Weak solution to the Navier-Stokes equations constructed by semi-discretization are suitable

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Dedicated to Hugo Beirão da Veiga on the occasion of his seventieth birthday

Abstract. We consider the three dimensional Navier-Stokes equations and we prove that weak solutions constructed by approximating the time-derivative by finite differences are suitable. The so-called method of semi-discretization is of fundamental importance in the numerical analysis and it is one of the building blocks for the full discretization of the equations.

1. Introduction

In this paper we consider the three dimensional Navier-Stokes equations with unit viscosity and zero external force (assumptions which are unessential since corresponding result can be obtained by scaling or by adding a smooth enough external force), namely

$$\begin{aligned} \partial_t u - \Delta u + (u\cdot\nabla)u + \nabla p &= 0 \qquad && (t,x) \in]0,T[\times]0,2\pi[^3,\\ \nabla\cdot u &= 0 && (t,x) \in]0,T[\times]0,2\pi[^3,\\ u(0,x) &= u_0(x) && x \in]0,2\pi[^3, \end{aligned} \tag{1.1}$$

and, for simplicity, we restrict to the space-periodic setting. More general, but more technical results in a bounded domain with Dirichlet boundary condition will appear in a forthcoming work, still in progress [**7**]. In this short paper we address just the space-periodic case to explain the main ideas needed to handle the time-discretization, a topic which seems not explored yet, in the context of construction of solutions satisfying the local energy inequality.

Starting from the results of Scheffer [**22**] and Caffarelli, Kohn, and Nirenberg [**9**] concerning the partial regularity for the Navier-Stokes equations, it turned out that *Suitable Weak Solution* (SWS in the sequel) are of paramount importance. We recall that the weak solutions constructed by Leray and Hopf (for the Cauchy problem and for the bounded domain with Dirichlet boundary conditions, respectively) satisfy the equations in a weak sense:

$$\int_0^{+\infty} (u,\partial_t\phi) - (\nabla u,\nabla\phi) - ((u\cdot\nabla)\,u,\phi)\,dt = -(u_0,\phi(0)),$$

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for all smooth, periodic, and divergence-free functions $\phi : [0,+\infty[\times]0,2\pi[^3\to \mathbb{R}^3$, such that $\phi(t,x)=0$, for all $t\geq T$; The initial datum is attained in the sense that

$$\lim_{t\to 0^+}\|u(t)-u_0\|=0;$$

The velocity u satisfies also the energy inequality

$$\frac{1}{2}\|u(t)\|^2+\int_0^t\|\nabla u(s)\|^2\,ds\leq\frac{1}{2}\|u_0\|^2\qquad\text{for all } t\in[0,T],$$

where $\|\,.\,\|$ denotes the $L^2(]0,2\pi[^3)$-norm, and $(\,.\,,\,.\,)$ the associated scalar product.

On the other hand, the celebrated results of partial regularity from [**9**], concern solutions with a more stringent condition: Solutions need to be also suitable weak solutions, that is certain regularity on the pressure p is requested and the local-energy inequality

$$\partial_t\left(\frac{1}{2}|u|^2\right)+\nabla\cdot\left(\left(\frac{1}{2}|u|^2+p\right)u\right)-\Delta\left(\frac{1}{2}|u|^2\right)+|\nabla u|^2\leq 0,$$

has to be satisfied in the sense of distributions.

In [**9**] an approximation scheme with retarded mollifiers, which resemble the "Tonelli approximation" for ordinary differential equations, has been introduced in order to prove existence of SWS. This opened immediately the question whether Leray-Hopf weak solutions are suitable or not, hence if the partial regularity is valid for them. Two different techniques producing SWS have been later introduced by Beirão da Veiga in 1985. In a couple of companion papers [**1, 2**] (but see also [**3**]) he addressed in two elegant ways the problem, showing that the approximation by the bi-Laplacian (Δ^2) and also another (very general) one for the convective term lead to SWS. At present it is known that most of the infinite-dimensional approximations produce suitable weak solutions, even if proofs can become very technical (or in some case are still unknown) in the case of a bounded domain with Dirichlet boundary conditions. In this respect we want to recall the results on Leray-α approximation [**16**], those for the compressible approximation in [**11**], and the forthcoming analysis of the Voigt approximation and the compressible approximation [**5, 6**], with particular emphasis on the treatment of solid boundaries. We also recall the existence of SWS via smoothing with Yosida approximation by Farwig, Kozono, and Sohr [**12**] in very general (also unbounded and with non-compact boundary) domains.

Probably even more importantly, in the introduction of [**2**], Beirão da Veiga highlighted the fact that:

> *There is no evidence that solutions obtained by the Faedo-Galerkin method verify the local energy inequality.*

This is a fundamental question, since approximations by finite dimensional spaces are the most relevant for the numerical analysis of the problem. The first partial solution to this problem came with the two companion papers by Guermond [**13, 14**], where he proved that if projectors over the finite element spaces used to discretize (with respect to x) velocity and pressure satisfy certain *commutation* properties, then the weak solutions constructed in the limit of vanishing mesh-size are suitable. In particular, these results cover the MINI element and the Taylor-Hood one. What is still missing is the case of the Fourier-Galerkin method in the three-dimensional torus. Further partial results in this case have been later obtained by Biryuk, Craig, and Ibrahim [**8**], but the problem –in its full-generality– is still unsolved. The links and connections between the local energy inequality, the numerical approximations, and also the theory of turbulence and its description, raised in the last decade. Beside the role for partial regularity results, the notion of SWS represents now also a paradigm in the behavior of reasonable Large Eddy Simulation approximations to the Navier-Stokes equations, see Guermond *et al.* [**15, 17**]. In fact, making a parallel with the notion of entropy solutions, it is suggested that LES models should select "physically relevant" solutions of the Navier-Stokes equations, namely those which satisfy the local energy inequality.

In this paper, we continue in the spirit of connecting results from analysis, with those requested to design efficient and stable schemes for the numerical approximation of the 3D Navier-Stokes equations. In particular, we focus on the problem whether approximations of the Navier-Stokes by numerical methods produce SWS in the limit of the mesh-size (or discretization parameter) going to zero. More precisely, we focus here only on the time-discretization and we consider approximation of the time-derivative obtained by finite differences. A time-step size $\kappa > 0$ is given and $M = [T/\kappa] \in \mathbb{N}$ is the total number of time-steps, while $I^M := \{t_m\}_{m=0}^M$, with $t_m := m\kappa$, is the corresponding net. We then make the following backward finite-difference approximation for the time-derivative in the interval $]t_{m-1}, t_m[$

$$\partial_t u \sim d_t u^m := \frac{u^m - u^{m-1}}{\kappa}.$$

This produces or each $1 \leq m \leq M$ an elliptic problem, which is properly determined by the type of numerical algorithm chosen. This technique of discretization in time, while the problem remains continuous in space, is called Rothe method (or method of lines, or method of semi-discretization), see for instance Rektorys [**21**] and Kačur [**18**]. A more specific analysis for the Navier-Stokes equations is done by Temam [**23**, Ch. III, § 4] and Rautmann [**20**]. The time discretization can be also coupled with the space one, in order to produce a full numerical approximation.

In particular, we first analyze the following scheme, briefly discussing related numerical methods.

Algorithm. (Euler implicit) Given a time-step-size $\kappa > 0$ and the corresponding net $I^M = \{t_m\}_{m=0}^M$, for $m \geq 1$ and for u^{m-1} given from the previous step with $u^0 = u_0$, compute the iterate u^m as follows: Solve

$$\text{(NS}^k\text{)} \qquad \begin{cases} d_t u^m - \Delta u^m + (u^m \cdot \nabla)\, u^m + \nabla p^m = 0, \\ \nabla \cdot u^m = 0, \end{cases}$$

endowed with periodic boundary conditions

REMARK 1.1. The scheme (NS^k) is an Euler implicit scheme and, at each step, a fully implicit equation has to be solved. Observe that for each given t_m we have to solve a problem very close to the stationary Navier-Stokes one, for which we have standard results of existence of weak solutions u^m, but the most important part is obtaining estimates on $\{u^m\}_{m=1,\dots,M}$ independent of κ.

For each m, we also associate to u^m a corresponding pressure p^m and then, to the sequence $\{u^m, p^m\}_{m=1,\dots,M}$ we can associate the functions (v_M, u_M, q_M) defined in $[0,T]$, as follows for $m=1,\dots,M$:

$$
\begin{cases}
v_M(t)=u^{m-1}+\dfrac{t-t_{m-1}}{\kappa}(u^m-u^{m-1}) & \text{for } t\in[t_{m-1},t_m[,\\
v_M(t)=u^M & \text{for } t=t_M,\\
\\
u_M(t)=u^0 & \text{for } t=t_0,\\
u_M(t)=u^m & \text{for } t\in]t_{m-1},t_m],\\
\\
q_M(t)=p^m & \text{for } t\in]t_{m-1},t_m],
\end{cases}
\tag{1.2}
$$

in such a way that $v_M(t_m)=u_M(t_m)$, for all $m=0,\dots,M$.

The main result of this paper is the following:

THEOREM 1.2. *Let be given $u_0\in H^1$. Then, there exists $(v,q)\in L^\infty(0,T;L^2)\cap L^2(0,T;H^1)\times L^{5/3}(0,T;L^{5/3})$ and a sequence $\kappa\to 0$ ($M\to+\infty$) such that the functions v_M and u_M both converge to u, and q_M converges to q. The functions v is a Leray-Hopf weak solutions to the Navier-Stokes equations* (1.1) *and moreover, for all $\phi\in C^\infty(\mathbb{T}\times[0,T])$ such that $\phi\geq 0$ and $\phi(0,x)=0$, the couple (v,q) satisfies the local energy inequality*

$$
\begin{aligned}
&\int_\Omega |v(x,t)|^2\phi(x,t)\,dx+2\int_0^t\int_\Omega|\nabla v(x,\tau)|^2\phi(x,\tau)\,dxd\tau\\
&\quad\leq\int_0^t\int_\Omega\Big(|v(x,\tau)|^2(\partial_t\phi(x,\tau)-\Delta\phi(x,\tau))+(|v(x,\tau)|^2\\
&\qquad+2q(x,\tau))v(x,\tau)\cdot\nabla\phi(x,\tau)\Big)\,dxd\tau.
\end{aligned}
\tag{1.3}
$$

The existence part is rather standard, see Temam [**23**, Ch. III, § 4], while the original contribution of this paper is the verification of the local energy inequality (1.3).

REMARK 1.3. The hypothesis on the initial datum can be relaxed to the more natural condition of square integrability. We are assuming more regularity to keep the proof as simple as possible and without technicalities. We also observe that this point (see also [**21**, Ch. 13]) is generally not treated or overlooked in the literature and requires to handle a further technical part, which will be appropriately detailed in [**7**].

Beside the proof being elementary, this result will represent the core of more general results which will correlate the full space-time discretization of the Navier-Stokes equations, with the notion of local energy inequality [**7**]. In particular, in a forthcoming paper we shall consider space-time discretizations as those in Quarteroni and Valli [**19**, § 13.4], but following a different path: Time-discretization will be performed before numerical discretization of the space variables.

2. Notation

We briefly fix the notation, which is typical of space-periodic problems. In the sequel we shall use the customary Lebesgue spaces $L^p(\Omega)$ and Sobolev spaces

$W^{k,p}(\Omega)$ and $H^s(\Omega) := W^{s,2}(\Omega)$, with $\Omega :=]0, 2\pi[^3$; for simplicity we shall do not distinguish between scalar and vector valued functions. Since we shall work with periodic boundary conditions the spaces are made of periodic functions. In the Hilbertian case $p = 2$ we can easily characterize the divergence-free spaces by using Fourier Series on the 3D torus. We denote by (e_1, e_2, e_3) the orthonormal basis of $\mathbb{R}^3$, and by $x := (x_1, x_2, x_3)$ the generic point in $\mathbb{R}^3$. Let $\mathbb{T}$ be the torus defined by $\mathbb{T} := \mathbb{R}^3/2\pi\mathbb{Z}^3$. We use $\|\cdot\|$ to denote the $L^2(\mathbb{T})$ norm and we impose the zero mean condition $\int_\Omega \phi\,dx = 0$ on velocity and pressure. We also define, for an exponent $s \geq 0$,

$$H^s := \left\{ w : \mathbb{T} \to \mathbb{R}^3,\ w \in H^s(\mathbb{T})^3, \quad \nabla \cdot w = 0, \quad \int_\mathbb{T} w\,dx = 0 \right\},$$

where $H^s(\mathbb{T})^3 := \left[H^s(\mathbb{T})\right]^3$ and if $0 \leq s < 1$ the condition $\nabla \cdot w = 0$ must be understood in a weak sense. For $w \in H^s$, we can expand the velocity field with Fourier series $w(x) = \sum_{k \neq 0} \widehat{w}_k e^{ik\cdot x}$, where k is the wave-vector and the Fourier coefficients are given by $\widehat{w}_k = \frac{1}{(2\pi)^3} \int_\mathbb{T} w(x) e^{-ik\cdot x} dx$. If $|k| = \sqrt{|k_1|^2 + |k_2|^2 + |k_3|^2}$, then the H^s norm is defined by

$$\|w\|^2_{H^s} := \sum_{k \neq 0} |k|^{2s} |\widehat{w}_k|^2,$$

where, as above, $\|w\|_{H^0} := \|w\|$. We finally characterize $H^s \subset H^s(\mathbb{T})$ as follows:

$$H^s := \left\{ w = \sum_{k \neq 0} \widehat{w}_k e^{ik\cdot x} : \quad \sum_{k \neq 0} |k|^{2s} |\widehat{w}_k|^2 < \infty,\ k \cdot \widehat{w}_k = 0, \widehat{w}_{-k} = \overline{\widehat{w}_k} \right\}.$$

Together with the classical Bochner spaces $L^p(0, T; X)$, endowed with the norm

$$\|f\|_{L^p(0,T;X)} := \begin{cases} \left(\int_0^T \|f(s)\|_X \right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \sup\limits_{0 \leq s \leq T \leq m \leq M} \|f(s)\|_X & \text{if } p = +\infty, \end{cases}$$

to deal with discrete problems we shall make use of the natural weighted spaces $l^p(I^M; X)$. The discrete counterpart of $L^p(0, T; X)$ consists of X-valued sequences $\{a_m\}_{m=0}^M$, endowed with the norm

$$\|a_m\|_{\ell^p(I^M;X)} := \begin{cases} \left(\kappa \sum\limits_{m=0}^M \|a_m\|_X^p \right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \max\limits_{0 \leq m \leq M} \|a_m\|_X & \text{if } p = +\infty. \end{cases}$$

We first recall a lemma about existence of solutions, see for instance Temam [**23**, Ch. III, Lem. 4.3].

Lemma 2.1. *There exists at least one sequence* $\{u^m\}_{m=0}^M$ *defined by the algorithm* (NS^k) *with* $u^0 = u_0$.

Proof. Observe that we have $M - 1$ coupled systems and in particular, given $u^{m-1} \in H^1$ and $\kappa > 0$, the function $u^m \in H^1$ can be obtained as solution of the modified steady Navier-Stokes system

$$\begin{cases} \dfrac{u^m}{\kappa} - \Delta u^m + (u^m \cdot \nabla)\, u^m + \nabla p^m = \dfrac{u^{m-1}}{\kappa}, \\ \nabla \cdot u^m = 0, \end{cases}$$

By testing with u^m itself (reasoning which can be made rigorous by a Faedo-Galerkin approximation) one obtains

$$\frac{\|u^m\|^2}{2\kappa} + \|\nabla u^m\|^2 \leq \frac{\|u^{m-1}\|^2}{2\kappa},$$

and it is enough to use the Brouwer fixed point theorem to infer existence of at least one solution $u^m \in H^1$. The existence of the associated p^m is then obtained by De Rham theorem. Observe that we do not have neither uniqueness, neither estimates independent of $\kappa > 0$. □

Remark 2.2. In many problems of numerical analysis it is unpleasant to have non-uniqueness for the approximate discrete problem. In the perspective of trying to have better properties of regularity on u^m, which could traduce in better convergence, it is also worth considering the following scheme:

Algorithm. (Euler semi-implicit) Given a time-step-size $\kappa > 0$ and the corresponding net $I^M = \{t_m\}_{m=0}^M$, for $m \geq 1$ and u^{m-1} given from the previous step with $u^0 = u_0$, compute the iterate u^m as follows: Solve

$$\begin{cases} d_t u^m - \Delta u^m + (u^{m-1} \cdot \nabla)\, u^m + \nabla p^m = 0, \\ \nabla \cdot u^m = 0, \end{cases} \tag{2.1}$$

endowed with periodic boundary conditions

The role of the semi-implicit approximation is emphasized for instance in [**4**], where it represents a critical tool to obtain optimal convergence rates for shear-dependent fluids with nonlinear viscosities (and consequently with the lack of the standard regularity results known for strong solutions to the Navier-Stokes equations). See also Diening, Ebmeyer, and Růžička [**10**] for general parabolic system.

For system (2.1) we have to solve for each $m = 1, \ldots, M$ the linear system

$$\begin{cases} \dfrac{u^m}{\kappa} - \Delta u^m + (u^{m-1} \cdot \nabla)\, u^m + \nabla p^m = \dfrac{u^{m-1}}{\kappa}, \\ \nabla \cdot u^m = 0, \end{cases}$$

for which the *a-priori* estimate is the same as for the implicit scheme (NS^k), since $\int_{\mathbb{T}} (u^{m-1} \cdot \nabla)\, u^m \cdot u^m \, dx = 0$, and uniqueness follows from linearity of the problem.

3. Proof of the main result

In order to show that solutions constructed by the algorithm (NS^k) converge to a SWS, we need to pass to the limit as $\kappa \to 0^+$ (and consequently $M \to +\infty$), hence estimates independent of κ are requested. To this end, we multiply the equations (NS^k) by u^m itself and we use a slightly different argument to prove the following lemma. Observe that the same argument will also work for the algorithm (2.1). We mainly consider (NS^k), since it is the first one found in many texts when proposing alternate proofs of existence of weak solutions by semi discretization.

LEMMA 3.1. *Let be given $u_0 \in H^1$. Then, there exists a constant $C > 0$, (independent of κ) such that*

$$
\begin{aligned}
&\|u^m\|_{\ell^\infty(I^M;H^0)\cap\ell^2(I^M;H^1)} \le C,\\
&\|p^m\|_{\ell^{5/3}(I^M;L^{5/3}(\mathbb{T}))} \le C.
\end{aligned}
\tag{3.1}
$$

PROOF. We test the equations (NS^k) by u^m. By integration by parts and with the elementary algebra equality

$$(a-b,a) = \frac{a^2-b^2}{2} + \frac{(a-b)^2}{2} \qquad a,\, b \in \mathbb{R},$$

we easily get

$$\frac{1}{2} d_t\|u^m\|^2 + \frac{\kappa}{2}\|d_t u^m\|^2 + \|\nabla u^m\|^2 = 0.$$

Next, by multiplying by $\kappa > 0$ and, summing up over $m = 1, \dots, M$, we obtain

$$\frac{1}{2}\|u^M\|^2 + \frac{\kappa^2}{2}\sum_{m=1}^{M}\|d_t u^m\|^2 + \kappa\sum_{m=1}^{M}\|\nabla u^m\|^2 \le \frac{1}{2}\|u_0\|^2. \tag{3.2}$$

From the last inequality (considering also sums up to any number smaller or equal than M) we deduce

$$u^m \in \ell^\infty(I^M;H^0)\cap\ell^2(I^M;H^1).$$

The definition of the weighted spaces $\ell^p(I^M;X)$ allows us to use the standard Hölder inequality. By using also the usual interpolation results one directly obtains

$$u^m \in \ell^{10/3}(I^M;L^{10/3}(\mathbb{T})).$$

As we previously observed in Lemma 2.1, to each u^m we can also associate a discrete pressure p^m, by using De Rham theorem. Here we want to have a more precise information about the regularity of p^m and this is the crucial point where we use the space periodicity. By taking the divergence of (NS^k) and since $\nabla\cdot u^m = 0$, we get the following Poisson equation (endowed with periodic boundary conditions)

$$-\Delta p^m = \nabla\cdot(u^m\cdot\nabla)\,u^m = \sum_{i,j=1}^{3}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}u_i^m\,u_i^m, \qquad m = 1,\dots,M, \tag{3.3}$$

where u_l^m denotes the l-component of the vector u^m. By using that $u_i^m\,u_j^m \in \ell^{5/3}(I^M;L^{5/3}(\mathbb{T}))$, the unique solution p^m with zero mean value of (3.3) satisfies

$$p^m \in \ell^{5/3}(I^M;L^{5/3}(\mathbb{T})),$$

since it is obtained just inverting the Laplace operator for each $m = 1,\dots,M$. □

From the estimate (3.2) we have the following inequality holds. We state it as a Lemma, because it will be crucial in the proof of the Theorem 1.2.

LEMMA 3.2. *The following estimate holds true*

$$\kappa^2\sum_{m=1}^{M}\|d_t u^m\|^2 = \sum_{m=1}^{M}\|u^m - u^{m-1}\|^2 \le \|u_0\|^2.$$

We observe that this estimate is obtained because the "*natural multiplier*" (the one which cancels out the convective term) is u^m, and the estimate comes from algebraic manipulation of the integral $\int_{\mathbb{T}} d_t u^m\, u^m\, dx$. In presence of different schemes this argument may fail and explains why a different treatment is required by other numerical methods, see Section 4. On the other hand, exactly the same estimates can be proved for the semi-implicit scheme (2.1), since again the correct multiplier is u^m.

We show now some properties of the step function u_M and of the piecewise linear function v_M.

LEMMA 3.3. *Let $u_0 \in H^1$ be given. Then, there exists a constant $C > 0$ (independent of κ) such that*

$$\begin{aligned}
&\|v_M\|_{L^\infty(0,T;H^0)\cap L^2(0,T;H^1)} \leq C,\\
&\|u_M\|_{L^\infty(0,T;H^0)\cap L^2(0,T;H^1)} \leq C,\\
&\|q_M\|_{L^{5/3}(0,T;L^{5/3}(\mathbb{T}))} \leq C,\\
&\|\partial_t v_M\|_{L^{4/3}(0,T;H^{-1})} \leq C, \ \text{where for any } s>0,\ H^{-s} := (H^s)'.
\end{aligned}$$

Moreover, we also have the following identity

$$\|u_M - w_M\|^2_{L^2(0,T;H^0)} = \frac{\kappa}{3}\sum_{m=1}^{M} \|u^m - u^{m-1}\|^2. \tag{3.4}$$

PROOF. The results of Lemma 3.1 imply that the functions v_M defined in (1.2) are, for each positive M, Lipschitz functions $[0,T] \mapsto H^1$. For each M, the function v_M satisfies, in the sense of distributions over $]0,T[$, the following equality

$$\frac{d}{dt}(v_M,\psi) + (\nabla u_M, \nabla\psi) + ((u_M\cdot\nabla)\,u_M,\psi)\,dt = 0 \qquad \forall\,\psi \in H^1. \tag{3.5}$$

Note that, by their definition (v_M, u_M, q_M) have the regularity stated in the lemma, which derive directly from the analogous one valid for the sequence $\{u^m, p^m\}$.

REMARK 3.4. Observe that the proof of existence of weak solutions in [**23**] is based on estimates for the discrete time-derivative in $L^2(H^{-s})$, for $s \geq 3/2$. The idea of obtaining such an estimates in negative spaces (but also considering negative Sobolev spaces in the time variable) represents also the core of the results in [**13, 14**]. Here, due to the particular setting, we can follow a more standard path.

The estimate $\partial_t v_M \in L^{4/3}(0,T;(H^{-1}))$ is obtained, as usual, by a comparison argument. It remains to prove estimate (3.4). We have

$$v_M(t) - u_M(t) = \frac{(t-t_{m-1})}{k}(u^m - u^{m-1}) + u^{m-1} - u^m \qquad \forall\, t \in]t_{m-1}, t_m].$$

Then

$$\begin{aligned}\int_0^T \|v_M - u_M\|^2\,dt &= \sum_{m=1}^M \int_{t_{m-1}}^{t_m} \|v_M(t) - u_M(t)\|^2\,dt\\ &= \sum_{m=1}^M \|u^m - u^{m-1}\|^2 \int_{t_{m-1}}^{t_m} \left(\frac{t - t_{m-1}}{k} - 1\right)^2 dt\\ &= \frac{\kappa}{3}\sum_{m=1}^M \|u^m - u^{m-1}\|^2.\end{aligned}$$

□

We are now ready to prove the main result of this paper.

Proof of Theorem 1.2. From a standard application of the Aubin-Lions compactness argument, we get that from the sequence $\{v_M\}_M$ bounded uniformly in $L^2(0,T;H^1)$ and such that $\partial_t v_M \in L^{4/3}(0,T;(H^{-1}))$, again with bound independent of M, we can extract a (relabelled) sub-sequence $v_M \to v$ in $L^2(0,T;H^0)$. Moreover, by using (3.4) and Lemma 3.2 we have also that

$$u_M - v_M \to 0 \text{ in } L^2(0,T;H^0),$$

hence also the sequence u_M converges strongly to v in $L^2(0,T;H^0)$. By using standard interpolation inequality and the previous strong convergence, it is common to show also that

$$\begin{aligned} v_M \to v &\quad \text{strongly in } L^3(0,T;L^3(\mathbb{T})),\\ u_M \to v &\quad \text{strongly in } L^3(0,T;L^3(\mathbb{T})).\end{aligned}$$

Moreover, since q_M is uniformly bounded in $L^{5/3}((0,T)\times\mathbb{T})$, up to extraction of a further sub-sequence, we have that

$$q_M \to q \text{ weakly in } L^{5/3}((0,T)\times\mathbb{T}).$$

Finally, as in [**23**] we have that v is a weak solutions of the Navier-Stokes equations with associated pressure q. We show now that (v,q) satisfies the local energy inequality. To this end we test the equations (3.5) by $u_M\phi$, where $\phi \geq 0$ is periodic in the space variable, smooth, and vanishes for $t = 0, T$. The first term regarding the time-derivative is the most relevant for our purposes. We have

$$\begin{aligned}\int_0^T (\partial_t v_M, u_M\phi)\,dt &= \int_0^T (\partial_t v_M, (v_M - v_M + u_M)\,\phi)\,dt\\ &= \int_0^T (\partial_t v_M, v_M)\,\phi\,dt + \int_0^T (\partial_t v_M, (u_M - v_M)\,\phi)\,dt\\ &= I_1 + I_2\end{aligned}$$

We start with the first term I_1. By splitting the integral over $[0,T]$ with the sum of integrals over $[t_{m-1}, t_m]$ and, by performing integration by parts, we immediately obtain

$$\int_0^T (\partial_t v_M, v_M\phi)\,dt = \sum_{m=1}^M \int_{t_{m-1}}^{t_m} (\partial_t v_M, v_M\phi)\,dt = \sum_{m=1}^M \int_{t_{m-1}}^{t_m} (\frac{1}{2}\partial_t |v_M|^2, \phi)\,dt$$
$$= \frac{1}{2}\sum_{m=1}^M (|u^m|^2, \phi(t_m,x)) - (|u^{m-1}|^2, \phi(t_{m-1},x)) - \sum_{m=1}^M \int_{t_{m-1}}^{t_m} (\frac{1}{2}|v_M|^2, \partial_t\phi)\,dt,$$

where we used that $\partial_t v_M(t) = \frac{u^m - u^{m-1}}{\kappa}$, for $t \in [t_{m-1}, t_m[$. Next, we observe that the sum telescopes and consequently we have

$$\int_0^T (\partial_t v_M, v_M\phi)\,dt$$
$$= \frac{1}{2}(|u^M|^2, \phi(T,x)) - \frac{1}{2}(|u_0|^2, \phi(0,x)) - \sum_{m=1}^M \int_{t_{m-1}}^{t_m} (\frac{1}{2}|v_M|^2, \partial_t\phi)\,dt$$
$$= -\int_0^T (\frac{1}{2}|v_M|^2, \partial_t\phi)\,dt.$$

By the strong convergence of $v_M \to v$ in $L^2(0,T;L^2(\mathbb{T}))$ we can conclude that

$$\lim_{M\to+\infty} \int_0^T \int_{\mathbb{T}} \partial_t v_M\, v_M\phi\,dxdt = -\frac{1}{2}\int_0^T \int_{\mathbb{T}} |v|^2\partial_t\phi\,dxdt.$$

Then, we consider the second term. Since u_M is constant on the interval $[t_{m-1}, t_m[$ we can write

$$\int_0^T (\partial_t v_M, (u_M - v_M)\phi)\,dt = -\sum_{m=1}^M \int_{t_{m-1}}^{t_m} (\partial_t(v_M - u_M), (v_M - u_M)\phi)\,dt$$
$$= -\sum_{m=1}^M \int_{t_{m-1}}^{t_m} (\partial_t(\frac{|v_M - u_M|^2}{2}), \phi)\,dt$$
$$= \sum_{m=1}^M \int_{t_{m-1}}^{t_m} (\frac{|v_M - u_M|^2}{2}, \partial_t\phi)\,dt,$$

where in the last line we have used the fact we do do not have boundary terms because $v_M(t_m) = u_M(t_m)$ for all $m = 0, \dots, M$. Then, since $u_M - v_M$ goes to 0 strongly in $L^2(0,T;H^0)$, we get that $I_2 \to 0$ as $M \to +\infty$ (or $\kappa \to 0$).

By the usual reasoning we have (in this term the integration by parts is in space, so there is no need for a special treatment) that

$$-\int_0^T \int_{\mathbb{T}} \Delta u_M\, u_M\phi\,dxdt = \int_0^T \int_{\mathbb{T}} |\nabla u_M|^2\phi\,dxdt + \frac{1}{2}\int_0^T \int_{\mathbb{T}} \nabla|u_M|^2\,\nabla\phi\,dxdt$$
$$= \int_0^T \int_{\mathbb{T}} |\nabla u_M|^2\phi\,dxdt - \frac{1}{2}\int_0^T \int_{\mathbb{T}} |u_M|^2\Delta\phi\,dxdt,$$

and integration by parts are possible due to the space-periodicity. By using the lower semi-continuity of the norm and $\phi \geq 0$, we obtain:

$$\lim_{M\to+\infty} \int_0^T \int_{\mathbb{T}} |\nabla u_M|^2\phi\,dx \geq \int_0^T \int_{\mathbb{T}} |\nabla v|^2\phi\,dx,$$

while again by the strong convergence in $L^2(0,T;L^2(\mathbb{T}))$

$$\lim_{M\to+\infty}\frac{1}{2}\int_0^T\int_{\mathbb{T}}|u_M|^2\Delta\phi\,dxdt=\frac{1}{2}\int_0^T\int_{\mathbb{T}}|v|^2\Delta\phi\,dxdt.$$

The convective term is treated again by integrating by parts. In fact, we obtain

$$\begin{aligned}\int_0^T\int_{\mathbb{T}}(u_M\cdot\nabla)\,u_M\,u_M\phi\,dx&=\frac{1}{2}\int_0^T\int_{\mathbb{T}}\nabla|u_M|^2\,u_M\phi\,dx\\&=-\frac{1}{2}\int_0^T\int_{\mathbb{T}}|u_M|^2\,u_M\nabla\phi\,dx,\end{aligned}$$

and, by the strong convergence $u_M\to v$ in $L^3(0,T;L^3(\mathbb{T}))$, we get

$$\lim_{M\to+\infty}\int_0^T\int_{\mathbb{T}}(u_M\cdot\nabla)\,u_M\,u_M\phi\,dx=\frac{1}{2}\int_0^T\int_{\mathbb{T}}|v|^2\,v\nabla\phi\,dx.$$

Finally, the term with the pressure is integrated by parts

$$\int_0^T\int_{\mathbb{T}}\nabla q_M\,u_M\phi\,dx=-\int_0^T\int_{\mathbb{T}}q_M\,u_M\nabla\phi\,dx,$$

and, thanks to the weak convergence $q_M\to q$ in $L^{5/3}(0,T;L^{5/3}(\mathbb{T}))$ and again the strong convergence of u_M in $L^3(0,T;L^3(\mathbb{T}))$, in we get

$$\lim_{M\to+\infty}\int_0^T\int_{\mathbb{T}}\nabla q_M\,u_M\phi\,dx=\int_0^T\int_{\mathbb{T}}q\,v\nabla\phi\,dx.$$

We finally proved that

$$\begin{aligned}2\int_0^T&\int_{\mathbb{T}}|\nabla v(x,\tau)|^2\phi(x,\tau)\,dxd\tau\\&\leq\int_0^T\int_{\mathbb{T}}\Big(|v(x,\tau)|^2(\partial_t\phi(x,\tau)-\Delta\phi(x,\tau))+(|v(x,\tau)|^2\\&\qquad+2q(x,\tau))v(x,\tau)\cdot\nabla\phi(x,\tau)\Big)\,dxd\tau,\end{aligned}$$

for all smooth and non-negative ϕ, which are space-periodic, and with compact support with respect to time.

By following the argument detailed in [**9**, p. 13], by with a further test-function only of the time variable and approximating a Dirac's delta at a given time $t\in]0,T]$, one can easily deduce from the latter the validity of (1.3) for a.e. $t\in[0,T]$. □

4. Remarks on other algorithms

The algorithm (NS^k) analyzed in the previous section is a particular case of a more general algorithm, the so called θ-*scheme*, defined below (cf. [**19**, § 13.4]):

Algorithm. **(θ-scheme)** Let $\theta\in[0,1]$. Given a time-step-size $\kappa>0$ and the corresponding net $I^M=\{t_m\}_{m=0}^M$, for $m\geq1$ and for u^{m-1} given from the previous step with $u^0=u_0$, compute the iterate u^m as follows:

$$(\text{NS}^k_\theta)\qquad\begin{cases}d_tu^m-\Delta u^m_\theta+(u^m_\theta\cdot\nabla)\,u^m_\theta+\nabla p^m_\theta=0,\\ \nabla\cdot u^m_\theta=0,\end{cases}$$

with $u^m_\theta = \theta u^m + (1-\theta)u^{m-1}$, $p^m_\theta = \theta p^m + (1-\theta)p^{m-1}$, and endowed with periodic boundary conditions. The algorithm (NS^k_θ) reduces to (NS^k) when $\theta = 1$. In particular the scheme with $\theta = \frac{1}{2}$, is very important because it is the only choice of the parameter θ which makes the scheme of second order with respect to the time-step κ.

Algorithm. **(1/2-scheme)** Given a time-step-size $\kappa > 0$ and the corresponding net $I^M = \{t_m\}_{m=0}^M$, for $m \geq 1$ and for u^{m-1} given from the previous step with $u^0 = u_0$, compute the iterate u^m as follows:

$$(\mathrm{NS}^k_{1/2}) \qquad \begin{cases} d_t u^m - \dfrac{1}{2}\Delta(u^m + u^{m-1}) + \dfrac{1}{4}((u^m + u^{m-1})\cdot\nabla)(u^m + u^{m-1}) \\ \hfill +\nabla(p^m + p^{m-1}) = 0, \\ \hfill \nabla\cdot u^m = 0. \end{cases}$$

The analysis performed in the previous section is not directly applicable to the general θ-scheme, with $\theta \neq 1$. The motivation relies on the fact that the "*natural a-priori*" estimate is obtained by using as test function u^m_θ, which produces the analogous of the energy estimate. In particular, in the case $\theta = 1/2$, we get the following identity

$$\frac{\|u^m\|^2 - \|u^{m-1}\|^2}{2} + 2\kappa \left\| \frac{\nabla u^m + \nabla u^{m-1}}{2} \right\|^2 \leq 0,$$

which is enough to show the energy estimates on u^m. What is missing is the so-called *stability estimate* as in Lemma 3.2. The crucial point is that, to obtain the estimate in Lemma 3.2, one needs to test with u^m, but at this point

$$\frac{1}{4}\int_{\mathbb{T}} \big((u^m + u^{m-1})\cdot\nabla\big)(u^m + u^{m-1})\, u^m\, dx \neq 0,$$

and an estimation of the nonlinear term involves regularity which is not available on the sequence $\{u^m\}_{m=0,\dots,M}$. A way to overcome this, and to have a good stability estimates is that of discretizing also in the space variables with finite elements and use available inverse estimates. This produces results of conditional stability, subject to a certain coupling of the time-step size κ and of the space mesh-size h. Typically one obtains the coupling

$$\kappa\, h^2 \leq C,$$

as in the analysis of the scheme

$$(\text{C-K}) \qquad \begin{cases} d_t u^m - \dfrac{1}{2}\Delta(u^m + u^{m-1}) + \dfrac{1}{2}(u^{m-1}\cdot\nabla)(u^m + u^{m-1}) \\ \hfill +\nabla(p^m + p^{m-1}) = 0, \\ \hfill \nabla\cdot u^m = 0, \end{cases}$$

which is a variant of the 1/2-θ scheme, see [**23**, Scheme (5.2), p. 334] and can be considered as an interpretation of the classical Crank-Nicholson scheme.

The analysis of this method in presence of certain space-discretization, and in connection with the local energy inequality will be done with full details in [**7**].

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