QUASI-SELECTIVE ULTRAFILTERS
AND ASYMPTOTIC NUMEROSITIES

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ABSTRACT. We isolate a new class of ultrafilters on $\mathbb{N}$, called "quasi-selective" because they are intermediate between selective ultrafilters and P-points. (Under the Continuum Hypothesis these three classes are distinct.) The existence of quasi-selective ultrafilters is equivalent to the existence of "asymptotic numerosities" for all sets of tuples $A \subseteq \mathbb{N}^k$. Such numerosities are hypernatural numbers that generalize finite cardinalities to countable point sets. Most notably, they maintain the structure of ordered semiring, and, in a precise sense, they allow for a natural extension of asymptotic density to all sequences of tuples of natural numbers.

INTRODUCTION

Special classes of ultrafilters over $\mathbb{N}$ have been introduced and variously applied in the literature, starting from the pioneering work by G. Choquet [8, 9] in the sixties.

In this paper we introduce a new class of ultrafilters, namely the *quasi-selective ultrafilters*, as a tool to generate a good notion of "equinumerosity" on the sets of tuples of natural numbers (or, more generally, on all point sets $A \subseteq \mathcal{L}^k$ over a countable line $\mathcal{L}$). By equinumerosity we mean an equivalence relation that preserves the basic properties of equipotency for finite sets, including the Euclidean principle that "the whole is greater than the part". More precisely, we require that – similarly as finite numbers – the corresponding numerosities be the non-negative part of a discretely ordered ring, where 0 is the numerosity of the empty set, 1 is the numerosity of every singleton, and sums and products of numerosities correspond to disjoint unions and Cartesian products, respectively.

This idea of numerosities that generalize finite cardinalities has been recently investigated by V. Benci, M. Di Nasso and M. Forti in a series of papers, starting from [1], where a numerosity is assigned to each pair $\langle A, \ell_A \rangle$, depending on the (finite-to-one) "labelling function" $\ell_A : A \to \mathbb{N}$. The existence of a numerosity function for labelled sets turns out to be equivalent to the existence of a selective ultrafilter. That research was then continued by investigating a similar notion of numerosity for sets of arbitrary cardinality, namely: sets of ordinals in [2], subsets of a superstructure in [3], point sets over the real line in [11]. A related notion of "fine density" for sets of natural numbers is introduced and investigated in [10]. In each of these contexts special classes of ultrafilters over large sets naturally arise.

Here we focus on subsets $A \subseteq \mathbb{N}^k$ of tuples of natural numbers, and we show that the existence of particularly well-behaved equinumerosity relations (which we

call "asymptotic") for such sets is equivalent to the existence of another special kind of ultrafilters, named quasi-selective ultrafilters. Such ultrafilters may be of independent interest, because they are closely related (but not equivalent) to other well-known classes of ultrafilters that have been extensively considered in the literature. In fact, on the one hand, all selective ultrafilters are quasi-selective and all quasi-selective ultrafilters are P-points. On the other hand, it is consistent that these three classes of ultrafilters are distinct.

The paper is organized as follows. In Section 1 we introduce the class of quasi-selective ultrafilters on $\mathbb{N}$ and we study their properties, in particular their relationships with P-points and selective ultrafilters. In Section 2, assuming the Continuum Hypothesis, we present a general construction of quasi-selective non-selective ultrafilters, that are also weakly Ramsey in the sense of [4, 12]. In Section 3 we introduce axiomatically a general notion of "equinumerosity" for sets of tuples of natural numbers. In Section 4 we show that the resulting numerosities, where sum, product and ordering are defined in the standard Cantorian way, are the non-negative part of an ordered ring. Namely this ring is isomorphic to the quotient of a ring of power-series modulo a suitable ideal. In Section 5 we introduce the special notion of "asymptotic" equinumerosity, which generalizes the fine density of [10]. We show that there is a one-to-one correspondence between asymptotic equinumerosities and quasi-selective ultrafilters, where equinumerosity is witnessed by a special class of bijections depending on the ultrafilter ("$\mathcal{U}$-congruences"). The corresponding semiring of numerosities is isomorphic to an initial cut of the ultrapower $\mathbb{N}_{\mathcal{U}}^{\mathbb{N}}$. In particular, asymptotic numerosities exist if and only if there exist quasi-selective ultrafilters. Final remarks and open questions are contained in the concluding Section 6.

In general, we refer to [7] for definitions and basic facts concerning ultrafilters, ultrapowers, and nonstandard models, and to [6] for special ultrafilters over $\mathbb{N}$.

The authors are grateful to Vieri Benci for many useful discussions and suggestions.

1. QUASI-SELECTIVE ULTRAFILTERS

We denote by $\mathbb{N}$ the set of all *nonnegative integers*, and by $\mathbb{N}^+$ the subset of all *positive* integers.

Recall that, if $\mathcal{F}$ is a filter on X, then two functions $f, g : X \to Y$ are called $\mathcal{F}$-*equivalent* if $\{x \in X \mid f(x) = g(x)\} \in \mathcal{F}$. In this case we write $f \equiv_{\mathcal{F}} g$.

Definition 1.1. A nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is called *quasi-selective* if every function f such that $f(n) \leq n$ for all $n \in \mathbb{N}$ is $\mathcal{U}$-equivalent to a nondecreasing one.

The name 'quasi-selective' recalls one of the characterizations of *selective* (or *Ramsey*) ultrafilters (see *e.g.* [1, Prop. 4.1]), namely

- *The ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is selective if and only if every $f : \mathbb{N} \to \mathbb{N}$ is $\mathcal{U}$-equivalent to a nondecreasing function.*

In particular all selective ultrafilters are quasi-selective.

Let us call "interval-to-one" a function $g : \mathbb{N} \to \mathbb{N}$ such that, for all n, $g^{-1}(n)$ is a (possibly infinite) interval of $\mathbb{N}$.

Proposition 1.2. *Let $\mathcal{U}$ be a quasi-selective ultrafilter. Then any partition of $\mathbb{N}$ is an* interval partition *when restricted to a suitable set in $\mathcal{U}$. Hence every $f : \mathbb{N} \to \mathbb{N}$*

is $\mathcal{U}$-equivalent to an "interval-to-one" function. In particular all quasi-selective ultrafilters are P-points.

Proof. Given a partition of $\mathbb{N}$, consider the function f mapping each number to the least element of its class. Then $f(n) \leq n$ for all $n \in \mathbb{N}$, so, by quasi-selectivity, there exists a set $U \in \mathcal{U}$ such that the restriction $f_{|U}$ is nondecreasing. Then the given partition is an interval partition when restricted to U. $\qquad\square$

We say that a function $f : \mathbb{N} \to \mathbb{N}$ has *polynomial growth* if it is eventually dominated by some polynomial, *i.e.* if there exist k, m such that for all $n > m$, $f(n) \leq n^k$.

We say that a function $f : \mathbb{N} \to \mathbb{N}$ has *minimal steps* if $|f(n+1) - f(n)| \leq 1$ for all $n \in \mathbb{N}$.

Proposition 1.3. *The following properties are equivalent for a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$:*

(1) *$\mathcal{U}$ is quasi-selective;*
(2) *every function of polynomial growth is $\mathcal{U}$-equivalent to a nondecreasing one;*
(3) *every function with minimal steps is $\mathcal{U}$-equivalent to a nondecreasing one.[1]*

Proof. (1)$\Longrightarrow$(2). We prove that if every function $f < n^k$ is $\mathcal{U}$-equivalent to a nondecreasing one, then the same property holds for every function $g < n^{2k}$. The thesis then follows by induction on k. Given g, let f be the integral part of the square root of g. So $g < f^2 + 2f + 1$, and hence $g = f^2 + f_1 + f_2$ for suitable functions $f_1, f_2 \leq f < n^k$. By hypothesis we can pick nondecreasing functions f', f_1', f_2' that are $\mathcal{U}$-equivalent to f, f_1, f_2, respectively. Then clearly g is $\mathcal{U}$-equivalent to the nondecreasing function $f'^2 + f_1' + f_2'$.

(2)$\Longrightarrow$(3) is trivial.

(3)$\Longrightarrow$(1). We begin by showing that (3) implies the following

Claim. *There exists $U = \{u_1, \ldots, u_n \ldots\} \in \mathcal{U}$ such that $u_{n+1} > 2u_n$.*

Define three minimal step functions f_0, f_1, f_2 as follows, for $k \in \mathbb{N}^+$:

$$f_0(m) = 2^{k-1} - |3 \cdot 2^{k-1} - m| \ \text{ for } \ 2^k \leq m \leq 2^{k+1}$$

$$f_1(m) = 9 \cdot 2^{2k-1} - |15 \cdot 2^{2k-1} - m| \ \text{ for } \ 3 \cdot 2^{2k} \leq m \leq 3 \cdot 2^{2k+2}$$

$$f_2(m) = 9 \cdot 2^{2k-2} - |15 \cdot 2^{2k-2} - m| \ \text{ for } \ 3 \cdot 2^{2k-1} \leq m \leq 3 \cdot 2^{2k+1}$$

The graphs of these functions are made up of the catheti of isosceles right triangles whose hypotenuses are placed on the horizontal axis. The function f_0 is decreasing in the intervals $[3 \cdot 2^{k-1}, 2^{k+1}]$, whereas in the intervals $[2^k, 3 \cdot 2^{k-1}]$ the function f_1 is decreasing for odd k, and f_2 is decreasing for even k. The ultrafilter $\mathcal{U}$ contains a set V on which all three of these functions are nondecreasing. Such a set V has at most one point in each interval. Starting from each one of the first four points of V, partition V into four parts by taking every fourth point, so as to obtain four sets satisfying the condition of the claim. Then exactly one of the resulting sets belongs to $\mathcal{U}$, and the claim follows.

Now remark that every function $f \leq n$ can be written as a sum $f_1 + f_2$, where $f_1, f_2 \leq \lceil \frac{n}{2} \rceil$. Pick a set $U \in \mathcal{U}$ as given by the claim. Then both functions f_1 and f_2 agree on U with suitable minimal step functions, because $u_{n+1} - u_n > \frac{u_{n+1}}{2}$, whereas $g \leq \lceil \frac{n}{2} \rceil$ implies $|g(u_{n+1}) - g(u_n)| \leq \lceil \frac{u_{n+1}}{2} \rceil$. So f_1, f_2 are equivalent

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[1] Ultrafilters satisfying this property are called *smooth* in [10].

modulo $\mathcal{U}$ to two nondecreasing functions f_1', f_2' respectively, and f is equivalent modulo $\mathcal{U}$ to their sum $f_1' + f_2'$, which is nondecreasing as the sum of nondecreasing functions. $\qquad\square$

Theorem 1.4. *Let $\mathcal{U}$ be a quasi-selective ultrafilter, and let $f : \mathbb{N} \to \mathbb{N}$ be nondecreasing. Then the following properties are equivalent:*

  (i) *for every function $g \le f$ there exists a nondecreasing function $h \equiv_{\mathcal{U}} g$;*
  (ii) *there exists $U = \{u_n \mid n \in \mathbb{N}\} \in \mathcal{U}$ such that $f(u_n) < u_{n+1} - u_n$.*

Proof. $(i) \Longrightarrow (ii)$. Define inductively the sequence $\langle x_n \mid n \in \mathbb{N} \rangle$ by putting $x_0 = 1$ and $x_{n+1} = f(x_n) + x_n$. Define $g : \mathbb{N} \to \mathbb{N}$ by $g(x_n + h) = f(x_n) - h$ for $0 \le h < f(x_n)$. Assuming (i), there is a set in $\mathcal{U}$ which meets each interval $[x_n, x_{n+1})$ in one point a_n. So by putting either $u_n = a_{2n}$ or $u_n = a_{2n+1}$ we obtain a set U satisfying the condition (ii). Namely, in the even case we have

$$u_{n+1} - u_n > x_{2n+2} - x_{2n+1} = f(x_{2n+1}) \ge f(u_n),$$

and similarly in the odd case.

$(ii) \Longrightarrow (i)$. By (ii) we may pick $U \in \mathcal{U}$ such that $x < y$ both in U implies $y > f(x)$. Given $g \le f$, partition U as follows

$$
\begin{aligned}
U_1 &= \{u \in U \mid \forall x \in U \, (x < u \Longrightarrow g(x) \le g(u)\,)\} \\
U_2 &= \{u \in U \mid \exists x \in U \, (x < u \,\&\, g(x) > g(u)\,)\}
\end{aligned}
$$

Then g is nondecreasing on U_1, so we are done when U_1 belongs to $\mathcal{U}$. Otherwise $U_2 \in \mathcal{U}$ and we have $g(u) < u$ for all $u \in U_2$. In fact, given $u \in U_2$, pick $x \in U$ such that $x < u$ and $g(x) > g(u)$: then

$$g(u) < g(x) \le f(x) < u.$$

Then (i) follows by quasi-selectivity of $\mathcal{U}$. $\qquad\square$

Let us denote by $F_{\mathcal{U}}$ the class of all functions f satisfying the equivalent conditions of the above theorem

$$F_{\mathcal{U}} = \{f : \mathbb{N} \to \mathbb{N} \text{ nondecreasing} \mid g \le f \Longrightarrow \exists h \text{ nondecreasing s.t. } h \equiv_{\mathcal{U}} g \}.$$

Recall that, if $\mathcal{U}$ is quasi-selective, then every function is $\mathcal{U}$-equivalent to an "interval-to-one" function. As a consequence, the class $F_{\mathcal{U}}$ "measures the selectivity" of quasi-selective ultrafilters, according to the following proposition.

Proposition 1.5. *Let $\mathcal{U}$ be a quasi-selective ultrafilter and let g be an unbounded interval-to-one function. Define the function g^+ by*

$$g^+(n) = \max \{x \mid g(x) = g(n)\},$$

and let e_g be the function enumerating the range of g^+. Then the following are equivalent:

  (1) *g is $\mathcal{U}$-equivalent to a one-to-one function;*
  (2) *g^+ belongs to $F_{\mathcal{U}}$;*
  (3) *there exists $U = \{u_0 < u_1 < \ldots < u_n < \ldots\} \in \mathcal{U}$ such that $u_n > e_g(n)$.*

Proof.

$(1) \Longrightarrow (2)$. Let g be one-to-one on $U \in \mathcal{U}$. Then each interval where g is constant contains at most one point of U. Hence g^+ is increasing on U. Let $h \le g^+$ be given, and put

$$U_1 = \{u \in U \mid h(u) \le u\}, \quad U_2 = \{u \in U \mid h(u) > u\}.$$

Then $u < h(u) \leq g^+(u)$ for $u \in U_2$, and hence h is increasing when restricted to U_2. So we are done when $U_2 \in \mathcal{U}$. On the other hand, when $U_1 \in \mathcal{U}$, the function h is $\mathcal{U}$-equivalent to a nondecreasing one, by quasi-selectivity.

$(2) \Longrightarrow (3)$. By Theorem 1.4, there exists a set $U = \{u_n \mid n \in \mathbb{N}\} \in \mathcal{U}$ such that $g^+(u_n) < u_{n+1} - u_n$. Suppose that $g(u_n) = g(u_{n+1})$ for some n; then $g^+(u_n) \geq u_{n+1} \geq u_{n+1} - u_n > g^+(u_n)$, a contradiction. Hence g is one-to-one on U, and so before u_{n+1} there are at least n intervals of the form $g^{-1}(k)$. Therefore $u_{n+1} > e_g(n)$, and $U \setminus \{u_0\}$ satisfies condition (3).

$(3) \Longrightarrow (1)$. For each $n \in \mathbb{N}$ let k be the unique number such that u_n lies in the interval $[e_g(n + k - 1), e_g(n + k))$. Then let h be the unique number such that $e_g(n + k)$ lies in the interval $(u_{n+h-1}, u_{n+h}]$. Thus we have

$$e_g(n + k - 1) \leq u_n \leq u_{n+h-1} < e_g(n + k) \leq u_{n+h}.$$

Then $k \geq h$ and $u_n \geq k$, and we can define the function f on U by $f(u_{n+i}) = k - i$, for $0 \leq i < h$. Since $f(u) \leq u$ for all $u \in U$ there exists a set $V \in \mathcal{U}$ on which f is nondecreasing. Then g is one-to-one on $V \cap U$.

$\square$

Recall that the ultrafilter $\mathcal{U}$ is *rapid* if for every increasing function f there exists $U = \{u_0 < u_1 < \ldots < u_n < \ldots\} \in \mathcal{U}$ such that $u_n > f(n)$. The equivalence of the conditions in the above corollary yields

Corollary 1.6. *A quasi-selective ultrafilter is selective if and only if it is rapid.*

$\square$

The class of functions $F_\mathcal{U}$ has the following closure properties:

Proposition 1.7. *Let $\mathcal{U}$ be a quasi-selective ultrafilter, and let*

$$F_\mathcal{U} = \{f : \mathbb{N} \to \mathbb{N} \text{ nondecreasing} \mid g \leq f \implies \exists h \text{ nondecreasing s.t. } h \equiv_\mathcal{U} g \}.$$

Then

   (1) *for all $f \in F_\mathcal{U}$ also $\widetilde{f} \in F_\mathcal{U}$, where*

$$\widetilde{f}(n) = f^{\circ f(n)}(n) = \underbrace{(f \circ f \circ \ldots \circ f)}_{f(n) \text{ times}}(n).^2$$

   (2) *Every sequence $\langle f_n \mid n \in \mathbb{N} \rangle$ in $F_\mathcal{U}$ is dominated by a function $f_\omega \in F_\mathcal{U}$, i.e. for all n there exists k_n such that $f_\omega(m) > f_n(m)$ for all $m > k_n$.*
     In particular the left cofinality of the gap determined by $F_\mathcal{U}$ in the ultrapower $\mathbb{N}_\mathcal{U}^\mathbb{N}$ is uncountable.

Proof. We prove first that $\widetilde{f}$ fulfills property (ii) of Theorem 1.4, provided f fulfills both properties (i) and (ii) of the same theorem. By possibly replacing f by $\max\{f, id\}$, we may assume without loss of generality that $f(n) \geq n$ for all $n \in \mathbb{N}$.

Let $U = \{u_0 < u_1 < \ldots < u_n \ldots\} \in \mathcal{U}$ be given by property (ii) for f. Define inductively the sequence $\sigma : \mathbb{N} \to \mathbb{N}$ by

$$\sigma(0) = 1 \text{ and } \sigma(n + 1) = \sigma(n) + f(u_{\sigma(n)}).$$

Define the function g on U by

$$g(u_{\sigma(n)+j}) = f(u_{\sigma(n)}) - j \quad \text{for } 0 \leq j < f(u_{\sigma(n)}).$$

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^{2} Here we agree that $f^{\circ 0}(n) = n$.

Since $g \leq f$ on U, there exists a subset $V \in \mathcal{U}$ on which g is nondecreasing. For each n, such a V contains at most one point $v_n = u_{\tau(n)}$ with $\sigma(n) \leq \tau(n) < \sigma(n+1)$. Assume without loss of generality that the set $\{v_{2n} \mid n \in \mathbb{N}\} \in \mathcal{U}$. We shall complete the proof by showing that $v_{2n+2} - v_{2n} > \widetilde{f}(v_{2n})$. Put $k = \tau(2n+2) - \tau(2n)$; then

$$
\begin{aligned}
v_{2n+2} - v_{2n} &= \sum_{i=0}^{k-1}(u_{\tau(2n)+i+1} - u_{\tau(2n)+i}) > \sum_{i=0}^{k-1} f(u_{\tau(2n)+i}) \geq \\
&\geq f(u_{\tau(2n)+k-1}) \geq f(f(u_{\tau(2n)+k-2})) \geq \ldots \geq f^{\circ k}(u_{\tau(2n)}).
\end{aligned}
$$

Now

$$
k = \tau(2n+2) - \tau(2n) > \sigma(2n+2) - \sigma(2n+1) = f(u_{\sigma(2n+1)}) \geq f(v_{2n}).
$$

Hence

$$
v_{2n+2} - v_{2n} > f^{\circ k}(u_{\tau(2n)}) \geq f^{\circ f(v_{2n})}(v_{2n}) = \widetilde{f}(v_{2n}).
$$

This completes the proof of (1).

In order to prove point (2), let sets $U_n \in \mathcal{U}$ be chosen so as to satisfy the property (ii) with respect to the function f_n. As $\mathcal{U}$ is a P-point, we can take $V \in \mathcal{U}$ almost included in every U_n, $i.e.$ $V \setminus U_n$ finite for all $n \in \mathbb{N}$. Define the function f_ω by

$$
f_\omega(m) = \min\{v' - v \mid v', v \in V, \ v' > v \geq m\}.
$$

Let k_n be such that, for all $v \in V$, $v \geq k_n$ implies $v \in U_n$. Given $m > k_n$ let $f_\omega(m) = v' - v$, with $m < v < v'$ as required by the definition of f_ω, and let u be the successor of v in U_n. Then

$$
f_n(m) \leq f_n(v) < u - v \leq v' - v = f_\omega(m),
$$

and so f_ω dominates every f_n. Now if $V = \{v_0 < v_1 < \ldots\}$, then $f_\omega(v_k) < v_{k+2} - v_k$. So either $U = \{v_{2n} \mid n \in \mathbb{N}\}$ or $U' = \{v_{2n+1} \mid n \in \mathbb{N}\}$ witnesses the property (ii) of Theorem 1.4 for f_ω, and (2) follows. $\square$

Remark that all *Ackermann functions*[3] $f_m(n) = A(m, n)$ belong to $F_\mathcal{U}$, because $f_{m+1} \leq \widetilde{f}_m$. Since every primitive recursive function is eventually dominated by some f_m, we obtain the following property of "primitive recursive rapidity":

Corollary 1.8. *Let $\mathcal{U}$ be a quasi-selective ultrafilter, and let $f : \mathbb{N} \to \mathbb{N}$ be primitive recursive. Then f is nondecreasing modulo $\mathcal{U}$, and there exists a set $U = \{u_0 < u_1 < \ldots < u_n \ldots\} \in \mathcal{U}$ with $u_{n+1} > f(u_n)$.* $\square$

As proved at the beginning of this section, one has the following implications

$$
selective \implies quasi\text{-}selective \implies P\text{-}point.
$$

Not even the existence of P-points can be proven in ZFC (see *e.g.* [16, 15]), so the question as to whether the above three classes of ultrafilters are distinct only makes sense under additional hypotheses.

However the following holds in ZFC:

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[3] Recall that $f_m(n) = A(m, n)$ can be inductively defined by

$$
f_0(n) = A(0, n) = n + 1, \quad f_{m+1}(n) = A(m+1, n) = \underbrace{(f_m \circ f_m \circ \ldots \circ f_m)}_{n+1 \ times}(1).
$$

Proposition 1.9. *Assume that the ultrafilter $\mathcal{U}$ is not a Q-point. Then there exists an ultrafilter $\mathcal{U}' \cong \mathcal{U}$ that is not quasi-selective.*

Proof. Let $\mathcal{U}$ be an ultrafilter that is not a Q-point, so there is a partition of $\mathbb{N}$ into finite sets F_n ($n \in \omega$) such that every set in $\mathcal{U}$ meets some F_n in more than one point. Inductively choose pairwise disjoint sets $G_n \subseteq \mathbb{N}$ such that, for each n,

$$\min(G_n) > |G_n| = |F_n|.$$

Let $f : \mathbb{N} \to \bigcup_n G_n$ be such that the restrictions $f \restriction F_n$ are bijections $f \restriction F_n : F_n \to G_n$ for all n. Since the G_n are pairwise disjoint, f is one-to-one, and therefore the ultrafilter $\mathcal{V} = f(\mathcal{U})$ is isomorphic to $\mathcal{U}$. We shall complete the proof by showing that $\mathcal{V}$ is not quasi-selective.

Define $g : \bigcup_n G_n \to \mathbb{N}$ by requiring that, for each n, the restriction $g \restriction G_n$ is the unique strictly decreasing bijection from G_n to the initial segment $[0, |G_n|)$ of $\mathbb{N}$. Notice that the values g takes on G_n are all $< \min(G_n)$; thus $g(x) < x$ for all $x \in G_n$. Extend g to all of $\mathbb{N}$ by setting $g(x) = 0$ for $x \notin \bigcup_n G_n$. Now $g : \mathbb{N} \to \mathbb{N}$ and $g(x) \leq x$ for all x. If $\mathcal{V}$ were quasi-selective, there would be a set $A \in \mathcal{V}$ on which g is non-decreasing. Since g is strictly decreasing on each G_n, each intersection $A \cap G_n$ would contain at most one point. Therefore each of the pre-images $f^{-1}(A) \cap F_n$ would contain at most one point. But from $A \in \mathcal{V}$, we infer that $f^{-1}(A)$ is in $\mathcal{U}$ and therefore meets some F_n in at least two points. This contradiction shows that $\mathcal{V}$ is not quasi-selective.

$\square$

Hence the mere existence of a non-selective P-point yields also the existence of non-quasi-selective P-points. So the second implication can be reversed only if the three classes are the same. Recall that this possibility has been shown consistent by Shelah (see [15, Section XVIII.4]).

We conclude this section by stating a theorem that settles the question under the Continuum Hypothesis CH:

Theorem 1.10. *Assume CH. Then there exist $2^{\mathfrak{c}}$ pairwise non-isomorphic P-points that are not quasi-selective, and $2^{\mathfrak{c}}$ pairwise non-isomorphic quasi-selective ultrafilters that are not selective.*

The first assertion of this theorem follows by combining Proposition 1.9 with the known fact that CH implies the existence of $2^{\mathfrak{c}}$ non-isomorphic non-selective P-points. The rather technical proof of the second assertion of the theorem is contained in the next section. A few open questions involving quasi-selective ultrafilters are to be found in the final Section 6.

2. A CONSTRUCTION OF QUASI-SELECTIVE ULTRAFILTERS

This section is entirely devoted to the proof of the following theorem, which in turn will yield the second assertion of Theorem 1.10 above.

Theorem 2.1. *Assume CH. For every selective ultrafilter $\mathcal{U}$, there is a non-selective but quasi-selective ultrafilter $\mathcal{V}$ above $\mathcal{U}$ in the Rudin-Keisler ordering.[4] Furthermore, $\mathcal{V}$ can be chosen to satisfy the partition relation $\mathbb{N} \to [\mathcal{V}]_3^2$.*

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[4] Recall that $\mathcal{V}$ is above $\mathcal{U}$ in the Rudin-Keisler (pre)ordering if there exists a function f such that $\mathcal{U} = f(\mathcal{V})$.

The square-bracket partition relation in the theorem means that, if $[\mathbb{N}]^2$ is partitioned into 3 pieces, then there is a set $H \in \mathcal{V}$ such that $[H]^2$ meets at most 2 of the pieces. This easily implies by induction that, if $[\mathbb{N}]^2$ is partitioned into any finite number of pieces, then there is a set $H \in \mathcal{V}$ such that $[H]^2$ meets at most 2 of the pieces. It is also known ([4]) to imply that $\mathcal{V}$ is a P-point and that $\mathcal{U}$ is, up to isomorphism, the only non-principal ultrafilter strictly below $\mathcal{V}$ in the Rudin-Keisler ordering.

It will be convenient to record some preliminary information before starting the proof of the theorem. Suppose $\mathcal{X}$ is an upward-closed (with respect to $\subseteq$) family of finite subsets of $\mathbb{N}$. Call $\mathcal{X}$ *rich* if every infinite subset of $\mathbb{N}$ has an initial segment in $\mathcal{X}$. (This notion resembles Nash-Williams's notion of a *barrier*, but it is not the same.) Define $\rho\mathcal{X}$ to be the family of those finite $A \subseteq \mathbb{N}$ such that $A \to (\mathcal{X})_2^2$, i.e., every partition of $[A]^2$ into two parts has a homogeneous set in $\mathcal{X}$. (The notation ρ stands for "Ramsey".)

Lemma 2.1. *If $\mathcal{X}$ is rich, then so is $\rho\mathcal{X}$.*

Proof. This is a standard compactness argument, but we present it for the sake of completeness. Let S be any infinite subset of $\mathbb{N}$, and, for each $n \in \mathbb{N}$, let S_n be the set of the first n elements of S. We must show that $S_n \in \rho\mathcal{X}$ for some n. Suppose not. Then, for each n, there are counterexamples, i.e., partitions $F : [S_n]^2 \to 2$ with no homogeneous set in $\mathcal{X}$. These counterexamples form a tree, in which the predecessors of any F are its restrictions to $[S_m]^2$ for smaller m. This tree is infinite but finitely branching, so König's infinity lemma gives us a path through it. The union of all the partitions along this path is a partition $G : [S]^2 \to 2$, and by Ramsey's theorem it has an infinite homogeneous set $H \subseteq S$. Since $\mathcal{X}$ is rich, it contains $H \cap S_n$ for some n. But then one of our counterexamples, namely $G \restriction [S_n]^2$, has a homogeneous set in $\mathcal{X}$, so it isn't really a counterexample. This contradiction completes the proof of the lemma. $\qquad\square$

Of course, we can iterate the operation ρ. The lemma implies that, if $\mathcal{X}$ is rich, then so is $\rho^n\mathcal{X}$ for any finite n. Notice also that we have $\mathcal{X} \supseteq \rho\mathcal{X} \supseteq \rho^2\mathcal{X} \supseteq \dots$.

We shall apply all this information to a particular $\mathcal{X}$, namely

$$\mathcal{L} = \{A \text{ finite, nonempty } \subseteq \mathbb{N} \mid \min(A) + 2 < |A|\},$$

which is obviously rich. Observe that any $A \in \mathcal{L}$ has $|A| \geq 3$. It easily follows that any $A \in \rho^n\mathcal{L}$ has $|A| \geq 3 + n$. (In fact the sizes of sets in $\rho^n\mathcal{L}$ grow very rapidly, but we don't need this fact here.) In particular, no finite set can belong to $\rho^n\mathcal{L}$ for arbitrarily large n, and so we can define a norm for finite sets by

$$\nu : [\mathbb{N}]^{<\omega} \to \mathbb{N} : A \mapsto \text{ least } n \text{ such that } A \notin \rho^n\mathcal{L}.$$

Because each $\rho^n\mathcal{L}$ is rich, we can partition $\mathbb{N}$ into consecutive finite intervals I_n such that $I_n \in \rho^n\mathcal{L}$ for each n. Define $p : \mathbb{N} \to \mathbb{N}$ to be the function sending all elements of any I_n to n; so $p^{-1}[B] = \bigcup_{n \in B} I_n$ for all $B \subseteq \mathbb{N}$.

For any $X \subseteq \mathbb{N}$, define its growth $\gamma(X) : \mathbb{N} \to \mathbb{N}$ to be the sequence of norms of its intersections with the I_n's:

$$\gamma(X)(n) = \nu(X \cap I_n).$$

Notice that, by our choice of the I_n, $\gamma(\mathbb{N})(n) > n$ for all n.

With these preliminaries, we are ready to return to ultrafilters and prove the theorem. The proof uses ideas from [4] and [14], but some modifications are needed, and so we present the proof in detail.

Proof of Theorem 2.1. Assume that CH holds, and let $\mathcal{U}$ be an arbitrary selective ultrafilter on $\mathbb{N}$. We adopt the quantifier notation for ultrafilters:

$(\mathcal{U}n)\,\varphi(n)$ means "for $\mathcal{U}$-almost all n, $\varphi(n)$ holds," *i.e.*, $\{n \mid \varphi(n)\} \in \mathcal{U}$.

Call a subset X of $\mathbb{N}$ *large* if

$$(\forall k \in \mathbb{N})(\mathcal{U}n)\,\gamma(X)(n) > \sqrt{n} + k;$$

equivalently, in the ultrapower of $\mathbb{N}$ by $\mathcal{U}$, $[\gamma(X)]$ is infinitely larger than $[\lceil\sqrt{n}\rceil]$. Since n is asymptotically much larger than $\sqrt{n}$, we have that $\mathbb{N}$ is large.

Using CH, list all partitions $F : [\mathbb{N}]^2 \to \{0,1\}$ in a sequence $\langle F_\alpha \mid \alpha < \aleph_1 \rangle$ of length $\aleph_1$. We intend to build a sequence $\langle A_\alpha \mid \alpha < \aleph_1 \rangle$ of subsets of $\mathbb{N}$ with the following properties:

(1) Each A_α is a large subset of $\mathbb{N}$.
(2) If $\alpha < \beta$, then $A_\beta \subseteq A_\alpha$ modulo $\mathcal{U}$, *i.e.*, $p[A_\beta - A_\alpha] \notin \mathcal{U}$, *i.e.*, $(\mathcal{U}n)\, A_\beta \cap I_n \subseteq A_\alpha$.
(3) For each n, F_α is constant on $[A_{\alpha+1} \cap I_n]^2$.

After constructing this sequence, we shall show how it yields the desired ultrafilter $\mathcal{V}$.

We construct A_α by induction on α, starting with $A_0 = \mathbb{N}$. We have already observed that requirement (1) is satisfied by $\mathbb{N}$; the other two requirements are vacuous at this stage.

Before continuing the construction, notice that the relation of inclusion modulo $\mathcal{U}$ is transitive.

At a successor step, we are given the large set A_α and we must find a large $A_{\alpha+1} \subseteq A_\alpha$ modulo $\mathcal{U}$ such that F_α is constant on each $[A_{\alpha+1} \cap I_n]^2$. Transitivity and the induction hypothesis then ensure that $A_{\alpha+1}$ is included in each earlier A_ξ modulo $\mathcal{U}$. (We shall actually get $A_{\alpha+1} \subseteq A_\alpha$, not just modulo $\mathcal{U}$, but the inclusions in the earlier A_ξ's will generally be only modulo $\mathcal{U}$.) We define $A_{\alpha+1}$ by defining its intersection with each I_n; then of course $A_{\alpha+1}$ will be the union of all these intersections.

If $\gamma(A_\alpha)(n) \leq 1$, then set $A_{\alpha+1} \cap I_n = \varnothing$. Note that the set of all such n's is not in $\mathcal{U}$, because A_α is large. If $\gamma(A_\alpha)(n) > 1$, then $A_\alpha \cap I_n$ is in $\rho^{\gamma(A_\alpha)(n)-1}\mathcal{L}$, so it has a subset that is homogeneous for $F_\alpha \restriction [A_\alpha \cap I_n]^2$ and is in $\rho^{\gamma(A_\alpha)(n)-2}\mathcal{L}$; let $A_{\alpha+1} \cap I_n$ be such a subset.

This choice of $A_{\alpha+1} \cap I_n$ (for each n) clearly ensures that requirements (2) and (3) are preserved. For requirement (1), simply observe that $(\mathcal{U}n)\,\gamma(A_{\alpha+1})(n) \geq \gamma(A_\alpha)(n) - 1$.

For the limit step of the induction, suppose β is a countable limit ordinal and we already have A_α for all $\alpha < \beta$. Choose an increasing $\mathbb{N}$-sequence of ordinals $\langle \alpha_i : i \in \mathbb{N} \rangle$ with limit β. Let $A'_n = \bigcap_{i \leq n} A_{\alpha_i}$. So the sets A'_n form a decreasing sequence. Because of induction hypothesis (2), each A'_n is equal modulo $\mathcal{U}$ to A_{α_n} (i.e., each includes the other modulo $\mathcal{U}$) and is therefore large. We shall find a large set A_β that is included modulo $\mathcal{U}$ in all of the A'_n, hence in all the A_{α_n}, and hence in all the A_α for $\alpha < \beta$. Thus, we shall preserve induction hypotheses (1) and (2); requirement (3) is vacuous at limit stages.

We shall obtain the desired A_β by defining its intersection with every I_n.

Because the ultrapower of $\mathbb{N}$ by $\mathcal{U}$ is countably saturated, we can fix a function $g : \mathbb{N} \to \mathbb{N}$ such that its equivalence class $[g]$ in the ultrapower is below each $[\gamma(A_n')]$ but above $[x \mapsto \lceil \sqrt{x} \rceil + k]$ for every $k \in \mathbb{N}$. For each $x \in \mathbb{N}$, define $h(x)$ to be the largest number $q \le x$ such that $g(x) \le \gamma(A_q')(x)$, or 0 if there is no such q. Finally, set $A_\beta \cap I_x = A_{h(x)}' \cap I_x$. We verify that this choice of A_β does what we wanted.

For any fixed n, $\mathcal{U}$-almost all x satisfy $g(x) \le \gamma(A_n')(x)$, and $x \ge n$, and therefore $h(x) \ge n$, and therefore $A_{h(x)}' \cap I_x \subseteq A_n' \cap I_x$. Thus, $A_\beta \subseteq A_n'$ modulo $\mathcal{U}$ for each n. As we saw earlier, this implies requirement (2).

Furthermore, for $\mathcal{U}$-almost all x,

$$\gamma(A_\beta)(x) = \gamma(A_{h(x)}')(x) \ge g(x),$$

and so our choice of g ensures that A_β is large.

This completes the construction of the sequence $\langle A_\alpha : \alpha < \aleph_1 \rangle$ and the verification of properties (1), (2), and (3). We shall now use this sequence to construct the desired ultrafilter.

We claim first that the sets $A_\alpha \cap p^{-1}[B]$, where α ranges over $\aleph_1$ and B ranges over $\mathcal{U}$, constitute a filter base. Indeed, the intersection of any two of them, say

$$A_\alpha \cap p^{-1}[B] \cap A_{\alpha'} \cap p^{-1}[B']$$

with, say, $\alpha \le \alpha'$, includes $A_{\alpha'} \cap p^{-1}[B \cap B' \cap C]$, where, thanks to requirement (2) above, C is a set in $\mathcal{U}$ such that $A_{\alpha'} \cap I_n \subseteq A_\alpha$ for all $n \in C$.

Let $\mathcal{V}$ be the filter generated by this filterbase. Because each A_α is large, because $[\gamma(A_\alpha \cap p^{-1}[B])] = [\gamma(A_\alpha)]$ in the $\mathcal{U}$-ultrapower for any $B \in \mathcal{U}$, and because largeness is obviously preserved by supersets, we know that every set in $\mathcal{V}$ is large.

We claim next that $\mathcal{V}$ is an ultrafilter. To see this, let $X \subseteq \mathbb{N}$ be arbitrary, and consider the following partition $F : [\mathbb{N}]^2 \to \{0, 1\}$. If X contains both or neither of x and y, then $F(\{x, y\}) = 0$; otherwise $F(\{x, y\}) = 1$. Requirement (3) of our construction provides an $A_\alpha \in \mathcal{V}$ such that F is constant on each $[A \cap I_n]^2$, say with value $f(n)$. A set on which F is constant with value 1 obviously contains at most two points, one in X and one outside X. As A is large, we infer that $(\mathcal{U}n)\, f(n) = 0$. That is, for $\mathcal{U}$-almost all n, $A \cap I_n$ is included in either X or $\mathbb{N} - X$. As $\mathcal{U}$ is an ultrafilter, it contains a set B such that either $A \cap I_n \subseteq X$ for all $n \in B$ or $A \cap I_n \subseteq \mathbb{N} - X$ for all $n \in B$. Then $A \cap p^{-1}[B] \in \mathcal{V}$ is either included in or disjoint from X. Since X was arbitrary, this completes the proof that $\mathcal{V}$ is an ultrafilter.

Since all sets in $\mathcal{V}$ are large, the finite-to-one function p is not one-to-one on any set in $\mathcal{V}$. Thus $\mathcal{V}$ is not selective, in fact not even a Q-point.

Our next goal is to prove that $\mathbb{N} \to [\mathcal{V}]_3^2$. Let an arbitrary $F : [\mathbb{N}]^2 \to \{0, 1, 2\}$ be given. We follow the custom of calling the values of F colors, and we use the notation $\{a < b\}$ to mean the set $\{a, b\}$ and to indicate the notational convention that $a < b$. We shall find two sets $X, Y \in \mathcal{V}$ such that all pairs $\{a < b\} \in [X]^2$ with $p(a) = p(b)$ have a single color and all pairs $\{a < b\} \in [Y]^2$ with $p(a) \ne p(b)$ have a single color (possibly different from the previous color). Then $X \cap Y \in \mathcal{V}$ has the weak homogeneity property required by $\mathbb{N} \to [\mathcal{V}]_3^2$.

To construct X, begin by considering the partition $G : [\mathbb{N}]^2 \to \{0, 1\}$ obtained from F by identifying the color 2 with 1. By our construction of $\mathcal{V}$, it contains a set A_α such that, for each n, all pairs in $[A_\alpha \cap I_n]^2$ are sent to the same color $g(n)$ by G. $\mathcal{U}$, being an ultrafilter, contains a set B on which g is constant. Then $A_\alpha \cap p^{-1}(B)$

is a set in $\mathcal{V}$ such that all pairs $\{a < b\}$ in $A_\alpha \cap p^{-1}(B)$ with $p(a) = p(b)$ have the same G-color. So the F-colors of these pairs are either all 0 or all in $\{1, 2\}$. If they are all 0, then $A_\alpha \cap p^{-1}(B)$ serves as the desired X. If they are all in $\{1, 2\}$, then we repeat the argument using G', obtained from F by identifying 2 with 0. We obtain a set $A_{\alpha'} \cap p^{-1}(B') \in \mathcal{V}$ such that the F-colors of its pairs with $p(a) = p(b)$ are either all 1 or all in $\{0, 2\}$. Then $A_\alpha \cap p^{-1}(B) \cap A_{\alpha'} \cap p^{-1}(B') \in \mathcal{V}$ serves as the desired X.

It remains to construct Y. It is well known that selective ultrafilters have the Ramsey property. So we can find a set $B \in \mathcal{U}$ such that all or none of the pairs $\{x < y\} \in [B]^2$ satisfy the inequality

$$9^{|\bigcup_{z \leq x} I_z|} < y.$$

If we had the "none" alternative here, then all elements $y \in B$ would be bounded by $9^{|\bigcup_{z \leq x} I_z|}$ where x is the smallest element of B. That is absurd, as B is infinite, so we must have the "all" alternative.

For each $b \in p^{-1}[B]$, let f_b be the function telling how b is related by F to elements in earlier fibers over B. That is, if $p(b) = n \in B$, let the domain of f_b be $\{a \in \mathbb{N} : p(a) < n \text{ and } p(a) \in B\}$, and define f_b on this domain by $f_b(a) = F(\{a, b\})$. Notice that the domain of f_b has cardinality at most $|\bigcup_{z \leq m} I_z|$ where m is the last element of B before n (or 0 if n is the first element of B). Since f_b takes values in 3, the number of possible f_b's, for $p(b) = n \in B$, is at most

$$3^{|\bigcup_{z \leq m} I_z|} < \sqrt{n},$$

by the homogeneity property of B.

Define a new partition, $H : [\mathbb{N}]^2 \to \{0, 1\}$ by setting $H(\{b < c\}) = 0$ if f_b and f_c are defined and equal, and $H(\{b < c\}) = 1$ otherwise. Proceeding as in the first part of the construction of X above, we obtain a set $Z \in \mathcal{V}$ such that all pairs $\{b < c\} \in [Z]^2$ with $p(b) = p(c)$ have the same H-color. That is, in each of the sets $Z \cap I_n$ $(n \in B)$, either all the points b have the same f_b or they all have different f_b's. But Z is large so the number of such points is, for $\mathcal{U}$-almost all n, larger than $\sqrt{n}$. So, by the estimate above of the number of f_b's, there are not enough of these functions for every b to have a different f_b. Thus, for $\mathcal{U}$-almost all $n \in B$, all $b \in Z \cap I_n$ have the same f_b. Shrinking B to a smaller set in $\mathcal{U}$ (which we still call B to avoid extra notation), we can assume that, for all $n \in B$, f_b depends only on $p(b)$ as long as $b \in Z$. Let us also shrink Z to $Z \cap p^{-1}[B]$, which is of course still in $\mathcal{V}$.

Going back to the original partition F, we have that the color of a pair $\{a < b\} \in [Z]^2$ with $p(a) < p(b)$ depends only on a and $p(b)$, because this color is $F(\{a < b\}) = f_b(a)$ and f_b depends only on $p(b)$.

For each $a \in Z$, let g_a be the function, with domain equal to the part of B after $p(a)$, such that $g_a(n)$ is the common value of $F(\{a < b\})$ for all $b \in Z \cap I_n$. This g_a maps a set in $\mathcal{U}$ (namely a final segment of B) into 3, so it is constant, say with value $j(a)$, on some set $C_a \in \mathcal{U}$.

Using again the Ramsey property of $\mathcal{U}$, we obtain a set $D \in \mathcal{U}$ such that $D \subseteq B$ and all or none of the pairs $\{x < y\} \in [D]^2$ satisfy

$$(\forall a \in B \cap I_x)\, y \in C_a.$$

If we had the "none" alternative, then, letting x be the first element of D, we would have that

$$D \cap \bigcap_{a \in B \cap I_x} C_a = \emptyset.$$

But this is the intersection of finitely many sets from $\mathcal{U}$, so it cannot be empty. This contradiction shows that we must have the "all" alternative. In view of the definition of C_a, this means that, when $a < b$ are in $Z \cap p^{-1}[D]$ and $p(a) < p(b)$,

the F-color of $\{a < b\}$ depends only on a, not on b. Since there are only finitely many possible colors and since $\mathcal{V}$ is an ultrafilter, we can shrink $Z \cap p^{-1}[D]$ to a set $Y \in \mathcal{V}$ on which the F-color of all such pairs is the same.

This completes the construction of the desired Y and thus the proof that $\mathbb{N} \to [\mathcal{V}]^2_3$.

This partition relation, together with an easy induction argument, gives the following slightly stronger-looking result. For any partition of $[\mathbb{N}]^2$ into any finite number of pieces, there is a set $H \in \mathcal{V}$ such that $[H]^2$ is included in the union of two of the pieces. We shall need this for a partition into 6 pieces.

Finally, we prove that $\mathcal{V}$ is quasi-selective. Consider an arbitrary $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) \leq n$ for all n; we seek a set in $\mathcal{V}$ on which f is non-decreasing. Define a partition $F : [\mathbb{N}]^2 \to 6$ by setting

$$F(\{a < b\}) = \begin{cases} 0 & \text{if } p(a) = p(b) \text{ and } f(a) < f(b) \\ 1 & \text{if } p(a) = p(b) \text{ and } f(a) = f(b) \\ 2 & \text{if } p(a) = p(b) \text{ and } f(a) > f(b) \\ 3 & \text{if } p(a) < p(b) \text{ and } f(a) < f(b) \\ 4 & \text{if } p(a) < p(b) \text{ and } f(a) = f(b) \\ 5 & \text{if } p(a) < p(b) \text{ and } f(a) > f(b) \end{cases}$$

Since p is non-decreasing, the six cases cover all the possibilities. Fix a set $H \in \mathcal{V}$ on whose pairs F takes only two values. The first of those two values must be in $\{0, 1, 2\}$ and the second in $\{3, 4, 5\}$ because p is neither one-to-one nor constant on any set in $\mathcal{V}$.

If the second value were 5, then by choosing an infinite sequence $a_0 < a_1 < \ldots$ in H with $p(a_0) < p(a_1) < \ldots$, we would get an infinite decreasing sequence $f(a_0) > f(a_1) > \ldots$ of natural numbers. Since this is absurd, the second value must be 3 or 4.

If the second value is 4, then any two elements of H from different I_n's have the same f value. But then the same is true also for any two elements of H from the same I_n, because we can compare them with a third element of H chosen from a different I_m. So in this case f is constant on H; in particular, it is non-decreasing, as desired.

So from now on, we may assume the second value is 3; f is increasing on pairs in H from different I_n's.

Thus, if the first value is either 0 or 1, then f is non-decreasing on all of H, as desired. It remains only to handle the case that the first value is 2; we shall show that this case is impossible, thereby completing the proof of the theorem.

Suppose, toward a contradiction, that the first value were 2. This means that, for each n, the restriction of f to $H \cap I_n$ is strictly decreasing. Temporarily fix n, and let b be the smallest element of $H \cap I_n$. Then $f(b)$ is the largest value

taken by f on $H \cap I_n$, and there are exactly $|H \cap I_n|$ such values. Therefore, $f(b) \geq |H \cap I_n| - 1$. On the other hand, by the hypothesis on f, we have $f(b) \leq b$, and therefore $b \geq |H \cap I_n| - 1$.

Now un-fix n. We have just shown that

$$\min(H \cap I_n) \geq |H \cap I_n| - 1,$$

and so $H \cap I_n \notin \mathcal{L}$. That is,

$$\gamma(H)(n) = \nu(H \cap I_n) = 0$$

for all n. That contradicts the fact that, like all sets in $\mathcal{V}$, H is large, and the proof of the theorem is complete.

$\square$

Now, in order to deduce Theorem 1.10, we have only to recall the well known fact that, under CH, there are $2^{\mathfrak{c}}$ pairwise non-isomorphic selective ultrafilters. For each of them, say $\mathcal{U}$, Theorem 2.1 provides a non-selective quasi-selective $\mathcal{V}$ that is Rudin-Keisler above $\mathcal{U}$. As remarked at the beginning of this section, according to [4], there is a unique class of selective ultrafilters Rudin-Keisler below $\mathcal{V}$, which has been chosen so as to satisfy the partition relation $\mathbb{N} \to [\mathcal{V}]_3^2$. So all $\mathcal{V}$s are pairwise non-isomorphic.

Finally, as remarked at the end of Section 1, Proposition 1.9 allows for associating to each $\mathcal{V}$ an isomorphic non-quasi-selective P-point $\mathcal{V}'$.

3. Equinumerosity of point sets

In this section we study a notion of "numerosity" for *point sets of natural numbers*, *i.e.* for subsets of the spaces of k-tuples $\mathbb{N}^k$. This numerosity will be defined by starting from an equivalence relation of "equinumerosity" that satisfies all the basic properties of equipotency between finite sets.

For simplicity we follow the usual practice and we identify Cartesian products with the corresponding "concatenations". That is, for every $A \subseteq \mathbb{N}^k$ and for every $B \subseteq \mathbb{N}^h$, we identify $A \times B = \{(\vec{a}, \vec{b}) \mid \vec{a} \in A, \vec{b} \in B\}$ with:

$$A \times B = \{(a_1, \ldots, a_k, b_1, \ldots, b_h) \mid (a_1, \ldots, a_k) \in A \text{ and } (b_1, \ldots, b_h) \in B\}.$$

Definition 3.1. We call *equinumerosity* an equivalence relation $\approx$ that satisfies the following properties for all point sets A, B of natural numbers:

(E1) $A \approx B$ if and only if $A \setminus B \approx B \setminus A$.

(E2) Exactly one of the following three conditions holds:
  (a) $A \approx B$;
  (b) $A' \approx B$ for some proper subset $A' \subset A$;
  (c) $A \approx B'$ for some proper subset $B' \subset B$.

(E3) $A \times \{P\} \approx \{P\} \times A \approx A$ for every point P.

(E4) $A \approx A'$ & $B \approx B' \implies A \times B \approx A' \times B'$.

Some comments are in order. Axiom (E1) is but a compact equivalent reformulation of the second and third common notions of Euclid's Elements (see [13]):

> "If equals be added to equals, the wholes are equal",

and

> "if equals be subtracted from equals, the remainders are equal".

(A precise statement of this equivalence is given in Proposition 3.2 below.) Notice that the first common notion

> "*Things which are equal to the same thing are also equal to one another*"

is already secured by the assumption that equinumerosity is an equivalence relation.

The *trichotomy property* of axiom (E2) combines two natural ideas: firstly that, given two sets, one is equinumerous to some subset of the other, and secondly that no proper subset is equinumerous to the set itself. So (E2) allows for a natural ordering of sizes that satisfies the implicit assumption of the classical theory that *(homogeneous) magnitudes are always comparable*, as well as the fifth Euclidean common notion

> "*The whole is greater than the part*".

We remark that both properties (E1) and (E2) hold for equipotency between finite sets, while both fail badly for equipotency between infinite sets.

The third axiom (E3) can be viewed as an instance of the fourth Euclidean common notion

> "*Things applying* [*exactly*] *onto one another are equal to one another*".[5]

In particular (E3) incorporates the idea that any set has equinumerous "lifted copies" in any higher dimension (see Proposition 3.4 below).

Axiom (E4) is postulated so as to allow for the natural definition of *multiplication* of numerosities, which admits the numerosity of every singleton as an identity by axiom (E3). This multiplication, together with the natural addition of numerosities, as given by *disjoint union*, satisfy the properties of *discretely ordered semirings* (see Theorem 4.2).

Remark that we do not postulate here *commutativity* of product. On the one hand, this assumption is unnecessary for the general treatment of numerosities; on the other hand, commutativity follows from the given axioms in the case of *asymptotic* numerosities (see Section 5).

Proposition 3.2. *The Axiom* (E1) *is equivalent to the conjunction of the following two principles:*[6]

(1) Sum Principle: *Let A, A', B, B' be such that $A \cap B = \emptyset$ and $A' \cap B' = \emptyset$. If $A \approx A'$ and $B \approx B'$, then $A \cup B \approx A' \cup B'$.*

(2) Difference Principle: *Let A, A', C, C' be such that $A \subseteq C$ and $A' \subseteq C'$. If $A \approx A'$ and $C \approx C'$, then $C \setminus A \approx C' \setminus A'$.*

Proof. We begin by proving that (E1) follows from the conjunction of (1) and (2). In fact, if $A \setminus B \approx B \setminus A$, then

$$A = (A \setminus B) \cup (A \cap B) \approx (B \setminus A) \cup (A \cap B) = B,$$

by (1). Conversely, if $A \approx B$, then

$$A \setminus B = A \setminus (A \cap B) \approx B \setminus (A \cap B) = B \setminus A,$$

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[5] Equicardinality is characterized by equipotency; similarly one might desire that equinumerosity be characterized by "isometry" witnessed by a suitable class of bijections. This assumption seems *prima facie* very demanding. However, we shall see that it is fulfilled by the *asymptotic equinumerosities* of Section 5.

[6] We implicitly assume that $A, B \subseteq \mathbb{N}^k$ and $A', B' \subseteq \mathbb{N}^{k'}$ are *homogeneous* pairs (*i.e.* sets of the same dimension); otherwise $A \cup B, A' \cup B', A \setminus B, A' \setminus B'$ are not point sets.

by (2).

Now remark that, if we put $A \cup B = C$ and $A' \cup B' = C'$ in (1), then both principles follow at once from the statement

(E1)* *Let A, A', C, C' be such that $A \subseteq C$ and $A' \subseteq C'$. If $A \approx A'$, then*

$$C \setminus A \approx C' \setminus A' \iff C \approx C'.$$

We are left to show that (E1) implies (E1)*.

Assume that the equivalence relation $\approx$ satisfies (E1), and put

$A_0 = A \setminus C'$, $A_0' = A' \setminus C$, $D = A \cap A'$, $A_1 = A \setminus (A_0 \cup D)$, $A_1' = A' \setminus (A_0' \cup D)$,
$C_0 = C \setminus (A \cup C')$, $C_0' = C' \setminus (A' \cup C)$, $E = (C \cap C') \setminus (A \cup A')$,

so as to obtain pairwise disjoint sets $A_0, A_1, A_0', A_1', D, C_0, C_0', E$ such that

$A = A_0 \cup A_1 \cup D$, $A' = A_0' \cup A_1' \cup D$,
$C \setminus C' = A_0 \cup C_0$, $C' \setminus C = A_0' \cup C_0'$,
$C \setminus A = C_0 \cup E \cup A_1'$, $C' \setminus A' = C_0' \cup E \cup A_1$.

Hence

$$C \approx C' \iff C_0 \cup A_0 \approx C_0' \cup A_0'$$

and

$$C \setminus A \approx C' \setminus A' \iff C_0 \cup A_1' \approx C_0' \cup A_1$$

Since $A \approx A'$ we have $A_0 \cup A_1 \approx A_0' \cup A_1'$, whence

$$C_0 \cup A_0 \cup A_1 \approx C_0 \cup A_0' \cup A_1'.$$

So $C \setminus A \approx C' \setminus A'$ implies $C_0 \cup A_0 \cup A_1 \approx C_0' \cup A_1 \cup A_0'$, whence $C \approx C'$.
Conversely, $C \approx C'$ implies $C_0 \cup A_0 \cup A_1 \approx C_0' \cup A_0' \cup A_1$, whence $C \setminus A \approx C' \setminus A'$.
$\square$

Thus equinumerosity behaves coherently with respect to the operations of *disjoint union* and *set difference*. We can now prove that our notion of equinumerosity satisfies the basic requirement that *finite point sets are equinumerous if and only if they have the same "number of elements"*.

Proposition 3.3. *Assume* (E1) *and* (E2), *and let A, B be finite sets. Then*

$$A \approx B \iff |A| = |B|.$$

Proof. We begin by proving that (E2) implies that any two singletons are equinumerous. In fact the only proper subset of any singleton is the empty set, and no nonempty set can be equinumerous to $\emptyset$, by (E2).

Given nonempty finite sets A, B, pick $a \in A$, $b \in B$ and put $A' = A \setminus \{a\}$, $B' = B \setminus \{b\}$. Then, by (E1)*, $A' \approx B' \iff A \approx B$, because $\{a\} \approx \{b\}$, and the thesis follows by induction on $n = |A|$.
$\square$

By the above proposition, we can identify each natural number $n \in \mathbb{N}$ with the equivalence class of all those point sets that have finite cardinality n.

Remark that trichotomy is not essential in order to obtain Proposition 3.3. In fact, let $\approx$ be nontrivial and satisfy (E1), (E3), and (E4). Then, by (E3), we have

$$\{x\} \approx \{y\} \times \{x\} \approx \{x\} \times \{y\} \approx \{y\}$$

for all $x \in \mathbb{N}^h$ and all $y \in \mathbb{N}^k$, and so all singletons are equinumerous. On the other hand, if any singleton is equinumerous to $\emptyset$, then all sets are, because of (E4).

Clearly (E3) formalizes the natural idea that singletons have "unitary" numerosity. A trivial but important consequence of this axiom, already mentioned above, is the existence of infinitely many pairwise disjoint equinumerous copies of any given point set in every higher dimension. Namely

Proposition 3.4. *Assume* (E3)*, and let $A \subseteq \mathbb{N}^k$ be a k-dimensional point set. If $h > k$ let $P \in \mathbb{N}^{h-k}$ be any $(h-k)$-dimensional point. Then $\{P\} \times A \subseteq \mathbb{N}^h$ is equinumerous to A.*   □

4. The algebra of numerosities

Starting from the equivalence relation of *equipotency*, Cantor introduced the algebra of cardinals by means of disjoint unions and Cartesian products. Our axioms have been chosen so as to allow for the introduction of an "algebra of numerosities" in the same Cantorian manner.

Definition 4.1. *Let $\mathbf{F} = \bigcup_{k \in \mathbb{N}^+} \mathcal{P}(\mathbb{N}^k)$ be the family of all point sets over $\mathbb{N}$, and let $\approx$ be an equinumerosity relation on $\mathbf{F}$.*
• *The* numerosity *of $A \in \mathbf{F}$ (with respect to $\approx$) is the equivalence class of all point sets equinumerous to A*

$$\mathfrak{n}_\approx(A) = [A]_\approx = \{B \in \mathbf{F} \mid B \approx A\}.$$

• *The* set of numerosities *of $\approx$ is the quotient set $\mathfrak{N}_\approx = \mathbf{F} / \approx$.*
• *The* numerosity function *associated to $\approx$ is the canonical map $\mathfrak{n}_\approx : \mathbf{F} \to \mathfrak{N}_\approx$.*

In the sequel we shall drop the subscript $\approx$ whenever the equinumerosity relation is fixed. Numerosities will be usually denoted by *Frakturen* $\mathfrak{x}, \mathfrak{y}, \mathfrak{z}$, *etc.*

The given axioms guarantee that numerosities are naturally equipped with a "nice" algebraic structure. (This has to be contrasted with the awkward cardinal algebra, where *e.g.* $\kappa + \mu = \kappa \cdot \mu = \max\{\kappa, \mu\}$ for all infinite κ, μ.)

Theorem 4.2. *Let $\mathfrak{N}$ be the set of numerosities of the equinumerosity relation $\approx$. Then there exist unique operations $+$ and $\cdot$, and a unique linear order $<$ on $\mathfrak{N}$, such that for all point sets A, B:*

(1) $\mathfrak{n}(A) + \mathfrak{n}(B) = \mathfrak{n}(A \cup B)$ *whenever $A \cap B = \emptyset$;[7]*
(2) $\mathfrak{n}(A) \cdot \mathfrak{n}(B) = \mathfrak{n}(A \times B)$ *;*
(3) $\mathfrak{n}(A) < \mathfrak{n}(B)$ *if and only if $A \approx B'$ for some proper subset $B' \subset B$.*

The resulting structure on $\mathfrak{N}$ is the non-negative part of a discretely ordered ring $(\mathfrak{R}, 0, 1, +, \cdot, <)$. Moreover, if the fundamental subring of $\mathfrak{R}$ is identified with $\mathbb{Z}$, then $\mathfrak{n}(A) = |A|$ for every finite point set A.

Proof. We begin with the ordering. Trivially $A \approx A$, and so the trichotomy property (E2) implies the irreflexivity $\mathfrak{n}(A) \not< \mathfrak{n}(A)$. In order to prove transitivity, we show first the following property:

($\star$) If $A \approx B$, then for any $X \subset A$ there exists $Y \subset B$ such that $X \approx Y$.

Since $A \approx B$, the proper subset $X \subset A$ cannot be equinumerous to B. Similarly, B cannot be equinumerous to a proper subset $X' \subset X \subset A$. We conclude that $X \approx Y$ for some $Y \subset B$, and ($\star$) is proved.

---

[7] Again, here we implicitly assume that $A, B \subseteq \mathbb{N}^k$ are homogeneous, as otherwise their union $A \cup B$ would not be a point set.

Now assume $\mathfrak{n}(A) < \mathfrak{n}(B) < \mathfrak{n}(C)$. Pick proper subsets $A' \subset B$ and $B' \subset C$ such that $A \approx A'$ and $B \approx B'$. By $(\star)$, there exists $A'' \subset B'$ such that $A' \approx A''$. So, $\mathfrak{n}(A) < \mathfrak{n}(C)$ holds because $A \approx A'' \subset C$, and transitivity follows. Finally, again by trichotomy, we get that the order $<$ is linear.

We now define a sum for numerosities. Given $\mathfrak{x}, \mathfrak{y} \in \mathfrak{N}$, there exist homogeneous disjoint point sets $A, B \subseteq \mathbb{N}^k$ such that $\mathfrak{n}(A) = \mathfrak{x}$ and $\mathfrak{n}(B) = \mathfrak{y}$, by Proposition 3.4. Then put $\mathfrak{x} + \mathfrak{y} = \mathfrak{n}(A \cup B)$. This addition is independent of the choice of A and B, by Proposition 3.2. Commutativity and associativity trivially follow from the corresponding properties of disjoint unions.

Clearly $0 = \mathfrak{n}(\emptyset)$ is the (unique) neutral element. Moreover Proposition 3.2 directly yields the cancellation law

$$\mathfrak{x} + \mathfrak{z} = \mathfrak{x} + \mathfrak{z}' \iff \mathfrak{z} = \mathfrak{z}'.$$

By definition, $\mathfrak{x} \le \mathfrak{y}$ holds if and only if there exists $\mathfrak{z}$ such that $\mathfrak{y} = \mathfrak{x} + \mathfrak{z}$. Such a $\mathfrak{z}$ is unique by the cancellation law, and so the monotonicity property with respect to addition follows from the equivalences

$$\mathfrak{w} + \mathfrak{x} \le \mathfrak{w} + \mathfrak{y} \iff \exists \mathfrak{z}\,(\mathfrak{w} + \mathfrak{x} + \mathfrak{z} = \mathfrak{w} + \mathfrak{y}) \iff \exists \mathfrak{z}\,(\mathfrak{x} + \mathfrak{z} = \mathfrak{y}) \iff \mathfrak{x} \le \mathfrak{y}.$$

The multiplication of numerosities given by condition (2) is well-defined by axiom (E4). Associativity follows from the corresponding property of concatenation (and this is the reason for our convention on Cartesian products). Distributivity is inherited from the corresponding property of Cartesian products with respect to disjoint unions. Moreover $1 = \mathfrak{n}(\{P\})$ is an identity, by (E3). Finally, by definition, $\mathfrak{x} \cdot \mathfrak{y} = 0$ if and only if $\mathfrak{x} = 0$ or $\mathfrak{y} = 0$.

Therefore $\langle \mathfrak{N}, +, \cdot, < \rangle$ is *the non-negative part of a linearly ordered ring* $\mathfrak{R}$, say.[8] By Proposition 3.3, $\mathbb{N}$ can be identified with the set of the numerosities of finite sets; hence it is an initial segment of $\mathfrak{N}$. It follows that the ordering of $\mathfrak{R}$ is discrete. □

Recall that we have not postulated an axiom of commutativity, and so in general the set of numerosities might be a *noncommutative* semiring. However numerosities can be given a nice algebraic characterization as *the non-negative part of the quotient of a ring of noncommutative formal power series with integer coefficients modulo a suitable prime ideal*. Let us fix our notation as follows:

- let $\mathbf{t} = \langle t_n \mid n \in \mathbb{N} \rangle$ be a sequence of *noncommutative* indeterminates and let $\mathbb{Z}\langle\!\langle \mathbf{t} \rangle\!\rangle$ be the corresponding ring of *noncommutative formal power series*;
- to each point $x = (x_1, \ldots, x_k) \in \mathbb{N}^k$ associate the *noncommutative monomial* $t_x = t_{x_1} \ldots t_{x_k}$;
- let S_X be the *characteristic series* of the point set $X \in \mathbf{F}$, *i.e.*

$$S_X = \sum_{x \in X} t_x \in \mathbb{Z}\langle\!\langle \mathbf{t} \rangle\!\rangle;$$

- let $\mathcal{R} \subseteq \mathbb{Z}\langle\!\langle \mathbf{t} \rangle\!\rangle$ be the ring of all formal series of *bounded degree* in $\mathbf{t}$ with *bounded integral coefficients*;
- let $\mathcal{R}^+$ be the multiplicative subset of the *bounded positive series*, *i.e.* the series in $\mathcal{R}$ having only positive coefficients;

[8] $\mathfrak{R}$ can be defined as usual by mimicking the construction of $\mathbb{Z}$ from $\mathbb{N}$. Take the quotient of $\mathfrak{N} \times \mathfrak{N}$ modulo the equivalence $(x, y) \equiv (x', y') \Leftrightarrow x + y' = x' + y$, and define the operations in the obvious way.

- let $\mathfrak{I}_0$ be the two-sided ideal of $\mathcal{R}$ generated by $\{t_n - 1 \mid n \in \mathbb{N}\}$.

Then

Proposition 4.3.

(1) *Characteristic series of nonempty point sets belong to $\mathcal{R}^+$;*

(2) *characteristic series behave well with respect to* unions *and* products, *i.e.:*

$$S_X + S_Y = S_{X \cup Y} + S_{X \cap Y} \quad and \quad S_{X \times Y} = S_X \cdot S_Y \text{ for all } X, Y \in \mathbf{F}.[9]$$

(3) *Let $S = a + \sum a_x t_x \in \mathcal{R}$, with $|a_x| \leq B$ and $\deg t_x \leq d$ for all x; then*

$$S = a + \sum_{k=1}^{d} \sum_{i=1}^{B} (S_{X_{ik}} - S_{Y_{ik}}),$$

where $X_{ik} = \{x \in \mathbb{N}^k \mid a_x \geq i\}$, $Y_{ik} = \{x \in \mathbb{N}^k \mid a_x \leq -i\}$.

(4) *For any $S \in \mathfrak{I}_0$ there exist finite sequences $\langle a_h \mid h \leq n \rangle$ of integers and $\langle X_i \mid 1 \leq i \leq m \rangle$ of point sets, such that*

$$\begin{aligned} S &= a_0(t_0 - 1) + \sum_{i,j=1}^{m} \varepsilon_{0ij}(S_{X_i \times \{0\} \times X_j} - S_{X_i \times X_j}) + \\ &+ \sum_{i=1}^{m} \delta_{0i}(S_{X_i \times \{0\}} - S_{X_i}) + \eta_{0i}(S_{\{0\} \times X_i} - S_{X_i}) + \\ &+ \sum_{h=1}^{n} \left(a_h(t_h - t_0) + \sum_{i,j=1}^{m} \varepsilon_{hij}(S_{X_i \times \{h\} \times X_j} - S_{X_i \times \{0\} \times X_j}) + \right. \\ &\left. + \sum_{i=1}^{m} \delta_{hi}(S_{X_i \times \{h\}} - S_{X_i \times \{0\}}) + \eta_{hi}(S_{\{h\} \times X_i} - S_{\{0\} \times X_i}) \right) \end{aligned}$$

with suitable coefficients $\varepsilon_{hij}, \delta_{hi}, \eta_{hi} \in \{0, \pm 1\}$.

(5) *Every positive series $P \in \mathcal{R}^+$ is equivalent modulo $\mathfrak{I}_0$ to the characteristic series S_X of some nonempty point set X.*

(6) *$\mathcal{R}$ is the subring with identity of $\mathbb{Z}\langle\!\langle \mathbf{t} \rangle\!\rangle$ generated by $\mathfrak{I}_0$ together with the set of the characteristic series of all point sets. More precisely*

$$\mathcal{R}/\mathfrak{I}_0 = \{S_X - S_Y + \mathfrak{I}_0 \mid X, Y \in \mathbf{F}\}$$

Proof. $(1), (2)$ and (3) directly follow from the definitions.

In order to prove (4), choose the binomials $t_n - t_0$ together with $t_0 - 1$ as generators of $\mathfrak{I}_0$. Recalling (3), we can write each element of $\mathfrak{I}_0$ as the sum of finitely many terms of one of the following types:

(a) $a(t_0 - 1)$ or $a(t_n - t_0)$, $n > 0$, with $a \in \mathbb{Z}$;

(b) $\pm S_X(t_0 - 1)$ or $\pm S_X(t_n - t_0)$, $n > 0$;

(c) $\pm(t_0 - 1)S_X$ or $\pm(t_n - t_0)S_X$, $n > 0$;

(d) $\pm S_X(t_0 - 1)S_Y$ or $\pm S_X(t_n - t_0)S_Y$, $n > 0$.

---

[9] *Caveat*: if X, Y have different dimensions their union does not belong to $\mathbf{F}$, so the first equality does not apply.

In order to get (4), list all the point sets appearing above, each with the appropriate number of repetitions, and remark that $S_X(t_0 - 1)S_Y = S_{X \times \{0\} \times Y} - S_{X \times Y}$, and similarly for the other types of summands.

(6) follows immediately from (5). As for (5), let a positive series $P = \sum n_x t_x$ be given, with coefficients not exceeding B, and degrees not exceeding d. Factorize each monomial as $t_x = t_{x'} t_{x''}$, where $t_{x'} \in \mathbb{Z}\langle\langle t_0, t_1 \rangle\rangle$ is the largest initial part of t_x containing only the variables t_0, t_1. Then

$$P = \sum_z \left(\sum_{\substack{x \\ t_{x''} = t_z}} n_x t_{x'} \right) t_z.$$

Put

$$N_z = \sum_{\substack{x \\ t_{x''} = t_z}} n_x.$$

Modulo $\mathfrak{I}_0$ we may replace each internal sum $\sum n_x t_{x'}$ by an arbitrary sum of N_z different monomials t_y in the variables t_0, t_1. The resulting series is the characteristic series of a k-dimensional point set, provided that all the resulting monomials $t_y t_z$ have the same degree k. Such a choice of the monomials t_y is clearly possible by taking k such that $2^k > B \cdot 2^d$.

$\square$

Once an equinumerosity relation has been fixed, the expression (4) shows that every element of $\mathfrak{I}_0$ is the sum of differences of characteristic series of equinumerous point sets, plus a multiple of $t_0 - 1$. So numerosities can be viewed as elements of the quotient ring of $\mathcal{R}$ modulo suitable ideals extending $\mathfrak{I}_0$, namely ideals $\mathfrak{I}$ such that the quotient $\mathcal{R}/\mathfrak{I}$ is a *discretely ordered ring* whose *positive elements* are the cosets $P + \mathfrak{I}$ for $P \in \mathcal{R}^+$. To this aim we define:

Definition 4.4. A two-sided ideal $\mathfrak{I}$ of $\mathcal{R}$ is a *gauge ideal* if

- $\mathfrak{I}_0 \subseteq \mathfrak{I}$,
- $\mathcal{R}^+ \cap \mathfrak{I} = \emptyset$, and
- for all $S \in \mathcal{R} \setminus \mathfrak{I}$ there exists $P \in \mathcal{R}^+$ such that $S \pm P \in \mathfrak{I}$.

Remark that any gauge ideal is a *prime* ideal of $\mathcal{R}$ that is *maximal* among the two-sided ideals disjoint from $\mathcal{R}^+$.

We can now give a purely algebraic characterization of equinumerosities.

Theorem 4.5. *There exists a biunique correspondence between equinumerosity relations on the space* $\mathbf{F}$ *of all point sets over* $\mathbb{N}$ *and gauge ideals of the ring* $\mathcal{R}$ *of all formal series of bounded degree in* $\mathbb{Z}\langle\langle \mathbf{t} \rangle\rangle$ *with bounded coefficients. In this correspondence, if the equinumerosity* $\approx$ *corresponds to the ideal* $\mathfrak{I}$, *then*

$$(*) \qquad\qquad X \approx Y \iff S_X - S_Y \in \mathfrak{I}.$$

More precisely, let $\mathfrak{n} : \mathbf{F} \to \mathfrak{N}$ *be the numerosity function associated to* $\approx$, *and let* $\pi : \mathcal{R} \to \mathcal{R}/\mathfrak{I}$ *be the canonical projection. Then there exists a unique isomorphism* τ *of ordered rings such that the following diagram commutes*

$$
\begin{array}{ccc}
\mathbf{F} & \xrightarrow{\;\;\Sigma\;\;} & \mathcal{R} \\
{\scriptstyle \mathfrak{n}}\Big\downarrow\!\!\!\!\!\nearrow & (**) & \searrow{\scriptstyle \pi} \\
\mathfrak{N} & \xrightarrow{\;\;j\;\;} \mathfrak{R} \xrightarrow[\cong]{\;\;\tau\;\;} & \mathcal{R}/\mathfrak{I}
\end{array}
$$

(where Σ maps any $X \in \mathbf{F}$ to its characteristic series $S_X \in \mathcal{R}$, $\mathfrak{R}$ is the ring generated by $\mathfrak{N}$, and j is the natural embedding.)

Proof. Given a gauge ideal $\mathfrak{I}$ on $\mathcal{R}$, define the equivalence $\approx$ on $\mathbf{F}$ by

$$ X \approx Y \iff S_X - S_Y \in \mathfrak{I}. $$

We show that $\approx$ satisfies conditions (E1)-(E4) of an equinumerosity relation. (E1) is trivial and (E4) follows from Proposition 4.3(2), because

$$ S_{X \times Y} - S_{X' \times Y'} = (S_X - S_{X'})S_Y + S_{X'}(S_Y - S_{Y'}). $$

(E3) holds because $\mathfrak{I}_0 \subseteq \mathfrak{I}$. We are left with the trichotomy condition (E2). First of all observe that at most one of the conditions $X \approx Y$, $X \approx X' \subset Y$, and $Y \approx Y' \subset X$ can hold. *E.g.*, by assuming the first two conditions one would get

$$ S_{Y \setminus X'} = S_Y - S_{X'} = S_Y - S_X + S_X - S_{X'} \in \mathfrak{I} \cap \mathcal{R}^+, $$

against the second property of gauge ideals. Finally, if $S_X - S_Y \notin \mathfrak{I}$, then by combining the third property of gauge ideals with Proposition 4.3(5), one obtains $S_X - S_Y \mp S_Z \in \mathfrak{I}$ for some nonempty point set Z. Hence either $X \approx Y \setminus Z$ or $Y \approx X \setminus Z$.

Conversely, given an equinumerosity relation $\approx$, let $\mathfrak{I}$ be the two-sided ideal of $\mathcal{R}$ generated by the set $\{S_X - S_Y \mid X \approx Y\} \cup \{t_0 - 1\}$. Then $\mathfrak{I}_0 \subseteq \mathfrak{I}$, by Proposition 4.3(4). Moreover, by Proposition 4.3.(6), every $S \in \mathcal{R}$ is congruent modulo $\mathfrak{I}$ to a difference $S_X - S_Y$, which in turn is congruent to $\pm S_Z$ for some Z, by property (E2). So $\mathfrak{I}$ is gauge, provided that it is disjoint from $\mathcal{R}^+$.

In order to prove that $\mathfrak{I} \cap \mathcal{R}^+ = \emptyset$, remark that, by points $(2), (3)$, and (4) of Proposition 4.3, every element $S \in \mathfrak{I}$ can be written as

$$ S = a(t_0 - 1) + \sum_{i=1}^{m}(S_{X_i} - S_{Y_i}) $$

for suitable (not necessarily different) equinumerous point sets $X_i \approx Y_i$.

If no term in S is negative, then $a \le 0$, and every element of each Y_i belongs to some X_j. Partition Y_1 as $Y_1 = \bigcup_{j=1}^{m} Y_{1j}$, with $Y_{1j} \subseteq X_j$. Put $X_i' = X_i \setminus Y_{1i}$: then

$$ S = a(t_0 - 1) + \sum_{i=1}^{m} S_{X_i'} - \sum_{i=2}^{m} S_{Y_i}, \quad \text{and} \quad \sum_{i=1}^{m} \mathfrak{n}(X_i') = \sum_{i=2}^{m} \mathfrak{n}(Y_i). $$

So $Y_2 \subseteq \bigcup_{i=1}^{m} X_i'$, and we can proceed as above by subtracting each element of Y_2 from an appropriate X_i', thus obtaining sets X_i'' that satisfy the conditions

$$ S = a(t_0 - 1) + \sum_{i=1}^{m} S_{X_i''} - \sum_{i=3}^{m} S_{Y_i}, \quad \text{and} \quad \sum_{i=1}^{m} \mathfrak{n}(X_i'') = \sum_{i=3}^{m} \mathfrak{n}(Y_i). $$

Continuing this procedure with $Y_3, \ldots, Y_m$ we reach the situation

$$S = a(t_0 - 1) + \sum_{i=1}^{m} S_{X_i^{(m)}}, \quad \text{and} \quad \sum_{i=1}^{m} \mathfrak{n}(X_i^{(m)}) = 0.$$

Hence all sets $X_i^{(m)}$ are empty, and $S = a(t_0 - 1)$. But then $a = 0$, because S has no negative terms. So $\mathfrak{I} \cap \mathcal{R}^+ = \emptyset$, and the ideal $\mathfrak{I}$ is gauge.

In particular one obtains the condition $(*)$

$$S_X - S_Y \in \mathfrak{I} \iff X \approx Y.$$

In fact, if $X \not\approx Y$, say $X \approx X' \subset Y$, then $S_X - S_{X'} = S_X - S_Y + S_{Y \setminus X'} \in \mathfrak{I}$. Hence $S_X - S_Y$ cannot be in $\mathfrak{I}$, because otherwise $S_{Y \setminus X'} \in \mathfrak{I} \cap \mathcal{R}^+$. So the equinumerosity $\approx$ corresponding to the ideal $\mathfrak{I}$ is precisely the one we started with, and the correspondence between equinumerosities and gauge ideals is biunique.

Given $\mathfrak{I}$ and $\approx$, let $\mathfrak{n} : \mathbf{F} \to \mathfrak{N}$ be the numerosity function associated to $\approx$. In order to make the diagram $(**)$ commutative, one has to put $\tau(\mathfrak{n}(X)) = S_X + \mathfrak{I}$. This definition is well posed by $(*)$, and it provides a semiring homomorphism of $\mathfrak{N}$ into $\mathcal{R}/\mathfrak{I}$, by Proposition 4.3(2). Moreover τ is one-to-one and onto $(\mathcal{R}^+ + \mathfrak{I})/\mathfrak{I}$ because of Proposition 4.3(5). So τ can be uniquely extended to the required ring isomorphism $\tau : \mathfrak{R} \to \mathcal{R}/\mathfrak{I}$, by Proposition 4.3(6).

$\square$

5. Asymptotic numerosities and quasi-selective ultrafilters

Various measures of size for infinite sets of positive integers $A \subseteq \mathbb{N}^+$, commonly used in number theory, are obtained by considering the sequence of ratios

$$\frac{|\{a \in A \mid a \le n\}|}{n}.$$

In fact, the *(upper, lower) asymptotic density* of A are defined as the limit (superior, inferior) of this sequence. This procedure might be viewed as measuring *the ratio between the numerosities of A and $\mathbb{N}^+$*. In this perspective, one should assume that $\mathfrak{n}(A) \le \mathfrak{n}(B)$ whenever the sequence of ratios for B dominates that for A. So one is led to introduce the following notion of "asymptotic" equinumerosity relation.

Definition 5.1. *For $X \subseteq \mathbb{N}^k$ put*

$$X_n = \{x \in X \mid x_i \le n \text{ for } i = 1, \ldots, k\}.$$

The equinumerosity relation $\approx$ is asymptotic *if:*

(E0)   *If $|X_n| \le |Y_n|$ for all $n \in \mathbb{N}$, then there exists $Z \subseteq Y$ such that $X \approx Z$.*

Remark that sets X and Y are not assumed to be of the *same dimension*. In fact, at the end of this section we shall use the condition (E0) to give a notion of "quasi-numerosity" that is defined on *all sets of tuples* of natural numbers.

According to this definition, if $\mathfrak{n} : \mathbf{F} \to \mathfrak{N}$ is the numerosity function associated to an equinumerosity $\approx$, then $\approx$ is asymptotic if and only if $\mathfrak{n}$ satisfies the following property for all $X, Y \in \mathbf{F}$:

$$|X_n| \le |Y_n| \text{ for all } n \in \mathbb{N} \implies \mathfrak{n}(X) \le \mathfrak{n}(Y).$$

In the following theorem we give a nice algebraic characterization of asymptotic numerosities. Namely

- Let $\mathbf{e}^n$ be the sequence made up of $(n+1)$-many ones. For $S \in \mathcal{R}$ let $S(\mathbf{e}^n)$ be the value taken by S when 1 is assigned to the variables t_j for $0 \le j \le n$, while 0 is assigned to the remaining variables. So

$$|X_n| = S_X(\mathbf{e}^n) \text{ for every point set } X.$$

Then we have

Theorem 5.2. *Define the map* $\Phi : \mathcal{R} \to \mathbb{Z}^{\mathbb{N}}$ *by* $\Phi(S) = \langle\, S(\mathbf{e}^n)\,\rangle_{n \in \mathbb{N}}$. *Then*

(i) Φ *is a ring homomorphism, whose kernel* $\mathfrak{K}$ *is disjoint from* $\mathcal{R}^+$. *The range of* Φ *is the subring* $\mathbb{P}$ *of* $\mathbb{Z}^{\mathbb{N}}$ *consisting of all polynomially bounded sequences*

$$\mathbb{P} = \{g : \mathbb{N} \to \mathbb{Z} \mid \exists k, m \,\forall n > m \ |g(n)| < n^k\} \subseteq \mathbb{Z}^{\mathbb{N}}.$$

(ii) *Let* $\mathfrak{n} : \mathbf{F} \to \mathfrak{N}$ *be the numerosity function associated to an asymptotic equinumerosity* $\approx$. *Then there exists a unique ring homomorphism* ψ *of* $\mathbb{P}$ *onto the ring* $\mathfrak{R}$ *generated by* $\mathfrak{N}$ *such that the following diagram commutes*

(where Σ *maps any* $X \in \mathbf{F}$ *to its characteristic series* $S_X \in \mathcal{R}^+$, *and* j *is the natural embedding.)*

(iii) *Under the hypotheses in* (ii), *there exists a quasi-selective ultrafilter* $\mathcal{U}$ *on* $\mathbb{N}$ *such that the sequence* $g \in \mathbb{P}$ *belongs to kernel of* ψ *if and only if its zero-set* $Z(g) = \{n \mid g(n) = 0\}$ *belongs to* $\mathcal{U}$. *In particular*

$$(\sharp) \qquad\qquad X \approx Y \iff \{n \in \mathbb{N} \mid |X_n| = |Y_n|\} \in \mathcal{U}.$$

Proof.

(i) The first assertion is immediate. In order to characterize the range of Φ, recall that there are n^k (noncommutative) monomials of degree k in n variables. On the one hand, if $\deg S < d$ and all the coefficients of S are bounded by B, then $|S(\mathbf{e}^n)| < Bn^d$. On the other hand, assume that $|g(n) - g(n-1)| < n^k$ for $n > m$, say. Pick a (homogeneous) polynomial $p(t_0, \ldots, t_m)$ such that $p(\mathbf{e}^0) = g(0), \ldots, p(\mathbf{e}^m) = g(m)$. Then, for each $n > m$, add (or subtract) exactly $|g(n) - g(n-1)|$ monomials of degree not exceeding k in the variables $t_0, \ldots, t_n$, where t_n actually appears. The resulting series $S \in \mathcal{R}$ clearly verifies $S(\mathbf{e}^n) = g(n)$. Notice that if all monomials are chosen of the same degree, then S is the difference of two characteristic series. Hence any $S \in \mathcal{R}$ is congruent modulo $\mathfrak{K}$ to a difference $S_X - S_Y$.

(ii) The gauge ideal $\mathfrak{I}$ corresponding to $\approx$ contains $\mathfrak{K}$ because the equinumerosity $\approx$ is asymptotic. Therefore $\mathfrak{R} \cong \mathcal{R}/\mathfrak{I} \cong (\mathcal{R}/\mathfrak{K})/(\mathfrak{I}/\mathfrak{K}) \cong \mathbb{P}/\Phi[\mathfrak{I}]$. So, in the notation of Theorem 4.5, ψ is the unique ring homomorphism such that $\tau \circ \psi \circ \Phi = \pi$.

(iii) The kernel $\ker \psi = \mathbb{I}$ is a prime ideal of $\mathbb{P}$, because $\mathfrak{R}$ is a domain. The idempotents of $\mathbb{P}$ are all and only the characteristic functions χ_A of subsets $A \subseteq \mathbb{N}$. Hence, for every $A \subseteq \mathbb{N}$, the ideal $\mathbb{I}$ contains exactly one of the two complementary

idempotents $\chi_A, 1 - \chi_A = \chi_{\mathbb{N}\setminus A}$. Moreover $1 - \chi_{A\cap B} = 1 - \chi_A\chi_B = 1 - \chi_A + \chi_A(1 - \chi_B)$, and $A \subseteq B$ implies $1 - \chi_B = (1 - \chi_A)(1 - \chi_B)$. Hence the set

$$\mathcal{U} = \{A \subseteq \mathbb{N} \mid 1 - \chi_A \in \mathbb{I}\}$$

is an ultrafilter on $\mathbb{N}$.

Now $g = g(1 - \chi_{Z(g)})$, so g belongs to $\mathbb{I}$ whenever $Z(g) \in \mathcal{U}$. Conversely, if $Z(g) \notin \mathcal{U}$, then $\psi(\chi_{Z(g)}) = 0$ and $\psi(g^2 + \chi_{Z(g)} - 1)$ is non-negative. It follows that $\psi(g) \neq 0$. In particular

$$X \approx Y \iff \Phi(S_X - S_Y) \in \ker\psi \iff \{n \in \mathbb{N} \mid |X_n| - |Y_n| = 0\} \in \mathcal{U}$$

It remains to show that $\mathcal{U}$ is quasi-selective. To this aim, remark that, modulo the gauge ideal $\mathfrak{I} = \ker(\psi \circ \Phi) = \Phi^{-1}[\mathbb{I}]$, each series in $\mathcal{R}$ is equivalent to $\pm P$ for some $P \in \mathcal{R}^+$. In turn such a P is mapped by Φ to a nondecreasing element of $\mathbb{P}$. As Φ is surjective, we may conclude that every polynomially bounded non-negative sequence is $\mathcal{U}$-equivalent to a nondecreasing one. Hence $\mathcal{U}$ is quasi-selective.

$\square$

This construction allows for classifying all asymptotic equinumerosity relations by means of quasi-selective ultrafilters on $\mathbb{N}$ as follows:

Corollary 5.3. *There exists a biunique correspondence between asymptotic equinumerosity relations on the space of point sets over $\mathbb{N}$ and quasi-selective ultrafilters on $\mathbb{N}$. In this correspondence, if the equinumerosity $\approx$ corresponds to the ultrafilter $\mathcal{U}$, then*

$(\sharp)$ $$X \approx Y \iff \{n \in \mathbb{N} \mid |X_n| = |Y_n|\} \in \mathcal{U}.$$

More precisely, let $\mathfrak{N}$ be the set of numerosities of $\approx$, and let $\mathfrak{n}$ be the corresponding numerosity function. Then the map

$$\varphi : \mathfrak{n}(X) \mapsto [\langle |X_n| \rangle_{n \in \mathbb{N}}]_{\mathcal{U}}$$

is an isomorphism of ordered semirings between $\mathfrak{N}$ and the initial segment $\mathbb{P}_{\mathcal{U}}$ of the ultrapower $\mathbb{N}_{\mathcal{U}}^{\mathbb{N}}$ that contains the classes of all polynomially bounded sequences. In particular, asymptotic numerosities are, up to isomorphism, nonstandard integers.

Proof. Given an asymptotic equinumerosity $\approx$, let Φ, ψ, and the ultrafilter $\mathcal{U} = \mathcal{U}_{\approx}$ be as in Theorem 5.2. Then $\mathcal{U}$ is quasi-selective, and

$$X \approx Y \iff \Phi(S_X - S_Y) \in \ker\psi \iff \{n \in \mathbb{N} \mid |X_n| - |Y_n| = 0\} \in \mathcal{U},$$

that is $(\sharp)$.

Conversely, given a quasi-selective ultrafilter $\mathcal{U}$ on $\mathbb{N}$, define the equivalence $\approx_{\mathcal{U}}$ on **F** by $(\sharp)$. Then (E1),(E3), and (E4) are immediate. We prove now the following strong form of (E0) that implies also (E2):

$$\{n \in \mathbb{N} \mid |X_n| \leq |Y_n|\} \in \mathcal{U} \implies \exists Z \subseteq Y \text{ s.t. } Z \approx_{\mathcal{U}} X.$$

Assume that $\{n \in \mathbb{N} \mid |X_n| \leq |Y_n|\} = U \in \mathcal{U}$: by quasi-selectivity there exists $V \in \mathcal{U}$ such that $|X_n|, |Y_n|$, and $|Y_n| - |X_n|$ are nondecreasing on V. Then one can isolate from Y a subset $Z \approx X$ in the following way: if m, m' are consecutive elements of $U \cap V$, put in Z exactly $|X_{m'}| - |X_m|$ elements of $Y_{m'} \setminus Y_m$.

So $\mathfrak{n}(X) < \mathfrak{n}(Y)$ if and only if $\varphi(\mathfrak{n}(X)) < \varphi(\mathfrak{n}(Y))$ in $\mathbb{P}_{\mathcal{U}}$, and the last assertion of the theorem follows. Finally, remark that every subset $A \subseteq \mathbb{N}$ can be written as $A = \{n \in \mathbb{N} \mid |X_n| = |Y_n|\}$ for suitable point sets X, Y. So the biconditional $(\sharp)$ uniquely determines the ultrafilter $\mathcal{U} = \mathcal{U}_{\approx}$, and one has

- $\mathcal{U}_{(\approx_{\mathcal{U}})} = \mathcal{U}$ for all quasi-selective ultrafilters $\mathcal{U}$;
- $\approx_{(\mathcal{U}_{\approx})} = \approx$ for all asymptotic equinumerosity relations $\approx$.

Hence the correspondence between $\approx$ and $\mathcal{U}$ is biunique.

$\square$

It is worth remarking that the property (E0) yields at once (E3) and commutativity of product, as well as many other natural instantiations of the fourth Euclidean common notion:

"*Things applying* [*exactly*] *to one another are equal to one another.*"

More precisely, if the *support* of a tuple is defined by

$$\operatorname{supp}(x_1, \ldots, x_k) = \{x_1, \ldots, x_k\},$$

then all *support preserving bijections* can be taken as "congruences" for asymptotic numerosities, because any such $\sigma : X \to Y$ maps X_n onto Y_n for all $n \in \mathbb{N}$.

Actually, in order to give a "Cantorian" characterization of asymptotic equinumerosities, we isolate a wider class of bijections, namely

Definition 5.4. Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$. A bijection $\sigma : X \to Y$ is a $\mathcal{U}$-*congruence* if $\{n \in \mathbb{N} \mid \sigma[X_n] = Y_n\} \in \mathcal{U}$.

When the ultrafilter $\mathcal{U}$ is quasi-selective, the $\mathcal{U}$-congruences determine an asymptotic equinumerosity:

Corollary 5.5. *Let $\approx$ be an asymptotic equinumerosity, and let $\mathcal{U}$ be the corresponding quasi-selective ultrafilter. Then $X \approx Y$ if and only if there exists a $\mathcal{U}$-congruence $\sigma : X \to Y$.* $\square$

This point of view allows for an interesting generalization of the notion of asymptotic equinumerosity to *all subsets* of $\mathbb{E} = \bigcup_{k \in \mathbb{N}^+} \mathbb{N}^k$, namely

- put $\mathbb{E}_n = \bigcup_{1 \le k \le n} \{0, \ldots, n\}^k$;
- let $\mathcal{U}$ be a filter on $\mathbb{N}$;
- call a map $\sigma : \mathbb{E} \to \mathbb{E}$ a $\mathcal{U}$-*isometry* if $\{n \mid \sigma[\mathbb{E}_n] = \mathbb{E}_n\} \in \mathcal{U}$;
- let $\mathfrak{S}_{\mathcal{U}}$ be the group of all $\mathcal{U}$-isometries, and for $X, Y \subseteq \mathbb{E}$ put

$$X \approx_{\mathcal{U}} Y \iff \text{there exists } \sigma \in \mathfrak{S}_{\mathcal{U}} \text{ such that } \sigma[X] = Y.$$

If $\mathcal{U}$ is a quasi-selective ultrafilter we can now assign a "quasi-numerosity" to every subset of $\mathbb{E}$, namely its orbit under $\mathfrak{S}_{\mathcal{U}}$:

$$\mathfrak{n}_{\mathcal{U}}(X) = [X]_{\approx_{\mathcal{U}}} = X^{\mathfrak{S}_{\mathcal{U}}} = \{\sigma[X] \mid \sigma \in \mathfrak{S}_{\mathcal{U}}\}.$$

Let $\mathfrak{N}_{\mathcal{U}} = \mathcal{P}(\mathbb{E})/\approx_{\mathcal{U}}$ be the set of quasi-numerosities, and let $\mathbb{S}_{\mathcal{U}}$ be the initial segment of the ultrapower $\mathbb{N}_{\mathcal{U}}^{\mathbb{N}}$ determined by $\mathfrak{n}_{\mathcal{U}}(\mathbb{E}) = [\langle (n+1)\frac{(n+1)^n - 1}{n} \mid n \in \mathbb{N} \rangle]_{\mathcal{U}}$.

Notice that if $X \subseteq \mathbb{N}^k$, then $X_n = X \cap \mathbb{E}_n$ for $n \ge k$. So we have

Theorem 5.6. *The relation $\approx_{\mathcal{U}}$ is an equivalence on $\mathcal{P}(\mathbb{E})$ that satisfies the properties (E0),(E1),(E2), and, when restricted to $\mathbf{F}$, agrees with the asymptotic equinumerosity corresponding to $\mathcal{U}$. Moreover the map*

$$\varphi_{\mathcal{U}} : \mathfrak{n}_{\mathcal{U}}(X) \mapsto [\langle |X \cap \mathbb{E}_n| \rangle_{n \in \mathbb{N}}]_{\mathcal{U}}$$

preserves sums and is an order isomorphism between $\mathfrak{N}_{\mathcal{U}}$ and $\mathbb{S}_{\mathcal{U}} \subseteq \mathbb{N}_{\mathcal{U}}^{\mathbb{N}}$. $\square$

It is worth mentioning that both the "multiplicative" properties (E3) and (E4) can fail for sets of infinite dimension. *E.g.*

$$\{(0,1)\} \times \mathbb{E} \subset \{0\} \times \mathbb{E} \subset \mathbb{E}$$

have increasing quasi-numerosities, thus contradicting both (E3) and (E4).

6. Final remarks and open questions

It is interesting to remark that the non-selective quasi-selective ultrafilter $\mathcal{V}$ defined in the proof of Theorem 2.1 satisfies the following "weakly Ramsey" property:[10]

for any finite coloring of $[\mathbb{N}]^2$ there is $U \in \mathcal{V}$ such that $[U]^2$ has only two colors.

It is easily seen that if $\mathcal{V}$ is weakly Ramsey, then every function is either one-to-one or nondecreasing modulo $\mathcal{V}$. So both weakly Ramsey and quasi-selective ultrafilters are P-points. However these two classes are different whenever there exists a non-selective P-point, because the former is closed under isomorphism, whereas the latter is not, by Proposition 1.9.

In ZFC, one can draw the following diagram of implications

Quasi-selective

$$\nearrow \qquad\qquad \searrow$$

Selective   P-point

$$\searrow \qquad\qquad \nearrow$$
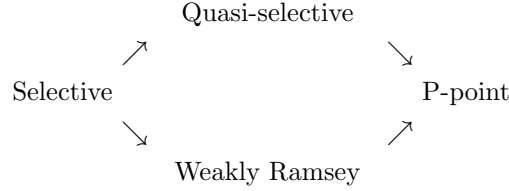
Weakly Ramsey

Recall that, assuming CH, the following facts hold:
- there exist quasi-selective weakly Ramsey ultrafilters that are not selective (Theorem 2.1);
- the class of quasi-selective non-selective ultrafilters is not closed under isomorphisms (Proposition 1.9);
- there are non-weakly-Ramsey P-points (see Theorem 2 of [4]).

It follows that, in the diagram above, no arrow can be reversed nor inserted

The relationships between quasi-selective and weakly Ramsey ultrafilters are extensively studied in [12]. In particular, it is proved there that both quasi-selective and weakly Ramsey ultrafilters are P-points of a special kind, since they share the property that every function is equivalent to an *interval-to-one* function. So the question naturally arises as to whether this class of "interval P-points" is distinct from either one of the other three classes.

Many weaker conditions than the Continuum Hypothesis have been considered in the literature, in order to get more information about special classes of ultrafilters on $\mathbb{N}$. Of particular interest are (in)equalities among the so called "combinatorial cardinal characteristics of the continuum". (*E.g.* one has that P-points or selective ultrafilters are generic if $\mathfrak{c} = \mathfrak{d}$ or $\mathfrak{c} = \mathbf{cov}(\mathcal{B})$, respectively.. Moreover if $\mathbf{cov}(\mathcal{B}) < \mathfrak{d} = \mathfrak{c}$ then there are filters that are included in P-points, but cannot be extended to selective ultrafilters. See the comprehensive survey [5].) We conjecture that similar hypotheses can settle the problems mentioned above.

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[10] Ultrafilters satisfying this property have been introduced in [4] under the name *weakly Ramsey*, and then generalized to $(n+1)$-Ramsey ultrafilters in [14].

It is worth mentioning that, given a quasi-selective ultrafilter $\mathcal{U}$, the corresponding asymptotic "quasi-numerosity" $\mathfrak{n}_{\mathcal{U}}$ of Theorem 5.6 can be extended to all subsets of the *algebraic Euclidean space*

$$\mathcal{Q} = \bigcup_{k \in \mathbb{N}^+} \overline{\mathbb{Q}}^k, \quad \text{where } \overline{\mathbb{Q}} \text{ is the field of all algebraic numbers.}$$

To this end, replace the sequence of finite sets $\mathbb{E}_n$ by

$$\mathcal{Q}_n = \{(\alpha_1, \ldots, \alpha_k) \in \mathcal{Q} \mid k \leq n, \text{ and } \exists a_{ih} \in \mathbb{Z}, |a_{ih}| \leq n, \sum_{0 \leq h \leq n} a_{ih}\alpha_i^h = 0\}.$$

Then extend the definition of $\mathcal{U}$-*isometry* to maps $\sigma : \mathcal{Q} \to \mathcal{Q}$ such that

$$\{n \mid \sigma[\mathcal{Q}_n] = \mathcal{Q}_n\} \in \mathcal{U},$$

and, for $X, Y \subseteq \mathcal{Q}$ put

$$X \approx_{\mathcal{U}} Y \iff \text{ there exists a } \mathcal{U}\text{-isometry } \sigma \text{ such that } \sigma[X] = Y.$$

Remark that the sequence $\langle |\mathcal{Q}_n| \rangle_{n \in \mathbb{N}}$ belongs to $F_{\mathcal{U}}$, since it is bounded by $\langle n^{3n^2} \rangle_{n \in \mathbb{N}}$, say. Hence one obtains the following natural extension of Theorem 5.6.

Theorem 6.1. *The relation* $\approx_{\mathcal{U}}$ *is an equivalence on* $\mathcal{P}(\mathcal{Q})$ *that satisfies the properties* (E0),(E1),(E2), *and, when restricted to* $\bigcup_{k \in \mathbb{N}} \mathcal{P}(\overline{\mathbb{Q}}^k)$, *also* (E3) *and* (E4). *The map*

$$\mathfrak{n}_{\mathcal{U}}(X) \mapsto [\langle |X \cap \mathcal{Q}_n| \rangle_{n \in \mathbb{N}}]_{\mathcal{U}}$$

preserves sums and is an order isomorphism between the set of "asymptotic quasi-numerosities" $\mathfrak{N}_{\mathcal{U}} = \mathcal{P}(\mathcal{Q})/\approx_{\mathcal{U}}$ *and the initial segment* $\mathbb{T}_{\mathcal{U}} \subseteq \mathbb{N}_{\mathcal{U}}^{\mathbb{N}}$ *determined by* $\mathfrak{n}_{\mathcal{U}}(\mathcal{Q}) = [\langle |\mathcal{Q}_n| \rangle_{n \in \mathbb{N}}]_{\mathcal{U}}$. $\square$

Similar results hold for point sets over any *countable* line $\mathcal{L}$ equipped with a *height function* h, provided that the corresponding function $g : \mathbb{N} \to \mathbb{N}$ defined by $g(n) = |\{x \in \mathcal{L} \mid h(x) \leq n\}|$ belongs to the class $F_{\mathcal{U}}$ of Section 1. If this is not the case, one can still maintain a biunique correspondence between asymptotic equinumerosities and ultrafilters, by restricting to ultrafilters $\mathcal{U}$ with the property that every function bounded by g is $\mathcal{U}$-equivalent to a nondecreasing one (*g-quasi-selective* ultrafilters).

The question as to whether there exist equinumerosities which are not asymptotic with respect to suitable height functions is still open. Of particular interest might be the identification of equinumerosities whose existence is provable in ZFC alone. Actually, the very notion of gauge ideal has been introduced in order to facilitate the investigation of these most general equinumerosity relations.

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