial_j\varphi(z)dz$$

$$= \int_{\text{supp }\varphi}\sum_{ij}|g_\xi(\varepsilon_n z)|^{1/2}g_\xi^{ij}(\varepsilon_n z)\partial_i\frac{v_{\varepsilon_n,\xi}(z)}{\varepsilon_n^2}\partial_j\varphi(z)dz$$

$$\to \int_{\mathbb{R}^3}\sum_i\partial_i\gamma(z)\partial_i\varphi(z)dz \text{ as } n \to \infty.$$

Thus by (51) and by (53) and because $\left\{\frac{1}{\varepsilon_n^2}\tilde{v}_{\varepsilon_n,\xi}\right\}_n$ converges to γ weakly in $L^6(\mathbb{R}^3)$ we get

$$\int_{\mathbb{R}^3}\sum_i\partial_i\gamma(z)\partial_i\varphi(z)dz = q\int_{\mathbb{R}^3}U^2(z)\varphi(z)dz \text{ for all } \varphi \in C_0^\infty(\mathbb{R}^3).$$

Thus, up to subsequences, $\left\{\dfrac{1}{\varepsilon_n^2}\tilde{v}_{\varepsilon_n,\xi}\right\}_n$ converges to γ, weakly in $L^6(\mathbb{R}^3)$ and the function $\gamma \in D^{1,2}(\mathbb{R}^3)$ is a weak solution of $-\Delta\gamma = qU^2$ in $\mathbb{R}^3$. $\hspace{2cm}\square$

Remark 22. We remark that γ is positive radially symmetric and decays exponentially at infinity with its first derivative because it solves $-\Delta\gamma = qU^2$ in $\mathbb{R}^3$.

References

[1] A. Ambrosetti, D. Ruiz,*Multiple bound states for the Schrödinger-Poisson problem*, Commun. Contemp. Math. **10** (2008) 391–404

[2] A. Azzollini, P. D'Avenia, A. Pomponio,*On the Schrödinger-Maxwell equations under the effect of a general nonlinear term*, Ann. Inst. H. Poincar Anal. Non Linaire **27** (2010), no. 2, 779–791

[3] A. Azzollini, A. Pomponio,*Ground state solutions for the nonlinear Schrödinger-Maxwell equations*, J. Math. Anal. Appl. **345** (2008) no. 1, 90–108

[4] A. Azzollini, A. Pomponio,*Ground state solutions for the nonlinear Klein-Gordon-Maxwell equations*, Topol. Methods Nonlinear Anal. **35** (2010), no. 1, 33–42

[5] J.Bellazzini, L.Jeanjean, T.Luo, *Existence and instability of standing waves with prescribed norm for a class of Schrödinger-Poisson equations* in press on Proc. London Math. Soc. (arXiv http://arxiv.org/abs/1111.4668)

[6] V.Benci, D.Fortunato, *An eigenvalue problem for the Schrödinger-Maxwell equations*, Topol. Methods Nonlinear Anal. **11** (1998), no. 2, 283–293

[7] V. Benci, D. Fortunato, *Solitary waves of the nonlinear Klein-Gordon field equation coupled with the Maxwell equations*, Rev. Math. Phys. **14** (2002), no. 4, 409–420.

[8] V. Benci, D. Fortunato, *Existence of hylomorphic solitary waves in Klein-Gordon and in Klein-Gordon-Maxwell equations*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **20** (2009), no. 3, 243–279.

[9] D. Cassani, *Existence and non-existence of solitary waves for the critical Klein-Gordon equation coupled with Maxwell's equations*, Nonlinear Anal. **58** (2004) no. 7-8, 733–747.

[10] Y. Choquet-Bruhat, *Solution globale des Equations de Maxwell-Dirac-Klein-Gordon*, Rend. Circ. Mat. Palermo **31** (1982), no. 2, 267–288

[11] T. D'Aprile, D. Mugnai, *Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations*, Proc. Roy. Soc. Edinburgh Sect. A **134** (2004), no. 5, 893–906.

[12] T. D'Aprile, D. Mugnai, *Non-existence results for the coupled Klein-Gordon-Maxwell equations*, Adv. Nonlinear Stud. **4** (2004), no. 3, 307–322.

[13] T. D'Aprile, J. Wei, *Layered solutions for a semilinear elliptic system in a ball*, J. Differential Equations **226** (2006) , no. 1, 269–294.

[14] T. D'Aprile, J. Wei, *Clustered solutions around harmonic centers to a coupled elliptic system*, Ann. Inst. H. Poincaré Anal. Non Linéaire **24** (2007), no. 4, 605–628.

[15] P. D'Avenia, L. Pisani, *Nonlinear Klein-Gordon equations coupled with Born-Infeld type equations*, Electron. J. Differential Equations **26** (2002), No. 26, 13 pp.

[16] P. D'Avenia, L. Pisani, G. Siciliano, *Klein-Gordon-Maxwell system in a bounded domain*, Discrete Contin. Dyn. Syst. **26** (2010), no. 1, 135–149.

[17] P. D'Avenia, L. Pisani, G. Siciliano, *Dirichlet and Neumann problems for Klein-Gordon-Maxwell systems*, Nonlinear Anal. **71** (2009), no. 12, e1985e1995.

[18] E. Deumens, *The Klein-Gordon-Maxwell nonlinear system of equations*, Phys. D. **18** (1986), no. 1-3, 371–373.

[19] O.Druet, E.Hebey, *Existence and a priori bounds for electrostatic Klein-Gordon-Maxwell systems in fully inhomogeneous spaces*, Commun. Contemp. Math. **12** (2010), no. 5, 831–869.

[20] M.Ghimenti, A.M.Micheletti, *Number and profile of low energy solutions for singularly perturbed Klein Gordon Maxwell systems on a Riemannian manifold*, work in preparation

[21] M.Ghimenti, A.M.Micheletti, *Low energy solutions for the semiclassical limit of Schrödinger Maxwell systems*, work in preparation

[22] B. Gidas, W.M. Ni, L. Nirenberg, *Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^n$*, Mathematical analysis and applications, Part A, pp. 369–402, Adv. in Math. Suppl. Stud., 7a, Academic Press, New York-London, 1981.

[23] D. Gilbarg, N.S. Trudinger, *Elliptic partial differential equations of second order.* Grundlehren der Mathematischen Wissenschaften, Vol. 224. Springer-Verlag, Berlin-New York, 1977

[24] I. Ianni, G. Vaira, *On concentration of positive bound states for the Schrödinger-Poisson problem with potentials*, Adv. Nonlinear Stud. **8** (2008), no. 3, 573–595.

[25] Kikuchi, *On the existence of solutions for a elliptic system related to the Maxwell-Schrödinger equations*, Nonlinear Anal. 67 (2007) 1445–1456.

[26] S. Klainerman, M. Machedon, *On the Maxwell-Klein-Gordon equation with finite energy*, Duke Math. J. **74** (1994)

[27] M.K.Kwong, *Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^n$*, Arch. Rational Mech. Anal. **105** (1989), 243–266.

[28] N. Masmoudi, K. Nakanishi, *Uniqueness of finite energy solutions for Maxwell-Dirac and Maxwell-Klein-Gordon equations*, Comm. Math. Phys. **243** (2003), no. 1, 123–136.

[29] N. Masmoudi, K. Nakanishi, *Nonrelativistic limit from Maxwell-Klein-Gordon and Maxwell-Dirac to Poisson-Schrdinger*, Int. Math. Res. Not. **2003**, no. 13, 697–734.

[30] A.M. Micheletti, A. Pistoia, *The role of the scalar curvature in a nonlinear elliptic problem on Riemannian manifolds*, Calc. Var. Partial Differential Equations **34** (2009), no. 2, 233-265.

[31] A.M. Micheletti, A. Pistoia, *Generic properties of critical points of the scalar curvature for a Riemannian manifold*, Proc. Amer. Math. Soc. **138** (2010), no. 9, 3277–3284.

[32] D. Mugnai, *Coupled Klein-Gordon and Born-Infeld-type equations: Looking for solitary waves*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **460** (2004), no. 2045, 1519–1527.

[33] L.Pisani, G.Siciliano, *Note on a Schrödinger-Poisson system in a bounded domain*, Appl. Math. Lett. **21** (2008), no. 5, 521–528.

[34] D. Ruiz, *Semiclassical states for coupled Schrödinger-Maxwell equations: Concentration around a sphere*, Math. Models Methods Appl. Sci. **15** (2005), no. 1, 141–164.

[35] G. Siciliano, *Multiple positive solutions for a Schrödinger-Poisson-Slater system*, J. Math. Anal. Appl. **365** (2010), no. 1, 288–299.

[36] Z. Wang, H.S. Zhou, *Positive solution for a nonlinear stationary Schrödinger-Poisson system in $\mathbb{R}^3$*, Discrete Contin. Dyn. Syst. 18 (2007) 809–816.

(Marco Ghimenti) Dipartimento di Matematica, Università di Pisa, via F. Buonarroti 1/c, 56127 Pisa, Italy

E-mail address: `marco.ghimenti@dma.unipi.it`.

(Anna Maria Micheletti) Dipartimento di Matematica, Università di Pisa, via F. Buonarroti 1/c, 56127 Pisa, Italy

E-mail address: `a.micheletti@dma.unipi.it`.

(Angela Pistoia) Dipartimento SBAI, Università di Roma "La Sapienza", via Antonio Scarpa 16, 00161 Roma, Italy

E-mail address: `pistoia@dmmm.uniroma1.it`