

THE COHOMOLOGY OF THE BRAID GROUP B_3 AND OF $SL_2(\mathbb{Z})$ WITH COEFFICIENTS IN A GEOMETRIC REPRESENTATION

F. CALLEGARO, F. R. COHEN*, AND M. SALVETTI

ABSTRACT. The purpose of this article is to describe the integral cohomology of the braid group B_3 and $SL_2(\mathbb{Z})$ with local coefficients in a classical geometric representation given by symmetric powers of the natural symplectic representation.

These groups have a description in terms of the so called “divided polynomial algebra”. The results show a strong relation between torsion part of the computed cohomology and fibrations related to loop spaces of spheres.

1. INTRODUCTION

This article addresses the cohomology of the third braid group B_3 or $SL_2(\mathbb{Z})$ with local coefficients in a classical geometric representation given by symmetric powers of the natural symplectic representation. Calculations of this type have been basic in several areas of mathematics such as number theory as well as algebraic geometry. These results can be regarded as giving the cohomology of certain mapping class groups with non-trivial local coefficients.

For example, the first characteristic zero computation of this type, important in number theory, is the classical computation of a certain ring of modular forms due to G. Shimura [Shi71]. A version for the “stabilized” mapping class group in characteristic zero was developed by Looijenga [Loo96]. The new feature of the current article is (1) a determination of the torsion in the cohomology of this special case of B_3 together with (2) an identification of this torsion as the cohomology of another space which has been basic within homotopy theory.

The main new results in this paper gives the cohomology of the braid group B_3 as well as $SL(2, \mathbb{Z})$ with natural coefficients over integral symmetric powers of the natural symplectic representation. One of the main new points is that the construction of Shimura which gives the ring of modular forms also “transforms” polynomial rings to torsion variations of divided power algebras closely connected to the cohomology of the loop space of a sphere and related fibrations.

The precise problem considered here is the computation of the cohomology of the braid group B_3 and of the special linear group $SL_2(\mathbb{Z})$ with certain non-trivial local coefficients. First consider a particularly natural geometric, classical representation. In general, let $M_{g,n}$ be an orientable surface of genus g with n connected components in its boundary. Isotopy classes of Dehn twists around simple loops c_1, \dots, c_{2g} such that

$$|c_i \cap c_{i+1}| = 1 \quad (i = 1, \dots, 2g - 1), \quad c_i \cap c_j = \emptyset \quad \text{otherwise}$$

Date: May 3, 2013.

*Partially supported by DARPA.

represents the braid group B_{2g+1} in the symplectic group $Aut(H_1(M_{g,n}); <, >)$ of all automorphisms preserving the intersection form. Such representations arise classically, for example in singularity theory, as it is related to some monodromy operators (see [Waj80] for example) as well as in number theory in the guise of modular forms (see [Shi71] for example), and geometry (see [Loo74] for example).

In this paper we restrict to the case $g = 1$, $n = 1$, where the symplectic group equals $SL_2(\mathbb{Z})$. We extend the above representation to the symmetric power $M = \mathbb{Z}[x, y]$. Notice that this representation splits into irreducibles $M = \bigoplus_{n \geq 0} M_n$, according to polynomial-degree.

Our result is the complete integral computation of

$$H^*(B_3; M) = \bigoplus_n H^*(B_3; M_n)$$

and

$$H^*(SL_2(\mathbb{Z}); M) = \bigoplus_n H^*(SL_2(\mathbb{Z}); M_n).$$

The free part of the cohomology of $SL_2(\mathbb{Z})$ is a classical computation due to G. Shimura ([Shi71], [FTY88]) and is related to certain spaces of modular forms.

In [FTY88], Furusawa, Tezuka and Yagita study the group $\text{Diff}_+ T^2$ of orientation preserving diffeomorphisms of the 2-dimensional torus T^2 . They compute its cohomology with coefficients in \mathbb{Q} and \mathbb{Z}_p . As an intermediate step they compute the cohomology of the group $SL_2(\mathbb{Z})$ with coefficients in the module $M \otimes \mathbb{Q}$ and $M \otimes \mathbb{Z}_p$, $p \neq 2, 3$, as this provides the E_2 -page of a spectral sequence converging to the cohomology $H^*(\text{Diff}_+ T^2; \mathbb{Z})$ (see equation (7)). The results in this paper extend the computations in [FTY88] for the cohomology of $SL_2(\mathbb{Z})$ to the primes $p = 2, 3$. Moreover we compute for all primes the higher order torsion of $H^*(SL_2(\mathbb{Z}); M)$, providing in Theorem 3.7 and Theorem 3.8 a description of the \mathbb{Z} -module and in Corollary 3.9 a description of the torsion of order different from 2 and 3 in $H^*(\text{Diff}_+ T^2; \mathbb{Z})$.

The groups $H^*(SL_2(\mathbb{Z}); M)$ and $H^*(B_3; M)$ have an interesting description in terms of a variation of the so called ‘‘divided polynomial algebra’’. Unexpectedly, we found a strong relation between the cohomology in this case and the cohomology of some spaces which were constructed in a completely different framework and which are basic to the growth of p -torsion in the homotopy groups of spheres.

An outline of the paper is as follows. In section 2, we give the general results and the main tools which are used here. Section 3 is devoted to giving all precise statements together with some additional details concerning a mod p variation of the divided polynomial algebra. In sections 4, 5, 6, 7 we develop all the technical tools for the computations. In particular, section 4 is devoted to the first cohomology of $SL_2(\mathbb{Z})$ and of B_3 , where we use some classical results on polynomial invariants under $SL_2(\mathbb{Z})$; in section 5 we compute polynomial invariants under the \mathbb{Z}_2 , \mathbb{Z}_4 , \mathbb{Z}_6 -subgroups of $SL_2(\mathbb{Z})$; section 6 is devoted to the p -torsion of the higher cohomology of $SL_2(\mathbb{Z})$; in section 7 we use the previous results and the associated spectral sequence for the full computation of the second cohomology of B_3 . Section 8 compares the cohomology determined here to the cohomology of some spaces which seem quite different. We are led to ask questions about possible

connections as well as questions about analogous results concerning congruence subgroups as well as extending these results to B_n for all n .

These structures are at the intersection of several basic topics during the special period on “Configuration Spaces”, which took place in Pisa. The authors started to work together to develop the problem considered here precisely during the special period on “Configuration Spaces” in Pisa.

The authors are grateful to the Ennio de Giorgi Mathematical Research Institute, the Scuola Normale Superiore as well as the University of Pisa for the opportunity to start this work together and the wonderful setting to do some mathematics ; they would like to thank the referee for many suggestions and for the great help in simplifying proofs.

The second author would like to thank many friends for an interesting, wonderful time in Pisa.

2. GENERAL RESULTS AND MAIN TOOLS

Let B_3 denote the braid group with 3 strands. We consider the geometric representation of B_3 into the group

$$\text{Aut}(H_1(M_{1,1}; \mathbb{Z}), \langle , \rangle) \cong SL_2(\mathbb{Z})$$

of the automorphisms which preserve the intersection form where $M_{1,1}$ is a genus-1 oriented surface with 1 boundary component. Explicitly, if the standard presentation of B_3 is

$$B_3 = \langle \sigma_1, \sigma_2 : \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle,$$

then one sends σ_1, σ_2 into the Dehn twists around one parallel and one meridian of $M_{1,1}$ respectively. Taking a natural basis for the H_1 , the previous map is explicitly given by

$$\lambda : B_3 \rightarrow SL_2(\mathbb{Z}) : \quad \sigma_1 \rightarrow \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \quad \sigma_2 \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Of course, any representation of $SL_2(\mathbb{Z})$ will induce a representation of B_3 by composition with λ . We will ambiguously identify σ_i with $\lambda(\sigma_i)$ when the meaning is clear from the context.

It is well known that λ is surjective and the kernel is infinite cyclic generated by the element $c := (\sigma_1\sigma_2)^6$ ([Mil71, Th. 10.5]) so that we have a sequence of groups

$$(1) \quad 1 \longrightarrow \mathbb{Z} \xrightarrow{j} B_3 \xrightarrow{\lambda} SL_2(\mathbb{Z}) \longrightarrow 1.$$

Let V be a rank-two free \mathbb{Z} -module and x, y a basis of V . The action of $SL_2(\mathbb{Z})$ over V

$$\sigma_1 : \begin{cases} x \rightarrow x - y \\ y \rightarrow y \end{cases}, \quad \sigma_2 : \begin{cases} x \rightarrow x \\ y \rightarrow x + y \end{cases}$$

extends to the symmetric algebra $S[V]$, which we identify with the polynomial algebra $M := \mathbb{Z}[x, y]$. If we fix $\deg x = \deg y = 1$, the homogeneous component M_n of degree n is an irreducible invariant free submodule of rank $n + 1$ so we have a decomposition $M = \bigoplus_n M_n$.

In this paper we describe the cohomology groups

$$H^*(B_3; M) = \bigoplus_{n \geq 0} H^*(B_3; M_n)$$

and

$$H^*(SL_2(\mathbb{Z}); M) = \bigoplus_{n \geq 0} H^*(SL_2(\mathbb{Z}); M_n)$$

where the action of B_3 over M is induced by the above map λ .

The free part of the cohomology of $SL_2(\mathbb{Z})$ tensored with the real numbers is well-known to be isomorphic to the ring of modular forms based on the $SL_2(\mathbb{Z})$ -action on the upper 1/2 plane by fractional linear transformations:

Theorem 2.1 (The Eichler-Shimura isomorphism [Eic57, Shi59]). *There exists an \mathbb{R} -linear isomorphism*

$$\begin{aligned} H^1(SL_2(\mathbb{Z}); M_{2n} \otimes \mathbb{R}) &\cong \text{Mod}_{2n+2}^0 \oplus \mathbb{R} & n \geq 1 \\ H^i(SL_2(\mathbb{Z}); M_n \otimes \mathbb{R}) &= 0 & i > 1 \text{ or } i = 1, \text{ odd } n \end{aligned}$$

where Mod_n^0 is the vector space of cuspidal modular forms of weight n .

In our computations, a basic first step is to rediscover the Poincaré series for the cohomology of $SL_2(\mathbb{Z})$ with coefficients in M .

Essentially, we use two "general" tools for the computations. The first one is the spectral sequence associated to the long exact sequence in (1), with

$$(2) \quad E_2^{s,t} = H^s(SL_2(\mathbb{Z}); H^t(\mathbb{Z}, M)) \Rightarrow H^{s+t}(B_3; M)$$

Notice that the element c above acts trivially on M , so we have

$$H^i(\mathbb{Z}; M) = H^i(\mathbb{Z}; \mathbb{Z}) \otimes M = \begin{cases} M & \text{for } i = 0, 1 \\ 0 & \text{for } i > 1 \end{cases}$$

and the spectral sequence has a two-row E_2 -page

$$(3) \quad E_2^{s,t} = \begin{cases} 0 & \text{for } t > 1 \\ H^s(SL_2(\mathbb{Z}), M) & \text{for } t = 0, 1. \end{cases}$$

The second main tool which we use comes from the following well-known presentation of $SL_2(\mathbb{Z})$ as an amalgamated product of torsion groups. We set for brevity $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$.

Proposition 2.2. ([MKS66, Ser03]) *The group $SL_2(\mathbb{Z})$ is an amalgamated product*

$$SL_2(\mathbb{Z}) = \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$$

where

$$\mathbb{Z}_4 \quad \text{is generated by the 4-torsion element } w_4 := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\mathbb{Z}_6 \quad \text{is generated by the 6-torsion element } w_6 := \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\mathbb{Z}_2 \quad \text{is generated by the 2-torsion element } w_2 := \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}. \quad \square$$

So, for every $SL_2(\mathbb{Z})$ -module N , we can use the associated a Mayer-Vietoris sequence

$$(4) \quad \rightarrow H^i(\mathbb{Z}_4; N) \oplus H^i(\mathbb{Z}_6; N) \rightarrow H^i(\mathbb{Z}_2; N) \rightarrow H^{i+1}(SL_2(\mathbb{Z}); N) \rightarrow$$

We start to deduce some "general" result on the cohomology of $SL_2(\mathbb{Z})$. From (4) and very well-known properties of cohomology groups, it immediately follows:

Proposition 2.3. *Assume that multiplication by 2 and 3 is invertible in the module N (equivalent, $6N = N$). Then*

$$H^i(SL_2(\mathbb{Z}); N) = 0 \quad \text{for } i > 1. \quad \square$$

Corollary 2.4. *If $6N = N$ and N has no $SL_2(\mathbb{Z})$ invariants (i.e. $H^0(SL_2(\mathbb{Z}); N) = 0$) then*

$$H^1(B_3; N) = H^1(SL_2(\mathbb{Z}); N) = H^2(B_3; N).$$

Proof. The corollary follows immediately from the above spectral sequence which degenerates at E_2 . \square

Remark 2.5. The group B_3 has cohomological dimension 2. This follows since the trefoil knot complement is a $K(B_3, 1)$ -space (see [Mil71, p.84]). From a slightly different point of view it can be seen also from the construction given in [Sal94] and [DCS96], where a $K(B_n, 1)$ is proven to be homotopy equivalent to a $(n - 1)$ -dimensional CW-complex. So we only need to determine cohomology up to H^2 . It also follows that the differential

$$d_2^{s,1} : E_2^{s,1} \rightarrow E_2^{s+2,0}$$

is an isomorphism for all $s \geq 2$, and is surjective for $s = 1$.

For a given finitely generated \mathbb{Z} -module L and for a prime p define the p -torsion component of L as follows:

$$L_{(p)} = \{x \in L \mid \exists k \in \mathbb{N} \text{ such that } p^k x = 0\}.$$

Moreover write FL for the free part of L , that is FL defines the isomorphism class of a maximal free \mathbb{Z} -submodule of L .

Return to the module M defined before. The next result follows directly from the previous discussion (including Remark 2.5).

Corollary 2.6. (1) For all primes p we have

$$H^1(B_3; M)_{(p)} = H^1(SL_2(\mathbb{Z}); M)_{(p)}.$$

(2) For all primes $p > 3$,

$$H^1(B_3; M)_{(p)} = H^1(SL_2(\mathbb{Z}); M)_{(p)} = H^2(B_3; M)_{(p)}.$$

(3) For the free parts for $n > 0$,

$$FH^1(B_3; M_n) = FH^1(SL_2(\mathbb{Z}); M_n) = FH^2(B_3; M_n). \quad \square$$

Notice that statement (1) of Corollary 2.6 fails for the torsion-free summand: see Theorem 3.7 below for the precise statement including it.

3. MAIN RESULTS

In this section we state the main results, namely the complete description of the cohomology of the braid group B_3 and of $SL_2(\mathbb{Z})$ with coefficients in the module M . First, we need some definitions.

Let $\mathbb{Q}[x]$ be the ring of rational polynomials in one variable. We define the subring

$$\Gamma[x] := \Gamma_{\mathbb{Z}}[x] \subset \mathbb{Q}[x]$$

as the \mathbb{Z} -submodule generated by the elements $x_n := \frac{x^n}{n!}$ for $n \in \mathbb{N}$. It follows from the next formula that $\Gamma[x]$ is a subring with

$$(5) \quad x_i x_j = \binom{i+j}{i} x_{i+j}.$$

The algebra $\Gamma[x]$ is usually known as the *divided polynomial algebra* over \mathbb{Z} (see for example chapter 3C of [Hat02] or the Cartan seminars as an early reference [Car56]). In the same way, by using equation (5), one can define $\Gamma_R[x] := \Gamma[x] \otimes R$ over any ring R .

Let p be a prime number. Consider the p -adic valuation $v := v_p : \mathbb{N} \setminus \{0\} \rightarrow \mathbb{N}$ such that $p^{v(n)}$ is the maximum power of p dividing n . Define the ideal I_p of $\Gamma[x]$ as

$$I_p := (p^{v(i)+1} x_i, \quad i \geq 1)$$

and call the quotient

$$\Gamma_p[x] := \Gamma[x]/I_p$$

the *p -local divided polynomial algebra (mod p)*.

Proposition 3.1. *The ring $\Gamma_p[x]$ is naturally isomorphic to the quotient*

$$\mathbb{Z}[\xi_1, \xi_p, \xi_{p^2}, \xi_{p^3}, \dots]/J_p$$

where J_p is the ideal generated by the polynomials

$$p\xi_1, \quad \xi_{p^i}^p - p\xi_{p^{i+1}} \quad (i \geq 1).$$

The element ξ_{p^i} corresponds to some invertible multiple of the generator $x_{p^i} \in \Gamma_p[x]$, $i \geq 0$.

Proof. The ideal I_p is graded and there is a direct sum decomposition of \mathbb{Z} -modules with respect to the degree

$$\Gamma_p[x] = \bigoplus_{i \geq 0} [\mathbb{Z}/p^{v(i)+1}\mathbb{Z}] (x_i).$$

We fix in this proof $\deg x = 1$, so the i -th degree component of I_p is given by the ideal $p\tilde{I}_i \subset \mathbb{Z}$, where

$$\tilde{I}_i := \left(p^{v(1)} \binom{i}{1}, \dots, p^{v(i-1)} \binom{i}{i-1}, p^{v(i)} \right).$$

Since $v\left(p^{v(k)} \binom{i}{k}\right) = v\left(i \binom{i-1}{k-1}\right) \geq v(i)$ for any $k \leq i$, we have $\tilde{I}_i = (p^{v(i)})$.

Now we show that the x_{p^i} 's are generators of $\Gamma_p[x]$ as a ring. We will make use of the following well known (and easy) lemma.

Lemma 3.2. *Let $n = \sum_j n_j p^j$ be the p -adic expansions of n . Then*

$$v(n!) = \frac{n - \sum_j n_j}{p-1} = \sum_j n_j \frac{p^j - 1}{p-1}. \quad \square$$

Let $n := \sum_{j=0}^k n_j p^j$ be the p -adic expansion of n . It follows from (5) that

$$(6) \quad \prod_{j=0}^k (x_{p^j})^{n_j} = \frac{n!}{(p!)^{n_1} (p^2!)^{n_2} \dots (p^k!)^{n_k}} x_n.$$

From Lemma 3.2 the coefficient of x_n in the latter expression is not divisible by p . This clearly shows that the x_{p^i} 's are generators.

Again by (5) we have

$$(x_{p^i})^p = \frac{p^{i+1}!}{(p^i!)^p} x_{p^{i+1}}.$$

From Lemma 3.2 the coefficient of $x_{p^{i+1}}$ in the latter expression is divisible by p to the first power. Therefore, up to multiplying each x_{p^i} by a number which is a unit in $\mathbb{Z}_{p^{v(i)+1}}$, we get the relations

$$(x_{p^i})^p = p x_{p^{i+1}}, \quad i \geq 1.$$

Notice that these relations, together with $px_1 = 0$, imply by induction that $p^{i+1}x_{p^i} = 0$. Using equation (6), we obtain $p^{v(i)+1}x_{p^i} = 0$. Therefore one has a well defined surjection between $\tilde{\Gamma}_p[x] := \mathbb{Z}[\xi_1, \xi_p, \xi_{p^2}, \xi_{p^3}, \dots]/J_p$ and $\Gamma_p[x]$.

In $\tilde{\Gamma}_p[x]$ there exists a unique normal form, which derives from the unique p -adic expansion of a natural number n : each class is uniquely represented as a combination of monomials $\prod_{j=0}^k (\xi_{p^j})^{n_j}$. Such monomials map to (6), an invertible multiple of x_n , as we have seen before.

This concludes the proof of Proposition 3.1. □

Remark 3.3. Note that if $\Gamma[x]$ is graded with $\deg x = k$, then also $\mathbb{Z}[\xi_1, \xi_p, \xi_{p^2}, \dots]/J_p$ is graded, with $\deg \xi_{p^i} = kp^i$.

Proposition 3.4. *Let $\mathbb{Z}_{(p)}$ be the local ring obtained by inverting numbers prime to p and let $\Gamma_{\mathbb{Z}_{(p)}}[x]$ be the divided polynomial algebra over $\mathbb{Z}_{(p)}$. One has an isomorphism:*

$$\Gamma_p[x] \cong \Gamma_{\mathbb{Z}_{(p)}}[x]/(px).$$

Proof. The ideal (px) is graded (here we also set $\deg x = 1$) with degree i component given by the ideal

$$(pi) = (p^{v(i)+1}) \subset \mathbb{Z}_{(p)}.$$

This statement follows directly from the natural isomorphism

$$\mathbb{Z}_{(p)}/(p^k) \cong \mathbb{Z}_{p^k}. \quad \square$$

We can naturally define a divided polynomial ring in several variables as follows:

$$\Gamma[x, x', x'', \dots] := \Gamma[x] \otimes_{\mathbb{Z}} \Gamma[x'] \otimes_{\mathbb{Z}} \Gamma[x''] \otimes_{\mathbb{Z}} \dots$$

with the ring structure induced as subring of $\mathbb{Q}[x, x', x'', \dots]$. In a similar way we have

$$\Gamma_p[x, x', x'', \dots] := \Gamma_p[x] \otimes_{\mathbb{Z}} \Gamma_p[x'] \otimes_{\mathbb{Z}} \Gamma_p[x''] \otimes_{\mathbb{Z}} \dots$$

In the following we need only to consider the torsion part of Γ_p , which is that generated in degree greater than 0. So, we define the submodule

$$\Gamma_p^+[x, x', x'', \dots] := \Gamma_p[x, x', x'', \dots]_{\deg > 0}.$$

For every prime p and for $k \geq 1$ define polynomials

$$\mathcal{P}_{p^k} := xy(x^{p^k-1} - y^{p^k-1}).$$

Set

$$\mathcal{P}_p := \mathcal{P}_{p^1}, \quad \mathcal{Q}_p := \mathcal{P}_{p^2}/\mathcal{P}_p = \sum_{h=0}^{p-1} (x^{p-1})^{p-h} (y^{p-1})^h \in \mathbb{Z}[x, y].$$

Remark 3.5. There is a natural action of $SL_2(\mathbb{Z})$ on $(\mathbb{C}\mathbb{P}^\infty)^2$ which induces on the cohomology ring

$$H^*((\mathbb{C}\mathbb{P}^\infty)^2; \mathbb{Z}) = \mathbb{Z}[x, y]$$

the same action defined in section 2 (see [FTY88]). For coherence with this geometrical action, from now on we fix on $\mathbb{Z}[x, y]$ the grading $\deg x = \deg y = 2$. Then we have

$$\deg \mathcal{P}_p = 2(p+1), \quad \deg \mathcal{Q}_p = 2p(p-1).$$

Remark 3.6. Let Diff_+T^2 the group of all orientation preserving diffeomorphisms of the 2-dimensional torus T^2 and let Diff_0T^2 be the connected component of the identity. As showed in [EE69] the inclusion $T^2 \subset \text{Diff}_0T^2$ is an homotopy equivalence. We recall from [FTY88] the exact sequence of groups

$$1 \rightarrow \text{Diff}_0T^2 \rightarrow \text{Diff}_+T^2 \rightarrow SL_2(\mathbb{Z}) \rightarrow 1$$

that induces a Serre spectral sequence

$$(7) \quad E_2^{i,j} = H^i(SL_2(\mathbb{Z}); H^j((\mathbb{C}\mathbb{P}^\infty)^2; \mathbb{Z})) \Rightarrow H^{i+j}(\text{Diff}_+T^2; \mathbb{Z}).$$

The spectral sequence above collapses with coefficients $\mathbb{Z}[1/6]$. The total space Diff_+T^2 is homotopy equivalent to the Borel construction

$$E \times_{SL_2(\mathbb{Z})} (\mathbb{CP}^\infty)^2$$

where E is a contractible space with free $SL_2(\mathbb{Z})$ -action.

Our cohomological computations turn out to give all the p -torsion in $H^*(\text{BDiff}_+T^2; \mathbb{Z})$ for a prime $p > 3$. **The result is stated in Corollary 3.9.**

Let G be B_3 or a subgroup of $SL_2(\mathbb{Z})$. For any prime p , we define the Poincaré series

$$(8) \quad P_{G,p}^i(t) := \sum_{n=0}^{\infty} \dim_{\mathbb{F}_p}(H^i(G; M_n) \otimes \mathbb{F}_p) t^n$$

and for rational coefficients

$$(9) \quad P_{G,0}^i(t) := \sum_{n=0}^{\infty} \dim_{\mathbb{Q}} H^i(G; M_n \otimes \mathbb{Q}) t^n.$$

We obtain:

Theorem 3.7. *Let $M = \bigoplus_{n \geq 0} M_n \simeq H^*((\mathbb{CP}^\infty)^2; \mathbb{Z})$ be the $SL_2(\mathbb{Z})$ -module defined in the previous section. Then*

- (1) $H^0(B_3; M) = H^0(SL_2(\mathbb{Z}), M) = \mathbb{Z}$ concentrated in degree $n = 0$;
- (2) $H^1(B_3; M_n)_{(p)} = H^1(SL_2(\mathbb{Z}); M_n)_{(p)} = \Gamma_p^+[\mathcal{P}_p, \mathcal{Q}_p]_{\text{deg}=n}$;
- (3) $H^1(SL_2(\mathbb{Z}); M_0) = 0$; $H^1(B_3; M_0) = \mathbb{Z}$;
- (4) $FH^1(B_3; M_n) = FH^1(SL_2(\mathbb{Z}); M_n) = \mathbb{Z}^{f_n}$ for $n > 0$

where the rank f_n is given by the Poincaré series

$$(10) \quad P_{B_3,0}^1(t) = \sum_{n=0}^{\infty} f_n t^n = \frac{t^4(1 + t^4 - t^{12} + t^{16})}{(1 - t^8)(1 - t^{12})}$$

and we fix gradings $\text{deg } \mathcal{P}_p = 2(p+1)$ and $\text{deg } \mathcal{Q}_p = 2p(p-1)$.

The proof of part 1 of Theorem 3.7 is given in Proposition 4.4; part 2 is proved in Theorem 4.5; the proof of part 3 and 4 follows from the results in section 5; in particular the Poincaré series $P_{B_3,0}^1(t)$ is computed in Remark 5.12.

Theorem 3.8. *For $n > 1$ the cohomology $H^n(SL_2(\mathbb{Z}), M)$ is 2-periodic. The free part is zero and there is no p -torsion for $p \neq 2, 3$. There is no 2^i -torsion for $i > 2$ and no 3^i -torsion for $i > 1$.*

There is a rank-1 module of 4-torsion in $H^{2n}(SL_2(\mathbb{Z}), M_{8m})$, all the others modules have no 4-torsion.

Finally the 2 and 3-torsion components are determined by the following Poincaré polynomials:

$$(11) \quad P_{SL_2(\mathbb{Z}),2}^2(t) = \frac{1 - t^4 + 2t^6 - t^8 + t^{12}}{(1 - t^2)(1 + t^6)(1 - t^8)}$$

$$(12) \quad P_{SL_2(\mathbb{Z}),2}^3(t) = \frac{t^4(2 - t^2 + t^4 + t^6 - t^8)}{(1 - t^2)(1 + t^6)(1 - t^8)}$$

$$(13) \quad P_{SL_2(\mathbb{Z}),3}^2(t) = \frac{1}{1 - t^{12}}$$

$$(14) \quad P_{SL_2(\mathbb{Z}),3}^3(t) = \frac{t^8}{1 - t^{12}}.$$

The proof of Theorem 3.8 follows from the results in section 6; equations (11) and (12) follow from Proposition 6.5; equations (13) and (14) follow from Lemma 6.1 and the computations of section 5.

Since the spectral sequence (7) collapses with coefficients $\mathbb{Z}[1/6]$, as a corollary of Theorem 3.7 and Theorem 3.8 we recover the following result (see [FTY88] for the computation of the cohomology with coefficients in \mathbb{Q} and \mathbb{Z}_p):

Corollary 3.9. *Let $p > 3$ be a prime.*

$$H^0(\text{Diff}_+ T^2; \mathbb{Z}) = H^1(\text{Diff}_+ T^2; \mathbb{Z}) = \mathbb{Z}$$

and for $i > 1$

$$H^i(\text{Diff}_+ T^2; \mathbb{Z})_{(p)} = \Gamma_p^+[\mathcal{P}_p, \mathcal{Q}_p]_{\text{deg}=i-1}$$

and

$$H^i(\text{Diff}_+ T^2; \mathbb{Q}) = \mathbb{Q}^{f_{i-1}}. \quad \square$$

We recall (Corollary 2.4) that for any $SL_2(\mathbb{Z})$ -module N the free part and part of p -torsion in $H^1(SL_2(\mathbb{Z}); N)$ and $H^2(B_3; N)$ are the same, with $p \neq 2$ or 3 . We obtain the following theorems.

Theorem 3.10. *The following equalities hold:*

- (1) $H^2(B_3; M)_{(p)} = H^1(B_3; M)_{(p)} = \Gamma_p^+[\mathcal{P}_p, \mathcal{Q}_p]$ for $p \neq 2, 3$;
- (2) $FH^2(B_3; M_n) = FH^1(B_3; M_n) = \mathbb{Z}^{f_n}$

where the ranks f_n have been defined in Theorem 3.7.

The proof of the isomorphisms of Theorem 3.10 follows from the description of the spectral sequence given in section 7; the last isomorphism of part 1 follows from Theorem 4.5.

For the 2 and 3-torsion components, we find that the second cohomology group of B_3 differs from the first cohomology group as follows. We consider the \mathbb{Z} -module structure, and notice that $\Gamma_p^+[\mathcal{Q}_p]$ and $\mathcal{P}_p \cdot \Gamma_p^+[\mathcal{Q}_p]$ are direct summand submodules of $\Gamma_p^+[\mathcal{P}_p, \mathcal{Q}_p]$, for all p . In the second cohomology group we have modifications of these submodules.

Theorem 3.11. *For the 2 and 3-torsion components of the module $H^2(B_3; M)$ one has the following expressions:*

$$\text{a) } H^2(B_3; M)_{(2)} = (\Gamma_2^+[\mathcal{P}_2, \mathcal{Q}_2] \oplus \mathbb{Z}[\overline{\mathcal{Q}}_2^2]) / \sim$$

Here \overline{Q}_2 is a new variable of the same degree ($= 4$) as Q_2 ; the quotient module is defined by the relations $\frac{Q_2^n}{n!} \sim 2\overline{Q}_2^n$ for n even and $\frac{Q_2^n}{n!} \sim 0$ for n odd.

$$\text{b) } H^2(B_3; M)_{(3)} = (\Gamma_3^+[\mathcal{P}_3, Q_3] \oplus \mathbb{Z}[\overline{Q}_3]) / \sim$$

Here \overline{Q}_3 is a new variable of the same degree ($= 12$) as Q_3 ; the quotient module is defined by the relations $\frac{Q_3^n}{n!} \sim 3\overline{Q}_3^n$ and $\mathcal{P}_3 \frac{Q_3^n}{n!} \sim 0$.

The proof of Theorem 3.11 follows from the results of section 7.

Remark 3.12. For $p = 2$ one has to kill all the submodules generated by elements of the form $\frac{Q_2^n}{n!}$ for odd n . All these submodules are isomorphic to $\mathbb{Z}/2$. For n even the submodule generated by $\frac{Q_2^n}{n!}$, that is isomorphic to $\mathbb{Z}/2^{m+1}$ where m is the greatest power of 2 that divides n , must be replaced by a submodule isomorphic to $\mathbb{Z}/2^{m+2}$.

For $p = 3$ one has to kill all the submodules generated by elements of the form $\mathcal{P}_3 \frac{Q_3^n}{n!}$, that are isomorphic to $\mathbb{Z}/3$. Moreover for all n the submodule generated by $\frac{Q_3^n}{n!}$, that is isomorphic to $\mathbb{Z}/3^{m+1}$ where m is the greatest power of 3 that divides n , must be replaced by a submodule isomorphic to $\mathbb{Z}/3^{m+2}$.

4. POLYNOMIAL INVARIANTS FOR $SL_2(\mathbb{Z})$

In this section we prove the part of the main Theorem 3 which concerns $H^1(B_3; M_n) = H^1(SL_2(\mathbb{Z}); M_n)$ for $n > 0$.

A classical result ([Dic11]) characterizes the polynomials in $\mathbb{F}_p[y_1, \dots, y_n]$ which are invariant under the action of $GL_n(\mathbb{F}_p)$ (or of $SL_n(\mathbb{F}_p)$). We state only the case $n = 2$, which is what we need here.

Theorem 4.1. ([Dic11],[Ste87]) *For p a prime number, the algebra of $SL_2(\mathbb{F}_p)$ -invariants in $\mathbb{F}_p[x, y]$ is the polynomial algebra*

$$\mathbb{F}_p[\mathcal{P}_p, Q_p],$$

where \mathcal{P}_p, Q_p are the polynomials defined in section 3.

Of course, the algebra $\mathbb{F}_p[\mathcal{P}_p, Q_p]$ is graded. We keep here $\deg x = \deg y = 2$, so the degree is that induced by $\deg \mathcal{P}_p = 2(p+1)$, and $\deg Q_p = 2p(p-1)$.

Theorem 4.2. ([Ste87, Th. J]) *For p a prime number, a polynomial $P \in \mathbb{Z}_{p^r}[x, y]$ is invariant for $SL_2(\mathbb{Z}_{p^r})$ if and only if*

$$P = \sum_{i=0}^{r-1} p^i F_i$$

where F_i is a polynomial in the p^{r-i-1} -th powers of the \mathcal{P}_p, Q_p .

For our purposes, we need a slight variation of the statement of this theorem.

Corollary 4.3. *A polynomial $P \in \mathbb{Z}_{p^r}[x, y]$, homogeneous of degree n , is invariant under the action of $SL_2(\mathbb{Z})$ on $\mathbb{Z}_{p^r}[x, y]$ induced by the action on M_n , if and only if*

$$P = \sum_{i=0}^{r-1} p^i F_i$$

where F_i is a polynomial in the p^{r-i-1} -th powers of the $\mathcal{P}_p, \mathcal{Q}_p$.

Proof. The corollary follows directly from Theorem 4.2 and from the surjectivity of the natural projection $SL_2(\mathbb{Z}) \rightarrow SL_2(\mathbb{Z}_{p^r})$. In order to prove the surjectivity consider a matrix

$$\bar{A} = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \in SL_2(\mathbb{Z}/p^r\mathbb{Z}).$$

We suppose $\bar{c} = 0$. There is a matrix A with integer coefficients such that A maps to \bar{A} and we have

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $ad - bc = 1 + kp^r$. Clearly d is not divisible by p since \bar{d} is invertible. We can choose $c = p^r$ and hence $(c, d) = 1$. Then we choose new values $a' = a + xp^r, b' = b + yp^r$ and the determinant of the matrix

$$A' = \begin{pmatrix} a' & b' \\ c & d \end{pmatrix}$$

is $1 + kp^r + p^r(xd + yp^r)$. Hence we can find values of x, y such that $\det A' = 1$. The cases when any of the other entries is zero is analogous and the surjectivity follows since triangular matrixes with a zero entry generate the group $SL_2(\mathbb{Z}_{p^r})$. \square

We also need

Proposition 4.4. *If $n > 0$, there are no invariants in M_n under the action of $SL_2(\mathbb{Z})$.*

Proof. One can prove this result directly. However, it easily derives from Theorem 4.1. In fact, let $P \in \mathbb{Z}[x, y]$ be an invariant, homogeneous of degree d . Then P must be invariant modulo all primes p , which is impossible by Theorem 4.1: the polynomial P should be 0 modulo p for every $p > d$. \square

Corollary 4.3 and Proposition 4.4 give us, by the Universal Coefficients Theorem, a complete description of

$$H^0(SL_2(\mathbb{Z}); M \otimes \mathbb{Z}_{p^r}) = \mathbb{Z}_{p^r} \oplus \text{Tor}(H^1(SL_2(\mathbb{Z}); M); \mathbb{Z}_{p^r})$$

(and similar for B_3). The result can be stated in the following way.

Theorem 4.5. *For all prime numbers p and all $n > 0$, the p -torsion component*

$$H^1(SL_2(\mathbb{Z}); M_n)_{(p)} \cong H^1(B_3; M_n)_{(p)}$$

is described as follows: each monomial

$$\mathcal{P}_p^k \mathcal{Q}_p^h, \quad 2k(p+1) + 2hp(p-1) = n$$

generates a $\mathbb{Z}_{p^{m+1}}$ -summand, where

$$p^m \mid \gcd(k, h), \quad p^{m+1} \nmid \gcd(k, h).$$

□

Recall that since we fix $\deg x = \deg y = 2$ we have that the module M is concentrated even degree.

The condition stated in the previous theorem over the number m is equivalent to saying that m is the minimum power of p which is present in the p -adic expansion of k and h .

It is easy to check the equivalence between Theorem 4.5 and the description of the cohomology given in section 3 by using divided powers.

5. COHOMOLOGY OF $\mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_6$

Recall from section 2, Proposition 2.2, that $w_2 = (s_1 s_2 s_1)^2 \in SL_2(\mathbb{Z})$ generates a group isomorphic to \mathbb{Z}_2 , acting on the module \mathbb{Z}^2 as multiplication by the matrix $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

Proposition 5.1. *For even n we have:*

$$H^i(\mathbb{Z}_2; M_{2n}) = \begin{cases} M_{2n} & \text{if } i = 0 \\ 0 & \text{if } i \text{ is odd} \\ M_{2n} \otimes \mathbb{Z}_2 & \text{if } i > 0 \text{ is even} \end{cases}$$

For odd n we have:

$$H^i(\mathbb{Z}_2; M_{2n}) = \begin{cases} 0 & \text{if } i \text{ is even} \\ M_{2n} \otimes \mathbb{Z}_2 & \text{if } i \text{ is odd} \end{cases}$$

For rational coefficients we have:

$$H^i(\mathbb{Z}_2; M_{2n} \otimes \mathbb{Q}) = \begin{cases} M_{2n} \otimes \mathbb{Q} & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We use the standard resolution for a cyclic group \mathbb{Z}_m (see [Bro94]):

$$(15) \quad N \xrightarrow{T-1} N \xrightarrow{T^{m-1}+\dots+1} N \xrightarrow{T-1} \dots$$

where N is any \mathbb{Z}_m -module with action given by a T such that $T^m = 1$.

In the present case the sequence becomes:

$$(16) \quad M_n \xrightarrow{T-1} M_n \xrightarrow{T+1} M_n \xrightarrow{T-1} \dots$$

Notice that T acts as Id in degree $2n$, n even, while it acts as $-Id$ in degree $2n$ with n odd. Then everything follows trivially from sequence (16). □

The following proposition describes the ring structure of the \mathbb{Z}_2 -invariants. It directly follows from Proposition 5.1.

Proposition 5.2. *The module of invariants $H^0(\mathbb{Z}_2; M)$ is isomorphic to the polynomial ring*

$$\mathbb{Z}[a_2, b_2, c_2]/(a_2b_2 = c_2^2)$$

under the correspondence $a_2 = x^2$, $b_2 = y^2$, $c_2 = xy$.

The module $M \otimes \mathbb{Z}_2$ is \mathbb{Z}_2 -invariant. □

Proposition 5.3. *The Poincaré series for $H^i(\mathbb{Z}_2, M)$ are given by*

$$P_{\mathbb{Z}_2, p}^{2i}(t) = \frac{(1+t^4)}{(1-t^4)^2} \text{ for any prime } p \text{ and } i = 0 \text{ or } p = 2 \text{ and } i \geq 0;$$

$$P_{\mathbb{Z}_2, 2}^{2i+1}(t) = \frac{2t^2}{(1-t^4)^2}.$$

Proof. By Proposition (5.1) all we have to consider are the series of the dimensions of the polynomials of degree $2n$ with n even and odd respectively. These are exactly the series written here. □

Recall from Proposition 2.2 that the element $w_4 = s_1s_2s_1 \in SL_2(\mathbb{Z})$ generates a group isomorphic to \mathbb{Z}_4 and the action on M is given by $\begin{cases} x \mapsto -y \\ y \mapsto x \end{cases}$.

The following proposition is quite trivial to verify.

Proposition 5.4. *The group M , as a \mathbb{Z}_4 module, is the direct sum of modules of the following kinds:*

- a) I_1 isomorphic to a submodule linearly generated by the monomials $x^{2i}y^{2j}$, $x^{2j}y^{2i}$ for fixed $i \neq j$;
- b) I_2 isomorphic to a submodule linearly generated by the monomial $x^{2i}y^{2i}$ for fixed i ;
- c) I_3 isomorphic to a submodule linearly generated by the monomial $x^{2i+1}y^{2i+1}$ for fixed i ;
- d) I_4 isomorphic to a submodule linearly generated by the monomials $x^{2i+1}y^{2j+1}$, $x^{2j+1}y^{2i+1}$ for fixed $i \neq j$;
- e) I_5 isomorphic to a submodule linearly generated by the monomials $x^{2i+1}y^{2j}$, $x^{2j}y^{2i+1}$ for fixed i, j . □

Proposition 5.5. *The cohomology $H^i(\mathbb{Z}_4; I_j)$ is 2-periodic for $i \geq 1$ and is given by the following table:*

	I_1	I_2	I_3	I_4	I_5
H^0	\mathbb{Z}	\mathbb{Z}	0	\mathbb{Z}	0
H^1	0	0	\mathbb{Z}_2	0	\mathbb{Z}_2
H^2	\mathbb{Z}_2	\mathbb{Z}_4	0	\mathbb{Z}_2	0

Referring to the notation of 5.4 for the generators of the modules, we give the following representatives for the generators of the cohomology:

	I_1	I_2	I_3	I_4	I_5
H^0	$x^{2i}y^{2j} + x^{2j}y^{2i}$	$(xy)^{2i}$	0	$x^{2i+1}y^{2j+1} - x^{2j+1}y^{2i+1}$	0
H^1	0	0	$(xy)^{2i+1}$	0	$x^{2i+1}y^{2j}$
H^2	$x^{2i}y^{2j} + x^{2j}y^{2i}$	$(xy)^{2i}$	0	$x^{2i+1}y^{2j+1} - x^{2j+1}y^{2i+1}$	0

Proof. It derives easily from the above standard 2-periodic resolution for cyclic groups (15), applied to the modules I_j 's. \square

Proposition 5.6. *The module of invariants $H^0(\mathbb{Z}_4, M)$ is isomorphic to the polynomial ring*

$$(17) \quad \mathbb{Z}[d_2, e_4, f_4]/(f_4^2 = (d_2^2 - 4e_4)e_4)$$

under the correspondence $d_2 = x^2 + y^2$, $e_4 = x^2y^2$, $f_4 = x^3y - xy^3$.

The module of mod 2-invariants $H^0(\mathbb{Z}_4, M \otimes \mathbb{Z}_2)$ is isomorphic to the polynomial ring

$$(18) \quad \mathbb{Z}[\sigma_1, \sigma_2]$$

under the correspondence $\sigma_1 = x + y$, $\sigma_2 = xy$.

Proof. In the mod 2-case, notice that \mathbb{Z}_4 acts exchanging the variables x , y , so the invariants are all the symmetric polynomials.

In the integral case, first notice that the given polynomials are clearly invariants for \mathbb{Z}_4 .

Second, each class in the ring (17) has a standard representative of the form

$$(19) \quad P(d_2, e_4) + Q(d_2, e_4)f_4.$$

A polynomial of this form is zero iff P and Q both vanish, because the variables appear at an odd power in f_4 . But notice that d_2 , e_4 are the standard generators for the symmetric polynomials in x^2 , y^2 , so they are algebraically independent. This means that P and Q are the zero polynomials. Therefore all the relations among d_2 , e_4 , f_4 derive from the give one and the form (19) is unique.

Finally, we have to show that the given polynomials generate all the invariants. It is sufficient to see that the polynomials given in the first row of the generators-table in Proposition (5.5) are generated.

The polynomial appearing in the second column is a power of e_4 . The one in the first column, after dividing by the maximum possible power of e_4 , becomes a polynomial of the form

$$x^{2i} + y^{2i}.$$

This is a symmetric polynomial in x^2 , y^2 so it is generated by d_2 , e_4 .

The polynomial appearing in the fourth column, after dividing by the maximum possible power of e_4 , becomes of the form (up to sign)

$$x^{2i+1}y - xy^{2i+1}, \quad i > 0.$$

This latter is divisible by f_4 , with quotient a symmetric polynomial in x^2 , y^2 . \square

Proposition 5.7. *With respect to the gradation of M the Poincaré series for $H^i(\mathbb{Z}_4, M)$ are given by*

$$\begin{aligned} P_{\mathbb{Z}_4, p}^{2i}(t) &= \frac{1 + t^8}{(1 - t^4)(1 - t^8)} \text{ for any prime } p \text{ and } i = 0 \text{ or } p = 2 \text{ and } i \geq 0; \\ P_{\mathbb{Z}_4, 2}^{2i+1}(t) &= \frac{t^2 + t^4 + t^6 - t^8}{(1 - t^4)(1 - t^8)}. \end{aligned}$$

Proof. By looking at the generators-table in Proposition 5.5, we see that in odd dimension one has terms

$$\frac{t^4}{(1-t^8)}, \quad \frac{t^2}{(1-t^4)^2}$$

coming from the generators for I_3, I_5 respectively. Such terms add up to the expression for $P_{\mathbb{Z}_4,2}^{2i+1}(t)$ given in the statement.

Similarly, in even dimension we have three terms

$$\frac{t^4}{(1-t^8)(1-t^4)}, \quad \frac{1}{1-t^8}, \quad \frac{t^8}{(1-t^8)(1-t^4)}$$

coming from the generators for I_1, I_2, I_4 respectively, which add up to the given expression for $P_{\mathbb{Z}_4,p}^{2i}(t)$. \square

The cohomology of \mathbb{Z}_4 (so also $H^*(SL_2(\mathbb{Z}); M)$) contains 4-torsion, as it follows from Proposition 5.5. We define in general the polynomial

$$(20) \quad P_{G,4}^i(t) := \sum_{n=0}^{\infty} \dim_{\mathbb{F}_2}(2H^i(G; M_n))t^n.$$

We have immediately from Proposition 5.5

Proposition 5.8.

$$P_{\mathbb{Z}_4,4}^2(t) = \frac{1}{1-t^8}. \quad \square$$

Recall (Proposition 2.2) that the element $w_6 = s_1s_2 \in SL_2(\mathbb{Z})$ generates a group isomorphic to \mathbb{Z}_6 , where the action over M is $T : \begin{cases} x \mapsto -y \\ y \mapsto x+y \end{cases}$.

Proposition 5.9. *The module of invariants $H^0(\mathbb{Z}_6, M)$ is isomorphic to the polynomial ring*

$$(21) \quad \mathbb{Z}[p_2, q_6, r_6]/(r_6^2 = q_6(p_2^3 - 13q_6 - 5r_6))$$

via the correspondence $p_2 = x^2 + xy + y^2$, $q_6 = x^2y^2(x+y)^2$, $r_6 = x^5y - 5x^3y^3 - 5x^2y^4 - xy^5$.

The module of mod 2-invariants $H^0(\mathbb{Z}_6, M \otimes \mathbb{Z}_2)$ is isomorphic to the polynomial ring

$$(22) \quad \mathbb{Z}_2[s_2, t_3, u_3]/(u_3^2 = s_2^3 + t_3^2 + t_3u_3)$$

with the correspondence $s_2 = x^2 + xy + y^2$, $t_3 = xy(x+y)$, $u_3 = x^3 + x^2y + y^3$.

The module of mod 3-invariants $H^0(\mathbb{Z}_6, M \otimes \mathbb{Z}_3)$ is isomorphic to the polynomial ring

$$(23) \quad \mathbb{Z}_3[v_2, w_4, z_6]/(w_4^2 = v_2z_6)$$

with the correspondence $v_2 = (x-y)^2$, $w_4 = xy(x+y)(x-y)$, $z_6 = x^2y^2(x+y)^2$.

Proof. It is an easy check that p_2 , q_6 and $r_6 = xy(x+y)(x^3 - x^2y - 4xy^2 - y^3)$ are invariant. Moreover it is easy to check the relation $r_6^2 = q_6(p_2^3 - 13q_6 - 5r_6)$ stated in the theorem.

Modulo this relation, each polynomial can be reduced to one of the form

$$(24) \quad P(p_2, q_6) + Q(p_2, q_6)r_6.$$

Notice that p_2 and q_6 are symmetric polynomials, while the last factor of r_6 is not. This implies that a polynomial of type (24) vanishes iff P and Q both vanish.

An homogeneous relation between p_2 and q_6 would be of the form

$$\begin{aligned} 0 &= \sum_{k \geq 0} a_k p_2^{n-3k} q_6^k = \\ &= \sum_{k \geq 0} a_k (x^{2n-2k} y^{2k} + \text{terms containing } x \text{ in lower degree}). \end{aligned}$$

Since each term in the sum contains x to a maximum power which is decreasing with k , all the a_k 's must vanish. Therefore, all relations follow from the given one and the form (24) is unique.

We now have to show that the given polynomials generate. First notice that the ring in (21) is a free \mathbb{Z} -module. The Poincaré series reproducing its ranks is given, by the normal form (24), by

$$(25) \quad \frac{1 + t^{12}}{(1 - t^4)(1 - t^{12})}.$$

Let ω be a primitive 6th-root of the unity. Let us extend the action of \mathbb{Z}_6 over the extension

$$\mathbb{Q}(\omega)[x, y] \supset \mathbb{Z}[x, y]$$

where $\mathbb{Q}(\omega)$ is the cyclotomic field generated by ω . One can diagonalize the action as follows: one sees immediately that the two polynomials

$$p(x, y) := \bar{\omega}x + y, \quad \bar{p}(x, y) = \omega x + y$$

are eigenvectors with eigenvalues $\omega, \bar{\omega}$ respectively. In degree $2n$, one has eigenvectors

$$p^a \bar{p}^b, \quad a + b = n.$$

with eigenvalue $\omega^a \bar{\omega}^b$. It follows that the invariants are generated by all the monomials $p^a \bar{p}^b$ such that $a \equiv b \pmod{6}$. It is quite easy to check that the Poincaré series of the dimensions of the vector space of the invariants is given by the same series (25). Since such dimensions do not depend on the extension, it follows that the given polynomials p_2, q_6, r_6 generate the invariants over the rational numbers. So, in each degree, they generate over \mathbb{Z} a lattice of the same rank as that of all the invariants, and we have to exclude that they generate a sublattice.

An homogeneous integral invariant of degree $4n$ is a rational combination (keeping into account (24)) of the form

$$\begin{aligned} t(x, y) &= \sum_{a+3b=n} \lambda_{a,b} p_2^a q_6^b + \sum_{a+3b=n-3} \mu_{a,b} p_2^a q_6^b r_6 = \\ &= \sum_{a+3b=n} \lambda_{a,b} (x^{2a+4b} y^{2b} + \text{terms containing } x \text{ in lower degree}) + \\ &+ \sum_{a+3b=n-3} \mu_{a,b} (x^{2a+4b+5} y^{2b+1} + \text{terms containing } x \text{ in lower degree}) \end{aligned}$$

with $\lambda_{a,b}, \mu_{a,b} \in \mathbb{Q}$. Since the maximum powers of x which are contained into each summand are pairwise different, and they appear with coefficient 1, all the λ 's and μ 's must be integer. This concludes the first part of the theorem on the integral case.

The proof in case of \mathbb{Z}_2 -coefficients is similar. First one verifies that the given polynomials s_2, t_3, u_3 are invariants and the given relation holds. Each class in the ring (22) has a unique representative of the kind

$$P(s_2, t_3) + Q(s_2, t_3)u_3.$$

In fact, since s_2, t_3 are symmetric while u_3 is not, an argument as in the integral case proves that all the relations among the given polynomials derive from that in (22).

Next, we consider the action over the extension

$$\mathbb{F}_4[x, y] \supset \mathbb{Z}_2[x, y]$$

where we identify the field F_4 with $\mathbb{Z}_2[x]/(x^2 + x + 1)$. Let $a := [x] \in \mathbb{F}_4$. There are eigenvectors

$$p(x, y) := (a + 1)x + y, \quad \bar{p}(x, y) := ax + y$$

with eigenvalues $a, a + 1$ respectively. In degree $2n$, there are eigenvectors $p^k \bar{p}^h$, $k + h = n$, with eigenvalues $a^k (a + 1)^h$. Similar to the integral case, we deduce that the invariants correspond to the case

$$k \equiv h \pmod{3}.$$

So, the Poincaré series of the dimensions of the invariants, gradated by the degree, is

$$(26) \quad \frac{1 + t^6}{(1 - t^4)(1 - t^6)}$$

which coincides with the Poincaré series of the ring (22).

In case of \mathbb{Z}_3 -coefficients, again one verifies that the given polynomials are invariant and the given relation holds. As in the previous cases, one sees that each class in the ring (23) has a unique representative of the kind

$$(27) \quad P(v_2, z_6) + Q(v_2, z_6)w_4.$$

Notice that, as in the integral case, there are no invariants in degree $2n$, n odd, since $T^3 = -Id$. On the contrary, the operator does not diagonalize in this case. In degree 2, with respect to $p = x - y, q = x$ the operator T is given by the matrix $\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$. In degree $4n$, with respect to the basis $p^{2n-k}q^k, 0 \leq k \leq 2n$, T is given by an upper triangular matrix with the only eigenvalue 1. Notice that in degree $4n$ we have $T^3 = Id$, so

$$(28) \quad (T - Id)^3 = T^3 - Id = 0.$$

Therefore, the Jordan blocks of T have dimension at most 3.

We need the following lemma. Denote for brevity by $M'_d := M_d \otimes \mathbb{Z}_3$.

Lemma 5.10. *Let P be an invariant polynomial of degree d (so $P \in \ker(T - Id) : M'_d \rightarrow M'_d$). Assume that $P \in \text{Im}(T - Id)^{r-1}$, $P \notin \text{Im}(T - Id)^r$. Then, for every invariant polynomial Q of degree l the product $QP \in M'_{l+d}$ is an invariant of degree $l + d$ such that $QP \in \text{Im}(T - Id)^{r-1}$.*

Proof. Let R be a polynomial such that $(T - Id)^{r-1}(R) = P$. Then, since Q is invariant we have

$$(T - Id)^{r-1}(QR) = Q(T - Id)^{r-1}(R) = QP. \quad \square$$

In degree 4, we have

$$(T - Id)^2(-x^2) = v_2$$

so here there is a single Jordan block of dimension 3 for $T - Id$. It follows from Lemma 5.10 and from equation (28) that each polynomial of the form v_2u , u invariant, satisfies:

$$v_2u \in \text{Im}(T - Id)^2, \quad v_2u \notin \text{Im}(T - Id)^3.$$

In degree 8, we have

$$(T - Id)(-x^3y - x^2y^2) = w_4, \quad w_4 \notin \text{Im}(T - Id)^2,$$

so v_2^2 , w_4 belong to a Jordan base where v_2 is contained into a 3-block and w_4 is contained into a 2-block. It also derives from Lemma 5.10 that an invariant of the form w_4u , u invariant, satisfies $w_4u \in \text{Im}(T - Id)$.

In degree 12, we have $z_6 \notin \text{Im}(T - Id)$, so we have two 3-dimensional blocks containing v_2^3 and v_2w_4 respectively, and a 1-dimensional block containing z_6 .

The Poincaré series for the ring (23) is given by

$$(29) \quad \frac{1 + t^8}{(1 - t^4)(1 - t^{12})}.$$

In M_n all monomials in the v_2 , w_4 , z_6 (reduced in normal form, with the exponent of w_4 not bigger than 1) are linearly independent. They can be taken as part of a Jordan basis for T , where all monomials containing v_2 belong to a Jordan block of dimension exactly 3. All monomials containing w_4 and not v_2 belong to a Jordan block of dimension ≥ 2 , and monomials containing only z_6 belong to Jordan blocks of dimension ≥ 1 . Let μ_n be the sum of the dimensions of the Jordan blocks which contain all monomials of degree n in the v_2, w_4, z_6 , where the exponent of w_4 is ≤ 1 . Let $\mu(t) := \sum_n \mu_n t^n$ be the associated series. By the above arguments we have that the series μ is bigger (term by term) than

$$3 \left[\frac{1 + t^8}{(1 - t^4)(1 - t^{12})} - \frac{1 + t^8}{(1 - t^{12})} \right] + 2 \frac{t^8}{(1 - t^{12})} + \frac{1}{(1 - t^{12})} = \frac{1 + t^4}{(1 - t^4)^2}$$

but this last series is exactly

$$\frac{d}{d\tau} \frac{\tau}{1 - \tau^2} = \sum_{k \geq 0} (2k + 1) \tau^{2k} \quad \text{with } \tau = t^2$$

which coincides with the series of the dimensions of even-degree polynomials. Therefore the two series coincide, which means that v_2, w_4, z_6 generate the invariants.

This concludes the proof of the proposition. □

Proposition 5.11. *With respect to the gradation of M the Poincaré series for $H^i(\mathbb{Z}_6, M)$ are given by*

$$\begin{aligned} P_{\mathbb{Z}_6,p}^{2i}(t) &= \frac{1+t^{12}}{(1-t^4)(1-t^{12})} \quad \text{for } p=2 \text{ or } i=0, \text{ any } p \\ P_{\mathbb{Z}_6,3}^{2i}(t) &= \frac{1}{1-t^{12}} \quad \text{for } i>0 \\ P_{\mathbb{Z}_6,2}^{2i+1}(t) &= \frac{2t^6}{(1-t^4)(1-t^{12})} \\ P_{\mathbb{Z}_6,3}^{2i+1}(t) &= \frac{t^8}{1-t^{12}} \end{aligned}$$

Proof. The first formula for $p=0$ is just the Poincaré series of (21) (see (25)). In general we use the resolution (15) and the Universal Coefficient Theorem in the form

$$(30) \quad \tilde{P}_{\mathbb{Z}_6,p}^i(t) = P_{\mathbb{Z}_6,p}^i(t) + P_{\mathbb{Z}_6,p}^{i+1}(t)$$

where we set

$$\tilde{P}_{\mathbb{Z}_6,p}^i(t) = \sum_{n \geq 0} \dim_{\mathbb{Z}_p} H^i(\mathbb{Z}_6; M_n \otimes \mathbb{Z}_p) t^n.$$

Let $T' := T^5 + \dots + T + 1$.

In characteristic 2 one has $T^3 = Id$ and $T' = 0$ in any degree. It follows

$$H^0(\mathbb{Z}_6; M \otimes \mathbb{Z}_2) = H^{2i}(\mathbb{Z}_6; M \otimes \mathbb{Z}_2) = H^{2i-1}(\mathbb{Z}_6; M \otimes \mathbb{Z}_2), \quad i > 0.$$

Therefore $\tilde{P}_{\mathbb{Z}_6,2}^0 = \tilde{P}_{\mathbb{Z}_6,2}^1$ and by (30) $P^0 = P^{2i}$, $i > 0$, which gives the wanted expression for $P_{\mathbb{Z}_6,2}^{2i}$.

By (30) and by the knowledge of the Poincaré series of the H^0 's, we recursively find, using (25):

$$P_{\mathbb{Z}_6,2}^1(t) = \frac{1+t^6}{(1-t^4)(1-t^6)} - \frac{1+t^{12}}{(1-t^4)(1-t^{12})}$$

which gives the written expression for $P_{\mathbb{Z}_6,2}^{2i+1}$, concluding the case of characteristic 2.

In a similar way we have in characteristic 3 (using equation (29)):

$$\begin{aligned} P_{\mathbb{Z}_6,3}^1(t) &= \frac{1+t^8}{(1-t^4)(1-t^{12})} - \frac{1+t^{12}}{(1-t^4)(1-t^{12})} = \\ &= \frac{t^8}{(1-t^{12})} = P_{\mathbb{Z}_6,3}^{2i+1}(t), \quad i > 0 \end{aligned}$$

which gives the above formula.

We now calculate $\tilde{P}_{\mathbb{Z}_6,3}^{2i+1}$. It derives from the discussion given in the proof of Proposition 5.9 that T decompose into Jordan blocks of dimension ≤ 3 , where the 3-blocks correspond to the monomials containing a v_2 generator. One sees immediately that

$$\text{rank}(T') = \text{number of 3-blocks in } T$$

and therefore (by (29))

$$\tilde{P}_{\mathbb{Z}_6,3}^{2i}(t) = \tilde{P}_{\mathbb{Z}_6,3}^{2i-1}(t) = \frac{1+t^8}{1-t^{12}}, \quad i > 0.$$

Therefore, by (30)

$$P_{\mathbb{Z}_6,3}^{2i}(t) = \frac{1+t^8}{1-t^{12}} - \frac{t^8}{1-t^{12}} = \frac{1}{1-t^{12}}$$

which is the given formula. \square

Remark 5.12. As a corollary of the given formulas we obtain part 3 of Theorem 3.7. In fact, by the Mayer-Vietoris sequence (4) and by Propositions 5.7, 5.11, 5.3 we have

$$\begin{aligned} P_{SL_2(\mathbb{Z}),0}^1(t) &= 1 - (P_{\mathbb{Z}_4,0}^0(t) + P_{\mathbb{Z}_6,0}^0(t)) + P_{\mathbb{Z}_2,0}^0(t) = \\ &= 1 - \left(\frac{1+t^8}{(1-t^4)(1-t^8)} + \frac{1+t^{12}}{(1-t^4)(1-t^{12})} \right) + \frac{1+t^4}{(1-t^4)^2} \end{aligned}$$

which reduces to the expression given in Theorem 3.7. \square

6. COHOMOLOGY OF $SL_2(\mathbb{Z}) \pmod p$

We use the results of the previous sections to compute the modules

$$H^i(SL_2(\mathbb{Z}); M \otimes \mathbb{Z}_p)$$

and the module of p -torsion $H^i(SL_2(\mathbb{Z}); M)_{(p)}$ for any prime p .

In dimension 0 we have that $H^0(SL_2(\mathbb{Z}); M)$ is isomorphic to \mathbb{Z} concentrated in degree 0. The module

$$H^0(SL_2(\mathbb{Z}); M \otimes \mathbb{Z}_p)$$

has already been described in section 4 as well as $H^1(SL_2(\mathbb{Z}); M)_{(p)}$ (Theorem 4.5).

We can understand the p -torsion in $H^1(SL_2(\mathbb{Z}); M)$ by means of the results cited in section 4 for Dickson invariants. In fact the Universal Coefficient Theorem gives us the following isomorphism:

$$H^1(SL_2(\mathbb{Z}); M)_{(p)} = \Gamma_p^+[\mathcal{P}_p, \mathcal{Q}_p].$$

We want to remark that the previous equation gives an isomorphism only at the level of \mathbb{Z} -modules. The Poincaré series for the previous module is:

$$P_{SL_2(\mathbb{Z}),p}^1(t) = \frac{t^{2(p+1)} + t^{2p(p-1)} - t^{2(p^2+1)}}{(1-t^{2(p+1)})(1-t^{2p(p-1)})}.$$

Moreover from the decomposition

$$SL_2(\mathbb{Z}) = \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$$

we have the associated Mayer-Vietoris cohomology long exact sequence given in equation (4).

Since $H^i(\mathbb{Z}_4; M)_{(p)} = H^i(\mathbb{Z}_2; M)_{(p)} = 0$ for a prime $p \neq 2$ and $i \geq 1$ we have

Lemma 6.1. *For $i \geq 2$, $H^i(SL_2(\mathbb{Z}); M \otimes \mathbb{Z}_3) = H^i(\mathbb{Z}_6; M \otimes \mathbb{Z}_3)$ and we have the isomorphism*

$$H^i(SL_2(\mathbb{Z}); M)_{(3)} = H^i(\mathbb{Z}_6; M)_{(3)} \quad i \geq 2.$$

For any prime $p \neq 2, 3$ and for $i \geq 2$ we have $H^i(SL_2(\mathbb{Z}); M \otimes \mathbb{Z}_p) = 0$ and $H^i(SL_2(\mathbb{Z}); M)_{(p)} = 0$. \square

For $p = 2$ we need to compute in detail the maps in the Mayer-Vietoris long exact sequence. As seen in section 5, the element $x^{2i}y^{2i}$ generates a module $\mathbb{Z}_4 \subset H^{2j}(\mathbb{Z}_4; M_{8i})$. Moreover the following relation holds in $H^{2j}(\mathbb{Z}_6; M_{8i})$ for the powers of the polynomial $x^2 + xy + y^2 = p_2$:

$$(31) \quad p_2^{2i} = (2k + 1)x^{2i}y^{2i} + \sum_{a \neq 2i} c_a(x^a y^{4i-a} + (-1)^a x^{4i-a} y^a)$$

for some integer coefficients k and c_a for $a = 0, \dots, 4i$. Let us call ρ_{2i} the right term of the equation. The summands in the summatory of ρ_{2i} generate \mathbb{Z}_2 submodules of $H^{2j}(\mathbb{Z}_4; M_{8i})$, hence ρ_{2i} represents a class in $H^{2j}(\mathbb{Z}_4; M_{8i})$ that generates a submodule isomorphic to \mathbb{Z}_4 . It follows that \mathbb{Z}_4 generated by (ρ_{2i}, p_2^{2i}) is in the kernel of the map

$$H^{2j}(\mathbb{Z}_4; M_{8i}) \oplus H^{2j}(\mathbb{Z}_6; M_{8i}) \rightarrow H^{2j}(\mathbb{Z}_2; M_{8i}).$$

These are the only \mathbb{Z}_4 subgroups of $H^j(SL_2(\mathbb{Z}); M_i)$. In fact we have:

Lemma 6.2. *Let $j \geq 2$. For j odd or $i \not\equiv 0 \pmod{4}$ the group $H^j(SL_2(\mathbb{Z}); M_{2i})$ has no 4-torsion while for j even the group $H^j(SL_2(\mathbb{Z}); M_{8i})$ has a subgroup isomorphic to \mathbb{Z}_4 that is uniquely determined modulo 2-torsion elements.*

Proof. From Propositions 5.1, 5.4, 5.5 and 5.11 we have that the module $H^j(\mathbb{Z}_2; M_{2i})$ is trivial for i odd and j even and for i even and j odd, while $H^{j+1}(\mathbb{Z}_4; M_{2i}) \oplus H^{j+1}(\mathbb{Z}_6; M_{2i})$ is trivial for i odd and j odd and for i even and j even. Hence from the Mayer-Vietoris long exact sequence we can see that for $j \geq 2$ the \mathbb{Z}_4 submodules previously described are the only possible \mathbb{Z}_4 submodules of $H^j(SL_2(\mathbb{Z}); M_i)_{(2)}$. \square

To determine completely the 2-torsion cohomology $H^j(SL_2(\mathbb{Z}); M)_{(2)}$ we need to compute the rank of the kernel and of the image of the maps

$$H^1(\mathbb{Z}_4; M \otimes \mathbb{Z}_2) \oplus H^1(\mathbb{Z}_6; M \otimes \mathbb{Z}_2) \xrightarrow{p_1} H^1(\mathbb{Z}_2; M \otimes \mathbb{Z}_2).$$

and

$$H^2(\mathbb{Z}_4; M \otimes \mathbb{Z}_2) \oplus H^2(\mathbb{Z}_6; M \otimes \mathbb{Z}_2) \xrightarrow{p_2} H^2(\mathbb{Z}_2; M \otimes \mathbb{Z}_2).$$

We recall that we have the following commuting diagrams for $k > 0$:

$$\begin{array}{ccc} H^k(\mathbb{Z}_4; M \otimes \mathbb{Z}_2) \oplus H^k(\mathbb{Z}_6; M \otimes \mathbb{Z}_2) & \xrightarrow{p_k} & H^k(\mathbb{Z}_2; M \otimes \mathbb{Z}_2) \\ \parallel & & \parallel \\ H^{2j+k}(\mathbb{Z}_4; M \otimes \mathbb{Z}_2) \oplus H^{2j+k}(\mathbb{Z}_6; M \otimes \mathbb{Z}_2) & \xrightarrow{p_{2j+k}} & H^{2j+k}(\mathbb{Z}_2; M \otimes \mathbb{Z}_2). \end{array}$$

Hence the informations about $\ker p_1$ and Imp_1 , together with the previous remarks will allow us to determine completely the 2-torsion cohomology $H^j(SL_2(\mathbb{Z}); M)_{(2)}$ by means of the Universal Coefficient Theorem.

It is convenient to start with the case p_2 . We have that

$$H^2(\mathbb{Z}_6; M \otimes \mathbb{Z}_2) = \mathbb{Z}_2[s_2, t_3, u_3]/(u_3^2 = s_2^3 + t_3^2 + t_3 u_3)$$

where $s_2 = x^2 + xy + y^2$, $t_3 = xy(x + y)$ and $u_3 = x^3 + x^2y + y^3$ (see equation (22));

$$H^2(\mathbb{Z}_4; M \otimes \mathbb{Z}_2) = \mathbb{Z}_2[\sigma_1, \sigma_2]$$

where $\sigma_1 = x + y$ and $\sigma_2 = xy$ (see equation (18));

$$H^2(\mathbb{Z}_2; M \otimes \mathbb{Z}_2) = \mathbb{Z}_2[x, y]$$

(see Proposition 5.2).

The map p_2 is the natural inclusion of submodules of M and hence the image is the submodule $\mathbb{Z}_2[\sigma_1, \sigma_2] \oplus u_3\mathbb{Z}_2[\sigma_1, \sigma_2]$. The kernel is isomorphic to the intersection of $\mathbb{Z}_2[s_2, t_3, u_3]$ and $\mathbb{Z}_2[\sigma_1, \sigma_2]$. Hence $\ker p_2 = \mathbb{Z}_2[s_2, t_3]$. This proves:

Lemma 6.3. *The Poincaré series for $\ker p_2$ is*

$$\frac{1}{(1-t^4)(1-t^6)}$$

while the Poincaré series for Imp_2 is

$$\frac{1+t^6}{(1-t^4)^2}.$$

□

In the case p_1 we have

$$H^1(\mathbb{Z}_6; M \otimes \mathbb{Z}_2) = \mathbb{Z}_2[s_2, t_3, u_3]/(u_3^2 = s_2^3 + t_3^2 + t_3u_3)$$

where $s_2 = x^2 + xy + y^2$, $t_3 = xy(x + y)$ and $u_3 = x^3 + x^2y + y^3$,

$$H^1(\mathbb{Z}_4; M \otimes \mathbb{Z}_2) = \mathbb{Z}_2[x, xy]$$

and

$$H^1(\mathbb{Z}_2; M \otimes \mathbb{Z}_2) = \mathbb{Z}_2[x, y].$$

The equalities above can be computed using the standard 2-periodic resolution for \mathbb{Z}_q . Since $\sum_{k=0}^{q-1} T_q^k = 0$ on $M \otimes \mathbb{Z}_2$ we have that the cohomology $H^i(\mathbb{Z}_q; M \otimes \mathbb{Z}_2)$ is 1-periodic in degree greater than 0. In particular

$$H^1(\mathbb{Z}_q; M \otimes \mathbb{Z}_2) = H^2(\mathbb{Z}_q; M \otimes \mathbb{Z}_2) \quad q = 2, 4, 6.$$

The generators that we provide are genuine representatives with respect to the standard periodic resolution of \mathbb{Z}_q . This is trivial in the case of \mathbb{Z}_2 , where the boundary maps of the complex are the null maps for our system of coefficients. For \mathbb{Z}_4 it is clear that we are providing a set of representatives for the quotient

$$\mathbb{Z}_2[x, y]/(T_4 - \text{Id})\mathbb{Z}_2[x, y]$$

where T_4 acts on $\mathbb{Z}_2[x, y]$ by

$$T_4 : \begin{cases} x \mapsto y \\ y \mapsto x \end{cases}.$$

For \mathbb{Z}_6 we show that $H^1(\mathbb{Z}_6; M \otimes \mathbb{Z}_2)$ and $H^2(\mathbb{Z}_6; M \otimes \mathbb{Z}_2)$ have the same generators. In fact, let

$$T_6 : \begin{cases} x \mapsto y \\ y \mapsto x + y \end{cases}$$

be the homomorphism corresponding to the action of a generator of \mathbb{Z}_6 on $\mathbb{Z}_2[x, y]$, it is clear that the T_6 -invariant elements in $\mathbb{Z}_2[x, y]$ are in the kernel of the map $d^1 = \text{Id} + T_6 + \cdots + T_6^5 = (1 + T_6 + T_6^2)(1 + T_6^3)$. Moreover if $\psi \in \mathbb{Z}_2[x, y]$ is T_6 -invariant and is in the image of $T_6 - \text{Id}$ we have that

$$3\psi = (\text{Id} + T_6 + T_6^2)\psi = (\text{Id} + T_6 + T_6^2)(T_6 - \text{Id})\psi' = (T_6^3 - \text{Id})\psi' = 0$$

and so the intersection $\text{Im}\{T_6 - \text{Id} : \mathbb{Z}_2[x, y] \rightarrow \mathbb{Z}_2[x, y]\} \cap \mathbb{Z}_2[x, y]^{T_6}$ is trivial. An argument of dimension implies that $H^1(\mathbb{Z}_6; M \otimes \mathbb{Z}_2) = \mathbb{Z}_2[s_2, t_3, u_3]/(u_3^2 = s_2^3 + t_3^2 + t_3u_3)$.

In order to compute explicitly the map p_1 we need to compute the lifting of the inclusion homomorphisms $\mathbb{Z}_2 \hookrightarrow \mathbb{Z}_4$ and $\mathbb{Z}_2 \hookrightarrow \mathbb{Z}_6$ to the corresponding resolutions. Let $T_2 = T_4^2 = T_6^3$ the automorphism of $\mathbb{Z}_2[x, y]$ given by the generator of \mathbb{Z}_2 , then have the following commuting diagrams for \mathbb{Z}_4 :

$$\begin{array}{ccc} \mathbb{Z}_2[x, y] & \xlongequal{\quad} & \mathbb{Z}_2[x, y] \\ \downarrow T_4 - \text{Id} & & \downarrow T_2 - \text{Id} \\ \mathbb{Z}_2[x, y] & \xrightarrow{T_4 + \text{Id}} & \mathbb{Z}_2[x, y] \end{array}$$

and for \mathbb{Z}_6 :

$$\begin{array}{ccc} \mathbb{Z}_2[x, y] & \xlongequal{\quad} & \mathbb{Z}_2[x, y] \\ \downarrow T_6 - \text{Id} & & \downarrow T_2 - \text{Id} \\ \mathbb{Z}_2[x, y] & \xrightarrow{T_6^2 + T_6 + \text{Id}} & \mathbb{Z}_2[x, y]. \end{array}$$

The map $i_4 = T_4 + \text{Id} : \mathbb{Z}_2[x, y]^{T_4}$ acts as follows:

$$x^a y^b \mapsto x^a y^b + x^b y^a$$

and the image is $\text{Im}i_4 = \sigma_1 \mathbb{Z}_2[\sigma_1, \sigma_2]$.

The map $i_6 = T_6^2 + T_6 + \text{Id}$ is the natural inclusion, as the map T_6 acts as the identity on the invariant elements and we are working modulo 2. The intersection $\text{Im}i_4 \cap \text{Im}i_6$ is the submodule

$$\sigma_1 \sigma_2 \mathbb{Z}_2[\sigma_1^2 + \sigma_2, \sigma_1 \sigma_2] \oplus \mathbb{Z}_2[\sigma_2].$$

Lemma 6.4. *The Poincaré series for the kernel of p_1 is*

$$\frac{1}{1-t^4} + \frac{t^6}{(1-t^4)(1-t^6)} = \frac{1}{(1-t^4)(1-t^6)}. \quad \square$$

The computation of the Poincaré series $P_{SL_2(\mathbb{Z}), 2}^j(t)$ is now straightforward. We resume our result in the next proposition. We write $T_{SL_2(\mathbb{Z}), p}^j(t)$ for the Poincaré series for $H^j(SL_2(\mathbb{Z}); M_i)_{(p)} \otimes \mathbb{Z}_p$.

Proposition 6.5. *Let us assume $j > 0$. The module $H^j(SL_2(\mathbb{Z}); M_i)_{(2)}$ is given as follows:*

- a) *for j even and $i \equiv 0 \pmod{8}$ $H^j(SL_2(\mathbb{Z}); M_i)_{(2)}$ is the direct sum of one module isomorphic to \mathbb{Z}_4 and some modules isomorphic to \mathbb{Z}_2 ; the generator of the module \mathbb{Z}_4 (up to elements of 2-torsion) maps to the couple*

$$(\rho_{2i}, p_2^{2i}) \in H^{i+1}(\mathbb{Z}_4; M_j) \oplus H^{i+1}(\mathbb{Z}_6; M_j)$$

in the Mayer-Vietoris long exact sequence;

b) for all others i, j $H^j(SL_2(\mathbb{Z}); M_i)_{(2)}$ is the direct sum of a certain number of copies of \mathbb{Z}_2 .

The series $T_{SL_2(\mathbb{Z}),2}^j(t)$ are given by:

$$\begin{aligned} T_{SL_2(\mathbb{Z}),2}^1(t) &= \frac{t^4 + t^6 - t^{10}}{(1-t^4)(1-t^6)} \\ T_{SL_2(\mathbb{Z}),2}^{2i}(t) &= \frac{1-t^4+2t^6-t^8+t^{12}}{(1-t^2)(1+t^6)(1-t^8)} & i > 0 \\ T_{SL_2(\mathbb{Z}),2}^{2i+1}(t) &= \frac{t^4(2-t^2+t^4+t^6-t^8)}{(1-t^2)(1+t^6)(1-t^8)} & i > 0. \end{aligned} \quad \square$$

The computations follow from the results of section 5. The modules $H^i(SL_2(\mathbb{Z}); M \otimes \mathbb{Z}_p)$ can be completely determined from the previous observations using the Universal Coefficient Theorem. In particular we have that the Poincaré series for $H^i(SL_2(\mathbb{Z}); M \otimes \mathbb{Z}_p)$ with respect to the grading of M is given by the formula

$$P_{SL_2(\mathbb{Z}),p}^i(t) = \sum_{j=0}^{\infty} \dim_{\mathbb{F}_p} H^i(SL_2(\mathbb{Z}); M \otimes \mathbb{Z}_p) t^j = T_{SL_2(\mathbb{Z}),p}^i(t) + T_{SL_2(\mathbb{Z}),p}^{i+1}(t) + T_{SL_2(\mathbb{Z}),0}^i(t).$$

7. COHOMOLOGY OF B_3

In order to compute the cohomology of B_3 we use the Serre spectral sequence already described in section 2 for the extension

$$0 \rightarrow \mathbb{Z} \rightarrow B_3 \rightarrow SL_2(\mathbb{Z}) \rightarrow 1.$$

We recall (equation (3)) that the E_2 -page is represented in the following diagram:

$$\begin{array}{ccccccc} H^0(SL_2(\mathbb{Z}), M) & & H^1(SL_2(\mathbb{Z}), M) & & H^2(SL_2(\mathbb{Z}), M) & & \cdots \\ & & \searrow^{d_2} & & \searrow^{d_2} & & \\ H^0(SL_2(\mathbb{Z}), M) & & H^1(SL_2(\mathbb{Z}), M) & & H^2(SL_2(\mathbb{Z}), M) & & \cdots \end{array}$$

where all the other rows are zero. Recall that $H^i(SL_2(\mathbb{Z}), M)$ is 2-periodic and with no free part for $i \geq 2$.

The free part of $H^*(B_3; M)$ can easily be determined since the $E_2^{s,t}$ is torsion for $s > 1$, hence the spectral sequence for $H^*(B_3; M \otimes \mathbb{Q})$ collapses at E_2 and we have:

- (1) $H^0(B_3; M) = H^0(SL_2(\mathbb{Z}), M) = \mathbb{Z}$ concentrated in degree $n = 0$;
- (2) $H^1(B_3; M_0) = H^0(SL_2(\mathbb{Z}), M_0) = \mathbb{Z}$;
- (3) $FH^1(B_3; M_n) = FH^1(SL_2(\mathbb{Z}); M_n) = \mathbb{Z}^{f_n}$ for $n > 0$;
- (4) $FH^2(B_3; M_n) = FH^1(SL_2(\mathbb{Z}); M_n) = \mathbb{Z}^{f_n}$.

This proves what we stated in Theorem 3.7 and 3.10 for the free part.

Since the cohomology group $H^i(SL_2(\mathbb{Z}), M)$ has only 2 and 3-torsion for $i \geq 2$, the same argument implies that for the p -torsion with $p > 3$ the spectral sequence collapses at E_2 . Hence for $p > 3$ we have the isomorphism

$$H^1(B_3; M)_{(p)} = H^2(B_3; M)_{(p)} = H^1(SL_2(\mathbb{Z}); M)_{(p)} = \Gamma_p^+[\mathcal{P}_p, \mathcal{Q}_p]$$

where $\deg \mathcal{P}_p = 2(p+1)$ and $\deg \mathcal{Q}_p = 2p(p-1)$ as in section 3.

Moreover, since the first column of the spectral sequence is concentrated in degree 0, we have that

$$H^1(B_3; M) = H^1(SL_2(\mathbb{Z}); M) \oplus \mathbb{Z}$$

and the contribute of the summand \mathbb{Z} is concentrated in degree 0.

For a complete description of the cohomology of $H^2(B_3; M)$ we have to study the behaviour of the spectral sequence for the primes $p = 2$ and $p = 3$.

We consider the subgroups \mathbb{Z}_2 , \mathbb{Z}_4 and \mathbb{Z}_6 of $SL_2(\mathbb{Z})$ from the presentation given in Proposition 2.2 and the induced extensions. One has diagrams

$$(32) \quad \begin{array}{ccc} \mathbb{Z} \xlongequal{\quad} \mathbb{Z} & \mathbb{Z} \xlongequal{\quad} \mathbb{Z} & \mathbb{Z} \xlongequal{\quad} \mathbb{Z} \\ \downarrow 2 & \downarrow 4 & \downarrow 6 \\ \mathbb{Z} \hookrightarrow B_3 & \mathbb{Z} \hookrightarrow B_3 & \mathbb{Z} \hookrightarrow B_3 \\ \downarrow & \downarrow & \downarrow \\ \mathbb{Z}_2 \hookrightarrow SL_2(\mathbb{Z}) & \mathbb{Z}_4 \hookrightarrow SL_2(\mathbb{Z}) & \mathbb{Z}_6 \hookrightarrow SL_2(\mathbb{Z}) \end{array}$$

and the corresponding maps of spectral sequences give the commuting diagram

$$\begin{array}{ccc} H^i(SL_2(\mathbb{Z}); M) & \xrightarrow{d_2} & H^{i+2}(SL_2(\mathbb{Z}); M) \\ \downarrow \Phi_m^i & & \downarrow \Phi_m^{i+2} \\ H^i(\mathbb{Z}_m; M) & \xrightarrow{d_2} & H^{i+2}(\mathbb{Z}_m; M) \end{array}$$

for $m = 2, 4, 6$. We recall that the spectral sequences for $\mathbb{Z} \xrightarrow{m} \mathbb{Z} \rightarrow \mathbb{Z}_m$ collapses at E_2 and the differential

$$H^i(\mathbb{Z}_m; M) \xrightarrow{d_2} H^{i+2}(\mathbb{Z}_m; M)$$

is surjective for $i = 0$ and isomorphisms for $i > 0$.

For odd n the d_2 differential in the spectral sequence for the cohomology of B_3 has a very simple behaviour:

Lemma 7.1. *For odd n the differential:*

$$d_2 : H^{2i+1}(SL_2(\mathbb{Z}); M_{2n}) \rightarrow H^{2i+3}(SL_2(\mathbb{Z}); M_{2n})$$

is an isomorphism.

Proof. For odd n the groups $H^{2i}(\mathbb{Z}_2; M_{2n})$ are trivial and we have the following commuting diagram:

$$\begin{array}{ccccccc} 0 & \rightarrow & H^{2i+1}(SL_2(\mathbb{Z}); M_{2n}) & \rightarrow & H^{2i+1}(\mathbb{Z}_4; M_{2n}) \oplus H^{2i+1}(\mathbb{Z}_6; M_{2n}) & \rightarrow & H^{2i+1}(\mathbb{Z}_2; M_{2n}) \\ & & \downarrow d_2 & & \downarrow d_2 & & \downarrow d_2 \\ 0 & \rightarrow & H^{2i+3}(SL_2(\mathbb{Z}); M_{2n}) & \rightarrow & H^{2i+3}(\mathbb{Z}_4; M_{2n}) \oplus H^{2i+3}(\mathbb{Z}_6; M_{2n}) & \rightarrow & H^{2i+3}(\mathbb{Z}_2; M_{2n}) \end{array}$$

The statement follows from the Five Lemma. \square

For a detailed computation of the differential d_2 for the degree $4n$ part of the spectral sequence we have to consider the representations of $SL_2(\mathbb{Z})$ and B_3 over the module $M_n \otimes R$. Here R can

be either $\mathbb{Z}_{(p)}$, the ring of integers localized at the prime p , or the cyclic group \mathbb{Z}_{p^i} . We need to consider only $p = 2$ or $p = 3$.

7.1. 2-torsion. Let us first focus on the 2-torsion. We need to compare the Serre spectral sequence for $\mathbb{Z} \xrightarrow{2} \mathbb{Z} \rightarrow \mathbb{Z}_2$ and the spectral sequence for B_3 through the connecting homomorphism of the Mayer-Vietoris sequence.

Since in the spectral sequences associated to the central extension $\mathbb{Z} \rightarrow B_3 \rightarrow SL_2(\mathbb{Z})$ and $\mathbb{Z} \xrightarrow{2} \mathbb{Z} \rightarrow \mathbb{Z}_2$ the differential d_2 is just the cup product by the Euler class, we have that for any $SL_2(\mathbb{Z})$ -module N the following diagram associated to the decomposition $SL_2(\mathbb{Z}) = \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$ is commutative:

$$(33) \quad \begin{array}{ccccccc} \cdots & \rightarrow & H^{i-1}(\mathbb{Z}_2; N) & \rightarrow & H^i(SL_2(\mathbb{Z}); N) & \rightarrow & H^i(\mathbb{Z}_4; N) \oplus H^i(\mathbb{Z}_6; N) \rightarrow \cdots \\ & & \downarrow d_2 & & \downarrow d_2 & & \downarrow d_2 \\ \cdots & \rightarrow & H^{i+1}(\mathbb{Z}_2; N) & \rightarrow & H^{i+2}(SL_2(\mathbb{Z}); N) & \rightarrow & H^{i+2}(\mathbb{Z}_4; N) \oplus H^{i+2}(\mathbb{Z}_6; N) \rightarrow \cdots \end{array}$$

Recall from section 5 that for $n \equiv 0 \pmod{4}$ we have

$$H^0(\mathbb{Z}_2; M_n) = M_n, H^1(\mathbb{Z}_2; M_n) = 0 \text{ and } H^2(\mathbb{Z}_2; M_n) = M_n \otimes \mathbb{Z}_2.$$

Let now $N = M_n \otimes \mathbb{Z}_{(2)}$. As $H^i(\mathbb{Z}_6; N)$ is trivial for i odd (see Proposition 5.11) the diagram (33) specializes to:

$$(34) \quad \begin{array}{ccccccc} 0 & \longrightarrow & N^{SL_2(\mathbb{Z})} & \longrightarrow & N^{\mathbb{Z}_4} \oplus N^{\mathbb{Z}_6} & \longrightarrow & \\ & & \downarrow d_2 & & \downarrow d_2 & & \\ 0 & \rightarrow & H^2(SL_2(\mathbb{Z}); N) & \rightarrow & H^2(\mathbb{Z}_4; N) \oplus H^2(\mathbb{Z}_6; N) & \rightarrow & \\ & & \downarrow d_2 & & \downarrow d_2 & & \downarrow d_2 \\ & \rightarrow & H^0(\mathbb{Z}_2; N) & \rightarrow & H^1(SL_2(\mathbb{Z}); N) & \rightarrow & H^1(\mathbb{Z}_4; N) \rightarrow 0 \\ & & \downarrow d_2 & & \downarrow d_2 & & \downarrow d_2 \\ & \rightarrow & H^2(\mathbb{Z}_2; N) & \rightarrow & H^3(SL_2(\mathbb{Z}); N) & \rightarrow & H^3(\mathbb{Z}_4; N) \rightarrow 0 \end{array}$$

Lemma 7.2. For even n and $N = M_{2n} \otimes \mathbb{Z}_{(2)}$ we have the following equalities:

$$(35) \quad \begin{aligned} & \ker(d_2 : H^1(SL_2(\mathbb{Z}); N) \rightarrow H^3(SL_2(\mathbb{Z}); N)) = \\ & = 2\text{Im}(H^0(\mathbb{Z}_2; N) \rightarrow H^1(SL_2(\mathbb{Z}); N)) = \\ & = 2\ker(H^1(SL_2(\mathbb{Z}); N) \rightarrow H^1(\mathbb{Z}_4; N)). \end{aligned}$$

Moreover $H^3(SL_2(\mathbb{Z}); N) = H^1(SL_2(\mathbb{Z}); N) \otimes \mathbb{Z}_2$ and the differential

$$d_2 : H^1(SL_2(\mathbb{Z}); N) \rightarrow H^3(SL_2(\mathbb{Z}); N)$$

is the projection map $H^1(SL_2(\mathbb{Z}); N) \rightarrow H^1(SL_2(\mathbb{Z}); N) \otimes \mathbb{Z}_2$.

Proof. For even n and odd i we have $H^i(\mathbb{Z}_4; M_{2n}) = 0$ if $n \equiv 0 \pmod{4}$ and $H^i(\mathbb{Z}_4; M_{2n}) = \mathbb{Z}_2$ if $n \equiv 2 \pmod{4}$. Moreover the differential

$$d_2 : H^1(\mathbb{Z}_4; M_{2n}) \rightarrow H^3(\mathbb{Z}_4; M_{2n})$$

is an isomorphism. The inclusion map $i_4 : \mathbb{Z}_4 \rightarrow \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6 = SL_2(\mathbb{Z})$ has a right inverse $\nu_4 : SL_2(\mathbb{Z}) \rightarrow \mathbb{Z}_4$ defined on \mathbb{Z}_4 as the identity and on \mathbb{Z}_6 as the multiplication by 2. Hence the exact sequences of diagram (34) split, respectively in $H^1(SL_2(\mathbb{Z}); N)$ and $H^3(SL_2(\mathbb{Z}); N)$. Finally it is easy to verify that the differential

$$d_2 : H^0(\mathbb{Z}_2; M_{2n}) \rightarrow H^2(\mathbb{Z}_2; M_{2n})$$

is the quotient mod-2 map $M_{2n} \rightarrow M_{2n} \otimes \mathbb{Z}_2$ and we recall that the differential

$$d_2 : N^{\mathbb{Z}_4} \oplus N^{\mathbb{Z}_6} \rightarrow H^2(\mathbb{Z}_4; N) \oplus H^2(\mathbb{Z}_6; N)$$

is also surjective. The lemma follows from diagram (34). \square

We write here β_{p^r} as the r^{th} -order Bockstein operator associated to the sequence of coefficients

$$0 \rightarrow \mathbb{Z}_{p^r} \xrightarrow{\times p^r} \mathbb{Z}_{p^{2r}} \rightarrow \mathbb{Z}_{p^r} \rightarrow 0.$$

We can decompose $H^1(SL_2(\mathbb{Z}); M_n \otimes \mathbb{Z}_{(2)})$, for $n \equiv 0 \pmod{4}$, as a direct sum of a free $\mathbb{Z}_{(2)}$ -module of rank f_n (see Theorem 3.7) and some modules of the form \mathbb{Z}_{2^k} . We can choose a decomposition

$$(36) \quad H^1(SL_2(\mathbb{Z}); M_n \otimes \mathbb{Z}_{(2)}) = \mathbb{Z}_{(2)}^{f_n} \oplus \bigoplus_k \mathbb{Z}_{2^k}^{d_k}$$

and an analogous decomposition for $H^1(\mathbb{Z}_4; M_n \otimes \mathbb{Z}_{(2)})$ such that the map

$$H^1(SL_2(\mathbb{Z}); M_n \otimes \mathbb{Z}_{(2)}) \rightarrow H^1(\mathbb{Z}_4; M_n \otimes \mathbb{Z}_{(2)})$$

is diagonal with respect to the two decompositions. According to Lemma 7.2 the decomposition of $H^1(SL_2(\mathbb{Z}); M_n \otimes \mathbb{Z}_{(2)})$ induces a decomposition of $H^3(SL_2(\mathbb{Z}); M_n \otimes \mathbb{Z}_{(2)}) = H^1(SL_2(\mathbb{Z}); M_n \otimes \mathbb{Z}_{(2)}) \otimes \mathbb{Z}_2$.

From the previous lemmas the differential d_2 is completely determined and hence also the term E_∞ , as the spectral sequence lives only on rows 0 and 1 and then collapses at E_3 .

Lemma 7.3. *In the Serre spectral sequence for $H^*(B_3; M_n \otimes \mathbb{Z}_{(2)})$ the E_∞ -term for $n = 0$ is given by:*

$\mathbb{Z}_{(2)}$	$2H^1(SL_2(\mathbb{Z}); M_0 \otimes \mathbb{Z}_{(2)})$	0
$\mathbb{Z}_{(2)}$	$H^1(SL_2(\mathbb{Z}); M_0 \otimes \mathbb{Z}_{(2)})$	$H^2(SL_2(\mathbb{Z}); M_0 \otimes \mathbb{Z}_{(2)})$

and for $n > 0$, $n \equiv 0 \pmod{4}$, by:

0	$2H^1(SL_2(\mathbb{Z}); M_n \otimes \mathbb{Z}_{(2)})$	0
0	$H^1(SL_2(\mathbb{Z}); M_n \otimes \mathbb{Z}_{(2)})$	$H^2(SL_2(\mathbb{Z}); M_n \otimes \mathbb{Z}_{(2)})$

\square

Now we consider the extension problem associated to the E_∞ -term, in order to determine the cohomology groups $H^*(B_3; M \otimes \mathbb{Z}_{(2)})$.

Theorem 7.4. *We consider the extension problem for the E_∞ -term of the spectral sequence of Lemma 7.3 above. We can choose a decomposition in the form of equation (36) such that for the induced decomposition of $2H^1(SL_2(\mathbb{Z}); M_n \otimes \mathbb{Z}_{(2)})$ we have the following cases:*

- a) *for $n \equiv 0 \pmod{8}$ there is a summand $\mathbb{Z}_{2^i} \subset 2H^1(SL_2(\mathbb{Z}); M_n \otimes \mathbb{Z}_{(2)})$ corresponding, via Universal Coefficient Theorem, to the generator $\Omega_2^{2^i h} \in H^0(SL_2(\mathbb{Z}); M \otimes \mathbb{Z}_2)$ where $n = 2^{i+1}h$ ($2 \nmid h$, $i > 0$); the summand has a lifting to the module $\mathbb{Z}_{2^{i+2}}$ through the extension with the module \mathbb{Z}_4 corresponding to the generator*

$$(x^2 y^2)^{2^{i-1}} = e_4^{2^{i-1}} \in H^2(\mathbb{Z}_4; M \otimes \mathbb{Z}_{(2)});$$

- b) *for all others torsion summands in $2H^1(SL_2(\mathbb{Z}); M_n \otimes \mathbb{Z}_{(2)})$, that we can suppose to be in the form $\mathbb{Z}_{2^{i-1}}$, there is a lifting to the module \mathbb{Z}_{2^i} through the extension with a submodule \mathbb{Z}_2 corresponding to a generator in $H^2(SL_2(\mathbb{Z}); M \otimes \mathbb{Z}_{(2)})$;*
- c) *for all the free summands $\mathbb{Z}_{(2)} \subset 2H^1(SL_2(\mathbb{Z}); M_n \otimes \mathbb{Z}_{(2)})$ there is no lifting and the extension is trivial.*

Proof. We study the spectral sequence \overline{E}_2

$$\overline{E}_2^{i,j} = H^i(SL_2(\mathbb{Z}); M_n \otimes \mathbb{Z}_{2^k}) \text{ for } j = 0, 1$$

and zero otherwise, that converges to $H^*(B_3; M_n \otimes \mathbb{Z}_{2^k})$ for any integer $k > 0$. The dependence on the integer k is understood in the notation. As before, the spectral sequence collapses at \overline{E}_3 . We compute the cardinality of $\overline{E}_3^{1,1} \oplus \overline{E}_3^{2,0}$ that is the cardinality of $H^2(B_3; M_n \otimes \mathbb{Z}_{2^k})$.

We will need to consider also the $\overline{E}(\mathbb{Z}_m)_2$ spectral sequences for $H^*(\mathbb{Z}; M_n \otimes \mathbb{Z}_{2^k})$ obtained by the extensions given in the diagrams of equation (32), induced by the inclusion $\mathbb{Z}_m \hookrightarrow SL_2(\mathbb{Z})$, $m = 2, 4, 6$.

The Mayer-Vietoris exact sequence gives us maps of spectral sequences:

$$\Phi_4 : \overline{E}_2 \rightarrow \overline{E}(\mathbb{Z}_4)_2,$$

$$\Phi_6 : \overline{E}_2 \rightarrow \overline{E}(\mathbb{Z}_6)_2,$$

$$\Delta_2 : \overline{E}(\mathbb{Z}_2)_2 \rightarrow \overline{E}_2$$

where the last map is the homomorphism **that appears in diagram (33)** and has a degree shift coherently with the Mayer-Vietoris exact sequence connecting homomorphism.

Moreover we need to consider the map

$$\pi_* : E_2 \rightarrow \overline{E}_2$$

induced by the morphism of coefficients $\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}_{2^k}$ and the Bockstein homomorphism

$$\beta_{2^k} : \overline{E}_2 \rightarrow \overline{E}_2$$

and, for $m = 2, 4, 6$,

$$\beta_{2^k} : \overline{E}_2(\mathbb{Z}_m) \rightarrow \overline{E}_2(\mathbb{Z}_m)$$

defined before.

Here we use $a \vee b$ for $\max(a, b)$ and $a \wedge b$ for $\min(a, b)$.

Let $\omega = \mathcal{P}_2^a \mathcal{Q}_2^b$ be any monomial in $\mathcal{P}_2, \mathcal{Q}_2$. Let 2^i be the greatest power of 2 that divides both a and b . The polynomial ω is $SL_2(\mathbb{Z})$ -invariant modulo 2^{i+1} , hence it generates a submodule $\mathbb{Z}_{2^{i+1}} \subset H^0(SL_2(\mathbb{Z}); M \otimes \mathbb{Z}_{2^{i+1}})$. More generally, for $k \leq i$ the polynomial ω generates a submodule $\mathbb{Z}_{2^k} \subset H^0(SL_2(\mathbb{Z}); M \otimes \mathbb{Z}_{2^k})$, while for $k \geq i+1$ the polynomial $2^{k-i-1}\omega$ is $SL_2(\mathbb{Z})$ -invariant modulo 2^k and generates a submodule $\mathbb{Z}_{2^{i+1}} \subset H^0(SL_2(\mathbb{Z}); M \otimes \mathbb{Z}_{2^k})$. We remark that all these submodules are also direct summands.

Define $\delta = \max(k - i - 1, 0)$.

The Bockstein homomorphism

$$\beta_{2^k} : H^0(SL_2(\mathbb{Z}); M \otimes \mathbb{Z}_{2^k}) \rightarrow H^1(SL_2(\mathbb{Z}); M \otimes \mathbb{Z}_{2^k})$$

maps the polynomial $2^\delta \omega$ as follows. For $k \geq i+1$ the image $\beta_{2^k}(2^\delta \omega)$ generates a $\mathbb{Z}_{2^{i+1}}$ that is a direct summand of $H^1(SL_2(\mathbb{Z}); M \otimes \mathbb{Z}_{2^k})$, while for $k \leq i$ the image $\beta_{2^k}(2^\delta \omega)$ is 2^{i+1-k} -times the generator of a module \mathbb{Z}_{2^k} that is a direct summand of $H^1(SL_2(\mathbb{Z}); M \otimes \mathbb{Z}_{2^k})$. In particular for $i+1 \geq 2k$, $\beta_{2^k}(2^\delta \omega) = 0$.

In both cases ($k \geq i+1$ or $k \leq i$), the module $\mathbb{Z}_{2^{k \wedge (i+1)}} \subset H^1(SL_2(\mathbb{Z}); M \otimes \mathbb{Z}_{2^k})$ considered above is the image through π_* of a module $\mathbb{Z}_{2^{i+1}}$ that is a direct summand of $H^1(SL_2(\mathbb{Z}); M \otimes \mathbb{Z}_{(2)})$; we call $\tilde{\omega} \in H^1(SL_2(\mathbb{Z}); M \otimes \mathbb{Z}_{(2)})$ the generator of this module and we have

$$2^{i+1-k-\delta} \pi_* \tilde{\omega} = \beta_{2^k}(2^\delta \omega).$$

Let us now focus on the polynomial $\omega = \mathcal{Q}_2^{2^i h}$ for odd h and $i > 0$. We need to prove the following:

Lemma 7.5. *The differential $d_2 : \overline{E}_2^{0,1} \rightarrow \overline{E}_2^{2,0}$ maps the module $\mathbb{Z}_{2^{k \wedge (i+1)}}$ generated by $2^\delta \mathcal{Q}_2^{2^i h}$ surjectively to a module isomorphic to \mathbb{Z}_2 if $k \geq i+2$ or $k = 1$ and to \mathbb{Z}_4 if $2 \leq k \leq i+1$.*

Proof. Let $k \leq i+1$ and set $\omega = \mathcal{Q}_2^{2^i h}$. Then $\delta = 0$ and it is easy to verify that monomial $(xy)^{2^i h}$ appears with an odd coefficient in the algebraic expansion of $2^\delta \omega = \mathcal{Q}_2^{2^i h}$. Hence the cocycle $\Phi_4(2^\delta \mathcal{Q}_2^{2^i h})$ generates a direct summand \mathbb{Z}_{2^k} in $\overline{E}(\mathbb{Z}_4)_2$ and for $k \neq 1$ the differential d_2 maps the cocycle $\Phi_4(2^\delta \mathcal{Q}_2^{2^i h})$ to the generator of a module \mathbb{Z}_4 in $\overline{E}(\mathbb{Z}_4)_2$. Since $H^2(SL_2(\mathbb{Z}); M \otimes \mathbb{Z}_{2^k})$ has no 8-torsion from Lemma 6.2, the claim for $2 \leq k \leq i+1$ follows from the commutativity of the maps d_2 and Φ_4 . See the diagram in table 1. The same argument shows that for $k = 1$ the cocycle $d_2(2^\delta \mathcal{Q}_2^{2^i h})$ generates a module $\mathbb{Z}_2 \subset \overline{E}_2^{2,0}$.

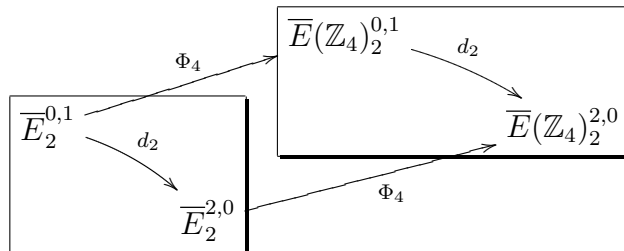


TABLE 1.

Now suppose $k \geq i + 2$. In this case $\delta > 0$ and by the same argument as before, we have that the class $d_2\Phi_4(2^\delta\Omega_2^{2^i h})$ has no 4-torsion. From the description of the cohomology of $SL_2(\mathbb{Z})$ given in Lemma 6.2 it follows that also the class $d_2(2^\delta\Omega_2^{2^i h})$ has no 4-torsion. In fact the morphism Φ_4 maps any class of 4-torsion in $\overline{E}_2^{2,0}$ isomorphically to a class of 4-torsion in $\overline{E}(\mathbb{Z}_4)_2^{2,0}$.

In order to complete the proof of the lemma we need to show that $d_2(2^\delta\Omega_2^{2^i h})$ is non-zero for $k \geq i + 2$. We consider the diagram of table 2. From the previous computations we have

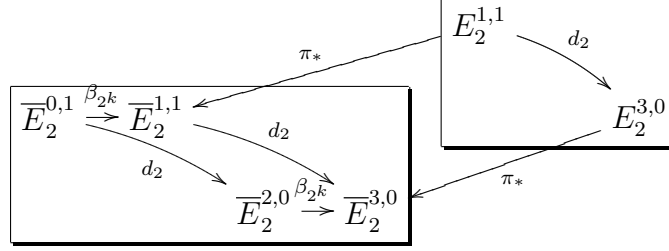


TABLE 2.

$\beta_{2^k}(2^\delta\omega) = \pi_*\tilde{\omega}$ and the Bockstein map β_{2^k} maps isomorphically the submodule $\mathbb{Z}_{2^{i+1}}$ generated by $2^\delta\omega$ to the submodule $\mathbb{Z}_{2^{i+1}}$ generated by $\pi_*\tilde{\omega}$. Moreover from Lemma 7.2 the differential d_2 maps $\pi_*\tilde{\omega}$ to the generator of a submodule \mathbb{Z}_2 . Hence, by the commutativity of the differential d_2 and the Bockstein map β_{2^k} , the cohomology class $d_2(2^\delta\omega)$ is non-zero. The lemma follows. \square

In case ω is different from an even power of Ω_2 , we have:

Lemma 7.6. *Let $\omega = \mathcal{P}_2^a\Omega_2^b$ be a monomial in \mathcal{P}_2 and Ω_2 different from an even power of Ω_2 . The differential $d_2 : \overline{E}_2^{0,1} \rightarrow \overline{E}_2^{2,0}$ maps the module $\mathbb{Z}_{2^{k \wedge (i+1)}}$ generated by $2^\delta\omega$ surjectively to a module isomorphic to \mathbb{Z}_2 .*

Proof. Let $k \geq i + 1$. In this case we can use the Bockstein morphism β_{2^k} and compare the spectral sequence \overline{E}_2 with E_2 by means of the map π_* . The Bockstein β_{2^k} maps the submodule $\mathbb{Z}_{2^{i+1}}$ generated by $2^\delta\omega$ isomorphically to the submodule $\mathbb{Z}_{2^{i+1}}$ generated by $\pi_*\tilde{\omega}$ and from Lemma 7.2 the differential d_2 maps $\pi_*\tilde{\omega}$ to the generator of a submodule \mathbb{Z}_2 . The argument in the proof of the previous lemma implies that the class $d_2(2^\delta\omega)$ has no 4-torsion.

Now we assume $k \leq i$ and hence $\delta = 0$. If a is odd then $i = 0$ and there is nothing to prove. If $a = 0$ then b is odd and again $i = 0$ and there is nothing to prove. If $a \neq 0$ and a even, a symmetry argument that exchanges x and y shows that the power of xy that appears in the algebraic expansion of ω has even coefficient. Hence the class $d_2\Phi_4(\omega)$ has at most 2-torsion and can be eventually zero. We need to prove that $d_2(\omega)$ is non-zero. It is easy to see that the monomials $x^{2a+2b}y^a$ and x^ay^{2a+2b} appear in the algebraic expansion of ω with coefficient 1. As we are assuming that a is even we have that $\Phi_4(\omega)$ has non-trivial projection to the \mathbb{Z}_4 -invariant submodule of $\mathbb{Z}_{2^k}[x, y]$ generated by $x^{2a+2b}y^a$ and x^ay^{2a+2b} and since $d_2(x^{2a+2b}y^a + x^ay^{2a+2b}) \neq 0$ we have also that $d_2\Phi_4(\omega) \neq 0$. \square

Lemma 7.7. *The differential $d_2 : \overline{E}_2^{0,1} \rightarrow \overline{E}_2^{2,0}$ induces an injective map $\overline{d}_2 : \overline{E}_2^{0,1} \otimes \mathbb{Z}_2 \rightarrow \overline{E}_2^{2,0} \otimes \mathbb{Z}_2$.*

Proof. We define the map $\bar{\beta}_{2^k} : \bar{E}_2^{0,1} \otimes \mathbb{Z}_2 \rightarrow \bar{E}_2^{1,1} \otimes \mathbb{Z}_2$ induced by the Bockstein β_{2^k} and we consider a splitting $\bar{E}_2^{0,1} \otimes \mathbb{Z}_2 = \ker \bar{\beta}_{2^k} \oplus \text{coker} \bar{\beta}_{2^k}$.

The kernel of $\bar{\beta}_{2^k}$ is generated by those classes $2^\delta \mathcal{P}_2^a \Omega_2^b$ such that $\delta = 0$, that is $k \leq i + 1$ where 2^i is the greatest power of 2 that divides both a and b . It follows that the restriction of Φ_4 to $\ker \bar{\beta}_{2^k}$ is injective. Moreover it is easy to verify that the induced map

$$\bar{d}_2 : \bar{E}(\mathbb{Z}_4)_2^{0,1} \otimes \mathbb{Z}_2 \rightarrow \bar{E}(\mathbb{Z}_4)_2^{2,0} \otimes \mathbb{Z}_2$$

is injective. Hence the restriction of d_2 to $\ker \bar{\beta}_{2^k} \subset \bar{E}_2^{0,1} \otimes \mathbb{Z}_2$ is injective and the image is contained in the kernel of the map

$$\bar{\beta}_{2^k} : \bar{E}_2^{2,0} \otimes \mathbb{Z}_2 \rightarrow \bar{E}_2^{3,0} \otimes \mathbb{Z}_2.$$

The image of $\bar{\beta}_{2^k} : \bar{E}_2^{0,1} \otimes \mathbb{Z}_2 \rightarrow \bar{E}_2^{1,1} \otimes \mathbb{Z}_2$ is contained in the image of the projection

$$\bar{\pi}_* : E_2^{1,1} \otimes \mathbb{Z}_2 \rightarrow \bar{E}_2^{1,1} \otimes \mathbb{Z}_2$$

and since the map

$$\bar{d}_2 : E_2^{1,1} \otimes \mathbb{Z}_2 \rightarrow E_2^{3,0} \otimes \mathbb{Z}_2$$

is an isomorphism it follows that the restriction of the differential d_2 to $\text{Im} \bar{\beta}_{2^k} \cap \bar{E}_2^{1,1}$ is injective and the image of d_2 is contained in the image of $\bar{\beta}_{2^k} : \bar{E}_2^{2,0} \otimes \mathbb{Z}_2 \rightarrow \bar{E}_2^{3,0} \otimes \mathbb{Z}_2$.

The claim of the lemma follows. \square

The previous arguments together with Lemmas 7.5 and 7.6 determine completely the differential $d_2 : \bar{E}_2^{0,1} \rightarrow \bar{E}_2^{2,0}$.

Now we focus on $d_2 : \bar{E}_2^{1,1} \rightarrow \bar{E}_2^{3,0}$.

We consider the splitting of the module $\bar{E}_2^{1,1} = T_1 \oplus F_1 \oplus T_2$, where T_1 is the image, through π_* of the torsion subgroup of $E_2^{1,1}$; F_1 is the image, through π_* of the torsion-free part of $E_2^{1,1}$ according to the chosen decomposition in equation (36); T_2 is a subgroup corresponding, through the Bockstein homomorphism and the map π_* , to the torsion subgroup in $E_2^{2,1}$.

We recall from the previous results that the differential $d_2 : E_2^{1,1} \rightarrow E_2^{3,0}$ is the quotient modulo 2 for degree multiple of 4 (see Lemma 7.2) and an isomorphism for degree $2n$, n odd, (see Lemma 7.1) with a 2-torsion module. Since the map π_* induces a surjection

$$E_2^{1,1} \rightarrow F^1 \oplus T^1 \subset \bar{E}_2^{1,1}$$

and the map $\pi_* : E_2^{3,0} \rightarrow \bar{E}_2^{3,0}$ is injective. We can also remark that $F^1 \oplus T^1$ is in the kernel of the Bockstein operator β_{2^k} . Hence we have the following

Lemma 7.8. *The differential $d_2 : \bar{E}_2^{1,1} \rightarrow \bar{E}_2^{3,0}$ acts on the submodule $F^1 \oplus T^1 \subset \bar{E}_2^{1,1}$ as quotient modulo 2:*

$$d_2 : F^1 \oplus T^1 \rightarrow (F^1 \oplus T^1) \otimes \mathbb{Z}_2 \subset \bar{E}_2^{3,0}.$$

The image $d_2(F^1 \oplus T^1)$ is contained in the kernel of the Bockstein operator β_{2^k} . \square

$$\begin{array}{ccc}
 \overline{E}_2^{1,1} & \xrightarrow{\beta_{2^k}} & \overline{E}_2^{2,1} \\
 & \searrow d_2 & \searrow d_2 \\
 & & \overline{E}_2^{3,0} \xrightarrow{\beta_{2^k}} \overline{E}_2^{4,0}
 \end{array}$$

TABLE 3.

Finally we consider the diagram of table 3 and we look at the submodule $T^2 \subset \overline{E}_2^{1,1}$. Note that T^2 is a \mathbb{Z}_4 -module. In particular for $k \geq 2$ the restriction of the Bockstein map $\beta_{2^k} : \overline{E}_2^{1,1} \rightarrow \overline{E}_2^{2,1}$ to the submodule $T^2 \subset \overline{E}_2^{1,1}$ is injective. Moreover, since B_3 has homological dimension 2, the differential $d_2 : \overline{E}_2^{2,1} \rightarrow \overline{E}_2^{4,0}$ is an isomorphism. It follows that for $k \geq 2$ the restriction of the differential $d_2 : \overline{E}_2^{1,1} \rightarrow \overline{E}_2^{3,0}$ to the submodule $T^2 \subset \overline{E}_2^{1,1}$ is injective. Since the Bockstein operator β_{2^k} is injective in $T^2 \subset \overline{E}_2^{1,1}$ we have also that the image $d_2(T^2)$ projects injectively to the cokernel of the Bockstein and hence for $k \geq 2$.

For $k = 1$ we can consider the long exact sequence of equation (33) for $N = M_n \otimes \mathbb{Z}_2$. For $i \geq 0$ and $m = 2, 4, 6$ the differential

$$d_2 : H^i(\mathbb{Z}_m; M_n \otimes \mathbb{Z}_2) \rightarrow H^{i+2}(\mathbb{Z}_m; M_n \otimes \mathbb{Z}_2)$$

is always an isomorphism. Hence, by the five Lemma, it follows that the differential

$$d_2 : H^i(SL_2(\mathbb{Z}); M_n \otimes \mathbb{Z}_2) \rightarrow H^{i+2}(SL_2(\mathbb{Z}); M_n \otimes \mathbb{Z}_2)$$

is an isomorphism too.

In general we have proved:

Lemma 7.9. *For any $k \geq 1$ the kernel of the map $d_2 : \overline{E}_2^{1,1} \rightarrow \overline{E}_2^{3,0}$ is equal to the kernel of the restriction*

$$d_2 : F^1 \oplus T^1 \rightarrow (F^1 \oplus T^1) \otimes \mathbb{Z}_2 \subset \overline{E}_2^{3,0}.$$

□

For the computation of the cardinality of $H^2(B_3; M_n \otimes \mathbb{Z}_{2^k})$ we define the function

$$f(k) := \log_2(|H^2(B_3; M_n \otimes \mathbb{Z}_{2^k})|)$$

and since $K(B_3, 1)$ is realized as a 2-dimensional complex, from the Universal Coefficient Theorem it follows

$$f(k) = \log_2(|H^2(B_3; M_n) \otimes \mathbb{Z}_{2^k}|)$$

and the function $f(k)$ determines the group $H^2(B_3; M_n) \otimes \mathbb{Z}_{(2)}$. If we suppose that

$$H^2(B_3; M_n) \otimes \mathbb{Z}_{(2)} \simeq \mathbb{Z}_{(2)}^{\oplus f_n} \oplus \bigoplus_t \mathbb{Z}_{2^t}^{\oplus \mu_t}$$

where f_n is the integer defined in Theorem 3.7 and the μ_t are non-negative integers, we have

$$f(k) = f_n k + \sum_t \mu_t (t \wedge k)$$

and we need to determine the function $f(k)$ only up to constants in order to get the integers μ_t . The spectral sequence \overline{E}_r collapses at the term \overline{E}_3 and we have $|H^2(B_3; M_n \otimes \mathbb{Z}_{2^k})| = |\overline{E}_3^{1,1} \oplus \overline{E}_3^{2,0}|$ and

$$(37) \quad f(k) = \log_2(|\overline{E}_3^{1,1}|) + \log_2(|\overline{E}_3^{2,0}|).$$

The term $\log_2(|\overline{E}_3^{1,1}|)$ can be computed using Lemma 7.8 and the description of $H^1(SL_2(\mathbb{Z}); M_n)_{(p)}$ given in Theorem 4.5. We obtain:

$$\log_2(|\overline{E}_3^{1,1}|) = kf_n + \sum_{\substack{a, b \in \mathbb{N}, \\ 6a+4b=n}} ((\lambda_2(a, b) + 1) \wedge k - 1)$$

where $\lambda_2(a, b)$ is the greatest integer such that $2^{\lambda_2(a, b)} \mid \gcd(a, b)$.

From the description of the differential $d_2 : \overline{E}_2^{0,1} \rightarrow \overline{E}_2^{2,0}$ given in Lemmas 7.5, 7.6 and 7.7 we have that, up to constants, the cardinality of $\overline{E}_3^{2,0}$ follows by:

$$\log_2(|\overline{E}_3^{1,1}|) = \sum_{2^{i+2}h=n} \chi_{\{k \geq i+2\}}.$$

Hence, up to constants,

$$f(k) = kf_n + \sum_1 ((\lambda_2(a, b) + 1) \wedge k - 1) + \sum_2 (i + 2) \wedge k - 1$$

where the sum \sum_1 is given for $a, b \in \mathbb{N}$ such that $6a + 4b = n$ and $a \neq 0$ or b odd and the sum \sum_2 is given for $i, h \in \mathbb{N}$ such that $i > 0$, h is odd and $2^{i+1}h = n$.

From the description of the spectral sequence it follows that the extension involves the generator described in the statement of the Theorem 7.4. \square

As a consequence of Theorem 7.4 we have the first part of Theorem 3.11 that is the result:

Theorem 7.10. *We have the following isomorphism:*

$$H^2(B_3; M)_{(2)} = (\Gamma_2^+[\mathcal{P}_2, \mathcal{Q}_2] \oplus \mathbb{Z}[\overline{\mathcal{Q}}_2^2]) / \sim$$

Here $\overline{\mathcal{Q}}_2$ is a new variable of degree 4; the quotient module is defined by the relations $\frac{\mathcal{Q}_2^n}{n!} \sim 2\overline{\mathcal{Q}}_2^n$ for n even and $\frac{\mathcal{Q}_2^n}{n!} \sim 0$ for n odd. \square

7.2. 3-torsion. Let now consider only the 3-torsion of $H^2(B_3; M_{2n})$. For odd n the 3-torsion part of $H^i(SL_2(\mathbb{Z}); M_{2n})$ is trivial, so we investigate the Serre spectral sequence and the differential

$$d_2 : H^1(SL_2(\mathbb{Z}); N) \rightarrow H^3(SL_2(\mathbb{Z}); N)$$

where now N is the module $M_{2n} \otimes \mathbb{Z}_{(3)}$ for n even. In this case from diagram (33) we get

$$(38) \quad \begin{array}{ccccccc} \cdots & \rightarrow & H^0(\mathbb{Z}_2; N) & \rightarrow & H^1(SL_2(\mathbb{Z}); N) & \rightarrow & H^1(\mathbb{Z}_6; N) \rightarrow 0 \\ & & & & \downarrow d_2 & & \downarrow d_2 \\ 0 & \longrightarrow & H^3(SL_2(\mathbb{Z}); N) & \rightarrow & H^3(\mathbb{Z}_6; N) & \rightarrow & 0 \end{array}$$

Again the spectral sequence collapses at E_3 and we have:

Lemma 7.11. *In the Serre spectral sequence for $H^*(B_3; M_n \otimes \mathbb{Z}_{(3)})$ the E_∞ -term is given by:*

$$\begin{array}{|ccc|} \hline \mathbb{Z}_{(3)} & K & 0 \\ \hline \mathbb{Z}_{(3)} & H^1(SL_2(\mathbb{Z}); M_n \otimes \mathbb{Z}_{(3)}) & H^2(SL_2(\mathbb{Z}); M_n \otimes \mathbb{Z}_{(3)}) \\ \hline \end{array}$$

where $K = \ker(H^1(SL_2(\mathbb{Z}); M_n \otimes \mathbb{Z}_{(3)}) \rightarrow H^1(\mathbb{Z}_6; M_n \otimes \mathbb{Z}_{(3)}))$. \square

Similar to the case $p = 2$, the final step is to consider the extension problem associated to the E_∞ -term. We can assume also for $p = 3$ to have a decomposition

$$(39) \quad H^1(SL_2(\mathbb{Z}); M_n \otimes \mathbb{Z}_{(3)}) = \mathbb{Z}_{(3)}^{f_n} \oplus \bigoplus_k \mathbb{Z}_{3^k}^{d_k}$$

and an induced decomposition on

$$H^1(\mathbb{Z}_6; M_n \otimes \mathbb{Z}_{(3)}) = H^3(SL_2(\mathbb{Z}); M_n \otimes \mathbb{Z}_{(3)})$$

such that the differential d_2 is diagonal with respect to the two decompositions.

As for the 2-torsion, we use \overline{E}_2 for the spectral sequence defined by

$$\overline{E}_2^{i,j} = H^j(SL_2(\mathbb{Z}); M_n \otimes \mathbb{Z}_{3^k}) \text{ for } i = 0, 1$$

and zero otherwise, that converges to $H^*(B_3; M_n \otimes \mathbb{Z}_{3^k})$ and also the $\overline{E}(\mathbb{Z}_m)_2$ spectral sequences for $H^*(\mathbb{Z}; M_n \otimes \mathbb{Z}_{3^k})$.

The two spectral sequences are connected by the map

$$\Phi_6 : \overline{E}_2 \rightarrow \overline{E}(\mathbb{Z}_6)_2$$

induced by the inclusion $\mathbb{Z}_6 \hookrightarrow SL_2(\mathbb{Z})$ given in the third diagram of equation (32).

Proposition 7.12. *Let us consider the generator $\mathcal{P}_3 \mathcal{Q}_3^{3^i k} \in H^0(SL_2(\mathbb{Z}); M \otimes \mathbb{Z}_3) = \overline{E}_2^{0,1}$. The corresponding module $\mathbb{Z}_3 \subset H^1(SL_2(\mathbb{Z}); M_n \otimes \mathbb{Z}_{(3)}) = E_2^{1,1}$ with $n = 6 \cdot 3^i k + 4$ is mapped isomorphically by the differential d_2 to the module $\mathbb{Z}_3 = H^3(SL_2(\mathbb{Z}); M_n \otimes \mathbb{Z}_{(3)}) = E_2^{3,0}$.*

Proof. The statement follows considering the spectral sequence $\overline{E}_2^{i,j}$ with coefficients in $M_n \otimes \mathbb{Z}_3$ and the spectral sequence $\overline{E}(\mathbb{Z}_6)_2^{i,j}$, both for $k = 1$ and the maps of the diagram in table 4.

We have the equalities $\mathcal{P}_3 = w_4$ and $\mathcal{Q}_3 \equiv v_2^3 + z_6 \pmod{3}$. The generator of the module $\mathbb{Z}_3 = H^3(\mathbb{Z}_6; M_n \otimes \mathbb{Z}_{(3)}) = E_2^{3,0}$ projects through π_* to the Bockstein class $\beta_3(w_4(v_2^3 + z_6)^{3^i k}) \in \overline{E}_2^{3,0}$ and hence the statement follows from comparing the differentials $d_2 : \overline{E}_2^{0,1} \rightarrow \overline{E}_2^{2,0}$, that is

$$d_2 : H^0(SL_2(\mathbb{Z}); M_n \otimes \mathbb{Z}_3) \rightarrow H^2(SL_2(\mathbb{Z}); M_n \otimes \mathbb{Z}_3)$$

and $d_2 : \overline{E}(\mathbb{Z}_6)_2^{0,1} \rightarrow \overline{E}(\mathbb{Z}_6)_2^{2,0}$, that is

$$(40) \quad d_2 : H^0(\mathbb{Z}_6; M_n \otimes \mathbb{Z}_3) \rightarrow H^2(\mathbb{Z}_6; M_n \otimes \mathbb{Z}_3).$$

\square

Theorem 7.13. *We consider the extension problem for the E_∞ -term of the spectral sequence of Lemma 7.11 above.*

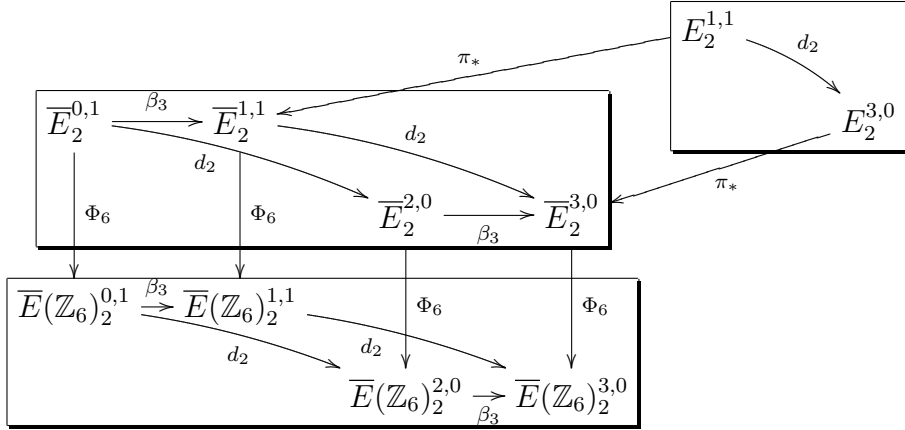


TABLE 4.

For $n \equiv 0 \pmod{12}$ the summand $\mathbb{Z}_{3^{i+1}} \subset H^1(SL_2(\mathbb{Z}); M_n \otimes \mathbb{Z}_{(3)}) = E_2^{1,1}$ corresponding, via Universal Coefficient Theorem, to the generator $\mathcal{Q}_3^{3^i h} \in H^0(SL_2(\mathbb{Z}); M \otimes \mathbb{Z}_3)$ ($3 \nmid h$, $i > 0$) has a lifting to the module $\mathbb{Z}_{3^{i+2}}$ through the extension with the module \mathbb{Z}_3 corresponding to the generator

$$(p_2^3 + q_6)^{3^i h} \in H^2(\mathbb{Z}_6; M \otimes \mathbb{Z}_{(3)}).$$

For all the other summands there is no lifting and the extension is trivial.

Proof. We study the spectral sequence \overline{E}_2 converging to $H^*(B_3; M_n \otimes \mathbb{Z}_{3^k})$ for any $k > 0$. As before, the spectral sequence collapses at \overline{E}_3 . We compute the cardinality of $\overline{E}_3^{1,1} \oplus \overline{E}_3^{2,0}$ that is the cardinality of $H^2(B_3; M_n \otimes \mathbb{Z}_{3^k})$. We start with an explicit description of $\overline{E}_2^{0,1} = H^0(SL_2(\mathbb{Z}); M_n \otimes \mathbb{Z}_{3^k})$, $\overline{E}_2^{1,1} = H^1(SL_2(\mathbb{Z}); M_n \otimes \mathbb{Z}_{3^k})$, $\overline{E}_2^{2,0} = H^2(SL_2(\mathbb{Z}); M_n \otimes \mathbb{Z}_{3^k})$ and $\overline{E}_2^{3,0} = H^3(SL_2(\mathbb{Z}); M_n \otimes \mathbb{Z}_{3^k})$ and of the corresponding $\overline{E}(\mathbb{Z}_6)_2$ terms of the spectral sequence for $H^2(\mathbb{Z}; M_n \otimes \mathbb{Z}_{3^k})$.

The term $\overline{E}(\mathbb{Z}_6)_2^{0,1} = H^0(\mathbb{Z}_6; M_n \otimes \mathbb{Z}_{3^k})$ is given by the ring of invariants $\mathbb{Z}_{3^k}[x, y]^{\mathbb{Z}_6}$. In particular we have the polynomial $q_6^{3^j h}$, for $h \not\equiv 0 \pmod{3}$, that generates a group \mathbb{Z}_3 and the polynomial $3^{k-1}w_4q_6^i$ that generates a group \mathbb{Z}_3 ; the groups are direct summands of $\mathbb{Z}_{3^k}[x, y]^{\mathbb{Z}_6}$.

The term $\overline{E}(\mathbb{Z}_6)_2^{1,1} = H^1(\mathbb{Z}_6; M_n \otimes \mathbb{Z}_{3^k})$ is a free \mathbb{Z}_3 -module generated by the polynomials $3^{k-1}q_6^{3^j h}$ and by the image $\beta_{3^k}(3^{k-1}w_4q_6^i)$, where β_{3^k} is the Bockstein map associated to the extension

$$\mathbb{Z}_{3^k} \rightarrow \mathbb{Z}_{3^{2k}} \rightarrow \mathbb{Z}_{3^k}.$$

The term $\overline{E}(\mathbb{Z}_6)_2^{2,0} = H^2(\mathbb{Z}_6; M_n \otimes \mathbb{Z}_{3^k})$ is a free \mathbb{Z}_3 -module generated by the polynomials $3^{k-1}w_4q_6^i$ and $q_6^{3^j h} = \beta_{3^k}(3^{k-1}q_6^{3^j h})$.

The term $\overline{E}(\mathbb{Z}_6)_2^{3,0} = H^3(\mathbb{Z}_6; M_n \otimes \mathbb{Z}_{3^k})$ is a free \mathbb{Z}_3 -module generated by the polynomials $3^{k-1}q_6^{3^j h}$ and $\beta_{3^k}(3^{k-1}w_4q_6^i)$.

The term $\overline{E}_2^{0,1} = H^0(SL_2(\mathbb{Z}); M_n \otimes \mathbb{Z}_{3^k})$ is given by the ring of invariants $\Gamma_3[\mathcal{P}_3, \mathcal{Q}_3] \otimes \mathbb{Z}_{3^k}$. In particular $3^{k-1}\mathcal{P}_3\mathcal{Q}_3^i$ generates a group \mathbb{Z}_3 and $3^{(k-j-1)\vee 0}\mathcal{Q}_3^{3^j h}$ generates a group $\mathbb{Z}_{3^{k \wedge (j+1)}}$; the groups are direct summands of $H^0(SL_2(\mathbb{Z}); M_n \otimes \mathbb{Z}_{3^k})$.

The term $\overline{E}_2^{1,1} = H^1(SL_2(\mathbb{Z}); M_n \otimes \mathbb{Z}_{3^k})$ is given by the direct sum of a free \mathbb{Z}_{3^k} -module $\mathbb{Z}_{3^k}^{f_n}$, the modules \mathbb{Z}_3 generated by classes $\overline{3^{k-1}q_6^i}$ such that $\Phi_6(\beta_{3^k}(\overline{3^{k-1}q_6^i})) = q_6^i \in H^2(\mathbb{Z}_6; M_n \otimes \mathbb{Z}_{3^k})$ and a module isomorphic to $\Gamma_3^+[\mathcal{P}_3, \mathcal{Q}_3] \otimes \mathbb{Z}_{3^k} \subset H^0(SL_2(\mathbb{Z}); M_n \otimes \mathbb{Z}_{3^k})$. In particular $\beta_{3^k}(3^{k-1}\mathcal{P}_3\mathcal{Q}_3^i)$ generates a group \mathbb{Z}_3 and for $k > j$ the cocycle $\beta_{3^k}(3^{(k-j-1)\vee 0}\mathcal{Q}_3^{3^j h}) = \beta_{3^k}(3^{(k-j-1)}\mathcal{Q}_3^{3^j h})$ generates a group $\mathbb{Z}_{3^{k \wedge (j+1)}} = \mathbb{Z}_{3^{j+1}}$ and all these groups are direct summands in $\overline{E}_2^{1,1}$.

The inclusion $\mathbb{Z}_6 \hookrightarrow SL_2(\mathbb{Z})$ induces for $i \geq 2$ the isomorphism $\Phi_6^i : H^i(SL_2(\mathbb{Z}), M_n \otimes \mathbb{Z}_{3^k}) \simeq H^i(\mathbb{Z}_6; M_n \otimes \mathbb{Z}_{3^k})$. Hence the terms $\overline{E}_2^{2,0} = H^2(SL_2(\mathbb{Z}); M_n \otimes \mathbb{Z}_{3^k})$ and $\overline{E}_2^{3,0} = H^3(SL_2(\mathbb{Z}); M_n \otimes \mathbb{Z}_{3^k})$ are generated respectively by the counterimages of the polynomials $3^{k-1}w_4q_6^i$ and $q_6^{3^j h} = \beta_{3^k}(3^{k-1}q_6^{3^j h})$ and by the counterimages of the polynomials $3^{k-1}q_6^{3^j h}$ and $\beta_{3^k}(3^{k-1}w_4q_6^i)$.

Now we compute the differentials $\overline{d}_2 : \overline{E}(\mathbb{Z}_6)_2^{0,1} \rightarrow \overline{E}(\mathbb{Z}_6)_2^{2,0}$ and $\overline{d}_2 : \overline{E}(\mathbb{Z}_6)_2^{1,1} \rightarrow \overline{E}(\mathbb{Z}_6)_2^{3,0}$ and the corresponding differentials for \overline{E}_2 .

We use the fact that the spectral sequence $\overline{E}(\mathbb{Z}_6)$ collapses after the differential \overline{d}_2 to the cohomology $H^*(\mathbb{Z}; M_n \otimes \mathbb{Z}_{3^k})$.

For the computation of $\overline{d}_2 : \overline{E}(\mathbb{Z}_6)_2^{0,1} \rightarrow \overline{E}(\mathbb{Z}_6)_2^{2,0}$ we have the maps $\overline{d}_2 : q_6^{3^j h} \mapsto q_6^{3^j h}$ and $\overline{d}_2 : 3^{k-1}w_4q_6^i \mapsto 3^{k-1}w_4q_6^i$ that is the quotient map $\mathbb{Z}_{3^k} \rightarrow \mathbb{Z}_3$ and the isomorphism $\mathbb{Z}_3 \rightarrow \mathbb{Z}_3$ for the corresponding generated submodules. From the Jordan blocks decomposition in the proof of Proposition 5.9 one can see that for the other monomials in $\overline{E}(\mathbb{Z}_6)_2^{0,1}$ the differential is trivial.

The differential $\overline{d}_2 : \overline{E}(\mathbb{Z}_6)_2^{1,1} \rightarrow \overline{E}(\mathbb{Z}_6)_2^{3,0}$ must be an isomorphism we have the maps $\overline{d}_2 : 3^{k-1}q_6^{3^j h} \mapsto 3^{k-1}q_6^{3^j h}$ and $\overline{d}_2 : \beta_{3^k}(3^{k-1}w_4q_6^i) \mapsto \beta_{3^k}(3^{k-1}w_4q_6^i)$.

Finally we can describe the differential \overline{d}_2 for \overline{E} . We use $\Phi_6 : \overline{E}_2 \rightarrow \overline{E}(\mathbb{Z}_6)_2$ to compute it.

Recall that we have the equalities $\mathcal{P}_3 = w_4$ and $\mathcal{Q}_3 \equiv q_6 + p_2^3 \pmod{3}$.

For the classes $3^{k-1}\mathcal{P}_3\mathcal{Q}_3^i \in \overline{E}^{0,1}$ we have $\Phi_6(3^{k-1}\mathcal{P}_3\mathcal{Q}_3^i) = 3^{k-1}w_4q_6^i$ and $\overline{d}_2 : 3^{k-1}w_4q_6^i \mapsto 3^{k-1}w_4q_6^i$. Hence $\overline{d}_2(3^{k-1}\mathcal{P}_3\mathcal{Q}_3^i) = \Phi_6^{-1}(3^{k-1}w_4q_6^i)$.

For the classes $3^{(k-j-1)\vee 0}\mathcal{Q}_3^{3^j h} \in \overline{E}^{0,1}$ we have $\Phi_6(3^{(k-j-1)\vee 0}\mathcal{Q}_3^{3^j h}) \equiv 3^{(k-j-1)\vee 0}(q_6 + p_2^3)^{3^j h} \pmod{3}$. Moreover $\overline{d}_2((q_6 + p_2^3)^{3^j h}) = q_6^{3^j h}$. Hence $\overline{d}_2(3^{(k-j-1)\vee 0}\mathcal{Q}_3^{3^j h}) = 3^{(k-j-1)\vee 0}\Phi_6^{-1}(q_6^{3^j h})$ and since the cocycle $q_6^{3^j h}$ generates a \mathbb{Z}_3 module, the map is surjective if and only if $k \leq j + 1$.

As the group B_3 had dimension 2 the differential $\overline{d}_2 : \overline{E}^{1,1} \rightarrow \overline{E}^{3,0}$ must be surjective.

As we did before for the 2-torsion of higher order, the computation of the cardinality of $H^2(B_3; M_n \otimes \mathbb{Z}_{3^k})$ is straightforward and we can define the function

$$f(k) := \log_3(|H^2(B_3; M_n \otimes \mathbb{Z}_{3^k})|)$$

As the spectral sequence \overline{E}_r collapses at the term \overline{E}_3 we have $|H^2(B_3; M_n \otimes \mathbb{Z}_{3^k})| = |\overline{E}_3^{1,1} \oplus \overline{E}_3^{2,0}|$ and hence:

$$(41) \quad f(k) = \epsilon_n(k) \log_3(|\mathbb{Z}_3|) + \log_3(|\mathbb{Z}_{3^k}^{f_n}|) + \log_3(|\bigoplus_{i,F} \mathbb{Z}_{3^{(i+1) \wedge k}}|)$$

The first summand in equation (41) corresponds to the submodule in $\overline{E}_2^{2,0}$ generated by $\Phi_6^{-1}(q_6^{3^j h})$ and since the generator survives to \overline{E}_∞ if and only if $k > j + 1$ we have that the coefficient $\epsilon_n(k)$ is given by

$$\epsilon_n(k) = \begin{cases} 1 & \text{if } n = 12 \cdot 3^j h \text{ with } 3 \nmid h \text{ and } k > j + 1, \\ 0 & \text{otherwise.} \end{cases}$$

The second summand in (41) corresponds to the submodule $\mathbb{Z}_{3^k}^{f_n} \subset \overline{E}_2^{1,1}$.

The third summand in (41) corresponds to the submodule of $\Gamma_3^+[\mathcal{P}_3, \mathcal{Q}_3] \otimes \mathbb{Z}_{3^k} \subset \overline{E}_2^{1,1}$ generated, for i running over all positive integers, by the monic monomials in 2 variables $F(x, y)$ with $\deg F(\mathcal{P}_3, \mathcal{Q}_3) = n$, such that $F(\mathcal{P}_3, \mathcal{Q}_3)$ can be written as a monomial in $\mathcal{P}_3^{3^i}, \mathcal{Q}_3^{3^i}$, but not as a monomial in $\mathcal{P}_3^{3^{i+1}}, \mathcal{Q}_3^{3^{i+1}}$ and $F(\mathcal{P}_3, \mathcal{Q}_3)$ is not in the form $\mathcal{P}_3 \mathcal{Q}_3^j$. The direct sum runs over all these polynomials.

From (41) we get

$$f(k) = \epsilon_n(k) + f_n k + \sum_{i,F} (i+1) \wedge k$$

where the sum runs for i, F as before. It is clear that this is the logarithm of the cardinality of the degree n part of a module $N \otimes \mathbb{Z}_{3^k}$ where N is given by

$$\bigoplus_n \mathbb{Z}^{f_n} \oplus (\Gamma_3^+[\mathcal{P}_3, \mathcal{Q}_3] \oplus \mathbb{Z}[\overline{\mathcal{Q}}_3]) / \sim$$

where $\overline{\mathcal{Q}}_3$ is a new variable of the same degree ($= 12$) as \mathcal{Q}_3 and the quotient module is defined by the relations $\frac{\mathcal{Q}_3^n}{n!} \sim 3\overline{\mathcal{Q}}_3^n$ and $\mathcal{P}_3 \frac{\mathcal{Q}_3^n}{n!} \sim 0$.

From the description of the spectral sequence we can see the the extension involves the generator described in the statement. \square

Hence we have proved also the following :

Theorem 7.14.

$$H^2(B_3; M)_{(3)} = (\Gamma_3^+[\mathcal{P}_3, \mathcal{Q}_3] \oplus \mathbb{Z}[\overline{\mathcal{Q}}_3]) / \sim$$

with $\deg \overline{\mathcal{Q}}_3 = 12$, where the quotient of modules is defined by $\frac{\mathcal{Q}_3^n}{n!} \sim 3\overline{\mathcal{Q}}_3^n$ and $\mathcal{P}_3 \frac{\mathcal{Q}_3^n}{n!} \sim 0$. \square

8. TOPOLOGICAL COMPARISONS AND SPECULATIONS

The purpose of this section is to describe a topological space with the features that the reduced integral cohomology is precisely the torsion in $H^*(SL_2(\mathbb{Z}); M \otimes \mathbb{Z}[1/6])$.

In particular, there are spaces $T_p(2n+1)$ which are total spaces of p -local fibrations, $p > 3$, of the form

$$S^{2n-1} \rightarrow T_p(2n+1) \rightarrow \Omega(S^{2n+1})$$

for which the integral cohomology is recalled next where it is assumed that $p > 3$ throughout this section ([Ani93], [GT10]).

The integral cohomology ring of $\Omega(S^{2n+1})$ is the divided power algebra $\Gamma[x(2n)]$ where the degree of $x(2n)$ is $2n$. The mod- p homology ring (as a coalgebra) of $T_p(2n+1)$ is

$$E[v] \otimes F_p[w]$$

where the exterior algebra $E[v]$ is generated by an element v of degree $2n-1$, and the mod- p polynomial ring $F_p[w]$ is generated by an element w of degree $2n$. Furthermore, the r -th higher order Bockstein defined by b_r is given as follows:

$$b_r(w^{p^s}) = \begin{cases} 0 & \text{if } r < s \\ vw^{-1+p^s} & \text{if } r = s. \end{cases}$$

This information specifies the integral homology groups of $T_p(2n+1)$ (and thus the integral cohomology groups) listed in the following theorem. Recall the algebra $\Gamma_p[x]$ of Proposition 3.1 In addition fix a positive n , and consider the graded ring $\Gamma_p[x(2n)]$ where $x(2n)$ has degree $2n > 0$.

Theorem 8.1. *Assume that p is prime with $p > 3$.*

(1) *The reduced integral homology groups of $T_p(2n+1)$ are given as follows:*

$$\bar{H}_i(T_p(2n+1)) = \begin{cases} \mathbb{Z}/p^r\mathbb{Z} & \text{if } i = 2np^{r-1}k - 1 \text{ where } p \nmid k, \\ \{0\} & \text{otherwise.} \end{cases}$$

(2) *The additive structure for the cohomology ring of $T_p(2n+1)$ with $\mathbb{Z}_{(p)}$ coefficients is given as follows:*

$$\bar{H}^i(T_p(2n+1)) = \begin{cases} \mathbb{Z}/p^r\mathbb{Z} & \text{if } i = 2np^{r-1}k \text{ where } p \nmid k, \\ \{0\} & \text{otherwise.} \end{cases}$$

(3) *The cohomology ring of $T_p(2n+1)$ with $\mathbb{Z}_{(p)}$ coefficients is isomorphic to $\Gamma_p[x(2n)]$ in degrees strictly greater than 0. \square*

The next step is to compare the cohomology of the spaces $T_p(2n+1)$ to the earlier computations.

Theorem 8.2. *Assume that p is prime with $p > 3$. The p -torsion in the cohomology of*

$$E_{B_3} \times_{B_3} (\mathbb{C}\mathbb{P}^\infty)^2$$

is the reduced cohomology of the following space:

$$\Sigma^2(T_p(2p+3) \times T_p(2p^2 - 2p + 1)) \vee \Sigma(T_p(2p+3) \vee T_p(2p^2 - 2p + 1)). \quad \square$$

REFERENCES

- [Ani93] David Anick, *Differential algebras in topology*, Research Notes in Mathematics, vol. 3, A K Peters Ltd., Wellesley, MA, 1993. MR 1213682 (94h:55020)
- [Bri73] Egbert Brieskorn, *Sur les groupes de tresses [d'après V. I. Arnold]*, Séminaire Bourbaki, 24ème année (1971/1972), Exp. No. 401, Springer, Berlin, 1973, pp. 21–44. Lecture Notes in Math., Vol. 317. MR 0422674 (54 #10660)
- [Bro94] Kenneth S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York, 1994, Corrected reprint of the 1982 original. MR 1324339 (96a:20072)
- [Car56] H. Cartan, *Puissances divisées*, Séminaire Henri Cartan; 7e année: 1954/55. Algèbre d'Eilenberg-MacLane et homotopie, Exposé no. 7, Secrétariat mathématique, Paris, 1956, p. 11.
- [Coh01] Frederick R. Cohen, *On genus one mapping class groups, function spaces, and modular forms*, Topology, geometry, and algebra: interactions and new directions (Stanford, CA, 1999), Contemp. Math., vol. 279, Amer. Math. Soc., Providence, RI, 2001, pp. 103–128. MR 1850743 (2002f:55037)
- [DCP95] C. De Concini and C. Procesi, *Wonderful models of subspace arrangements*, Selecta Math. (N.S.) **1** (1995), no. 3, 459–494. MR 1366622 (97k:14013)
- [DCS96] C. De Concini and M. Salvetti, *Cohomology of Artin groups: Addendum: “The homotopy type of Artin groups”* [Math. Res. Lett. **1** (1994), no. 5, 565–577; MR1295551 (95j:52026)] by Salvetti, Math. Res. Lett. **3** (1996), no. 2, 293–297. MR 1386847 (97b:52015)
- [Dic11] Leonard Eugene Dickson, *A fundamental system of invariants of the general modular linear group with a solution of the form problem*, Trans. Amer. Math. Soc. **12** (1911), no. 1, 75–98. MR 1500882

- [EE69] Clifford J. Earle and James Eells, *A fibre bundle description of Teichmüller theory*, J. Differential Geom. **3** (1969), 19–43. MR 0276999 (43 #2737a)
- [Eic57] M. Eichler, *Eine Verallgemeinerung der Abelschen Integrale*, Math. Z. **67** (1957), 267–298. MR 0089928 (19,740a)
- [Fra33] Hermann Frasch, *Die Erzeugenden der Hauptkongruenzgruppen für Primzahlstufen*, Math. Ann. **108** (1933), no. 1, 229–252. MR 1512847
- [FTY88] Masaaki Furusawa, Michishige Tezuka, and Nobuaki Yagita, *On the cohomology of classifying spaces of torus bundles and automorphic forms*, J. London Math. Soc. (2) **37** (1988), no. 3, 520–534. MR 939127 (89f:57060)
- [Gro50] Emil Grosswald, *On the structure of some subgroups of the modular group*, Amer. J. Math. **72** (1950), 809–834. MR 0039753 (12,591c)
- [GT10] Brayton Gray and Stephen Theriault, *An elementary construction of Anick’s fibration*, Geom. Topol. **14** (2010), no. 1, 243–275. MR 2578305 (2011a:55013)
- [Han06] Juncheol Han, *The general linear group over a ring*, Bull. Korean Math. Soc. **43** (2006), no. 3, 619–626. MR 2264921 (2007i:11042)
- [Hat02] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR 1867354 (2002k:55001)
- [Kur34] A. G. Kurosh, *Die untergruppen der freien produkte von beliebigen gruppen*, Math. Ann. **109** (1934), no. 1, 647–660. MR 1512914
- [Loo74] Eduard Looijenga, *The complement of the bifurcation variety of a simple singularity*, Invent. Math. **23** (1974), 105–116. MR 0422675 (54 #10661)
- [Loo96] ———, *Stable cohomology of the mapping class group with symplectic coefficients and of the universal Abel-Jacobi map*, J. Algebraic Geom. **5** (1996), no. 1, 135–150. MR 1358038 (97g:14026)
- [Mil71] John Milnor, *Introduction to algebraic K-theory*, Princeton University Press, Princeton, N.J., 1971, Annals of Mathematics Studies, No. 72. MR 0349811 (50 #2304)
- [MKS66] Wilhelm Magnus, Abraham Karrass, and Donald Solitar, *Combinatorial group theory: Presentations of groups in terms of generators and relations*, Interscience Publishers [John Wiley & Sons, Inc.], New York-London-Sydney, 1966. MR 0207802 (34 #7617)
- [PR10] Jonathan Pakianathan and Nicholas Rogers, *Higher torsion in p -groups, Casimir operators and the classifying spectral sequence of a Lie algebra*, arXiv: 1007.2683v1, 2010.
- [Sal94] Mario Salvetti, *The homotopy type of Artin groups*, Math. Res. Lett. **1** (1994), no. 5, 565–577. MR 1295551 (95j:52026)
- [Sch27] O. Schreier, *Die Untergruppen der freien Gruppen.*, Abhandlungen Hamburg **5** (1927), 161–183 (German).
- [Ser03] Jean-Pierre Serre, *Trees*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003, Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation. MR 1954121 (2003m:20032)
- [Shi59] Goro Shimura, *Sur les intégrales attachées aux formes automorphes*, J. Math. Soc. Japan **11** (1959), 291–311. MR 0120372 (22 #11126)
- [Shi71] ———, *Introduction to the arithmetic theory of automorphic functions*, Publications of the Mathematical Society of Japan, No. 11. Iwanami Shoten, Publishers, Tokyo, 1971, Kanô Memorial Lectures, No. 1. MR 0314766 (47 #3318)
- [Ste87] Robert Steinberg, *On Dickson’s theorem on invariants*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **34** (1987), no. 3, 699–707. MR 927606 (89c:11177)
- [Waj80] Bronislaw Wajnryb, *On the monodromy group of plane curve singularities*, Math. Ann. **246** (1979/80), no. 2, 141–154. MR 564684 (81d:14016)

SCUOLA NORMALE SUPERIORE, PISA, ITALY

E-mail address: `f.callegaro@sns.it`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ROCHESTER, ROCHESTER NY 14627, USA

E-mail address: `cohf@math.rochester.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PISA, PISA ITALY

E-mail address: `salvetti@dm.unipi.it`