On the pseudoconvexity and pseudolinearity of some classes of fractional functions

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Abstract

The aim of the paper is to study the pseudoconvexity (pseudoconcavity) of the ratio between a quadratic function and the square of an affine function. Applying the Charnes-Cooper transformation of variables the function is transformed in a quadratic one defined on a suitable halfspace. The characterization of the pseudoconvexity of such a quadratic function allows us to give necessary and sufficient conditions for the pseudoconvexity and the pseudolinearity of the ratio in terms of the initial data.

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1 Introduction

Pseudoconvexity and pseudolinearity of functions are widely studied in the literature for their nice properties and for their economic applications [?, ?, ?]. In particular, these classes of functions play an important role in Optimization because of the fundamental property that a local minimum is also global and it is reached at an extremum point in case of pseudolinearity. Since many applications give rise to multi-ratio fractional programs [?], some approaches for studying pseudoconvexity and pseudolinearity for particular classes of fractional functions have been recently suggested ([?, ?, ?, ?]). In this framework, the Charnes-Cooper transformation has been shown to be an useful tool because of its property to preserve pseudoconvexity and pseudolinearity

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([?, ?]).
In this paper we consider the ratio between a quadratic function and the square of an affine function and we give a complete characterization of pseudoconvexity and pseudolinearity for it. More precisely, by means of the Charnes-Cooper transformation, the ratio is transformed in a quadratic function defined on a suitable halfspace. The study of pseudoconvexity (pseudolinearity) of the transformed function allows to give a characterization of the pseudoconvexity (pseudolinearity) of the ratio in terms of the initial data. Based on this characterization, a procedure for testing pseudoconvexity is given and it is illustrated by several numerical examples.

2 Statement of the problem

The aim of this paper is to study the pseudoconvexity of the function

$$f(x) = \frac{\frac{1}{2}x^T Ax + a^T x + a_0}{(b^T x + b_0)^2} \tag{1}$$

on the halfspace $S = \{x \in \Re^n : b^T x + b_0 > 0\}$, $b_0 \neq 0$.
We recall that a differentiable function h defined on an open convex set X is pseudoconvex if for $x^1, x^2 \in X$

$$h(x^1) > h(x^2) \Rightarrow \nabla h(x^1)^T (x^2 - x^1) < 0$$

In order to find conditions which ensure the pseudoconvexity of f, we first study the pseudoconvexity of a quadratic function defined on an halfspace. Trough the paper we will use the following notations:

- $\nu_-(C)$ ($\nu_+(C)$) denotes the number of negative (positive) eigenvalues of a matrix C;
- $r(C)$ denotes the rank of a matrix C
- $\ker C$ denotes the kernel of C that is $\ker C = \{v : Cv = 0\}$;
- $\operatorname{Im} C$ denotes the set $\operatorname{Im} C = \{z = Cv,\ v \in \Re^s\}$;
- $v^\perp$ denotes the orthogonal space to the vector v, that is $v^\perp = \{w : v^T w = 0\}$.
- $\dim W$ denotes the dimension of the vector space W.

It is well known that a quadratic function is pseudoconvex if and only if it is convex, so that pseudoconvexity can differ from convexity only if it is restricted on a proper subset of $\Re^n$ (see for instance [?]).
A necessary condition for the pseudoconvexity of f is given by the following theorem.

Theorem 2.1 *If f is pseudoconvex on S then the matrix A has at most one negative eigenvalue.*

Proof. Suppose by contradiction $v_-(A) > 1$ and let v_1 and v_2 be two linearly independent eigenvectors associated with two negative eigenvalues of A, such that $v_1^T v_2 = 0$. Let W be the linear subspace generated by v_1 and v_2. Let us note that $\dim(\ker A) \leq n-2$ and $\dim(b^\perp) = n-1$ so that $\ker A \neq b^\perp$. Moreover since either $W \subset b^\perp$ or $\dim(W + b^\perp) = n$, we have $\dim(W \cap b^\perp) = \dim W + \dim b^\perp - \dim(W + b^\perp) = 1$ and then $W \cap b^\perp \neq \emptyset$. Let $v \in W \cap b^\perp,\ v \neq 0$. Since v is a linear combination of v_1 and v_2, we have $v^T A v < 0$. Consider the line $x = x_0 + tv,\ x_0 \in S,\ t \in \Re$ which is contained in S since $b^T x + b_0 = b^T x_0 + b_0 > 0$. It is easy to verify that the restriction $\varphi(t) = f(x_0 + tv)$ is of the kind $\varphi(t) = \alpha t^2 + \beta t + \gamma$ with $\alpha < 0$ and this contradicts the pseudoconvexity of f. ■

Performing the Charnes-Cooper transformation $y = \frac{x}{b^T x + b_0}$, whose inverse is $x = \frac{b_0 y}{1 - b^T y}$ (see [?]), function f is transformed in the following quadratic function

$$f(x(y)) = Q(y) = y^T Q y + q^T y + q_0$$

where:

$$Q = \frac{1}{2}A - \frac{ab^T + ba^T}{2b_0} + \frac{a_0}{b_0^2} bb^T \tag{2}$$

$$q = \frac{1}{b_0}\left(a - 2\frac{a_0}{b_0} b\right),\quad q_0 = \frac{a_0}{b_0^2} \tag{3}$$

Taking into account that the previous transformation preserves pseudoconvexity and pseudoconcavity [?, ?], we have the following result.

Theorem 2.2 *The function $f(x)$ is pseudoconvex (pseudoconcave) on the halfspace S if and only if the quadratic function $Q(y)$ is pseudoconvex (pseudoconcave) on the halfspace $S^* = \left\{ y \in \Re^n : \frac{1 - b^T y}{b_0} > 0 \right\}$.*

The following theorem characterizes the pseudoconvexity of $Q(y)$ on the halfspace $H = \{y \in \Re^n :\ c^T y + c_0 > 0\}$.

Theorem 2.3 *The function $Q(y)$ is pseudoconvex on the halfspace H if and only if one of the following conditions holds:*
i) $\nu_-(Q) = 0$;
ii) $\nu_-(Q) = 1,\ \ker Q = c^\perp,\ q = \beta c,\ c_0 \leq \frac{\|c\|^4 \beta}{2c^T Qc}$.

Proof. [1] In [?] it is shown that the quadratic function $Q(y)$ defined on the halfspace H is pseudoconvex if and only if either it is convex (i.e. $\nu_-(Q) = 0$) or the following conditions hold:

a) $\nu_-(Q) = 1$;

b) $r(Q) = r(Q|q) = 1$;

c) $H \subseteq A_1$ where $A_1 = \{x \in \Re^n : u^T y + \gamma > 0\}$ is the maximal domain where $Q(y)$ is pseudoconvex. This domain A_1 can be characterized in terms of eigenvectors and eigenvalues of Q; more precisely u is a normalized eigenvector associated with the negative eigenvalue μ.

Furthermore when $Q(y)$ is not convex, it can be written as follows

$$Q(y) = \mu \left(u^T y + \gamma\right)^2 + \sigma. \tag{4}$$

Now we prove that condition ii) is equivalent to conditions a), b), c). With this aim, observe that condition c) is equivalent to $c = \|c\|\, u$ and $\frac{c_0}{\|c\|} \leq \gamma$ and hence condition b) is equivalent to $\ker Q = c^\perp$ and $q = \beta c$. From (**??**) we have that

$$2\mu\gamma u^T y = 2\gamma\mu \frac{c^T y}{\|c\|} = q^T y = \beta c^T y$$

hence $\gamma = \dfrac{\beta\,\|c\|}{2\mu}$. Therefore $\dfrac{c_0}{\|c\|} \leq \gamma$ is equivalent to $c_0 \leq \dfrac{\beta\,\|c\|^2}{2\mu}$ and the proof is complete. ■

Corollary 2.1 *Consider the function $h(y) = y^T Qy$. Then $h(y)$ is pseudoconvex on H if and only if Q is positive semidefinite or $Q = \mu cc^T$ with $\mu < 0$ and $c_0 < 0$.*

[1] A direct proof of the theorem can be found in [?].

3 Pseudoconvexity of the function $f(x)$

In order to characterize the pseudoconvexity of the function $f(x)$ in terms of the initial data A, a, a_0, b, b_0, we distinguish two exhaustive cases, that is $\ker A = b^{\perp}$ (Theorem **??**) and $\ker A \neq b^{\perp}$ (Theorem **??**).

Observe that condition $\ker A = b^{\perp}$ is equivalent to say that the matrix A can be written as $A = \delta bb^T$ where δ is the unique eigenvalue different from zero and the vector b is an associated eigenvector.

Theorem 3.1 *Consider the function $f(x)$ with $A = \delta bb^T,\ \delta \in \Re$. Then $f(x)$ is pseudoconvex on $S = \{x \in \Re^n : b^T x + b_0 > 0\}$ if and only if there exists $\gamma \in \Re$ such that $a = \gamma b$, and one of the following conditions holds*
i) $\delta b_0^2 - 2\gamma b_0 + 2a_0 \geq 0$
ii) $\delta b_0^2 - 2\gamma b_0 + 2a_0 < 0$ and $\gamma \leq \delta b_0$.

Proof. It can be easily proved that if a and b are linearly independent, conditions i) and ii) of Theorem **??** do not hold.
Since there exists $\gamma \in \Re$ such that $a = \gamma b$ and taking into account (**??**) and (**??**), we have $Q = (\frac{1}{2}\delta b_0^2 - \gamma b_0 + a_0)\frac{bb^T}{b_0^2}$, $q = (\frac{2a_0}{b_0} - \gamma)(-\frac{b}{b_0})$. Setting $c = -\frac{b}{b_0}$, $c_0 = \frac{1}{b_0}$, from Theorem **??** $f(x)$ is pseudoconvex on S if and only if $\mu = \frac{1}{2}\delta b_0^2 - \gamma b_0 + a_0$ is non negative or $\mu < 0$ and $c_0 \leq \frac{\beta}{2\mu}$ with $\beta = \frac{2a_0}{b_0} - \gamma$. This last inequality is equivalent to $\frac{1}{b_0} \leq \frac{\frac{2a_0}{b_0} - \gamma}{2(\frac{1}{2}\delta b_0^2 - \gamma b_0 + a_0)}$, that is $\frac{\gamma - \delta b_0}{\delta b_0^2 - 2\gamma b_0 + 2a_0} \geq 0$. Since $\mu < 0$ necessarily we have $\gamma - \delta b_0 \leq 0$ and the thesis is achieved. ■

Corollary 3.1 *The function $f(x)$ with $A = \delta bb^T,\ a = \gamma b,\ \delta, \gamma \in \Re$, is pseudoconvex on the halfspace S if and only if it can be reduced in the following canonical form*

$$f(x) = \frac{B}{b^T x + b_0} + \frac{C}{(b^T x + b_0)^2} + D \tag{5}$$

where $C \geq 0$ or $C < 0$ and $B \leq 0$.

The following theorem gives a complete characterization of the pseudoconvexity of f in the general case $\ker A \neq b^{\perp}$.

Theorem 3.2 *When $\ker A \neq b^{\perp}$, the function f is pseudoconvex on the halfspace S if and only if A is positive semidefinite on $b^{\perp}$ and one of the following conditions holds:*
i) there exists $\alpha \in \Re$ such that $Ab - \frac{\|b\|^2}{b_0}a = \alpha b$ with

$$\alpha \geq \frac{b_0 b^T a - 2\|b\|^2 a_0}{b_0^2} \tag{6}$$

ii) $Ab - \frac{\|b\|^2}{b_0} a \neq \alpha b$ *for every* $\alpha \in \Re$*, there exist* $a^*, b^* \in \Re^n$ *such that* $Ab^* = b$*,* $Aa^* = a$*,* $b^* \in b^\perp$*,* $b^T a^* = b_0$ *and*

$$a^{*T} a \leq 2a_0 \tag{7}$$

iii) $Ab - \frac{\|b\|^2}{b_0} a \neq \alpha b$ *for every* $\alpha \in \Re$*, there exist* $a^*, b^* \in \Re^n$ *such that* $Ab^* = b$*,* $Aa^* = a$*,* $b^{*T} b \neq 0$ *and*

$$a_0 - \frac{a^{*T} a}{2} + \frac{1}{2b^T b^*} \left(b_0 - b^T a^*\right)^2 \geq 0 \tag{8}$$

iv) $Ab - \frac{\|b\|^2}{b_0} a \neq \alpha b$ *for every* $\alpha \in \Re$ *and there exist* $\mu^* \in \Re$*,* $a^* \in \Re^n$ *such that* $a = Aa^* + \mu^* b$*,* $b \notin \operatorname{Im} A$ *and*

$$a_0 - \mu^* b_0 - \frac{1}{2} a^{*T} A a^* \geq 0 \tag{9}$$

Proof. From Theorem **??**, $f(x)$ is pseudoconvex on S if and only if the function $Q(y)$ is pseudoconvex on $S^* = \left\{y \in \Re^n : c^T y + c_0 > 0\right\}$, with $c = -\frac{1}{b_0} b,\ c_0 = \frac{1}{b_0}$.
The case *ii*) of Theorem **??** corresponds to the case $\ker A = b^\perp,\ a = \gamma b$ and the characterization of the pseudoconvexity of f is given in Theorem **??**.
When $\ker A \neq b^\perp$, f is pseudoconvex if and only if the matrix Q is positive semidefinite, with $Q = \frac{1}{2} A - \frac{ab^T + ba^T}{2b_0} + \frac{a_0}{b_0^2} bb^T$.
Let us note that for every $u \in b^\perp$ we have $u^T Q u = \frac{1}{2} u^T A u$, so that Q is positive semidefinite on $b^\perp$ if and only if A is positive semidefinite on $b^\perp$.
Let $\Re^n$ be decomposed as the direct sum between the space generated by vector b and its orthogonal space, so that every $x \in \Re^n$ can be written as $x = kb + w$ where $k \in \Re$ and $w \in b^\perp$. We have

$$x^T Q x = k^2 b^T Q b + k \left(Ab - \frac{\|b\|^2}{b_0} a\right)^T w + \frac{1}{2} w^T A w \tag{10}$$

where

$$b^T Q b = \frac{1}{2} b^T A b - \frac{\|b\|^2}{b_0} a^T b + \frac{a_0}{b_0^2} \|b\|^4 \tag{11}$$

Consequently, the matrix Q is positive semidefinite if and only if

$$\varphi(k, w) = k^2 b^T Q b + k \left(Ab - \frac{\|b\|^2}{b_0} a\right)^T w + \frac{1}{2} w^T A w \geq 0,\ \forall w \in b^\perp, \forall k \in \Re. \tag{12}$$

We are going to distinguish two exhaustive cases:
Case 1. $\left(Ab - \frac{\|b\|^2}{b_0}a\right)^T w = 0$ for every $w \in b^\perp$.
Case 2. There exists $w \in b^\perp$ such that $\left(Ab - \frac{\|b\|^2}{b_0}a\right)^T w \neq 0$.
Case 1. It is equivalent to say that there exists $\alpha \in \Re$, such that

$$\left(Ab - \frac{\|b\|^2}{b_0}a\right) = \alpha b \tag{13}$$

and condition (**??**) becomes

$$k^2 b^T Q b + \frac{1}{2} w^T A w \geq 0, \ \forall w \in b^\perp, \ \forall k \in \Re. \tag{14}$$

Since $w^T A w \geq 0$ for every $w \in b^\perp$, (**??**) is verified $\forall k \in \Re$ if and only if

$$b^T Q b = \frac{1}{2} b^T A b - \frac{\|b\|^2}{b_0} a^T b + \frac{a_0}{b_0^2} \|b\|^4 \geq 0. \tag{15}$$

From (**??**) we obtain $b^T A b - \frac{\|b\|^2}{b_0} b^T a = \alpha \|b\|^2$, so that $b^T A b = \frac{\|b\|^2}{b_0} b^T a + \alpha \|b\|^2$ and consequently $b^T Q b = \frac{1}{2}\frac{\|b\|^2}{b_0} b^T a + \frac{1}{2}\alpha \|b\|^2 - \frac{\|b\|^2}{b_0} a^T b + \frac{a_0}{b_0^2}\|b\|^4 = \frac{1}{2}\|b\|^2 \left(\alpha - \frac{1}{b_0} b^T a + \frac{2a_0}{b_0^2}\|b\|^2\right)$. So condition (**??**) is satisfied if and only if

$$\alpha \geq \frac{b_0 b^T a - 2\|b\|^2 a_0}{b_0^2}$$

Consequently, if A is positive semidefinite on $b^\perp$ and $\left(Ab - \frac{\|b\|^2}{b_0}a\right)^T w = 0$ for every $w \in b^\perp$, Q is positive semidefinite if and only if (**??**) is verified.
Case 2. Let us note that, corresponding to an element $w \in b^\perp$ such that $\left(Ab - \frac{\|b\|^2}{b_0}a\right)^T w \neq 0$, necessarily we have $w^T A w > 0$, otherwise (**??**) is not verified $\forall k \in \Re$. Furthermore, (**??**) is equivalent to

$$\inf_{(k,w)\in\Re\times b^\perp} \varphi(k,w) = \inf_{k\in\Re} \inf_{w\in b^\perp} \varphi(k,w) \geq 0.$$

It is well known that a convex quadratic function either has minimum value or its infimum is equal to $-\infty$ and consequently Q is positive semidefinite if and only if $\inf_{w\in b^\perp} \varphi(k,w) = \min_{w\in b^\perp} \varphi(k,w)$ and $\inf_{k\in\Re} \min_{w\in b^\perp} \varphi(k,w) \geq 0$.
Now, for any given $k \in \Re$, consider the following minimization problem

$$\begin{cases} \min \left[\varphi(k,w) = k^2 b^T Q b + k\left(Ab - \frac{\|b\|^2}{b_0}a\right)^T w + \frac{1}{2} w^T A w\right] \\ b^T w = 0 \end{cases} \tag{16}$$

Since A is positive semidefinite on the orthogonal space $b^\perp$, w^* is the solution of Problem (**??**) if and only if there exists (w^*, λ^*) which satisfies the following necessary and sufficient optimality conditions

$$\begin{cases} Aw^* + kAb - k\frac{\|b\|^2}{b_0}a = \lambda^* b & (1) \\ b^T w^* = 0 & (2) \end{cases} \tag{17}$$

Let us note that (**??**) implies $w^{*T}Aw^* + kw^{*T}(Ab - \frac{\|b\|^2}{b_0}a) = 0$, so that

$$\varphi(k, w^*) = k^2 b^T Q b - \frac{1}{2} w^{*T} A w^* \tag{18}$$

Furthermore, from (**??**.1), we have

$$k\frac{\|b\|^2}{b_0}a = A(w^* + kb) - \lambda^* b. \tag{19}$$

We are going to distinguish the two cases: $b \in \operatorname{Im} A$, $b \notin \operatorname{Im} A$.
If $b \in \operatorname{Im} A$, there exists b^* such that $Ab^* = b$, so that condition (**??**) implies $a \in \operatorname{Im} A$, i.e. there exists a^* such that $Aa^* = a$. Therefore equation (**??**.1) can be written as follows

$$A\left(w^* + kb - \lambda^* b^* - k\frac{\|b\|^2}{b_0}a^*\right) = 0.$$

As a consequence $w^* + kb - \lambda^* b^* - k\frac{\|b\|^2}{b_0}a^* \in \ker A$, so that

$$w^* = \lambda^* b^* + k\frac{\|b\|^2}{b_0}a^* - kb + e, \tag{20}$$

with $e \in \ker A$. Substituting (**??**) and (**??**) in (**??**) and (**??**.2) we get

$$\varphi(k, w^*) = \frac{\|b\|^4}{b_0^2}\left(a_0 - \frac{1}{2}a^{*T}a\right)k^2 + \lambda^* \frac{\|b\|^2}{b_0}\left(b_0 - b^T a^*\right)k - \frac{1}{2}\lambda^2 b^T b^* \tag{21}$$

$$b^T w^* = \lambda^* b^T b^* + k\frac{\|b\|^2}{b_0}\left(b^T a^* - b_0\right) = 0 \tag{22}$$

If $b^T b^* = 0$, from (**??**) $b^T w^* = 0$ for every k and necessarily we have $\left(b_0 - b^T a^*\right) = 0$ and therefore

$$\inf_{k\in\Re}\min_{w\in b^\perp}\varphi(k, w) = \inf_{k\in\Re}\varphi(k, w^*) = \inf_{k\in\Re}\left[k^2\frac{1}{2b_0^2}\|b\|^4\left(2a_0 - a^{*T}a\right)\right] \geq 0$$

if and only if $\frac{1}{2b_0^2}\|b\|^4\left(2a_0 - a^{*T}a\right) \geq 0$. Thus, Q is positive semidefinite if and only if $ii)$ holds.

If $b^Tb^* \neq 0$; from (??) we obtain

$$\lambda^* = \frac{k}{b_0}\frac{\|b\|^2}{b^Tb^*}\left(b_0 - b^Ta^*\right)$$

and substituting λ^* in (??) we get

$$\varphi\left(k, w^*\right) = k^2\frac{\|b\|^4}{b_0^2}\left(a_0 - \frac{a^{*T}a}{2} + \frac{1}{2b^Tb^*}\left(b_0 - b^Ta^*\right)^2\right)$$

Therefore $\inf\limits_{k\in\Re}\min\limits_{w\in b^\perp}\varphi\left(k, w\right) = \inf\limits_{k\in\Re}\varphi\left(k, w^*\right) \geq 0$ if and only if $a_0 - \frac{a^{*T}a}{2} + \frac{1}{2b^Tb^*}\left(b_0 - b^Ta^*\right)^2 \geq 0$. Consequently, Q is positive semidefinite if and only if $iii)$ holds.

Finally we deal with the case $b \notin \operatorname{Im} A$. From (??), system (??) has solutions if and only if there exist $a^* \in \Re^n$ and μ^* such that $a = Aa^* + \mu^*b$ and hence equation (??.1) can be written as follows

$$k\frac{\|b\|^2}{b_0}\left(Aa^* + \mu^*b\right) = A\left(w^* + kb\right) - \lambda^*b$$

or equivalently

$$A\left(w^* + kb - k\frac{\|b\|^2}{b_0}a^*\right) = \left(k\frac{\|b\|^2}{b_0}\mu^* + \lambda^*\right)b$$

Since $b \notin \operatorname{Im} A$, the above equation holds if and only if $k\frac{\|b\|^2}{b_0}\mu^* + \lambda^* = 0$ and hence $\left(\lambda^*, w^*\right)$ is the solution of system (??) if and only if

$$\begin{aligned}\lambda^* &= -k\frac{\|b\|^2}{b_0}\mu^* \\ w^* &= k\frac{\|b\|^2}{b_0}a^* - kb + e,\ e \in \ker A\end{aligned}$$

Therefore

$$\varphi\left(k, w^*\right) = k^2\frac{\|b\|^4}{b_0^2}\left(a_0 - b_0\mu^* - \frac{1}{2}a^{*T}Aa^*\right)$$

so that

$$\inf_{k\in\Re}\min_{w\in b^\perp}\varphi\left(k, w\right) = \inf_{k\in\Re}\varphi\left(k, w^*\right) \geq 0$$

if and only if $a_0 - b_0\mu^* - \frac{1}{2}a^{*T}Aa^* \geq 0$. Consequently, Q is positive semidefinite if and only if $iv)$ holds and the proof is complete. ■

Remark 3.1 *Let us note that in ii) and iii) of Theorem* **??**, *necessarily we have* $\ker A \subset a^{\perp} \cap b^{\perp}$. *In fact,* $Aa^* = a$, $Ab^* = b$, *imply* $z^T Aa^* = z^T a = 0$, $z^T Ab^* = z^T b = 0 \quad \forall z \in \ker A$. *Consequently, relations (**??**) and (**??**) are independent from the particular choice of* a^*, b^*.
With respect to iv) of Theorem **??**, *let* $\mu^*, \mu_1^* \in \Re$ *and* $a^*, a_1^* \in \Re^n$ *such that* $a = Aa^* + \mu^* b = Aa_1^* + \mu_1^* b$; *then* $A(a^* - a_1^*) = (\mu_1^* - \mu^*)b$. *Since* $b \notin \operatorname{Im} A$, *necessarily we have* $\mu_1^* = \mu^*$ *and* $a_1^* \in a^* + \ker A$. *As a consequence, in (**??**)* μ^* *is unique and* $a^{*T} Aa^*$ *is independent from the particular choice of* a^*.

4 Special cases

When the matrix A is not singular (in particular when A is positive definite) the characterization of the pseudoconvexity of the function f assumes a very simple form as it is stated in the following results.

Theorem 4.1 *Assume that* A *is not singular. The function* f *is pseudoconvex on the halfspace* S *if and only if* A *is positive semidefinite on* $b^{\perp}$ *and one of the following conditions holds:*
i) $b^T A^{-1} b = 0$ *and* $2a_0 \geq a^T A^{-1} a$;
ii) $b^T A^{-1} b \neq 0$ *and* $2a_0 - a^T A^{-1} a + \frac{\left(b_0 - b^T A^{-1} a\right)^2}{b^T A^{-1} b} \geq 0$.

Proof. Let us note that case *iv)* of Theorem **??** does not occur since the non singularity of A implies $b \in \operatorname{Im} A$.
Consider case *i)* of Theorem **??**. We have

$$b = \frac{\|b\|^2}{b_0} A^{-1} a + \alpha A^{-1} b \tag{23}$$

so that

$$a^T b = \frac{\|b\|^2}{b_0} a^T A^{-1} a + \alpha a^T A^{-1} b \tag{24}$$

Substituting (**??**) in (**??**), we obtain

$$2a_0 - a^T A^{-1} a + \frac{\alpha}{\|b\|^2} \left(b_0^2 - b_0 b^T A^{-1} a\right) \geq 0 \tag{25}$$

If $b^T A^{-1} b = 0$, from (**??**), we have $b^T A^{-1} a = b_0$, so that (**??**) becomes $2a_0 - a^T A^{-1} a \geq 0$ and thus *i)* is verified.
If $b^T A^{-1} b \neq 0$, from (**??**), we have

$$\frac{\alpha}{\|b\|^2} = \frac{b_0 - b^T A^{-1} a}{b_0 b^T A^{-1} b} \tag{26}$$

Substituting (**??**) in (**??**), we obtain condition *ii*).
Consider now condition *ii*) of Theorem **??**.
We have $b^* = A^{-1}b,\ a^* = A^{-1}a,\ b^TA^{-1}b = 0,\ b^TA^{-1}a = b_0$, so that (**??**) reduces to condition *i*).
At last consider condition *iii*) of Theorem **??**.
We have $b^* = A^{-1}b,\ a^* = A^{-1}a,\ b^TA^{-1}b \neq 0$, so that (**??**) reduces to condition *ii*). ■

Corollary 4.1 *Consider the function*

$$h(x) = \frac{\frac{1}{2}x^TAx + a_0}{(b^Tx + b_0)^2}$$

on the halfspace S, where A is not singular. Then h is pseudoconvex if and only if A is positive semidefinite on $b^\perp$ and one of the following conditions holds:
i) $b^TA^{-1}b = 0$ *and* $a_0 \geq 0$*;*
ii) $b^TA^{-1}b \neq 0$ *and* $2a_0 \geq -\frac{b_0^2}{b^TA^{-1}b}$.

Theorem 4.2 *Assume that A is positive definite on $\Re^n$. Then the function h is pseudoconvex on the halfspace S if and only if*

$$2a_0 - a^TA^{-1}a + \frac{\left(b_0 - b^TA^{-1}a\right)^2}{b^TA^{-1}b} \geq 0 \qquad (27)$$

Corollary 4.2 *Consider the function*

$$h(x) = \frac{\frac{1}{2}x^TAx + a_0}{(b^Tx + b_0)^2}$$

on the halfspace S, where A is positive definite. Then h is pseudoconvex if and only if $2a_0 \geq -\frac{b_0^2}{b^TA^{-1}b}$.

The following example shows that the function f may be not pseudoconvex even if A is positive definite.

Example 4.1 *Consider the function*

$$f(x_1, x_2) = \frac{x_1^2 + 2x_2^2 + 2x_1x_2 + 3x_1 + 2x_2 + 1}{(x_1 + x_2 + 1)^2}$$

Even if the matrix $A = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}$ *is positive definite, f is not pseudoconvex on S since (**??**) does not hold. The non pseudoconvexity of f on S can be also verified performing a restriction of f on the half line $x_2 = 0$, $x_1 > -1$.*

The following corollaries present other cases where conditions for the pseudoconvexity are very easy to be checked .

Corollary 4.3 *Consider the function*

$$h(x) = \frac{a^T x + a_0}{(b^T x + b_0)^2} \tag{28}$$

on the halfspace S.
Then h is pseudoconvex on S if and only if $a = \gamma b$ with $a_0 - \gamma b_0 \geq 0$ or with $a_0 - \gamma b_0 < 0$ and $\gamma \leq 0$.
Moreover when $a = 0$, h is pseudoconvex on S for every $a_0 \in \Re$.

Corollary 4.4 *Consider the function*

$$h(x) = \frac{\frac{1}{2} x^T A x}{(b^T x + b_0)^2}$$

on the halfspace S. Then h is pseudoconvex if and only if A is positive semidefinite or $A = \delta b b^T$ with $\delta < 0$ and $b_0 < 0$.

5 An algorithm to test for pseudoconvexity

The results obtained in the previous sections allow to state a simple algorithm for testing the pseudoconvexity of the function

$$f(x) = \frac{\frac{1}{2} x^T A x + a^T x + a_0}{(b^T x + b_0)^2}, \; x \in S = \{x \in \Re^n : b^T x + b_0 > 0\}, \; b_0 \neq 0$$

Step 0. If $A = \delta b b^T$ go to step 8, otherwise go to step 1.
Step 1. If A is not positive semidefinite on $b^\perp$, Stop: f is not pseudoconvex; otherwise calculate $Ab - \frac{\|b\|^2}{b_0} a$. If $Ab - \frac{\|b\|^2}{b_0} a = \alpha b$ go to step 2, otherwise go to step 3.
Step 2. If $\alpha \geq \frac{b_0 b^T a - 2\|b\|^2 a_0}{b_0^2}$, Stop: f is pseudoconvex otherwise Stop: f is not pseudoconvex.
Step 3. If the system $Ax = b$ has no solutions, go to step 7, otherwise go to step 4.
Step 4. If the system $Ax = a$ has no solutions Stop: f is not pseudoconvex, otherwise let a^* such that $Aa^* = a$ and let b^* such that $Ab^* = b$. If $b^T b^* = 0$ go to step 5, otherwise go to step 6.
Step 5. If $b^T a^* = b_0$ and $a^T a^* \leq 2a_0$, Stop: f is pseudoconvex, otherwise Stop: f is not pseudoconvex.

Step 6. If $a_0 - \frac{a^{*T}a}{2} + \frac{1}{2b^Tb^*}\left(b_0 - b^Ta^*\right)^2 \geq 0$, Stop: f is pseudoconvex, otherwise Stop: f is not pseudoconvex.
Step 7. If there exist μ^*, a^* such that $a = Aa^* + \mu^* b$ and $a_0 - \mu^* b_0 - \frac{1}{2}a^{*T}Aa^* \geq 0$, Stop: f is pseudoconvex, otherwise Stop: f is not pseudoconvex.
Step 8. If $a \neq \gamma b$, Stop: f is not pseudoconvex, otherwise go to step 9.
Step 9. If $\delta b_0^2 - 2\gamma b_0 + 2a_0 \geq 0$, Stop: f is pseudoconvex, otherwise go to step 10.
Step 10. If $\gamma \leq \delta b_0$, Stop: f is pseudoconvex, otherwise Stop: f is not pseudoconvex.
The following examples point out different cases that can occur applying the previous algorithm.

Example 5.1 *Consider the function*

$$f(x_1, x_2, x_3) = \frac{\frac{1}{2}x_1^2 + x_2^2 + \frac{3}{2}x_3^2 + 2x_1x_2 + x_1 + 2x_2 + a_0}{(x_1 + 1)^2}$$

Case i) of Theorem **??** *occurs and it can be easy verified that f is pseudoconvex for every $a_0 \geq \frac{1}{2}$.*

Example 5.2 *Consider the function*

$$f(x_1, x_2, x_3, x_4) = \frac{\frac{1}{2}x_1^2 + 2x_2^2 - x_3^2 + \frac{1}{2}x_4^2 + 2x_1x_2 + x_1 + 2x_2 + x_4 + a_0}{(2x_1 + 4x_2 - 2\sqrt{2}x_3 + 2)^2}$$

Case ii) of Theorem **??** *occurs and it can be easy verified that f is pseudoconvex for every $a_0 \geq 1$.*

Example 5.3 *Consider the function*

$$f(x_1, x_2, x_3) = \frac{x_1^2 + x_2^2 - x_3^2 + 2x_1x_2 + x_1 + x_2 - x_3 + 1}{(x_3 + b_0)^2}$$

Case iii) of Theorem **??** *occurs and it can be easy verified that f is pseudoconvex for every $b_0 \in [-\frac{1}{2}, \frac{3}{2}]$, $b_0 \neq 0$,*

Example 5.4 *Consider the function*

$$f(x_1, x_2, x_3) = \frac{x_1^2 + x_3^2 + x_1 + a_2x_2 + x_3 + 1}{(x_2 + 1)^2}$$

Case iv) of Theorem **??** *occurs and it can be easy verified that the function $f(x)$ is pseudoconvex for every $a_2 \leq \frac{1}{2}$.*

6 Pseudolinearity of the function $f(x)$.

It is well known that a function is pseudolinear if and only if it is both pseudoconvex and pseudoconcave. Taking into account that a function is pseudoconcave if and only if its opposite is pseudoconvex, from Theorem **??** we obtain

Theorem 6.1 *The function $Q(y)$ is pseudoconcave on H if and only if one of the following conditions holds:*
i) $\nu_+(Q) = 0$;
ii) $\nu_+(Q) = 1,\ \ker Q = c^\perp,\ q = \beta c,\ c_0 \le \frac{\|c\|^4 \beta}{2c^T Qc}$.

Combining *i*) and *ii*) of Theorem **??** with *i*) and *ii*) of Theorem **??** and taking into account that *ii*) of Theorem **??** and *ii*) of Theorem **??** cannot occur simultaneously, we achieve the following result.

Theorem 6.2 *The function $Q(y)$ is pseudolinear on H if and only if one of the following conditions hold:*
i) $Q = 0$;
ii) $Q = \mu cc^T, \quad \mu \ne 0, \quad q = \beta c, \quad \beta \in \Re, \quad c_0 \le \frac{\beta}{2\mu}$.

In terms of the data A, a, a_0, b, b_0, taking into account that the function $f(x)$ is pseudolinear on S if and only if $Q(y)$ is pseudolinear on H (see Theorem **??**), we have the following theorem.

Theorem 6.3 *The function $f(x)$ is pseudolinear on S if and only if one of the following conditions holds:*
i) $A = \frac{ab^T + ba^T}{b_0} - \frac{2a_0}{b_0^2} bb^T$;
ii) $A = \delta bb^T,\ a = \gamma b,\ \delta, \gamma \in \Re$ *with* $\delta b_0^2 - 2\gamma b_0 + 2a_0 > 0$ *and* $\gamma \ge \delta b_0$ *or* $\delta b_0^2 - 2\gamma b_0 + 2a_0 < 0$ *and* $\gamma \le \delta b_0$.

Proof. Condition *i*) is equivalent to *i*) of Theorem **??** taking into account relation (**??**), while *ii*) is equivalent to *ii*) of Theorem **??** taking into account the following relationships: $\mu = \frac{1}{2}\delta b_0^2 - \gamma b_0 + a_0$, $\beta = \frac{2a_0}{b_0} - \gamma$, $c_0 = \frac{1}{b_0}$. ∎

Corollary 6.1 *The function $f(x)$ is pseudolinear on S if and only if it can be reduced to a linear fractional function or to the following canonical form*

$$f(x) = \frac{B}{b^T x + b_0} + \frac{C}{(b^T x + b_0)^2} + D \tag{29}$$

where $C > 0$ and $B \ge 0$ or $C < 0$ and $B \le 0$.

Proof. Corresponding to case $i)$ of Theorem **??**, it results $f(x) = \frac{b_0 a^T x - a_0 b^T x + a_0 b_0}{b_0^2 (b^T x + b_0)}$ so that $f(x)$ is a linear fractional function; the canonical form (**??**) follows by $ii)$ of Theorem **??** taking into account Corollary **??**. ■

Corollary 6.2 *Consider function* $h(x) = \frac{a^T x + a_0}{(b^T x + b_0)^2}$. $h(x)$ *is pseudolinear on* S *if and only if* $a = \gamma b$ *with* $\gamma \geq 0$ *and* $a_0 - \gamma b_0 > 0$ *or* $\gamma \leq 0$ *and* $a_0 - \gamma b_0 < 0$. *Moreover when* $a = 0$, $h(x)$ *is pseudolinear on* S *for every* $a_0 \in \Re$.

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