2013 European Control Conference (ECC)
July 17-19, 2013, Zürich, Switzerland.

A parsimonious algorithm for the solution of continuous-time constrained LQR problems with guaranteed convergence

Gabriele Pannocchia and David Q. Mayne and James B. Rawlings and Giulio M. Mancuso

***Abstract*— This paper presents an efficient computational method for solving the input-constrained, continuous time, infinite horizon, linear quadratic regulator problem to within a user specified tolerance. The infinite dimensional input trajectory is approximated with a piecewise linear function on a finite time discretization to ensure input constraint satisfaction. This approximate problem is then a standard finite dimensional quadratic program and is solved by conventional methods, generating an upper bound for the optimal value function. The finite time discretization is then refined by subdividing the intervals estimated to cause the largest decrease in the cost function. Convergence of the solution of this discretized problem towards the optimal continuous-time solution, as the discretization is refined, is proved. Exploiting the strict convexity of the original infinite dimensional problem, the gradient of the cost function with respect to the continuous-time input can be computed to generate a lower bound for the optimal cost. For computational efficiency, a lower bound for the solution of the discretized control at a very fine discretization can be used instead. The algorithm terminates when the difference between the upper and lower bounds meets a user supplied tolerance.**

I. INTRODUCTION

The efficient and accurate solution of constrained optimal control problems is a fundamental topic in both the theory and application of optimal control, and has been studied intensely by the optimization and control communities for the last fifty years. See [1], for example, for a general overview of the problem. In particular, model predictive control requires a robust, online solution of a sequence of optimal control problems as the state, or state estimate, of the system becomes available after each measurement. The general optimal control problem for nonlinear models is nonconvex, and it is therefore difficult to establish optimality in the online, or indeed even offline, settings. Most industrial applications of model predictive control, however, employ linear models; for linear discrete-time models, the constrained optimal control problem, although infinite dimensional, is (strictly) convex, and optimality can be achieved with finite computation for any feasible initial state.

A brief, current review of the numerical methods applied to the continuous-time constrained linear quadratic regulator problem is given in [2]. In this paper, we present a new algorithm having guaranteed accuracy for solving this optimal control problem. We show that the algorithm is guaranteed to converge, and we compute a novel lower bound on the continuous-time optimal cost so that the algorithm can be terminated with a guarantee on the distance to optimality. To the best of our knowledge no other algorithm provides such a guarantee. To increase the algorithm's efficiency when used in an online environment, we compute and store quadrature formulas for the integration of the dynamic model and the stage cost on the infinite horizon time interval, and we base termination on a lower bound for the discretized optimal cost at a very fine discretization. The algorithm is shown through numerical examples to be efficient, so that its use in model predictive control of fast or large-scale processes is feasible. Due to space limitations we omit all proofs, which are available in a companion publication [3].

Notation: Given two reals (integers) a, b with $a < b$, $\mathbb{R}_{a:b}$ ($\mathbb{I}_{a:b}$) denote all reals (integers) x such that $a \leq x \leq b$. $|x|$ is the Euclidean norm of x. We denote $(x, y) \triangleq \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ and $\langle x, y\rangle$ is the inner product. Given a matrix A and positive integers a, b, c, d, the symbol $A_{a:b,c:d}$ denotes the selection of rows a to b and columns c to d, and $A_{a:b,:}$ denotes the selection of rows a to b and all columns. Given a set S, $\mathrm{int}(S)$ denotes its interior.

II. BACKGROUND

A. Optimal control problem

In this paper we address the computation of the optimal solution to the continuous-time infinite-horizon input-constrained linear quadratic regulation problem:

$$\mathbb{P}(x): \quad \min_{u(\cdot)} V(x, u(\cdot)) \triangleq \int_0^\infty \ell(x(t), u(t))dt, \tag{1a}$$

subject to $x(0) = x$ and

$$\dot{x} = f(x, u) \triangleq Ax + Bu, \quad \text{for all } t \in [0, \infty), \tag{1b}$$

$$u(t) \in \mathbb{U} \quad \text{for all } t \in [0, \infty). \tag{1c}$$

The state (x) and input (u) have dimension n and m, respectively. The cost function is: $\ell(x, u) \triangleq \frac{1}{2}(x'Qx+u'Ru)$.

Assumption 1: The pair (A, B) is stabilizable and the pair (C, A), $C \triangleq Q^{\frac{1}{2}}$, is detectable. Q and R are symmetric positive definite matrices. The constraint set $\mathbb{U}$ has the form:

$$\mathbb{U} \triangleq \prod_{i=1}^{m} \mathbb{U}_i, \qquad \text{where} \qquad \mathbb{U}_i \triangleq [u_i^{\min}, u_i^{\max}],$$

and contains the origin in its interior.

We define $\mathbb{X}_\infty$ as the set of initial states such that $\mathbb{P}(x)$ has a solution. In order to rewrite the *infinite-horizon* problem

G. Pannocchia is with Dept. of Civil and Industrial Engineering (DICI), Univ. of Pisa, Italy (g.pannocchia@diccism.unipi.it)

D.Q. Mayne is with Dept. of Electric & Electronic Engineering, Imperial College London, UK (d.mayne@imperial.ac.uk)

J.B. Rawlings is with Dept. of Chemical and Biological Engineering, Univ. of Wisconsin, Madison, USA (rawlings@engr.wisc.edu)

G.M. Mancuso is with Scuola Superiore Sant'Anna, Pisa, Italy (g.mancuso@sssup.it)

$\mathbb{P}(x)$ as an equivalent *finite-horizon* problem, we define a suitable ellipsoid invariant set as follows. Let P be the unique symmetric positive definite solution to the Riccati equation:

$$0 = Q + A'P + PA - PBR^{-1}B'P. \tag{2}$$

Given a positive scalar α, we consider the following set:

$$\mathbb{X}_f \triangleq \{x \in \mathbb{R}^n \mid x'Px \leq \alpha\}. \tag{3}$$

Clearly, $\mathbb{X}_f$ is an invariant (ellipsoidal) set for the *unconstrained* closed-loop system: $\dot{x} = Ax + Bu$, $u = Kx$, with $K = -R^{-1}B'P$. Because $\mathbb{U}$ contains the origin in its interior, if α is sufficiently small, then for any $x \in \mathbb{X}_f$ there holds $Kx \in \mathbb{U}$. Hence, $u(t) = Kx(t)$ remains feasible at all times with respect to the constraint (1c) once $x(t)$ has entered $\mathbb{X}_f$. Finding the largest ellipsoid (of arbitrary shape, i.e. with P not fixed) contained in the polytopic set $\mathcal{P} \triangleq \{x \in \mathbb{R}^n \mid Kx \in \mathbb{U}\}$ can be posed as an LMI problem [4]. However, because P is fixed, the largest α such that $\mathbb{X}_f \subseteq \mathcal{P}$ can be found directly as shown in [3].

Given $T > 0$, we replace $\mathbb{P}(x)$ by the following finite-horizon optimal control problem:

$$\mathbb{P}_T(x): \quad \min_{u(\cdot)} V_T(x, u(\cdot)) \triangleq V_f(x(T)) + \int_0^T \ell(x(t), u(t))dt, \tag{4a}$$

subject to $x(0) = x$ and

$$\text{model (1b) and constraint (1c) for all } t \in [0, T], \tag{4b}$$

in which $V_f(x) \triangleq \frac{1}{2}x'Px$ with P computed from (2). We denote by $u^0(\cdot)$ the input trajectory solution to $\mathbb{P}_T(x)$ and by $x^0(\cdot)$ the associated state trajectory.

Proposition 2: For any $x \in \mathbb{X}_\infty$, there exists a finite $T > 0$ such that the solution to $\mathbb{P}_T(x)$ satisfies $x^0(T) \in \mathbb{X}_f$.

If T is chosen large enough to ensure that $x^0(T) \in \mathbb{X}_f$, then it is possible to show (see, e.g., [5]–[7] for equivalent arguments in discrete-time constrained LQR problems) that $\mathbb{P}(x)$ and $\mathbb{P}_T(x)$ yield the same minimum and the infinite-horizon input trajectory defined as:

$$u^0_\infty(t) \triangleq \begin{cases} u^0(t) & \text{if } t \in [0, T], \\ Ke^{(A+BK)(t-T)}x^0(T) & \text{if } t > T, \end{cases} \tag{5}$$

is the minimizer of $\mathbb{P}(x)$.

B. Input parameterizations (holds)

Let γ be a *discretization* of the interval $[0, T]$, defined as a sequence of $J_\gamma \in \mathbb{I}_{>0}$ intervals $\{I_j \triangleq [t_j, t_{j+1}] \mid j \in \mathbb{I}_{0:J_\gamma-1}\}$ having the following properties:

- $\text{int}(I_j) \cap \text{int}(I_k) = \emptyset$ for any $j \neq k$,
- $\bigcup_{j \in \mathbb{I}_{0:J_\gamma-1}} I_j = [0, T]$.

Hence, $t_0 = 0$ and $t_{J_\gamma} = T$. We also denote the length of I_j as $\Delta_j \triangleq t_{j+1} - t_j$, and we assume that each Δ_j satisfies $\Delta_j = 2^{q_j}\Delta$ with $q_j \in \mathbb{I}_{\geq 0}$ and $\Delta > 0$, in which case we say that $\gamma \in \Gamma_\Delta$. In order to consider a finite parameterization of the function $u : [0, T] \to \mathbb{R}^m$, it is customary in sampled-data control of continuous-time systems (see, e.g. [8]) to assume that the input is constant in each interval I_j, i.e.

$$u(t) = u_j \qquad \text{for all } t \in I_j. \tag{6}$$

Formally, given a discretization γ we define $\mathcal{U}^\gamma_{\text{ZOH}}$ as the set of all measurable functions $u : [0, T] \to \mathbb{R}^m$ satisfying the zero-order hold (ZOH) parameterization (6) in which $u_j \in \mathbb{U}$ for all $j \in \mathbb{I}_{0:J_\gamma-1}$.

Besides the fact that restricting $u(\cdot)$ to the set $\mathcal{U}^\gamma_{\text{ZOH}}$ makes problem $\mathbb{P}_T(x)$ finite dimensional, it also ensures that $u(t) \in \mathbb{U}$ for all $t \in [0, T]$. In [2], we argued that a better choice is to assume the input piece-wise linear in each interval, i.e.

$$u(t) = (1 - \eta_j(t))u_j + \eta_j(t)v_j \qquad \text{for all } t \in I_j, \tag{7}$$

in which $\eta_j(t) \triangleq \frac{t - t_j}{\Delta_j}$ for all $j \in \mathbb{I}_{0:J_\gamma-1}$. Formally, given a discretization γ we define $\mathcal{U}^\gamma_{\text{PWLH}}$ as the set of all measurable functions $u : [0, T] \to \mathbb{R}^m$ satisfying the piece-wise linear hold (PWLH) parameterization (7) in which $(u_j, v_j) \in \mathbb{U}^2$ for all $j \in \mathbb{I}_{0:J_\gamma-1}$. Notice that for all $j \in \mathbb{I}_{0:J_\gamma-1}$, we have that $\eta_j(t_j) = 0$ and $\eta_j(t_{j+1}) = 1$. Thus, if $(u_j, v_j) \in \mathbb{U}^2$, then $u(t) \in \mathbb{U}$ for all $t \in I_j$.

C. Discretized optimal control problem

Given a discretization γ and choosing either ZOH or PWLH, i.e. defining $\mathcal{U}^\gamma \triangleq \mathcal{U}^\gamma_{\text{ZOH}}$ or $\mathcal{U}^\gamma \triangleq \mathcal{U}^\gamma_{\text{PWLH}}$, we can obtain a suboptimal solution to $\mathbb{P}_T(x)$ by solving the following discretized optimal control problem:

$$\mathbb{P}^\gamma_T(x): \quad \min_{u(\cdot) \in \mathcal{U}^\gamma} V_T(x, u(\cdot)) \quad \text{subject to } x(0) = x \text{ and model (1b)} . \tag{8}$$

We rewrite $\mathbb{P}^\gamma_T(x)$ as an equivalent discrete-time constrained LQR problem and solve it via Quadratic Programming (QP). Then, under certain conditions we accept the achieved solution or we refine the discretization γ.

III. PROPOSED ALGORITHM: THEORY

A. LQR discretization for ZOH via matrix exponential

Given an interval I_j, assuming to use the ZOH parameterization (6), it is well-known [8], [9] that we can compute an equivalent discrete-time system evolution as:

$$x_{j+1} = A_j x_j + B_j u_j, \tag{9}$$

where: $x_j \triangleq x(t_j)$, $A_j = e^{A\Delta_j}$, $B_j = \int_0^{\Delta_j} e^{As}Bds$. Moreover: $V_T(x, u(\cdot)) = \sum_{j=0}^{J_\gamma-1} \ell_j(x_j, u_j) + V_f(x(T))$, where $\ell_j(x_j, u_j) \triangleq \int_{t_j}^{t_{j+1}} \ell(x, u)dt = \frac{1}{2}(x_j Q_j x_j + u_j' R_j u_j + 2x_j' M_j u_j)$, in which

$$\begin{bmatrix} Q_j & M_j \\ M_j' & R_j \end{bmatrix} = \int_0^{\Delta_j} e^{\left[\begin{smallmatrix} A & B \\ 0 & 0 \end{smallmatrix}\right]'s} \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} e^{\left[\begin{smallmatrix} A & B \\ 0 & 0 \end{smallmatrix}\right]s} ds. \tag{10}$$

The above formulas allow one to compute all matrices $(A_j, B_j, Q_j, R_j, M_j)$ by solving a system of ordinary differential equations (ODE). However, Van Loan [10] showed that all above matrices can be found by means of a single matrix exponentiation as follows. First, define a block upper triangular matrix C and partition its exponential as follow:

$$C \triangleq \begin{bmatrix} -A' & I & 0 & 0 \\ & -A' & Q & 0 \\ & & A & B \\ & & & 0 \end{bmatrix} \quad e^{C\tau} \triangleq \begin{bmatrix} F_1(\tau) & G_1(\tau) & H_1(\tau) & K_1(\tau) \\ & F_2(\tau) & G_2(\tau) & H_2(\tau) \\ & & F_3(\tau) & G_3(\tau) \\ & & & F_4(t) \end{bmatrix}. \tag{11}$$

Then, obtain:

$$\begin{aligned} A_j = F_3(\Delta_j), \quad B_j = G_3(\Delta_j), \quad Q_j = F_3'(\Delta_j)G_2(\Delta_j), \\ R_j = R\Delta_j + \left[B'F_3'(\Delta_j)K_1(\Delta_j)\right] + \left[B'F_3'(\Delta_j)K_1(\Delta_j)\right]', \\ M_j = F_3'(\Delta_j)H_2(\Delta_j). \end{aligned} \tag{12}$$

B. LQR discretization for PWLH via matrix exponential

Computation of $(A_j, B_j, Q_j, R_j, M_j)$ for ZOH via matrix exponential formulas (11)-(12) is typically faster and more accurate than via ODE solver. We devise here a similar procedure for PWLH. To this aim, in each interval I_j, we (formally) consider an augmented system with state $z \triangleq (z^{(1)}, z^{(2)}) \in \mathbb{R}^{2n}$, in which $z^{(1)}(t) \triangleq x(t)$ and $z^{(2)}(t) \triangleq u(t) - u_j = \eta_j(t)(v_j - u_j)$, and constant input $w_j = (u_j, v_j) \in \mathbb{R}^{2m}$. This system evolves in I_j as:

$$\dot{z} = \left[\begin{smallmatrix} A & B \\ 0 & 0 \end{smallmatrix}\right] z + \begin{bmatrix} B & 0 \\ -\frac{\mathbf{I}_m}{\Delta_j} & \frac{\mathbf{I}_m}{\Delta_j} \end{bmatrix} w_j, \tag{13}$$

where $\mathbf{I}_m$ is the identity matrix of dimension $m \times m$. If we set $A^* \triangleq \left[\begin{smallmatrix} A & B \\ 0 & 0 \end{smallmatrix}\right]$, $B^* \triangleq \begin{bmatrix} B & 0 \\ -\frac{\mathbf{I}_m}{\Delta_j} & \frac{\mathbf{I}_m}{\Delta_j} \end{bmatrix}$, $Q^* \triangleq \left[\begin{smallmatrix} Q & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and define C and its partitioned exponential as in (11) with (A, B, Q) replaced by (A^*, B^*, Q^*), under PWLH (7) we obtain that:

$$\begin{aligned} z_{j+1} = A_j^* z_j + B_j^* w_j, \qquad \ell_j^*(z_j, w_j) \triangleq \int_{t_j}^{t_{j+1}} \ell(x, u) dt \\ = \frac{1}{2}(z_j' Q_j^* z_j + w_j' R_j^* w_j + 2z_j' M_j^* w_j), \end{aligned} \tag{14}$$

where

$$\begin{aligned} A_j^* = F_3(\Delta_j),\ B_j^* = G_3(\Delta_j),\ Q_j^* = F_3'(\Delta_j)G_2(\Delta_j), \\ R_j = \begin{bmatrix} \frac{R}{3} & \frac{R}{6} \\ \frac{R}{6} & \frac{R}{3} \end{bmatrix} \Delta_j + \left[B'F_3'(\Delta_j)K_1(\Delta_j)\right] + \\ \left[B'F_3'(\Delta_j)K_1(\Delta_j)\right]',\ M_j^* = F_3'(\Delta_j)H_2(\Delta_j). \end{aligned} \tag{15}$$

Finally, by noticing that $z^{(2)}(t_j) = 0$, in the discrete-time evolution and cost function we can remove $z^{(2)}$ to obtain:

$$x_{j+1} = A_j x_j + B_j w_j, \tag{16}$$

$$V_T(x, u(\cdot)) = \sum_{j=0}^{J_\gamma - 1} \ell_j(x_j, w_j) + V_f(x(T)), \tag{17}$$

where $\ell_j(x_j, u_j) = \frac{1}{2}(x_j' Q_j x_j + w_j' R_j w_j + 2x_j' M_j w_j)$, in which $A_j = A^*_{j\,1:n,1:n}$, $B_j = B^*_{j\,1:n,:}$, $Q_j = Q^*_{j\,1:n,1:n}$, $M_j = Q^*_{j\,1:n,:}$, and R_j is defined in (15). We observe that in (16) the discrete-time evolution of the system under PWLH is still described by a linear system with the *original state* x_j and *augmented input* $w_j = (u_j, v_j)$. In the sake of space, from now on we will focus only on PWLH, but all derivations and results will apply directly to ZOH, which can be seen as a particular PWLH in which $w_j = (u_j, u_j)$.

Given the above premises, problem $\mathbb{P}_T^\gamma(x)$ can be rewritten as a conventional discrete-time constrained LQR problem. Let $\mathbf{u} \triangleq (w_0, w_1, \ldots, w_{J_\gamma - 1})$ be an augmented input sequence of length J_γ. Then, $\mathbb{P}_T^\gamma(x)$ can be re-written as:

$$\mathbb{P}_T^\gamma(x): \quad \min_{\mathbf{u} \in \mathbb{U}^{2J_\gamma}} V_T^\gamma(x, \mathbf{u}) \triangleq \sum_{j=0}^{J_\gamma - 1} \ell_j(x_j, w_j) + V_f(x_{J_\gamma}),$$
$$\text{subject to } x_0 = x \text{ and model (16)}. \tag{18}$$

Proposition 3: For each γ, each x, the map $\mathbf{u} \mapsto V_T^\gamma(x, \mathbf{u})$ is Lipschitz continuous, differentiable and convex.

C. Gradient of the optimal cost and optimality functions

Given $x \in \mathbb{R}^n$ and $u(\cdot) \in \mathcal{U}^\gamma$ (or equivalently $\mathbf{u} \in \mathbb{U}^{2J_\gamma}$), let $\phi(t; x, u(\cdot))$ be the solution at time t of (1b) with initial condition $x(0) = x$, and let $\{\lambda_0, \lambda_1, \ldots, \lambda_{J_\gamma}\}$ be the solution of the discrete-time adjoint system defined by:

$$\lambda_{J_\gamma} = P x_{J_\gamma}, \qquad \lambda_j = A_j' \lambda_{j+1} + M_j' w_j + Q_j x_j, \tag{19}$$

in which $x_j = \phi(t_j; x, u(\cdot))$. Notice that (19) defines a backward recursion $j = J_\gamma - 1, J_\gamma - 2, \ldots, 0$. Let $H_j : \mathbb{R}^n \times \mathbb{R}^{2m} \times \mathbb{R}^n \to \mathbb{R}$ be defined as:

$$H_j(x, w, \lambda) \triangleq \ell_j(x, w) + \lambda'(A_j x + B_j w). \tag{20}$$

Then, the gradient of the cost function with respect to the augmented input sequence $\mathbf{u}$ is given by:

$$\nabla_{\mathbf{u}} V_T^\gamma(x, \mathbf{u}) \triangleq g^\gamma(x, u(\cdot)) = \mathrm{vec}\{g_0^\gamma, g_1^\gamma, \ldots, g_{J_\gamma - 1}^\gamma\},$$

in which, for $j = 0, 1, \ldots, J_\gamma - 1$:

$$\begin{aligned} g_j^\gamma(x, u(\cdot)) \triangleq \nabla_{w_j} H_j(x_j, w_j, \lambda_{j+1}) = \\ M_j' x_j + R_j w_j + B_j' \lambda_{j+1}, \end{aligned} \tag{21}$$

and $\mathrm{vec}\{\cdot\}$ forms a column vector of given vectors g_j^γ.

In the algorithm described in the next section, we solve a sequence of problems $\mathbb{P}_T^\gamma(x)$ with varying γ. We need an optimality function [11] for each problem, i.e. a continuous function of the control $u(\cdot)$ that is strictly negative when $u(\cdot)$ is not optimal for $\mathbb{P}_T^\gamma(x)$ and is zero when $u(\cdot)$ is optimal for $\mathbb{P}_T^\gamma(x)$. Since the initial state x does not vary, we omit it in future notation for simplicity. Thus, the optimality function $\theta^\gamma : \mathcal{U}^\gamma \to \mathbb{R}_{\leq 0}$ for problem $\mathbb{P}_T^\gamma(x)$ is defined by: $\theta^\gamma(u(\cdot)) \triangleq \sum_{j=0}^{J_\gamma - 1} \theta_j^\gamma(u(\cdot))$, in which $\theta_j^\gamma(u(\cdot)) \triangleq \langle g_j^\gamma(x, u(\cdot)), w_j^*(x, u(\cdot)) - w_j \rangle$, and $w_j^*(x, u(\cdot))$ is:

$$w_j^*(x, u(\cdot)) \triangleq \arg\min_z \{\langle g_j^\gamma(x, u(\cdot)), z \rangle \mid z \in \mathbb{U}^2\}. \tag{22}$$

Proposition 4: For any $x \in \mathbb{R}^n$, the function $\theta^\gamma(u(\cdot))$ is an optimality function for $\mathbb{P}_T^\gamma(x)$. Moreover, $V_T^\gamma(x, u(\cdot)) + \theta^\gamma(u(\cdot))$ is a lower bound for $V_T^{\gamma,0}(x)$.

Finally, let $\theta^\Delta(u(\cdot))$ denote $\theta^\gamma(u(\cdot))$ for the special case when $\gamma = \gamma^\Delta \triangleq \{I_0, I_1, \ldots, I_{J_\gamma - 1}\}$ where each $I_j = [t_j, t_j + \Delta]$, i.e. each constituent interval I_j has length Δ. We will refer to γ^Δ as the *finest* discretization.

Proposition 5: $\theta^\Delta(u(\cdot)) \leq \theta^\gamma(u(\cdot))$ and $V_T(x, u(\cdot)) + \theta^\Delta(u(\cdot))$ is a lower bound for $V_T^{\gamma,0}(x)$ for all $\gamma \in \Gamma_\Delta$.

IV. PROPOSED ALGORITHM: IMPLEMENTATION

A. Conceptual algorithm

We refer to $\gamma' \in \Gamma_\Delta$ as a *refinement* of $\gamma \in \Gamma_\Delta$ if some of the intervals $\{I'_j\}$ defining γ' are obtained by bisecting one or more intervals in the set $\{I_j\}$ that defines γ and if the remaining intervals in γ' are the same as the corresponding intervals in γ. If $V_T^0(x)$ and $V_T^{\gamma,0}$(x) are, respectively, the optimal value functions of $\mathbb{P}_T(x)$ and $\mathbb{P}_T^\gamma(x)$ then, clearly

$$V_T^{\gamma,0}(x) \geq V_T^0(x), \tag{23}$$

for all $x \in \mathbb{R}^n$, $\gamma \in \Gamma_\Delta$, and admissible $\Delta \in (0, T)$. We now state the (conceptual) optimization algorithm to solve $\mathbb{P}_T(x)$.

Algorithm 6: **Require:** $\Delta, \epsilon > 0$, $\gamma \in \Gamma_\Delta$, $c \in (0, 1)$.

1: Solve $\mathbb{P}_T^\gamma(x)$ yielding control $u(\cdot) \in \mathcal{U}^\gamma$, and compute $\theta^\Delta(u(\cdot))$.
2: Refine γ (repeatedly) until $\theta^\gamma(u(\cdot)) \leq c\theta^\Delta(u(\cdot))$.
3: If $\theta^\Delta(u(\cdot)) \leq -\epsilon$, go to Step 1. Else, go to Step 4.
4: Replace $\epsilon \leftarrow \epsilon/2$, $\Delta \leftarrow \Delta/2$. Bisect largest interval in γ. Go to Step 1.

A procedure for refining γ (repeatedly) is given below. In Step 4, $\epsilon \leftarrow \epsilon/2$ and $\Delta \leftarrow \Delta/2$ may be replaced, respectively, by $\epsilon \leftarrow c_1\epsilon$ and $\Delta \leftarrow c_2\Delta$ where $c_1, c_2 \in (0, 1)$. The control $u(\cdot)$ obtained in Step 1 satisfies $\theta^\gamma(u(\cdot)) = 0$; if γ' is the refined discretization obtained in Step 2, and $u(\cdot)$ is not optimal for $\mathbb{P}_T^{\gamma'}(x)$, then $\theta^{\gamma'}(u(\cdot)) < 0$.

B. Refinement strategy

Step 2 of Algorithm 6 requires repeated refinement of the discretization γ until the condition $\theta^\gamma(u(\cdot)) \leq c\theta^\Delta(u(\cdot))$ is satisfied. Since the length of each interval in the current discretization γ is an even multiple of the current Δ and since the length of all intervals in the refined discretization should also be a multiple of Δ, the refinement strategy consists of bisecting each interval with length greater than or equal to 2Δ and selecting a subset whose bisection satisfies the condition in Step 2, as detailed.

Suppose the current discretization γ consists of the intervals $\{I_0, I_1, ..., I_{J_\gamma - 1}\}$. Because the current $u(\cdot)$ is optimal for $\mathbb{P}_T^\gamma(x)$, then $\theta_j^\gamma(u(\cdot)) = 0$ for all $j \in \mathcal{J}_\gamma \triangleq \{0, 1, ..., J_\gamma - 1\}$. If I_j is bisected, yielding $I_{j1} = [t_j, t_{j1}]$ and $I_{j2} = [t_{j1}, t_{j+1}]$, let w_j be replaced by $w_{j1} = w_j$ in I_{j1} and $w_{j2} = w_j$ in I_{j2}, and let x_{j1} and λ_{j1} denote the value of $x(\cdot)$ (the current state trajectory) and λ at time t_{j1}. If this is done for each $j \in \mathcal{J} \subseteq \mathcal{J}_\gamma$, then the gradients $g_{j1}^\gamma(x, u(\cdot))$ and $g_{j2}^\gamma(x, u(\cdot))$ of the cost with respect to w_{j1} and w_{j2} may be computed from (21) yielding (γ' is the refined discretization):

$$\theta^{\gamma'}(u(\cdot)) = \sum_{j \in \mathcal{J}} \theta_j^{\gamma'}(u(\cdot)), \tag{24}$$

where $\theta_j^{\gamma'}(u(\cdot)) \triangleq \langle g_{j1}^\gamma(x, u(\cdot)), w_{j1}^*(x, u(\cdot)) - w_{j1}\rangle + \langle g_{j2}^\gamma(x, u(\cdot)), w_{j2}^*(x, u(\cdot)) - w_{j2}\rangle$ and $w_{j1}^*(x, u(\cdot))$ is:

$$w_{j1}^*(x, u(\cdot)) \triangleq \arg\min_z \{\langle g_{j1}^\gamma(x, u(\cdot)), z\rangle \mid z \in \mathbb{U}^2\}, \tag{25}$$

and a similar definition for $w_{j2}^*(x, u(\cdot))$. By ordering the intervals of $\mathcal{J}_\gamma$ in increasing value of $\theta_j^{\gamma'}(u(\cdot))$, i.e. starting from the most negative, $\mathcal{J}$ is chosen as the subset of $\mathcal{J}_\gamma$ with smallest cardinality such that the condition in Step 2 is satisfied by $\theta^{\gamma'}(u(\cdot))$. If no such $\mathcal{J}$ can be found, the procedure is repeated with γ replaced by the discretization with *every* I_j bisected.

C. Practical algorithm with stopping condition

All discrete-time matrices appearing in the various steps of Algorithm 6 can be computed and stored offline for a (finite) number of possible interval sizes, in geometric sequence of ratio 2, using the formulas derived in Section III. It can also be noticed that the computation of $w_j^*(\cdot)$ in (22) does not require solving a linear program because $\mathbb{U}$ (and hence $\mathbb{U}^2$ also) is a box constrained set. Thus, each component of $w_j^*(\cdot)$ is either the minimum value, zero, or the maximum value, respectively, when the corresponding component of $g_j^\gamma(\cdot)$ is positive, zero, or negative. The same considerations apply to the computation of $w_{j1}^*(\cdot)$ in (25) and $w_{j2}^*(\cdot)$.

For a given Δ, the loop in Steps 1-3 is always exited in a finite number of iterations because, otherwise, the refinement of γ would reach γ^Δ and then we would have $\theta^\gamma(u(\cdot)) = \theta^\Delta(u(\cdot)) = 0$, which makes the condition to proceed to Step 4 true. However, as written, Algorithm 6 never terminates because it would keep entering Step 4, reducing Δ and then going to Step 1. A practical variant would include a stopping condition such as $V_T^\gamma(u(\cdot)) - V_T^{LB}(u(\cdot)) \leq \rho$ in Step 4 where $V_T^{LB}(u(\cdot))$ is a lower bound to the optimal cost and ρ is suitably small. Such a lower bound is discussed in Appendix A. However, since computation of $V_T^{LB}(u(\cdot))$ is possibly expensive, for repeated use e.g. in MPC implementation, we suggest to use the stopping condition:

$$-\theta^\Delta(u(\cdot)) \leq \rho. \tag{26}$$

Thus, the algorithm terminates when the solution to $\mathbb{P}_T^\gamma$ is a close approximation to that at the finest discretization γ^Δ.

V. PROPERTIES OF THE ALGORITHM

Assumption 7: T is large enough that $x^0(T)$ obtained by solving $\mathbb{P}_T^\gamma(x)$ in Step 1 satisfies $x^0(T) \in \mathbb{X}_f$.

A. The space of control and state trajectories

We need to define the following space:

$$\mathcal{U} \triangleq \{u : [0, T] \to \mathbb{R}^m \mid u(\cdot) \text{ measurable}, \\ u(t) \in \mathbb{U}\ \forall t \in [0, T]\}. \tag{27}$$

Since $u(\cdot) \in \mathcal{U}$, it follows that $u(\cdot) \in L_p \triangleq \mathcal{L}_p([0, T], \mathbb{R}^m)$ for all $1 \leq p \leq \infty$ where

$$\mathcal{L}_p([0, T], \mathbb{R}^m) = \{u : [0, T] \to \mathbb{R}^m \mid u(\cdot) \text{ measurable}, \\ ||u(\cdot)||_p < \infty\}, \tag{28}$$

and $||u||_p \triangleq \left[\int_0^T |u(t)|^p dt\right]^{1/p}$, $||u(\cdot)||_\infty \triangleq \text{ess sup}_{[0,T]} |u(t)|$. The space L_p is a Banach space. The spaces L_p, $p = 1, 2, \ldots, \infty$ are nested; i.e. $p < q$ implies $L_q \subset L_p$. In fact, since $[0, T]$ has finite measure T,

$$\|u(\cdot)\|_p \leq T^{(1/p - 1/q)} \|u(\cdot)\|_q, \tag{29}$$

so that $\|u(\cdot)\|_q \to 0$ implies $\|u(\cdot)\|_p \to 0$ for all $p, q \in \mathbb{I}_{\geq 1} \cup \infty$, $p < q$ and all $u \in \mathcal{U}$. It is also possible to show that, for all $p, q \in \mathbb{I}_{\geq 1} \cup \infty$, $\|u(\cdot)\|_p \to 0$, $u(\cdot) \in \mathcal{U}$, implies $\|u(\cdot)\|_q \to 0$. Since $u(\cdot)$ is bounded and T is finite, it follows that the solution $x(\cdot) = \phi(\cdot; x, u(\cdot))$ of (1b) is absolutely continuous for any $(x, u(\cdot)) \in \mathbb{R}^n \times \mathcal{U}$.

Proposition 8: Problem $\mathbb{P}_T(x)$ has a solution $u^0(x) \in \mathcal{U}$ at each $x \in \mathbb{R}^n$.

Proposition 9: The function $\mathbf{u} \mapsto V_T(x, \mathbf{u})$ is continuous and Frechet differentiable in $\mathcal{U}$, where $\mathcal{U}$ is endowed with the L_p metric for any $p \in \mathbb{I}_{\geq 1} \cup \infty$.

The next result follows from Theorem 3.1 in [13].

Theorem 10: For each $x \in \mathbb{R}^n$, $u^0(x) : [0, T] \to \mathbb{U}$ is Lipschitz continuous.

B. Convergence of Algorithm 6

As discussed before, the loop in Steps 1-3 is always exited in a finite number of iterations. Let $\mathcal{I}$ index the sub-sequence (of iterations) in which Step 4 is entered and let $u_i(\cdot)$, ϵ_i, γ_i and Δ_i denote, respectively, the values of $u(\cdot)$, ϵ, γ and Δ at iteration i of the algorithm, $i \in \mathcal{I}$. Clearly ϵ_i and Δ_i both converge to 0 as $i \xrightarrow{\mathcal{I}} \infty$ (i.e. $i \to \infty$, $i \in \mathcal{I}$). Let δ_i denote the length of the largest interval in the discretization $\gamma_i \in \Gamma_{\Delta_i}$; clearly $\delta_i \to 0$ as $i \xrightarrow{\mathcal{I}} \infty$.

From Theorem 10, $u^0(\cdot)$, the optimal control for $\mathbb{P}_T(x)$, is Lipschitz continuous. Let κ denote the Lipschitz constant for $u^0(\cdot)$ and let $u_i^* : [0, T] \to \mathcal{U}$ denote the sample-hold version of $u^0(\cdot)$, in a PWLH sense, obtained by sampling $u^0(\cdot)$ at the times $\{t_j\}$ specified by the discretization γ_i. That is, $u_i^*(\cdot)$ is defined in (7) with $u_j = u^0(t_j)$ and $v_j = u^0(t_{j+1})$.

Proposition 11: $u_i^*(\cdot) \xrightarrow{\mathcal{I}} u^0(\cdot)$ in L_p (any $p \in \{1, 2, \ldots, \infty\}$) as $i \to \infty$.

We can now state the main result of this section:

Theorem 12: For all $x \in \mathbb{R}^n$, $V_T(x, u_i(\cdot)) \xrightarrow{\mathcal{I}} V_T^0(x)$ as $i \to \infty$.

Corollary 13: Let $\{u_i(\cdot) \mid i \in \mathcal{I}\}$ be a sequence of controls generated by the algorithm. Then $u_i(\cdot) \xrightarrow{\mathcal{I}} u^0(\cdot)$ in L_p (any $p \in \{1, 2 \ldots, \infty\}$) as $i \to \infty$.

VI. SIMULATION RESULTS

We present a few illustrative simulation results. Other results are reported in [3]. Computations are performed in Matlab (R2011b) on a MacBook Air (1.8 GHz Intel Core i7, 4 GB of RAM). The discretized constrained LQR problems are solved using `quadprog.m` [1], using both input and state sequences, $\{w_j\}$ and $\{x_j\}$, as decision variables (see, e.g., [14], [15]). Timing is measured with `tic` and `toc`.

We consider the double integrator:

$$\dot{x} = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] x + \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] u \qquad (30)$$

with: $Q = C'C$, $C = [\,1\ 0\,]$, $R = 0.1$ and input constraints $\mathbb{U} = [-1,\ 1]$. We consider an initial state of $x = \left[\begin{smallmatrix} 1 \\ -2.5 \end{smallmatrix}\right]$ as in [16] and we use $T = 10$. The initial discretization is made by five (equal) intervals of length $\Delta_j = 2$. The initial values for Δ and ϵ are, respectively, 0.125 and 0.10; the parameter c is chosen equal to 0.8.

[1] With options: `'interior-point-convex'` algorithm, function tolerance of 10^{-10} and variable tolerance of 10^{-8}.

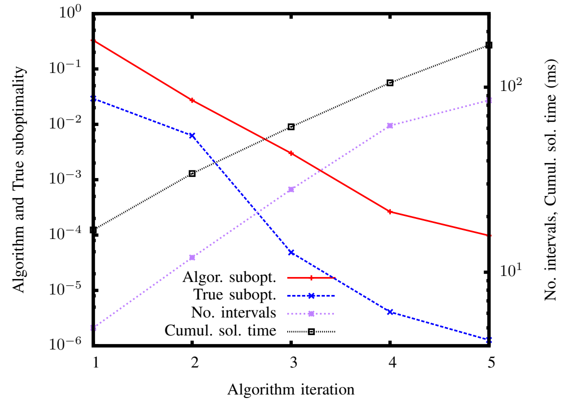

Fig. 1. Left axis: Algorithm sub-optimality, $-\theta^{\Delta}(u(\cdot))$, and true sub-optimality, $V_T^{\gamma,0}(x) - V_T^0(x)$. Right axis: number of intervals, J_γ, and cumulative solution time (in ms.) at the first five iterations

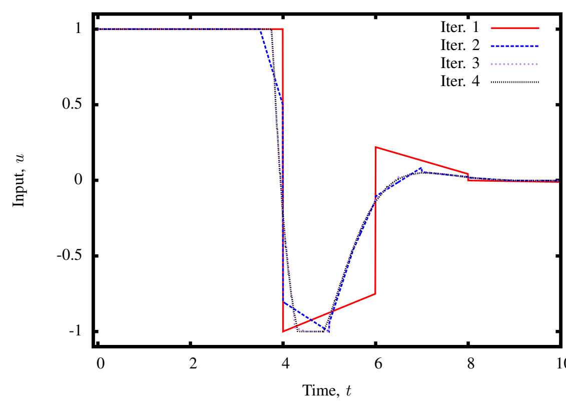

Fig. 2. Optimal input $u(\cdot) \in \mathcal{U}^\gamma$ achieved during the first four iterations

We report in Figure 1 the following performance indicators obtained during the first five iterations of Algorithm 6: (i) algorithm sub-optimality, $-\theta^{\Delta}(u(\cdot))$; (ii) true sub-optimality, $V_T^{\gamma,0}(x) - V_T^0(x)$; (iii) number of intervals, J_γ; (iv) cumulative solution time (in ms). Notice that the *true* optimal value was considered $V_T^0(x) = 5.3298957$, value that would be achieved at the sixth iteration of the algorithm. We notice that both the algorithm sub-optimality and the *true* sub-optimality decrease rapidly at each iteration, with the *true* one being smaller than the algorithm one. The (cumulative) solution time ranges from 10 ms of the first iteration to about 150 ms of the fifth one. In practice, the achieved solution is already very accurate at the end of the third iteration, which would be achieved after about 50 ms.

We also present in Figure 2 the input function $u(\cdot) \in \mathcal{U}^\gamma$ that is achieved during the first four iterations of the algorithm. We notice the rapid *improvement* of $u(\cdot)$ even after the second iteration, and practically no difference is appreciable from the third to the four iteration.

Next, we present in Figure 3 the performance indicators achieved with Algorithm 6 and stopping condition (26), for different values of the stopping tolerance, ρ. We can observe that a stopping tolerance of 10^{-2} appears sufficient

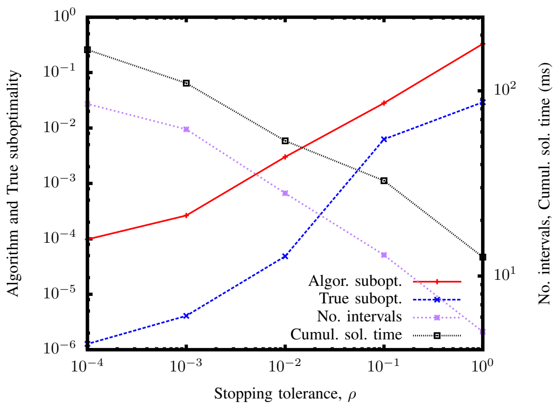

Fig. 3. Left axis: Algorithm sub-optimality, $-\theta^{\Delta}(u(\cdot))$, and true sub-optimality, $V_T^{\gamma,0}(x) - V_T^0(x)$. Right axis: number of intervals, J_γ, and solution time (in ms.) using stopping condition (26) for different ρ

to achieve an accurate solution in about 50 ms.

VII. CONCLUSIONS

This paper has presented a computational procedure for the input-constrained, infinite horizon, linear quadratic regulator problem. A novel feature of the procedure is the computation of a lower bound on the original infinite dimensional problem so that a termination criterion can be established. As far as we are aware, this is the first algorithm that guarantees the accuracy of the solution of the continuous time, infinite horizon problem. For online computational efficiency, a lower bound for the solution of the discretized control at a very fine discretization can be used instead. Another novel feature is the use of matrix exponentiation formulas to perform all required integrations of the state and adjoint differential equations. The required exponentiation can be performed offline and stored (using a small amount of memory), which significantly increases both the accuracy and speed of the approach. The convergence of the algorithm was established, and numerical examples show that the computation is efficient. We see no impediment to solving larger dimensional problems with this approach.

REFERENCES

[1] A. E. Bryson and Y. Ho, *Applied Optimal Control*. New York: Hemisphere Publishing, 1975.

[2] G. Pannocchia, J. B. Rawlings, D. Q. Mayne, and W. Marquardt, “On computing solutions to the continuous time constrained linear quadratic regulator,” *IEEE Trans. Auto. Contr.*, vol. 55, no. 9, pp. 2192–2198, 2010.

[3] G. Pannocchia, D. Q. Mayne, J. B. Rawlings, and G. Mancuso, “Addendum to ‘A parsimonious algorithm for the solution of continuous-time constrained lqr problems with guaranteed convergence’,” Available at http://twccc.che.wisc.edu/techreports.htm, TWCCC, Texas-Wisconsin-California Modeling and Control Consortium, Tech. Rep. 2013–01, May 2013.

[4] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in Control Theory*. Philadelphia: SIAM, 1994.

[5] D. Chmielewski and V. Manousiouthakis, “On constrained infinite-time linear quadratic optimal control,” *Sys. Cont. Let.*, vol. 29, pp. 121–129, 1996.

[6] P. O. M. Scokaert and J. B. Rawlings, “Infinite horizon linear quadratic control with constraints,” in *13th Triennial IFAC World Congress*, San Francisco, CA (USA), 1996, pp. 109–114.

[7] P. O. Scokaert and J. B. Rawlings, “Constrained linear quadratic regulation,” *IEEE Trans. Auto. Contr.*, vol. 43, no. 8, pp. 1163–1169, 1998.

[8] J. Yuz, G. Goodwin, A. Feuer, and J. De Doná, “Control of constrained linear systems using fast sampling rates,” *Syst. Contr. Lett.*, vol. 54, pp. 981–990, 2005.

[9] H. Kwakernaak and R. Sivan, *Linear Optimal Control Systems*. John Wiley & Sons, 1972.

[10] C. F. Van Loan, “Computing integrals involving the matrix exponential,” *IEEE Trans. Auto. Contr.*, vol. 23, no. 3, pp. 395–404, 1978.

[11] E. Polak, *Optimization: Algorithms and Consistent Approximations*, ser. Applied Mathematical Sciences. Springer Verlag, 1997.

[12] E. B. Lee and L. Markus, *Foundations of Optimal Control Theory*. New York: John Wiley & Sons, 1967.

[13] G. N. Galbraith and R. B. Vinter, “Lipschitz continuity of optimal controls for state constrained problems,” *SIAM J. Control Optim.*, vol. 42, no. 5, pp. 1727–1744, 2003.

[14] M. Diehl, H. J. Ferreau, and N. Haverbeke, “Efficient numerical methods for nonlinear MPC and moving horizon estimation,” in *Proceedings of the International Workshop on Assessment and Future Directions of NMPC*, Pavia, Italy, September 2008.

[15] C. V. Rao, S. J. Wright, and J. B. Rawlings, “Application of interior-point methods to model predictive control,” *J. Optim. Theory Appl.*, vol. 99, pp. 723–757, 1998.

[16] R. Goebel and M. Subbotin, “Continuous time linear quadratic regulator with control constraints via convex duality,” *IEEE Trans. Auto. Contr.*, vol. 52, no. 5, pp. 886–892, 2007.

APPENDIX

A. A lower bound to the (continuous-time) optimal cost

Given x and $u(\cdot) \in \mathcal{U}$, let $x(t) = \phi(t; x, u(\cdot))$ and let $\lambda : [0, T] \to \mathbb{R}^n$ denote the solution of the continuous-time adjoint system defined by:

$$-\dot{\lambda}(t) = A'\lambda(t) + Qx(t) \tag{31}$$
$$\lambda(T) = Px(T). \tag{32}$$

Then, as is well known:

$$\nabla_x V_T(x, u(\cdot)) = \lambda(0). \tag{33}$$

Let $H : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ be defined as

$$H(x, u, \lambda) \triangleq \ell(x, u) + \lambda'(Ax + Bu), \tag{34}$$

and let $D_u V_T(x, u(\cdot)) : [0, T] \to \mathbb{R}^m$ denote the Frechet derivative of $V_T(\cdot)$ at $(x, u(\cdot))$ with respect to $u(\cdot)$. Then, for all $t \in [0, T]$:

$$\begin{aligned} g(x, u(\cdot))(t) &\triangleq [D_u V_T(x, u(\cdot))(t)]' \\ &= \nabla_u H(x(t), u(t), \lambda(t)) = Ru(t) + B'\lambda(t). \end{aligned} \tag{35}$$

Thus, the optimality function $\theta : \mathcal{U} \to \mathbb{R}_{\leq 0}$ (θ does depend on x) for problem $\mathbb{P}_T(x)$ is defined by

$$\theta(u(\cdot)) \triangleq \int_0^T \langle g(x, u(\cdot))(t), u^*(t; x, u(\cdot)) - u(t)\rangle dt, \tag{36}$$

where $u^*(\cdot)$ is defined by

$$u^*(t; x, u(\cdot)) \triangleq \arg\min_z \{\langle g(x, u(\cdot))(t), z\rangle \mid z \in \mathbb{U}\}. \tag{37}$$

Proposition 14: For any $x \in \mathbb{R}^n$, the function $\theta(u(\cdot))$ is an optimality function for $\mathbb{P}_T(x)$. Moreover, $V_T^{LB}(u(\cdot)) \triangleq V_T(x, u(\cdot)) + \theta(u(\cdot))$ is a lower bound for $V_T^0(x)$.