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THE SUM OF A LINEAR AND A LINEAR FRACTIONAL FUNCTION: PSEUDOCONVEXITY ON THE NONNEGATIVE ORTHANT AND SOLUTION METHODS

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Abstract

The aim of the paper is to present sequential methods for a pseudoconvex optimization problem whose objective function is the sum of a linear and a linear fractional function and the feasible region is a polyhedron, not necessarily compact. Since the sum of a linear and a linear fractional function is not in general pseudoconvex, we first derive conditions characterizing its pseudoconvexity on the nonnegative orthant. We prove that the sum of a linear and a linear fractional function is pseudoconvex if and only if it assumes particular canonical forms. Then, theoretical properties regarding the existence of a minimum point and its location are established, together with necessary and sufficient conditions for the infimum to be finite. The obtained results allow us to suggest simplex-like sequential methods for solving optimization problems having as objective function the proposed canonical forms.

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1 Introduction

During the last decades, the topic of generalized fractional programming and in particular quadratic and multiplicative fractional programming has attracted a sizable number of researchers, both in mathematics and in applied disciplines such as economics/management and engineering. More precisely there are several applicative problems that can be formulated in term of optimizing the ratio between a quadratic and a linear function or the sum of linear ratios. This happens for instance whenever we try to find a compromise between absolute and relative terms like profit and return on investment, or return and return/risk. Applications arise for example in portfolio theory (see among others [11, 12]), in location theory (see for example [2]) in transportation problems ([10, 13]) and in problems of optimizing firm capital, production development fund and social, cultural and construction fund (see [15]). Both theoretical and algorithmic aspects have been investigated as the huge number of contributes in the recent literature witnesses (see for all the recent surveys on this topic [9, 14, 16]). Generalized fractional problems have been studied even in the broader context of generalized convex programming. Ratios of convex and concave functions as well as the sum of such ratios are not convex, in general, even in the case of linear ratios. Nevertheless they are generalized convex in some sense and there are many papers related to the problem of finding conditions under which a certain class of fractional functions verifies some generalized convexity properties. Among the different classes of generalized convexity, the pseudoconvexity occupies a leading position for its nice properties in optimization.
For the sake of completeness, we recall that a differentiable function g, defined on an open convex set

$X \subseteq \Re^n$, is pseudoconvex if the following logical implication holds:

$$x, y \in X,\ g(y) < g(x) \Rightarrow\ (y-x)^T \nabla g(x) < 0.$$

Moreover, for a pseudoconvex function a critical point is a global minimum and a local minimum is also global.
Beside the well known characterizations of pseudoconvexity for fractional functions ([1, 8]), there are several contributions suggesting "operative" necessary and sufficient conditions which are useful to test if a generalized fractional function is pseudoconvex or not (see for all [5] and references there). Pseudoconvexity is often characterized over the halfspace associated with the positivity of the denominator of the function; on the other hand, in many economic applications it is crucial to study the behavior of the objective function over a smaller set. This has lead ([4, 5]) to investigate the maximal domains of pseudoconvexity. Since in many optimization problems decision variables are nonnegative, in this paper we characterize the pseudoconvexity of a function f which is the sum of a linear and a linear fractional function on the nonnegative orthant. More precisely, in Section 2, we derive "very easy to be checked" conditions which ensure that f is pseudoconvex just on the nonnegative orthant. We get that f is pseudoconvex if and only if it assumes particular canonical forms. According with the key role of pseudoconvexity in optimization theory, it seems natural conceiving algorithms for generalized fractional problems which directly benefit from the nice properties of pseudoconvex functions. For such a reason in Section 3, we consider minimization problems whose the objective function f is pseudoconvex on the nonnegative orthant and the feasible region is a polyhedral set, not necessarily compact. We establish theoretical properties regarding the existence of a minimum point and its location, and we propose necessary and sufficient conditions for the infimum to be finite. At last, in Section 4, we suggest simplex-like algorithms, based on the so called optimal level solution method proposed in [3].

2 Statement of the problem and preliminary results

Consider the following problem

$$P: \inf_{x \in S} \left[f(x) = a^T x + \frac{c^T x + c_0}{d^T x + d_0} \right]$$

where $a, c, d \in \Re^n,\ d \neq 0,\ c_0, d_0 \in \Re, d_0 \neq 0$ and $S \subseteq H = \{x \in \Re^n : d^T x + d_0 > 0\}$ is a polyhedral set.
First of all we study the pseudoconvexity of f on the nonnegative orthant. With this aim we recall the following theorem (see [5]) which characterizes the maximal domains $D_{\max}$ of pseudoconvexity of f, in the sense that f is pseudoconvex on a convex set C with nonempty interior if and only if $C \subseteq D_{\max}$.

Theorem 1 *Consider the function f. The following conditions hold:*
i) if $a = \alpha d,\ \alpha \geq 0$, then f is pseudoconvex on H;
ii) if $c = \gamma d,\ c_0 - \gamma d_0 \geq 0$, then f is pseudoconvex on H;
iii) if $a = \alpha d,\ \alpha < 0$, and $c = \gamma d,\ c_0 - \gamma d_0 < 0$, then f is pseudoconvex on every open convex set C such that:

$$C \subseteq \{x \in \Re^n : d^T x + d_0 > d_0^*\} \ \text{or}\ C \subseteq \{x \in \Re^n : 0 < d^T x + d_0 < d_0^*\}$$

where $d_0^* = \sqrt{\dfrac{c_0 - \gamma d_0}{\alpha}}$*;*
iv) if $c = \beta a + \gamma d,\ \beta > 0$ and $\operatorname{rank}[a, d] = 2$, then f is pseudoconvex on every open convex set C such that:

$$C \subseteq \{x \in \Re^n : \beta a^T x + c_0 - \gamma d_0 > 0,\ d^T x + d_0 > 0\};$$

v) if $c = \beta a + \gamma d,\ \beta < 0$ and $\operatorname{rank}[a, d] = 2$, then f is pseudoconvex on every open convex set C such that:

$$C \subseteq \{x \in \Re^n : \beta a^T x + c_0 - \gamma d_0 > 0,\ d^T x + d_0 + \beta > 0\}$$

or

$$C \subseteq \{x \in \Re^n : \beta a^T x + c_0 - \gamma d_0 < 0,\ 0 < d^T x + d_0 < -\beta\}.$$

In any other case f is not pseudoconvex on $C \subseteq H$ whatever the open convex set C is.

Specifying the previous result we get the following characterization of the pseudoconvexity of f on the nonnegative orthant.

Theorem 2 *The function $f(x) = a^T x + \frac{c^T x + c_0}{d^T x + d_0}$ is pseudoconvex on $\Re^n_+$ if and only if $d \in \Re^n_+ \setminus \{0\}$, $d_0 > 0$ and one of the following conditions holds:*
i) $a = \alpha d$, $\alpha \geq 0$;
ii) $c = \gamma d$, $c_0 - \gamma d_0 \geq 0$;
iii) $a = \alpha d$, $\alpha < 0$, $c = \gamma d$, $c_0 - \gamma d_0 < 0$ and $d_0 > \sqrt{\frac{c_0 - \gamma d_0}{\alpha}}$;
iv) $c = \beta a + \gamma d$, $\beta > 0$, $\mathrm{rank}[a, d] = 2$, $a \in \Re^n_+$ and $c_0 - \gamma d_0 > 0$;
v) $c = \beta a + \gamma d$, $\beta < 0$, $\mathrm{rank}[a, d] = 2$, $a \in \Re^n_-$, $c_0 - \gamma d_0 > 0$ and $d_0 + \beta > 0$.

Proof. It is sufficient to note that a halfspace $\{x \in \Re^n :\ v^T x + v_0 > 0\} \supset \Re^n_+$ if and only if $v \in \Re^n_+ \setminus \{0\}$ and $v_0 > 0$. □

Let us note that when i) or ii) holds f is pseudoconvex on the whole halfspace H; theoretical properties and sequential methods for Problem P in such cases have been already established in [7]. Therefore, we limit ourselves to deal with the following problems corresponding to the cases iii)-v) of Theorem 2.

- **Case iii)** Function f assumes the form $f(x) = \alpha d^T x + \frac{\gamma d^T x + c_0}{d^T x + d_0}$, so that Problem P is equivalent to Problem
$$P_1 : \inf_{x \in S} \left[f_1(x) = \alpha d^T x + \frac{c_0^*}{d^T x + d_0} \right]$$
where $d \in \Re^n_+ \setminus \{0\}$, $d_0 > 0$, $\alpha < 0$, $c_0^* < 0$, $d_0 > \sqrt{\frac{c_0^*}{\alpha}}$ and $S \subseteq \Re^n_+$.

- **Case iv)** Function f assumes the form $f(x) = a^T x + \frac{(\beta a + \gamma d)^T x + c_0}{d^T x + d_0}$, so that Problem P is equivalent to Problem
$$P_2 : \inf_{x \in S} \left[f_2(x) = a^T x + \frac{\beta a^T x + c_0^*}{d^T x + d_0} \right]$$
where $d \in \Re^n_+ \setminus \{0\}$, $d_0 > 0$, $\beta > 0$, $\mathrm{rank}[a, d] = 2$, $a \in \Re^n_+$, $c_0^* > 0$ and $S \subseteq \Re^n_+$.

- **Case v)** Function f assumes the form $f(x) = a^T x + \frac{(\beta a + \gamma d)^T x + c_0}{d^T x + d_0}$, so that Problem P is equivalent to the Problem
$$P_3 : \inf_{x \in S} \left[f_3(x) = a^T x + \frac{\beta a^T x + c_0^*}{d^T x + d_0} \right],$$
where $d \in \Re^n_+ \setminus \{0\}$, $d_0 > 0$, $\beta < 0$, $\mathrm{rank}[a, d] = 2$, $a \in \Re^n_-$, $c_0^* > 0$, $d_0 + \beta > 0$ and $S \subseteq \Re^n_+$.

3 Theoretical properties

In this section we establish some theoretical properties for problems $P_1 - P_3$ which will allow us to suggest suitable sequential methods.
In general, when the minimum is not attained, problem P may have finite or not finite infimum. In any case there exists an extreme direction of the polyhedral feasible set S on which f reaches the infimum, as it is stated in the following theorem whose proof can be found in [7].

Theorem 3 *Let ℓ be the infimum of problem P.*
i) ℓ is attained as a minimum if and only if there exists a feasible point x_0 belonging to an edge of S such that $f(x_0) = \ell$.
ii) If ℓ is not attained as a minimum, then there exist a feasible point x_0 and an extreme direction u such that $\ell = \lim_{t \to +\infty} f(x_0 + tu)$.
iii) The infimum ℓ is not finite if and only if there exist a feasible point x_0 and an extreme direction u such that $\lim_{t \to +\infty} f(x_0 + tu) = -\infty$.

Remark 4 *Recall that $u \in \Re^n$ is an extreme direction for S if and only if for every $x_0 \in S$ we have $x_0 + tu \in S,\ \forall t \geq 0$. Consequently, since $d^T x + d_0 > 0, \forall x \in S$, necessarily we have $d^T u \geq 0$.*

The particular structure of the pseudoconvex objective function f allows us to specify the properties of an extreme direction along which the infimum is reached.
As regards to problem P_1 we have the following results.

Theorem 5 *Consider Problem P_1.*
i) A feasible point x_0 is a minimum for Problem P_1 if and only if x_0 is a maximum point for the linear problem $\max_{x \in S} d^T x$.
ii) $\inf_{x \in S} f_1(x) = -\infty$ if and only if there exists an extreme direction u such that $d^T u > 0$ or, equivalently, $\sup_{x \in S} d^T x = +\infty$.

Proof. It is sufficient to prove that a feasible direction is a decreasing direction for function f_1 if and only if it is an increasing direction for the linear function $d^T x$. With this aim, consider the restriction $\varphi(t) = f_1(x_0 + tu)$ where $x_0 \in S$ and u is a feasible direction.
We have: $\varphi'(t) = d^T u \left[\alpha - \dfrac{c_0^*}{\left(d^T x_0 + t d^T u + d_0\right)^2} \right]$, $\varphi'(0) = \dfrac{\alpha d^T u}{\left(d^T x_0 + d_0\right)^2} \left(\left(d^T x_0 + d_0\right)^2 - \dfrac{c_0^*}{\alpha} \right)$.
The assumptions $\alpha < 0$ and $d_0 > \sqrt{\frac{c_0^*}{\alpha}}$ imply $\left(d^T x_0 + d_0\right)^2 - \frac{c_0^*}{\alpha} > 0$. Consequently, we have $\varphi'(0) < 0$ if and only if $d^T u > 0$. The thesis follows. □

Regarding Problem P_2 note that $f(x) > a^T x \geq 0\ \forall x \in S \subseteq \Re^n_+$. This allows us to prove that if the infimum is not attained as a minimum, then it coincides with the minimum value of the linear function $a^T x$. To get this result we first establish the following theorem.

Theorem 6 *Consider Problem P_2. The infimum is not attained as a minimum if and only if there exists an extreme direction u such that $a^T u = 0$ and $d^T u > 0$.*

Proof. From ii) of Theorem 3, there exist an extreme direction u and $x_0 \in S$ such that:

$$\ell = \lim_{t \to +\infty} f_2(x_0 + tu) = \lim_{t \to +\infty} \left[a^T x_0 + t a^T u + \frac{\beta a^T x_0 + t \beta a^T u + c_0^*}{d^T x_0 + d_0 + t d^T u} \right].$$

Since f_2 is lower bounded, the limit ℓ is finite so that necessarily we have $a^T u = 0$ and $d^T u > 0$. Conversely, let u be an extreme direction such that $a^T u = 0$ and $d^T u > 0$. For every $x \in S$ it results $f_2(x) > f_2(x + tu),\ \forall t > 0$ and, consequently, f_2 does not assume minimal value on S. □

Theorem 7 *Consider Problem P_2. The infimum is not attained as a minimum if and only if $\inf_{x \in S} f_2(x) = \min_{x \in S} a^T x$.*

Proof. Since $f_2(x) > a^T x,\ \forall x \in S$, the equality $\inf_{x\in S} f_2(x) = \min_{x\in S} a^T x$ implies that the infimum is not attained. Conversely, from Theorem 6, there exist $x_0 \in S$ and an extreme direction u with $a^T u = 0$, $d^T u > 0$. Let x^* be the minimum point for $\min_{x\in S} a^T x$ and consider the restriction of f on the feasible half-line $x^* + tu$. We have $a^T x^* \le \ell = \lim_{t\to+\infty} f_2(x_0 + tu) \le \lim_{t\to+\infty} f_2(x^* + tu) = a^T x^*$, so that the thesis follows. □

Unlike problem P_2, problem P_3 may have finite or not finite infimum as it is stated in the following theorem.

Theorem 8 *Consider Problem* P_3.
i) We have $\inf_{x\in S} f_3(x) = -\infty$ *if and only if there exists an extreme direction* u, *such that* $a^T u < 0$ *and* $d^T u \ge 0$ *or, equivalently, if and only if* $\inf_{x\in S} a^T x = -\infty$.
ii) The infimum is finite and not attained as a minimum if and only there exists an extreme direction u, *such that* $a^T u = 0$ *and* $d^T u > 0$ *or, equivalently, if and only if* $\inf_{x\in S} f_3(x) = \min_{x\in S} a^T x$.

Proof. Let u be an extreme direction u and $x_0 \in S$; we have $\lim_{t\to+\infty} f_3(x_0 + tu) = \lim_{t\to+\infty} ta^T u$ if $d^T u > 0$, $\lim_{t\to+\infty} f_3(x_0+tu) = \lim_{t\to+\infty} ta^T u(d^T x_0+d_0+\beta)$ if $d^T u = 0$. Consequently, $\inf_{x\in S} f_3(x_0+tu) = -\infty$ if and only if $a^T u < 0$ and $d^T u \ge 0$, while $\inf_{x\in S} f_3(x_0 + tu)$ is finite if and only if $a^T u = 0$, $d^T u > 0$. Moreover, with a proof similar to the one given in Theorem 7, we get $\inf_{x\in S} f_3(x) = \min_{x\in S} a^T x$ and the proof is complete. □

4 Sequential methods

Taking into account Theorem 5, problem P_1 can be solved by means of any linear programming algorithm. In particular an optimal solution, if one exists, is attained at a vertex of S.
Regarding Problem $P_i, i = 2, 3$, the idea of the sequential method that we are going to describe, is to transform the problem into a linear parametric one where the parameter denotes the level of the linear function $d^T x + d_0$.
More exactly, set $d^T x + d_0 = \theta$, $S(\theta) = S \cap \{x \in \Re^n :\ d^T x + d_0 = \theta\}$, $\bar{\Theta} = \{\theta \in \Re : S(\theta) \ne \emptyset\}$ and consider the following parametric problem

$$P_i(\theta) : \inf_{x\in S(\theta)} f_i(x),\ \ i \in \{2, 3\}.$$

Obviously we have $\inf_{x\in S} f_i(x) = \inf_{\theta\in\bar{\Theta}} z_i(\theta)$, where $z_i(\theta)$ is the optimal value function of the parametric problem $P_i(\theta)$, i.e., $z_i(\theta) = \inf_{x\in S(\theta)} f_i(x)$.
The pseudoconvexity of f_i allows us to state a basic relationship between the optimal solution for problem P_i and a local minimum point of $z_i(\theta)$.

Theorem 9 *If* θ_0 *is a local minimum for* $z_i(\theta)$, *then an optimal solution for problem* $P_i(\theta_0)$ *is a global minimum for Problem* P_i.

Proof. By assumption there exists $\epsilon > 0$ such that $z_i(\theta_0) \le z_i(\theta)$, $\forall\theta \in (\theta_0 - \epsilon, \theta_0 + \epsilon)$. Let x_0 be an optimal solution for $P_i(\theta_0)$, so that $f_i(x_0) = z_i(\theta_0)$. Assume, by contradiction, that x_0 is not a local minimum for f_i. Consequently, $\forall\delta > 0$, there exists $x^* \in (x_0 + \delta\mathcal{B}) \cap S$, with $f_i(x^*) < f_i(x_0)$, where $\mathcal{B}$ is the unit ball.

Choosing $\delta = \frac{\epsilon}{2\|d\|}$, we have $x^* \in (x_0 + \delta\mathcal{B}) \cap S \subset \{x : \theta_0 - \epsilon < d^T x + d_0 < \theta_0 + \epsilon\}$. In fact $d^T x^* = d^T x_0 + \frac{\epsilon}{2\|d\|} d^T u$ and hence $d^T x_0 - \epsilon < d^T x^* < d^T x_0 + \epsilon$, or, equivalently, $\theta_0 - \epsilon < \theta^* < \theta_0 + \epsilon$, where $\theta^* = d^T x^* + d_0$.
We have $z_i(\theta_0) = f(x_0) > f_i(x^*) \geq \min\limits_{x \in S(\theta^*)} f_i(x) = z_i(\theta^*) \geq z_i(\theta_0)$ and we get a contradiction. Consequently, x_0 is a local minimum for f_i and the thesis follows from the pseudoconvexity of the function f_i. □

As a consequence of Theorem 9, an optimal solution for problem P_i can be found by looking for a local minimum for the function $z_i(\theta)$. The following theorem points out that a local minimum for $z_i(\theta)$, if one exists, belongs to the half-line $[\theta_0, +\infty)$, where θ_0 is the level of the denominator corresponding to an optimal solution for the linear problem $\min\limits_{x \in S} a^T x$.

Theorem 10 *Let x_0 be an optimal solution for the linear problem $\min\limits_{x \in S} a^T x$ and set $\theta_0 = d^T x_0 + d_0$. We have $\inf\limits_{x \in S} f_i(x) = \inf\limits_{\theta \geq \theta_0} z_i(\theta)$.*

Proof. Since $\inf\limits_{x \in S} f_i(x) = \inf\limits_{\theta \in \bar{\Theta}} z_i(\theta)$, we must prove that $f_i(x) < f_i(x_0)$ implies $\theta > \theta_0$. We have $f_i(x) - f_i(x_0) = \dfrac{\theta + \beta}{\theta}(a^T x - a^T x_0) + \dfrac{\theta_0 - \theta}{\theta_0 \theta}(\beta a^T x_0 + c_0^*)$.
Taking into account that $\dfrac{\theta + \beta}{\theta} = \dfrac{d^T x + d_0 + \beta}{d^T x + d_0} > 0$, $\beta a \in \Re^n_+$, $c_0^* > 0$, $f_i(x) - f_i(x_0) < 0$ necessarily implies $\theta > \theta_0$. □

For every fixed feasible level θ of the parameter, we have $f_i(x) = \dfrac{\theta + \beta}{\theta} a^T x + \dfrac{c_0^*}{\theta}$. Since $\dfrac{\theta + \beta}{\theta} > 0$, an optimal solution for $P_i(\theta)$ is found by solving the linear parametric problem

$$P_i^*(\theta) : \min_{x \in S(\theta)} a^T x$$

With this regard, let $S = \{x \in \Re^n : Ax = b, x \geq 0\}$ be the feasible region of problem P_i, where A is a $m \times n$ matrix, rank $A = m < n$, $b \in \Re^m$ and set $A^* = \begin{bmatrix} A \\ d^T \end{bmatrix}$, $b^* = \begin{pmatrix} b \\ -d_0 \end{pmatrix} + \theta e^{m+1}$, where $e^{m+1} \in \Re^{m+1}$ is the unit vector having the $(m+1)$-th component equal to 1 and all others equal to 0.
We have $S(\theta) = \{x \in \Re^n : A^* x = b^*, x \geq 0\}$.
Note that rank A^* = rank A if and only if there exists $\lambda \in \Re^n$ such that $d = \lambda^T A$. This implies $d^T x = \lambda^T A x = \lambda^T b$, $\forall x \in S$, so that problem P_i reduces to a linear programming problem. For such a reason in what follows we will assume rank $A^* = m + 1$.
We will use the following standard notations.
Let B be the basis corresponding to a basic feasible solution $x(\theta) \in S(\theta)$ which is also a vertex of S. We will partition the matrix A^* and the vectors a, x, as $A^* = [A_B^* | A_N^*]$, $a^T = (a_B^T, a_N^T)$, $x^T = (x_B^T, x_N^T)$.
Now we are able to find a relationship between an optimal basic solution $x(\bar{\theta}) = (\bar{x}_B, 0)$ for problem $P_i^*(\bar{\theta})$ and the optimal value function $z_i(\theta)$ evaluated on the stability interval $\mathcal{F} = \{\theta \in \Re : x_B(\theta) = x_B + \theta u_B \geq 0\} = [\theta_{\min}, \theta_{\max}]$, where $u_B = (A_B^*)^{-1} e^{m+1}$, $x_B(\bar{\theta}) = \bar{x}_B$ and $\theta_{\max}$ may be also $+\infty$.
We have

$$z_i(\theta) = \frac{\theta + \beta}{\theta} a_B^T (x_B + \theta u_B) + \frac{c_0^*}{\theta}, \ \theta \in \mathcal{F}$$

$$z_i'(\theta) = \frac{1}{\theta^2} \left(a_B^T u_B \theta^2 - c_0^* - \beta a_B^T x_B \right), \ \theta \in \mathcal{F}.$$

The idea of the sequential method that we are going to describe is the following.
Let x_0 be an optimal solution for $\min\limits_{x \in S} a^T x$, set $\theta_0 = d^T x_0 + d_0$ and let $x(\theta_0)$ be an optimal basic

solution for $P_i^*(\theta_0)$. Starting from θ_0, we will consider increasing level θ of the denominator up to find a local minimum of $z_i(\theta)$. More precisely, by applying sensitivity analysis, we find the stability interval $\mathcal{F} = [\theta_{\min}, \theta_{\max}]$, $\theta_{\min} = \theta_0$, associated with $x(\theta_0)$. If $z_i(\theta)$ has a critical point $\hat{\theta} \in \mathcal{F}$, then $x(\hat{\theta})$ is a global minimum for problem P_i; if $z_i(\theta)$ is decreasing on $\mathcal{F}$, the feasibility is lost for $\theta > \theta_{\max}$, and it is restored by applying a dual simplex iteration. In this last case we find a new stability interval and we repeat the analysis.
With respect to a stability interval $\mathcal{F} = [\theta_{\min}, \theta_{\max}]$, taking into account that $c_0^* + \beta a_B^T x_B > 0$, we have the following exhaustive cases.

- If $a_B^T u_B \leq 0$, then $z_i(\theta)$ is non-increasing on $\mathcal{F}$.
- If $a_B^T u_B > 0$, then $\hat{\theta} = \sqrt{\dfrac{c_0^* + \beta a_B^T x_B}{a_B^T u_B}}$ is a strict local minimum for $z_i(\theta)$. If $\hat{\theta} \in \mathcal{F}$, then $x(\hat{\theta})$ is a global minimum for problem P_i, otherwise $z_i(\theta)$ is decreasing on $\mathcal{F}$.

Now we are able to state the main steps of the algorithm.

Step 0 Solve $\min_{x \in S} a^T x$. If there do not exist solutions, STOP: $\inf_{x \in S} f_3(x) = -\infty$, otherwise let x_0 be an optimal solution, calculate $\theta_0 = d^T x_0 + d_0$ and let $x(\theta_0)$ be an optimal basic solution for $P_i^*(\theta_0)$ (see Remark 11). Set $\theta_0 = \theta_{\min}$ and go to Step 1.

Step 1 Determine the stability interval $\mathcal{F} = [\theta_{\min}, \theta_{\max}]$ associated with the optimal solution $x(\theta_{\min}) = (x_B + \theta_{\min} u_B, 0)$ of $P_i^*(\theta_{\min})$. If $a_B^T u_B \leq 0$, then go to Step 2, otherwise compute $\hat{\theta} = \sqrt{\frac{c_0^* + \beta a_B^T x_B}{a_B^T u_B}}$. If $\hat{\theta} \in \mathcal{F}$, then $x(\hat{\theta})$ is an optimal solution for problem P_i. If $\hat{\theta} > \theta_{\max}$, go to Step 3, otherwise STOP: $x(\theta_{\min})$ is an optimal solution for P_i.

Step 2 If $\theta_{max} = +\infty$, STOP: $\inf_{x \in S} f_i(x) = \min_{x \in S} a^T x$; otherwise go to Step 3.

Step 3 Let k be such that $x_{B_k} + \theta_{\max} u_{B_k} = 0$. Perform a pivot operation by means of the dual simplex algorithm, set $\theta_{\max} = \theta_{\min}$ and go to Step 1; if such pivot operation cannot be performed, STOP: $x(\theta_{\max})$ is an optimal solution for P_i.

Remark 11 *In order to find $x(\theta_0)$, we must extend the basis associated with x_0. With respect to such a basis, let $A = [A_B | A_N]$, $\bar{a}_N^T = a_N^T - a_B^T A_B^{-1} A_N$, $\bar{d}_N^T = d_N^T - d_B^T A_B^{-1} A_N$. The new variable which enters the basis, corresponding to the parametric constraint, is x_{N_k}, where N_k is such that $\dfrac{\bar{a}_{N_k}}{\bar{d}_{N_k}} = \min_{\bar{d}_{N_j} > 0} \left\{ \dfrac{\bar{a}_{N_j}}{\bar{d}_{N_j}} \right\}$.*

The following example shows the main steps of the algorithm.

Example 12 *Consider problem P_2 where $f(x) = 2x_1 + 3x_2 + \dfrac{4x_1 + 6x_2 + 76}{x_1 + x_2 + 1}$ and the feasible region is $S = \{x \in \Re^4 : 22x_1 - 9x_2 + x_3 = 44,\ 2x_1 + x_2 + x_4 = 1,\ x_i \geq 0, i = 1, .., 4\}$.
The vertex $x_0 = \left(\frac{1}{2}, 0, 33, 0\right)^T$ is the minimum for $a^T x = 2x_1 + 3x_2$ on S; by introducing the parametric constraint $x_1 + x_2 = \theta - 1$, we get $x(\theta) = (-1 + \theta, 0, 66 - 22\theta, -3 + 2\theta)^T$, so that the stability interval is $\mathcal{F} = \left[\frac{3}{2}, 3\right]$. The critical point of the optimal value function $z(\theta)$ is $\widehat{\theta} = 6$; since $\widehat{\theta} \notin \mathcal{F}$, $z(\theta)$ is decreasing on $\mathcal{F}$. For $\theta > 3$ the fesibility is lost and it is restored by means of an iteration of the dual simplex algorithm. We have $x(\theta) = \left(\dfrac{35}{31} + \dfrac{9}{31}\theta, -\dfrac{66}{31} + \dfrac{22}{31}\theta, 0, -\dfrac{27}{31} + \dfrac{40}{31}\theta\right)^T$, $\mathcal{F} = [3, +\infty]$. The critical point of $z(\theta)$ is $\widehat{\theta} = 5$, since $\widehat{\theta} \in \mathcal{F}$, then the optimal solution for P_2 is $\bar{x} = \left(\dfrac{80}{31}, \dfrac{44}{31}, 0, \dfrac{173}{31}\right)^T$ obtained by setting $\theta = 5$ in $x(\theta)$. Note that $\bar{x}$ belongs to an edge of S.*

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