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Branched covers of the sphere and the prime-degree conjecture

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Abstract

To a branched cover $\widetilde{\Sigma} \to \Sigma$ between closed, connected and orientable surfaces one associates a *branch datum*, which consists of Σ and $\widetilde{\Sigma}$, the total degree d, and the partitions of d given by the collections of local degrees over the branching points. This datum must satisfy the Riemann-Hurwitz formula. A *candidate surface cover* is an abstract branch datum, a priori not coming from a branched cover, but satisfying the Riemann-Hurwitz formula. The old *Hurwitz problem* asks which candidate surface covers are realizable by branched covers. It is now known that all candidate covers are realizable when Σ has positive genus, but not all are when Σ is the 2-sphere. However a long-standing conjecture asserts that candidate covers with prime degree are realizable. To a candidate surface cover one can associate one $\widetilde{X} \dashrightarrow X$ between 2-orbifolds, and in [18] we have completely analyzed the candidate surface covers such that either X is bad, spherical, or Euclidean, or both X and $\widetilde{X}$ are rigid hyperbolic orbifolds, thus also providing strong supporting evidence for the prime-degree conjecture. In this paper, using a variety of different techniques, we continue this analysis, carrying it out completely for the case where X is hyperbolic and rigid and $\widetilde{X}$ has a 2-dimensional Teichmüller space. We find many more realizable and non-realizable candidate covers, providing more support for the prime-degree conjecture.

In this paper we push one step forward the approach via geometric 2-orbifolds developed and first exploited in [18] to face the Hurwitz existence problem for branched covers between surfaces. See [8] for the original source concerning this problem, the classical [9, 23, 22, 4, 5, 3, 11, 2, 6, 12, 10], the more recent [1, 14, 15, 18, 20, 21, 16, 25], and below. In [18] we have determined the realizability of all candidate surface branched covers having associated candidate cover between 2-orbifolds with non-negative Euler characteristic or between rigid hyperbolic 2-orbifolds. These results provided in particular strong support for the long-standing conjecture [2] that a candidate cover with prime total degree is always realizable. In this paper we consider the case where in the associated candidate cover between 2-orbifolds the covered 2-orbifold is hyperbolic and rigid while the covering 2-orbifold has a 2-dimensional Teichmüller space. As a result we exhibit many new realizable and non-realizable candidate surface branched covers, finding a confirmation of the validity of the prime-degree conjecture for the cases under consideration. To discuss realizability we use a variety of techniques developed over the time by several authors, and some new ones. Some of the results proved in this paper are also contained, with minor variations, in the PhD thesis [17] of the first named author (University of Rome I, 2010).

The paper is organized as follows. In Section 1 we precisely state the Hurwitz problem, fixing the notation we employ to treat it, and we outline the classical results that motivate us to restrict our attention to one specific instance of the problem. In Section 2 we discuss how geometric 2-orbifolds relate to the problem and allow to split it into some more specific subproblems, stating the results of [18] where the easiest of these subproblems were solved. In Section 3 we describe the next easiest subproblem of the Hurwitz existence problem, which is treated in this paper, and we state the corresponding solution. This subproblem itself splits into two different cases, for each of which one needs accomplish two tasks, namely to enumerate the relevant candidate covers and to discuss their realizability. The required enumeration processes for the two cases are carried out in Section 4. Next, Section 5 contains an overview of the different techniques later used to discuss the realizability of the candidate covers found. The discussion itself, again separately for the two relevant cases, is finally carried out in Section 6.

1 (Candidate) surface branched covers, and some known results

In this section we state the Hurwitz existence problem using the same language and notation as in [18], and we very briefly review some by now classical results.

Branched covers Let $\widetilde{\Sigma}$ and Σ be closed, connected, and orientable surfaces, and $f : \widetilde{\Sigma} \to \Sigma$ be a branched cover, *i.e.*, a map locally modelled on functions of the form $(\mathbb{C},0) \xrightarrow{z\mapsto z^k} (\mathbb{C},0)$ with $k \geqslant 1$. If $k > 1$ then 0 in the target $\mathbb{C}$ is a branching point, and k is the local degree at 0 in the source $\mathbb{C}$. There is a finite number n of branching points, and, removing all of them from Σ and their preimages from $\widetilde{\Sigma}$, we see that f induces a genuine cover of some degree d. The collection $(d_{ij})_{j=1}^{m_i}$ of the local degrees at the preimages of the i-th branching point is a partition Π_i of d. We now define:

- $\ell(\Pi_i)$ to be the length m_i of Π_i;
- Π as the set $\{\Pi_1, \ldots, \Pi_n\}$ of all partitions of d associated to f;
- $\ell(\Pi)$ to be the total length $\ell(\Pi_1) + \ldots + \ell(\Pi_n)$ of Π.

Then multiplicativity of the Euler characteristic χ under genuine covers for surfaces with boundary implies the classical Riemann-Hurwitz formula

$$\chi(\widetilde{\Sigma}) - \ell(\Pi) = d \cdot \big(\chi(\Sigma) - n\big). \tag{1}$$

Candidate branched covers and the realizability problem Consider again two closed, connected, and orientable surfaces $\widetilde{\Sigma}$ and Σ, integers $d \geqslant 2$ and $n \geqslant 1$, and a set of partitions $\Pi = \{\Pi_1, \ldots, \Pi_n\}$ of d, with $\Pi_i = (d_{ij})_{j=1}^{m_i}$, such that condition (1) is satisfied. We associate to these data the symbol

$$\widetilde{\Sigma} \underset{(d_{11},\ldots,d_{1m_1}),\ldots,(d_{n1},\ldots,d_{nm_n})}{\overset{d:1}{\dashrightarrow}} \Sigma$$

that we will call a *candidate surface branched cover*. A classical (and still not completely solved) problem, known as the *Hurwitz existence problem*, asks which candidate surface branched covers are actually *realizable*, namely induced by some existent branched cover $f : \widetilde{\Sigma} \to \Sigma$. A non-realizable candidate surface branched cover will be called *exceptional*.

Over the last 50 years the Hurwitz existence problem was the object of a wealth of papers, many of which were listed above. The combined efforts of several mathematicians led in particular to the following results [9, 2]:

- If $\chi(\Sigma) \leqslant 0$ then any candidate surface branched cover is realizable, *i.e.*, the Hurwitz existence problem has a positive solution in this case;

- If $\chi(\Sigma) > 0$, *i.e.*, if Σ is the 2-sphere S, there exist exceptional candidate surface branched covers.

Remark 1.1. A version of the Hurwitz existence problem exists also for possibly non-orientable $\widetilde{\Sigma}$ and Σ. Condition (1) must be complemented in this case with a few more requirements (some obvious, and one slightly less obvious, see [20]). However it has been shown [3, 2] that again this generalized problem always has a positive solution if $\chi(\Sigma) \leqslant 0$, and that the case where Σ is the projective plane reduces to the case where Σ is the 2-sphere S.

According to the two facts stated, in order to face the Hurwitz existence problem, it is not restrictive to *assume the candidate covered surface* Σ *is the* 2*-sphere* S, which we will do henceforth. Considerable energy has been devoted over the time to a general understanding of the exceptional candidate surface branched covers in this case, and quite some progress has been made (see for instance the survey of known results contained in [20], together with the later papers [21, 16, 25]), but the global pattern remains elusive. In particular the following conjecture proposed in [2] appears to be still open:

Conjecture 1.2. *If* $\widetilde{\Sigma} \underset{\Pi}{\overset{d:1}{\dashrightarrow}} S$ *is a candidate surface branched cover and the degree* d *is a prime number then the candidate is realizable.*

The following fact, again established in [2], will serve to us as a motivation:

Proposition 1.3. *If Conjecture* 1.2 *is true for candidate surface branched covers having* $n = 3$ *branching points, then it is true in general.*

We conclude this section by mentioning that all exceptional candidate surface branched covers with $n = 3$ and $d \leqslant 20$ were determined by computer in [25]. There are very many of them, but none occurs for prime d.

2 Surface covers vs. 2-orbifold covers, and more known results

A 2-orbifold $X = \Sigma(p_1, \ldots, p_n)$ is a closed orientable surface Σ with n cone points of orders $p_i \geqslant 2$, at which X has a singular differentiable structure given by the quotient $\mathbb{C}/_{\langle \mathrm{rot}(2\pi/p_i)\rangle}$, where $\mathrm{rot}(\vartheta) : z \mapsto e^{i\vartheta} \cdot z$.

Geometric 2-orbifolds W. Thurston [24] introduced the notion of orbifold Euler characteristic

$$\chi^{\rm orb}\big(\Sigma(p_1,\ldots,p_n)\big)=\chi(\Sigma)-\sum_{i=1}^{n}\left(1-\frac{1}{p_i}\right),$$

and showed that:

- If $\chi^{\rm orb}(X)>0$ then X is either *bad* (not covered by a surface in the sense of orbifolds, see below) or *spherical*, namely the quotient of the metric 2-sphere $\mathbb{S}^2$ under a finite isometric action;
- If $\chi^{\rm orb}(X)=0$ (respectively, $\chi^{\rm orb}(X)<0$) then X is *Euclidean* (respectively, *hyperbolic*), namely the quotient of the Euclidean plane $\mathbb{E}^2$ (respectively, the hyperbolic plane $\mathbb{H}^2$) under a discrete isometric action.

In addition, Thurston proved that, for a hyperbolic X with n cone points and underlying surface of genus g, the Teichmüller space $\tau(X)$, namely the space of hyperbolic structures on X up to isometries isotopic to the identity, has real dimension $6(g-1)+2n$.

Orbifold covers Again following Thurston [24] we call *degree-d orbifold cover* a map $f:\widetilde{X}\to X$ between 2-orbifolds such that $f^{-1}(x)$ generically consists of d points and locally making a diagram of the following form commutative:

$$\begin{array}{ccc}(\mathbb{C},0) & \stackrel{\rm id}{\longrightarrow} & (\mathbb{C},0)\\ \downarrow & & \downarrow\\ (\widetilde{X},\widetilde{x}) & \stackrel{f}{\longrightarrow} & (X,x)\end{array}$$

where $\widetilde{x}$ and x have cone orders $\widetilde{p}$ and $p=k\cdot\widetilde{p}$ respectively, and the vertical arrows are the projections corresponding to the actions of $\langle{\rm rot}(2\pi/\widetilde{p})\rangle$ and $\langle{\rm rot}(2\pi/p)\rangle$, namely the maps defining the (possibly singular) local differentiable structures at $\widetilde{x}$ and x. Since this local model can be described by the map $z\mapsto z^k$, we see that f induces a branched cover between the underlying surfaces of $\widetilde{X}$ and X. Using the orbifold language one can then state the Riemann-Hurwitz formula (1) in the following equivalent fashion:

$$\chi^{\rm orb}(\widetilde{X})=d\cdot\chi^{\rm orb}(X). \qquad (2)$$

From surface to 2-orbifold candidate covers As one easily sees, distinct orbifold covers can induce the same surface branched cover (in the local model, the two cone orders can be multiplied by one and the same integer). However, as pointed out in [20], a surface branched cover has an "easiest" associated orbifold cover, *i.e.*, that with the smallest possible cone orders. This carries over to *candidate* covers, as we will now spell out. Consider a candidate surface branched cover

$$\widetilde{\Sigma} \xrightarrow[(d_{11},\ldots,d_{1m_1}),\ldots,(d_{n1},\ldots,d_{nm_n})]{d:1} \Sigma$$

and define

$$\begin{aligned} p_i &= \text{l.c.m.}\{d_{ij}: \ j=1,\ldots,m_i\}, & p_{ij} &= p_i/d_{ij}, \\ X &= \Sigma(p_1,\ldots,p_n), & \widetilde{X} &= \widetilde{\Sigma}\big((p_{ij})_{i=1,\ldots,n}^{j=1,\ldots,m_i}\big) \end{aligned}$$

where "l.c.m." stands for "least common multiple." Then we have a *preferred associated candidate 2-orbifold cover* $\widetilde{X} \stackrel{d:1}{\dashrightarrow} X$ satisfying $\chi^{\text{orb}}(\widetilde{X}) = d\cdot\chi^{\text{orb}}(X)$. Note that the original candidate surface branched cover cannot be reconstructed from $\widetilde{X}, X, d$ alone, but it can if $\widetilde{X} \stackrel{d:1}{\dashrightarrow} X$ is complemented with the *covering instructions*

$$(p_{11},\ldots,p_{1m_1}) \dashrightarrow p_1, \qquad \ldots \qquad (p_{n1},\ldots,p_{nm_n}) \dashrightarrow p_n$$

that one can include in the symbol $\widetilde{X} \stackrel{d:1}{\dashrightarrow} X$ itself, omitting the p_{ij}'s equal to 1. Of course a candidate surface branched cover is realizable if and only if the associated candidate 2-orbifold cover with appropriate covering instructions is realizable.

Splitting the Hurwitz problem according to orbifold geometry We have just shown that to each candidate surface branched cover one can attach a preferred candidate orbifold cover $\widetilde{X} \dashrightarrow X$. Following Thurston's geometric picture of 2-orbifolds one can then split the Hurwitz existence problem by restricting to the analysis of those candidate surface branched covers for which in the associated $\widetilde{X} \dashrightarrow X$ the geometry of X and $\widetilde{X}$ is prescribed, and in the hyperbolic case the dimension of $\tau(X)$ and $\tau(\widetilde{X})$ also is. Note that a candidate orbifold cover $\widetilde{X} \dashrightarrow X$ by definition satisfies the orbifold version (2) of the Riemann-Hurwitz formula; therefore, X and $\widetilde{X}$ have the same geometry, except possibly when one of them is bad and the other one is spherical.

Having in mind to attack Conjecture 1.2 and taking into account Proposition 1.3, one can actually restrict to the case where

- X is a triangular orbifold $S(p,q,r)$, whence geometrically rigid,

and split the Hurwitz existence problem as described in the following table.

Subproblem	**Description**
bad/$\mathbb{S}$	X is bad or spherical, namely $\chi^{\mathrm{orb}}(X) > 0$
$\mathbb{E}$	X is Euclidean, namely $\chi^{\mathrm{orb}}(X) = 0$
$\mathbb{H}(j)$ for $j \in \mathbb{N}$	X is hyperbolic and $\dim(\tau(\widetilde{X})) = 2j$

Known result In [18] we have completely solved the subproblems of the Hurwitz existence problem described above as bad/$\mathbb{S}$, as $\mathbb{E}$, and as $\mathbb{H}(0)$, finding the results described in the following table.

Subproblem	**Findings**
bad/$\mathbb{S}$	20 realizable isolated candidate covers, two infinite families of realizable candidate covers, 11 exceptional isolated candidate covers, and one infinite family of exceptional candidate covers
$\mathbb{E}$	14 infinite families of realizable candidate covers, two infinite families of exceptional candidate covers, and 12 infinite families of candidate covers for which the realizability was shown to be equivalent to an arithmetic condition on the degree
$\mathbb{H}(0)$	9 realizable isolated candidate covers and two isolated exceptional candidate covers

We mention that the arithmetic conditions on the degree d in the solution of subproblem $\mathbb{E}$ are given by congruences and/or by the fact that d belongs to the image of some quadratic form $\mathbb{N}\times\mathbb{N} \to \mathbb{N}$. Analyzing these conditions, for three of the families in the solution of subproblem $\mathbb{E}$ we were able to show that the candidate cover is "exceptional with probability 1," even though it is realizable when its degree is prime, which we view as a strong supporting evidence for Conjecture 1.2.

3 New results

We describe here the new contribution offered by the present paper, namely the solution of subproblem $\mathbb{H}(1)$ of the Hurwitz existence problem. This consists of the following steps:

- Enumeration of all the candidate surface branched covers having an associated candidate orbifold cover of the form $\widetilde{X} \dashrightarrow X$ where $X = S(p,q,r)$ is a rigid hyperbolic 2-orbifold, namely such that $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} < 1$, and $\widetilde{X}$ is hyperbolic with $\dim(\tau(\widetilde{X})) = 2$, namely $\widetilde{X} = S(\alpha,\beta,\gamma,\delta)$ with $(\alpha,\beta,\gamma,\delta) \neq (2,2,2,2)$, or $\widetilde{X} = T(\alpha)$ with $\alpha > 1$, where T is the torus;
- Discussion of realizability or exceptionality of all the candidate surface covers enumerated.

The next statements summarize our results. Before giving them we underline that all the exceptional candidates we have found occur for composite degree, which means that our results provide further supporting evidence for Conjecture 1.2. We also mention that the tables of the next pages contain a numbering of all the candidate covers we have found; these numbers will be referred to throughout the paper.

Theorem 3.1. *There exist precisely* 146 *candidate surface branched covers* $\widetilde{\Sigma} \underset{(\Pi_1,\Pi_2,\Pi_3)}{\overset{d:1}{\dashrightarrow}} S$ *for which in the associated candidate orbifold cover* $\widetilde{X} \dashrightarrow X$ *one has that* X *and* $\widetilde{X}$ *are hyperbolic of the form* $X = S(p,q,r)$ *and* $\widetilde{X} = S(\alpha,\beta,\gamma,\delta)$*. Precisely* 29 *of these candidate surface covers are exceptional, and the other* 117 *are realizable. The complete description of these candidate covers, including the associated candidate orbifold covers and information on their realizability, is contained in Tables 1 to 5.*

Theorem 3.2. *There exist precisely* 22 *candidate surface branched covers* $\widetilde{\Sigma} \underset{(\Pi_1,\Pi_2,\Pi_3)}{\overset{d:1}{\dashrightarrow}} S$ *for which in the associated candidate orbifold cover* $\widetilde{X} \dashrightarrow X$

one has that X and $\widetilde{X}$ are hyperbolic of the form $X = S(p,q,r)$ and $\widetilde{X} = T(\alpha)$. Precisely 5 *of these candidate surface covers are exceptional, and the other* 17 *are realizable. The complete description of these candidate covers, including the associated candidate orbifold covers and information on their realizability, is contained in Table 6.*

Remark 3.3. The issue of enumerating the relevant candidate covers for Theorems 3.1 and 3.2 is an elementary, though complicated, combinatorial problem, and its solution presented below in Section 4 does not employ sophisticated techniques. On the other hand, to discuss realizability of the candidates found, we describe in general terms in Section 5 and then we exploit in Section 6 a variety of different geometric methods. As a matter of fact, in some cases we offer new proofs of the known realizability or exceptionality of some candidates, and we establish the previously unknown realizability or exceptionality for other candidates using two or more different methods. Our aim here is to show that a wealth of different techniques are already in place and should allow one to attack more and more advanced instances of the Hurwitz existence problem, according to its splitting we have proposed above in Section 2.

d	Π_1	Π_2	Π_3	Associated $\widetilde{X} \dashrightarrow X$	Realizable?	#
5	(2,1,1,1)	(4,1)	(5)	$S(2,2,2,4) \dashrightarrow S(2,4,5)$	✓	1
	(3,1,1)	(3,1,1)	(5)	$S(3,3,3,3) \dashrightarrow S(3,3,5)$	✓	2
	(3,1,1)	(4,1)	(4,1)	$S(3,3,4,4) \dashrightarrow S(3,4,4)$	✓	3
	(2,2,1)	(3,2)	(4,1)	$S(2,2,3,4) \dashrightarrow S(2,4,6)$	✓	4
6	(2,2,1,1)	(4,1,1)	(6)	$S(2,2,4,4) \dashrightarrow S(2,4,6)$	✓	5
	(2,2,1,1)	(5,1)	(5,1)	$S(2,2,5,5) \dashrightarrow S(2,5,5)$	✓	6
	(2,2,1,1)	(4,2)	(5,1)	$S(2,2,2,5) \dashrightarrow S(2,4,5)$	✓	7
	(3,1,1,1)	(3,3)	(5,1)	$S(3,3,3,5) \dashrightarrow S(3,3,5)$	✓	8
	(3,1,1,1)	(3,3)	(4,2)	$S(2,3,3,3) \dashrightarrow S(3,3,4)$	✓	9
	(3,3)	(4,1,1)	(4,1,1)	$S(4,4,4,4) \dashrightarrow S(3,4,4)$	✓	10
	(2,2,2)	(3,2,1)	(5,1)	$S(2,3,5,6) \dashrightarrow S(2,5,6)$	✓	11
	(2,2,2)	(3,2,1)	(4,2)	$S(2,2,3,6) \dashrightarrow S(2,4,6)$	✓	12

Table 1. Candidate surface branched covers $S \overset{d:1}{\underset{(\Pi_1,\Pi_2,\Pi_3)}{\dashrightarrow}} S$ with associated hyperbolic $S(\alpha,\beta,\gamma,\delta) \dashrightarrow S(p,q,r)$; continued in Tables 2 to 5.

d	Π_1	Π_2	Π_3	Associated $\widetilde{X} \dashrightarrow X$	Realizable?	#
7	(2,2,1,1,1)	(3,3,1)	(7)	$S(2,2,2,3) \dashrightarrow S(2,3,7)$	✓	13
	(2,2,2,1)	(4,1,1,1)	(7)	$S(2,4,4,4) \dashrightarrow S(2,4,7)$	✓	14
	(3,3,1)	(3,3,1)	(5,1,1)	$S(3,3,5,5) \dashrightarrow S(3,3,5)$	✓	15
	(3,3,1)	(3,3,1)	(4,2,1)	$S(2,3,3,4) \dashrightarrow S(3,3,4)$	✓	16
	(2,2,2,1)	(3,3,1)	(5,2)	$S(2,2,3,5) \dashrightarrow S(2,3,10)$	✓	17
	(2,2,2,1)	(3,3,1)	(4,3)	$S(2,3,3,4) \dashrightarrow S(2,3,12)$	✓	18
	(2,2,2,1)	(4,2,1)	(6,1)	$S(2,2,4,6) \dashrightarrow S(2,4,6)$	✓	19
	(2,2,2,1)	(5,1,1)	(6,1)	$S(2,5,5,6) \dashrightarrow S(2,5,6)$	✓	20
8	(2,2,2,2)	(4,1,1,1,1)	(8)	$S(4,4,4,4) \dashrightarrow S(2,4,8)$	✓	21
	(2,2,2,1,1)	(3,3,1,1)	(8)	$S(2,2,3,3) \dashrightarrow S(2,3,8)$	✓	22
	(2,2,2,2)	(3,2,2,1)	(4,4)	$S(2,3,3,6) \dashrightarrow S(2,4,6)$	**Excep**	23
	(2,2,2,2)	(5,1,1,1)	(7,1)	$S(5,5,5,7) \dashrightarrow S(2,5,7)$	✓	24
	(2,2,2,2)	(5,1,1,1)	(6,2)	$S(3,5,5,5) \dashrightarrow S(2,5,6)$	**Excep**	25
	(2,2,2,2)	(4,2,1,1)	(7,1)	$S(2,4,4,7) \dashrightarrow S(2,4,7)$	✓	26
	(2,2,2,2)	(4,2,1,1)	(6,2)	$S(2,3,4,4) \dashrightarrow S(2,4,6)$	✓	27
	(2,2,2,2)	(3,3,1,1)	(5,3)	$S(3,3,3,5) \dashrightarrow S(2,3,15)$	**Excep**	28
	(3,3,1,1)	(3,3,1,1)	(4,4)	$S(3,3,3,3) \dashrightarrow S(3,3,4)$	✓	29
	(2,2,2,2)	(6,1,1)	(6,1,1)	$S(6,6,6,6) \dashrightarrow S(2,6,6)$	✓	30
	(2,2,2,2)	(4,2,2)	(6,1,1)	$S(2,2,6,6) \dashrightarrow S(2,4,6)$	**Excep**	31
	(2,2,2,1,1)	(4,4)	(6,1,1)	$S(2,2,6,6) \dashrightarrow S(2,4,6)$	✓	32
9	(2,2,2,1,1,1)	(3,3,3)	(8,1)	$S(2,2,2,8) \dashrightarrow S(2,3,8)$	✓	33
	(2,2,2,2,1)	(3,3,1,1,1)	(9)	$S(2,3,3,3) \dashrightarrow S(2,3,9)$	✓	34
	(3,3,3)	(3,3,3)	(5,1,1,1,1)	$S(5,5,5,5) \dashrightarrow S(3,3,5)$	**Excep**	35
	(3,3,3)	(3,3,3)	(4,2,1,1,1)	$S(2,4,4,4) \dashrightarrow S(3,3,4)$	**Excep**	36
	(3,3,1,1,1)	(3,3,3)	(4,4,1)	$S(3,3,3,4) \dashrightarrow S(3,3,4)$	✓	37
	(2,2,2,2,1)	(4,4,1)	(7,1,1)	$S(2,4,7,7) \dashrightarrow S(2,4,7)$	✓	38
	(2,2,2,2,1)	(4,4,1)	(6,2,1)	$S(2,3,4,6) \dashrightarrow S(2,4,6)$	✓	39
	(2,2,2,2,1)	(3,3,3)	(5,3,1)	$S(2,3,5,15) \dashrightarrow S(2,3,15)$	✓	40
	(2,2,2,2,1)	(3,3,3)	(5,2,2)	$S(2,2,5,5) \dashrightarrow S(2,3,10)$	**Excep**	41
	(2,2,2,2,1)	(3,3,3)	(4,3,2)	$S(2,3,4,6) \dashrightarrow S(2,3,12)$	✓	42
10	(2,2,2,2,2)	(3,3,1,1,1,1)	(10)	$S(3,3,3,3) \dashrightarrow S(2,3,10)$	✓	43
	(2,2,2,2,1,1)	(4,4,1,1)	(5,5)	$S(2,2,4,4) \dashrightarrow S(2,4,5)$	✓	44
	(2,2,2,2,1,1)	(3,3,3,1)	(8,2)	$S(2,2,3,4) \dashrightarrow S(2,3,8)$	✓	45
	(2,2,2,2,1,1)	(3,3,3,1)	(9,1)	$S(2,2,3,9) \dashrightarrow S(2,3,9)$	✓	46
	(2,2,2,2,2)	(5,5)	(6,1,1,1,1)	$S(6,6,6,6) \dashrightarrow S(2,5,6)$	✓	47

Table 2. Continued from Table 1.

d	Π_1	Π_2	Π_3	Associated $\widetilde{X} \dashrightarrow X$	Realizable?	#
10	(2,2,2,2,2)	(4,2,2,1,1)	(5,5)	$S(2,2,4,4) \dashrightarrow S(2,4,5)$	✓	48
	(2,2,2,2,2)	(4,4,2)	(7,1,1,1)	$S(2,7,7,7) \dashrightarrow S(2,4,7)$	**Excep**	49
	(2,2,2,2,2)	(4,4,2)	(6,2,1,1)	$S(2,3,6,6) \dashrightarrow S(2,4,6)$	**Excep**	50
	(2,2,2,2,2)	(4,4,1,1)	(8,1,1)	$S(4,4,8,8) \dashrightarrow S(2,4,8)$	✓	51
	(2,2,2,2,2)	(4,4,1,1)	(6,3,1)	$S(2,4,4,6) \dashrightarrow S(2,4,6)$	✓	52
	(2,2,2,2,2)	(4,4,1,1)	(6,2,2)	$S(3,3,4,4) \dashrightarrow S(2,4,6)$	✓	53
	(2,2,2,2,2)	(3,3,3,1)	(7,2,1)	$S(2,3,7,14) \dashrightarrow S(2,3,14)$	✓	54
	(2,2,2,2,2)	(3,3,3,1)	(5,4,1)	$S(3,4,5,20) \dashrightarrow S(2,3,20)$	✓	55
	(2,2,2,2,2)	(3,3,3,1)	(5,3,2)	$S(3,6,10,15) \dashrightarrow S(2,3,30)$	✓	56
	(2,2,2,2,2)	(3,3,3,1)	(4,3,3)	$S(3,3,4,4) \dashrightarrow S(2,3,12)$	**Excep**	57
	(3,3,3,1)	(3,3,3,1)	(4,4,1,1)	$S(3,3,4,4) \dashrightarrow S(3,3,4)$	✓	58
11	(2,2,2,2,2,1)	(3,3,3,1,1)	(10,1)	$S(2,3,3,10) \dashrightarrow S(2,3,10)$	✓	59
	(2,2,2,2,2,1)	(4,4,2,1)	(5,5,1)	$S(2,2,4,5) \dashrightarrow S(2,4,5)$	✓	60
12	(2,2,2,2,2,1,1)	(3,3,3,3)	(10,1,1)	$S(2,2,10,10) \dashrightarrow S(2,3,10)$	✓	61
	(2,2,2,2,2,1,1)	(3,3,3,3)	(8,2,2)	$S(2,2,4,4) \dashrightarrow S(2,3,8)$	✓	62
	(2,2,2,2,2,1,1)	(4,4,4)	(5,5,1,1)	$S(2,2,5,5) \dashrightarrow S(2,4,5)$	✓	63
	(2,...,2)	(4,4,1,1,1,1)	(6,6)	$S(4,4,4,4) \dashrightarrow S(2,4,6)$	✓	64
	(2,...,2)	(3,3,3,1,1,1)	(11,1)	$S(3,3,3,11) \dashrightarrow S(2,3,11)$	✓	65
	(2,...,2)	(3,3,3,1,1,1)	(10,2)	$S(3,3,3,5) \dashrightarrow S(2,3,10)$	✓	66
	(2,...,2)	(3,3,3,1,1,1)	(9,3)	$S(3,3,3,3) \dashrightarrow S(2,3,9)$	✓	67
	(2,...,2)	(3,3,3,1,1,1)	(8,4)	$S(2,3,3,3) \dashrightarrow S(2,3,8)$	✓	68
	(2,...,2)	(4,4,4)	(8,1,1,1,1)	$S(8,8,8,8) \dashrightarrow S(2,4,8)$	**Excep**	69
	(2,...,2)	(4,4,4)	(6,3,1,1,1)	$S(2,6,6,6) \dashrightarrow S(2,4,6)$	✓	70
	(2,...,2)	(4,4,4)	(6,2,2,1,1)	$S(3,3,6,6) \dashrightarrow S(2,4,6)$	✓	71
	(2,...,2)	(3,3,3,3)	(7,3,1,1)	$S(3,7,21,21) \dashrightarrow S(2,3,21)$	**Excep**	72
	(2,...,2)	(3,3,3,3)	(7,2,2,1)	$S(2,7,7,14) \dashrightarrow S(2,3,14)$	**Excep**	73
	(2,...,2)	(3,3,3,3)	(6,4,1,1)	$S(2,3,12,12) \dashrightarrow S(2,3,12)$	**Excep**	74
	(2,...,2)	(3,3,3,3)	(5,4,2,1)	$S(4,5,10,20) \dashrightarrow S(2,3,20)$	**Excep**	75
	(2,...,2)	(3,3,3,3)	(5,3,3,1)	$S(3,5,5,15) \dashrightarrow S(2,3,15)$	**Excep**	76
	(2,...,2)	(3,3,3,3)	(5,3,2,2)	$S(6,10,15,15) \dashrightarrow S(2,3,30)$	**Excep**	77
	(2,...,2)	(3,3,3,3)	(4,3,3,2)	$S(3,4,4,6) \dashrightarrow S(2,3,12)$	**Excep**	78
	(2,...,2)	(3,3,3,3)	(4,4,3,1)	$S(3,3,4,12) \dashrightarrow S(2,3,12)$	**Excep**	79
	(3,3,3,3)	(3,3,3,3)	(4,4,1,1,1,1)	$S(4,4,4,4) \dashrightarrow S(3,3,4)$	✓	80
	(2,...,2)	(5,5,1,1)	(5,5,1,1)	$S(5,5,5,5) \dashrightarrow S(2,5,5)$	✓	81
	(2,...,2)	(4,4,2,2)	(5,5,1,1)	$S(2,2,5,5) \dashrightarrow S(2,4,5)$	✓	82

Table 3. Continued from Table 2.

d	Π_1	Π_2	Π_3	Associated $\widetilde{X} \dashrightarrow X$	Realizable?	#
13	(2,...,2,1)	(3,3,3,3,1)	(8,4,1)	$S(2,2,3,8) \dashrightarrow S(2,3,8)$	✓	83
	(2,...,2,1)	(3,3,3,3,1)	(11,1,1)	$S(2,3,11,11) \dashrightarrow S(2,3,11)$	✓	84
	(2,...,2,1)	(3,3,3,3,1)	(10,2,1)	$S(2,3,5,10) \dashrightarrow S(2,3,10)$	✓	85
	(2,...,2,1)	(3,3,3,3,1)	(9,3,1)	$S(2,3,3,9) \dashrightarrow S(2,3,9)$	✓	86
14	(2,...,2)	(3,3,3,3,1,1)	(10,2,2)	$S(3,3,5,5) \dashrightarrow S(2,3,10)$	✓	87
	(2,...,2)	(3,3,3,3,1,1)	(8,4,2)	$S(2,3,3,4) \dashrightarrow S(2,3,8)$	✓	88
	(2,...,2)	(3,3,3,3,1,1)	(12,1,1)	$S(3,3,12,12) \dashrightarrow S(2,3,12)$	✓	89
	(2,...,2)	(4,4,4,1,1)	(6,6,1,1)	$S(4,4,6,6) \dashrightarrow S(2,4,6)$	✓	90
	(2,...,2,1,1)	(3,3,3,3,1,1)	(7,7)	$S(2,2,3,3) \dashrightarrow S(2,3,7)$	✓	91
15	(2,...,2,1)	(3,3,3,3,3)	(12,1,1,1)	$S(2,12,12,12) \dashrightarrow S(2,3,12)$	✓	92
	(2,...,2,1)	(3,3,3,3,3)	(8,4,2,1)	$S(2,2,4,8) \dashrightarrow S(2,3,8)$	✓	93
	(2,...,2,1)	(3,3,3,3,3)	(10,2,2,1)	$S(2,5,5,10) \dashrightarrow S(2,3,10)$	✓	94
	(2,...,2,1)	(4,4,4,1,1,1)	(5,5,5)	$S(2,4,4,4) \dashrightarrow S(2,4,5)$	✓	95
	(2,...,2,1,1,1)	(3,3,3,3,3)	(7,7,1)	$S(2,2,2,7) \dashrightarrow S(2,3,7)$	✓	96
16	(2,...,2)	(3,3,3,3,3,1)	(10,2,2,2)	$S(3,5,5,5) \dashrightarrow S(2,3,10)$	**Excep**	97
	(2,...,2)	(3,3,3,3,3,1)	(8,4,2,2)	$S(2,3,4,4) \dashrightarrow S(2,3,8)$	**Excep**	98
	(2,...,2)	(3,3,3,3,1,1,1,1)	(8,8)	$S(3,3,3,3) \dashrightarrow S(2,3,8)$	✓	99
	(2,...,2)	(3,3,3,3,3,1)	(13,1,1,1)	$S(3,13,13,13) \dashrightarrow S(2,3,13)$	✓	100
	(2,...,2)	(3,3,3,3,3,1)	(12,2,1,1)	$S(3,6,12,12) \dashrightarrow S(2,3,12)$	✓	101
	(2,...,2)	(3,3,3,3,3,1)	(9,3,3,1)	$S(3,3,3,9) \dashrightarrow S(2,3,9)$	✓	102
	(2,...,2)	(4,4,4,4)	(6,6,1,1,1,1)	$S(6,6,6,6) \dashrightarrow S(2,4,6)$	✓	103
	(2,...,2)	(4,4,4,2,1,1)	(5,5,5,1)	$S(2,4,4,5) \dashrightarrow S(2,4,5)$	✓	104
17	(2,...,2,1)	(3,3,3,3,3,1,1)	(8,8,1)	$S(2,3,3,8) \dashrightarrow S(2,3,8)$	✓	105
	(2,...,2,1)	(4,4,4,4,1)	(5,5,5,1,1)	$S(2,4,5,5) \dashrightarrow S(2,4,5)$	✓	106
18	(2,...,2,1,1)	(3,...,3)	(8,8,1,1)	$S(2,2,8,8) \dashrightarrow S(2,3,8)$	✓	107
	(2,...,2)	(3,3,3,3,3,1,1,1)	(8,8,2)	$S(3,3,3,4) \dashrightarrow S(2,3,8)$	✓	108
	(2,...,2)	(3,...,3)	(14,1,1,1,1)	$S(14,14,14,14) \dashrightarrow S(2,3,14)$	✓	109
	(2,...,2)	(3,...,3)	(12,2,2,1,1)	$S(6,6,12,12) \dashrightarrow S(2,3,12)$	✓	110
	(2,...,2)	(3,...,3)	(12,3,1,1,1)	$S(4,12,12,12) \dashrightarrow S(2,3,12)$	✓	111
	(2,...,2)	(3,...,3)	(10,5,1,1,1)	$S(2,10,10,10) \dashrightarrow S(2,3,10)$	✓	112
	(2,...,2)	(3,...,3)	(8,4,4,1,1)	$S(2,2,8,8) \dashrightarrow S(2,3,8)$	**Excep**	113
	(2,...,2)	(3,...,3)	(8,4,2,2,2)	$S(2,4,4,4) \dashrightarrow S(2,3,8)$	**Excep**	114
	(2,...,2)	(3,...,3)	(10,2,2,2,2)	$S(5,5,5,5) \dashrightarrow S(2,3,10)$	**Excep**	115
	(2,...,2)	(4,4,4,4,2)	(5,5,5,1,1,1)	$S(2,5,5,5) \dashrightarrow S(2,4,5)$	**Excep**	116

Table 4. Continued from Table 3.

d	Π_1	Π_2	Π_3	Associated $\widetilde{X} \dashrightarrow X$	Realizable?	#
19	(2,...,2,1)	(3,...,3,1)	(8,8,2,1)	$S(2,3,4,8) \dashrightarrow S(2,3,8)$	✓	117
20	(2,...,2)	(3,...,3,1,1)	(8,8,2,2)	$S(3,3,4,4) \dashrightarrow S(2,3,8)$	✓	118
	(2,...,2)	(3,...,3,1,1)	(9,9,1,1)	$S(3,3,9,9) \dashrightarrow S(2,3,9)$	✓	119
	(2,...,2)	(4,4,4,4,1,1,1,1)	(5,5,5,5)	$S(4,4,4,4) \dashrightarrow S(2,4,5)$	✓	120
21	(2,...,2,1)	(3,...,3)	(9,9,1,1,1)	$S(2,9,9,9) \dashrightarrow S(2,3,9)$	✓	121
	(2,...,2,1)	(3,...,3)	(8,8,2,2,1)	$S(2,4,4,8) \dashrightarrow S(2,3,8)$	**Excep**	122
	(2,...,2,1)	(3,...,3,1,1,1)	(7,7,7)	$S(2,3,3,3) \dashrightarrow S(2,3,7)$	✓	123
22	(2,...,2,1,1)	(3,...,3,1)	(7,7,7,1)	$S(2,2,3,7) \dashrightarrow S(2,3,7)$	✓	124
	(2,...,2)	(3,...,3,1)	(8,8,4,1,1)	$S(2,3,8,8) \dashrightarrow S(2,3,8)$	**Excep**	125
	(2,...,2)	(3,...,3,1)	(8,8,2,2,2)	$S(3,4,4,4) \dashrightarrow S(2,3,8)$	**Excep**	126
	(2,...,2)	(4,4,4,4,4,1,1)	(5,5,5,5,1,1)	$S(4,4,5,5) \dashrightarrow S(2,4,5)$	✓	127
24	(2,...,2)	(3,...,3)	(8,8,2,2,2,2)	$S(4,4,4,4) \dashrightarrow S(2,3,8)$	✓	128
	(2,...,2)	(3,...,3)	(8,8,4,2,1,1)	$S(2,4,8,8) \dashrightarrow S(2,3,8)$	✓	129
	(2,...,2)	(3,...,3)	(10,10,1,1,1,1)	$S(10,10,10,10) \dashrightarrow S(2,3,10)$	✓	130
	(2,...,2)	(3,...,3)	(9,9,3,1,1,1)	$S(3,9,9,9) \dashrightarrow S(2,3,9)$	**Excep**	131
	(2,...,2)	(4,...,4)	(5,5,5,5,1,1,1,1)	$S(5,5,5,5) \dashrightarrow S(2,4,5)$	✓	132
26	(2,...,2)	(3,...,3,1,1)	(8,8,8,1,1)	$S(3,3,8,8) \dashrightarrow S(2,3,8)$	✓	133
27	(2,...,2,1)	(3,...,3)	(8,8,8,1,1,1)	$S(2,8,8,8) \dashrightarrow S(2,3,8)$	✓	134
28	(2,...,2)	(3,...,3,1)	(8,8,8,2,1,1)	$S(3,4,8,8) \dashrightarrow S(2,3,8)$	✓	135
	(2,...,2)	(3,...,3,1,1,1,1)	(7,7,7,7)	$S(3,3,3,3) \dashrightarrow S(2,3,7)$	✓	136
29	(2,...,2,1)	(3,...,3,1,1)	(7,7,7,7,1)	$S(2,3,3,7) \dashrightarrow S(2,3,7)$	✓	137
30	(2,...,2,1,1)	(3,...,3)	(7,7,7,7,1,1)	$S(2,2,7,7) \dashrightarrow S(2,3,7)$	✓	138
	(2,...,2)	(3,...,3)	(8,8,8,2,2,1,1)	$S(4,4,8,8) \dashrightarrow S(2,3,8)$	✓	139
36	(2,...,2)	(3,...,3)	(8,8,8,8,1,1,1,1)	$S(8,8,8,8) \dashrightarrow S(2,3,8)$	✓	140
	(2,...,2)	(3,...,3,1,1,1)	(7,7,7,7,7,1)	$S(3,3,3,7) \dashrightarrow S(2,3,7)$	✓	141
37	(2,...,2,1)	(3,...,3,1)	(7,7,7,7,7,1,1)	$S(2,3,7,7) \dashrightarrow S(2,3,7)$	✓	142
44	(2,...,2)	(3,...,3,1,1)	(7,...,7,1,1)	$S(3,3,7,7) \dashrightarrow S(2,3,7)$	✓	143
45	(2,...,2,1)	(3,...,3)	(7,...,7,1,1,1)	$S(2,7,7,7) \dashrightarrow S(2,3,7)$	✓	144
52	(2,...,2)	(3,...,3,1)	(7,...,7,1,1,1)	$S(3,7,7,7) \dashrightarrow S(2,3,7)$	✓	145
60	(2,...,2)	(3,...,3)	(7,...,7,1,1,1,1)	$S(7,7,7,7) \dashrightarrow S(2,3,7)$	✓	146

Table 5. Continued from Table 4.

d	Π_1	Π_2	Π_3	Associated $\widetilde{X} \dashrightarrow X$	Realizable?	#
4	(3,1)	(4)	(4)	$T(3) \dashrightarrow S(3,4,4)$	✓	147
5	(2,2,1)	(5)	(5)	$T(2) \dashrightarrow S(2,5,5)$	✓	148
6	(2,2,2)	(5,1)	(6)	$T(5) \dashrightarrow S(2,5,6)$	✓	149
	(2,2,2)	(4,2)	(6)	$T(2) \dashrightarrow S(2,4,6)$	✓	150
	(3,3)	(3,3)	(5,1)	$T(5) \dashrightarrow S(3,3,5)$	✓	151
	(3,3)	(3,3)	(4,2)	$T(2) \dashrightarrow S(3,3,4)$	**Excep**	152
8	(2,2,2,2)	(4,4)	(7,1)	$T(7) \dashrightarrow S(2,4,7)$	✓	153
	(2,2,2,2)	(4,4)	(6,2)	$T(3) \dashrightarrow S(2,4,6)$	✓	154
9	(2,2,2,2,1)	(3,3,3)	(9)	$T(2) \dashrightarrow S(2,3,9)$	✓	155
	(3,3,3)	(3,3,3)	(4,4,1)	$T(4) \dashrightarrow S(3,3,4)$	✓	156
10	(2,2,2,2,2)	(4,4,2)	(5,5)	$T(2) \dashrightarrow S(2,4,5)$	✓	157
	(2,2,2,2,2)	(3,3,3,1)	(10)	$T(3) \dashrightarrow S(2,3,10)$	✓	158
12	(2,...,2)	(3,3,3,3)	(11,1)	$T(11) \dashrightarrow S(2,3,11)$	✓	159
	(2,...,2)	(3,3,3,3)	(10,2)	$T(5) \dashrightarrow S(2,3,10)$	✓	160
	(2,...,2)	(3,3,3,3)	(9,3)	$T(3) \dashrightarrow S(2,3,9)$	✓	161
	(2,...,2)	(3,3,3,3)	(8,4)	$T(2) \dashrightarrow S(2,3,8)$	✓	162
16	(2,...,2)	(3,3,3,3,3,1)	(8,8)	$T(3) \dashrightarrow S(2,3,8)$	**Excep**	163
	(2,...,2)	(4,4,4,4)	(5,5,5,1)	$T(5) \dashrightarrow S(2,4,5)$	**Excep**	164
18	(2,...,2)	(3,...,3)	(8,8,2)	$T(4) \dashrightarrow S(2,3,8)$	✓	165
21	(2,...,2,1)	(3,...,3)	(7,7,7)	$T(2) \dashrightarrow S(2,3,7)$	**Excep**	166
28	(2,...,2)	(3,...,3,1)	(7,7,7,7)	$T(3) \dashrightarrow S(2,3,7)$	✓	167
36	(2,...,2)	(3,...,3)	(7,7,7,7,7,1)	$T(7) \dashrightarrow S(2,3,7)$	**Excep**	168

Table 6. Candidates $T \overset{d:1}{\underset{(\Pi_1,\Pi_2,\Pi_3)}{\dashrightarrow}} S$ with associated hyperbolic $T(\alpha) \dashrightarrow S(p,q,r)$.

4 Enumeration of relevant candidate covers

In this section we establish the following two results:

Theorem 4.1. *The candidate surface branched covers with associated hyperbolic orbifold candidate* $S(\alpha,\beta,\gamma,\delta) \dashrightarrow S(p,q,r)$ *are precisely the* 146 *items listed in Tables 1 to 5.*

Theorem 4.2. *The candidate surface branched covers with associated hyperbolic orbifold candidate* $T(\alpha) \dashrightarrow S(p,q,r)$ *are precisely the* 22 *items listed in Table 6.*

Proof of 4.1. Given a degree d and three partitions Π_1,Π_2,Π_3 of d we recall that $\ell(\Pi_i)$ denotes the length of Π_i, and we note that the Riemann-Hurwitz formula (1) reads

$$\ell(\Pi_1)+\ell(\Pi_2)+\ell(\Pi_3)=d+2 \tag{3}$$

in this case, because $\widetilde{\Sigma}=\Sigma$ is the sphere S. In addition we define $c(\Pi_i)$ as the number of entries in Π_i which are different from l.c.m.(Π_i). We must then find those d and Π_1,Π_2,Π_3 satisfying (3), the relation

$$c(\Pi_1)+c(\Pi_2)+c(\Pi_3)=4 \tag{4}$$

and such that for the associated candidate $\widetilde{X} \dashrightarrow X$ one has that X (or, equivalently, $\widetilde{X}$) is hyperbolic. We begin with the following:

Proposition 4.3. *The relevant degrees* d *and partitions* Π_1,Π_2,Π_3 *with* $d \leqslant 12$ *are the* 82 *items listed in Tables 1 to 3.*

Proof. For each d between 2 and 12 we must:

(**a**) List all the partitions Π of d such that $c(\Pi)\leqslant 4$, excluding $(1,\ldots,1)$;

(**b**) Select an unordered triple Π_1,Π_2,Π_3 meeting conditions (3) and (4);

(**c**) Discard the triples such that the sum of reciprocals of l.c.m.(Π_i) for $i=1,2,3$ is greater than or equal to 1.

Note that excluding $(1,\ldots,1)$ in (**a**) guarantees that in the orbifold cover $\widetilde{X}\to X$ associated to $S \underset{(\Pi_1,\Pi_2,\Pi_3)}{\overset{d:1}{\dashrightarrow}} S$ one has $X=S(p,q,r)$. Then (**c**) reads $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$ and means that X is hyperbolic.

Π	(10)	(9,1)	(8,2)	(7,3)	(6,4)	(5,5)
ℓ	1	2	2	2	2	2
c	0	1	1	2	2	0
Π	(8,1,1)	(7,2,1)	(6,3,1)	(6,2,2)	(5,4,1)	(5,3,2)
ℓ	3	3	3	3	3	3
c	2	3	2	2	3	3
Π	(4,4,2)	(4,3,3)	(7,1,1,1)	(6,2,1,1)	(5,3,1,1)	(5,2,2,1)
ℓ	3	3	4	4	4	4
c	1	3	3	3	4	4
Π	(4,4,1,1)	(4,3,2,1)	(4,2,2,2)	(3,3,3,1)	(3,3,2,2)	(6,1,1,1,1)
ℓ	4	4	4	4	4	5
c	2	4	3	1	4	4
Π	(4,2,2,1,1)	(2,...,2)	(3,3,1,1,1,1)	(2,2,2,2,1,1)	(2,2,2,1,1,1,1)	
ℓ	5	5	6	6	7	
c	4	0	4	2	4	

Table 7. The partitions Π of $d = 10$ with $c(\Pi) \leqslant 4$

c	0			1			2				3		4			
ℓ	1	2	5	2	3	4	2	3	4	6	3	4	4	5	6	7
#	1	1	1	2	1	1	2	3	1	1	4	3	4	2	1	1

Table 8. Values of (c, ℓ) for the partitions Π of $d = 10$ with $c(\Pi) \leqslant 4$, and numbers of Π's giving each (c, ℓ).

Achieving tasks (**a**), (**b**), and (**c**) is a matter that only requires a little time and care, and that can also safely be carried out by computer. As an only example, we make the argument explicit for $d = 10$, addressing the reader to [19] for the other cases. We first show in Table 7 the 28 partitions Π of $d = 10$ with $c(\Pi) \leqslant 4$.

Analyzing these partitions we see that the possible values of the pairs (c, ℓ) and the numbers of partitions giving each of them are those in Table 8.

To achieve task (**b**) we must now select all possible unordered triples of partitions such that the corresponding (c, ℓ)'s sum up to $(4, 12)$, which can be done in the ways described in Table 10. The table also contains the type geometry of X and $\widetilde{X}$ in the corresponding candidate orbifold cover. Task

(c_1,ℓ_1)	(c_2,ℓ_2)	(c_3,ℓ_3)	#
(0,5)	(0,2)	(4,5)	2
(0,5)	(0,1)	(4,6)	1
(0,5)	(1,4)	(3,3)	4
(0,5)	(1,3)	(3,4)	3
(0,5)	(2,4)	(2,3)	3
(0,2)	(2,4)	(2,6)	1
(1,4)	(1,4)	(2,4)	1
(1,2)	(1,4)	(2,6)	2
(1,3)	(1,3)	(2,6)	1

Table 9. Triples (c,ℓ) summing up to $(4,12)$, and numbers of different choices for the corresponding partitions.

Π_1	Π_2	Π_3	Associated cover	Geometry
(2,2,2,2,2)	(5,5)	(6,1,1,1,1)	$S(6,6,6,6) \dashrightarrow S(2,5,6)$	$\mathbb{H}$
(2,2,2,2,2)	(5,5)	(4,2,2,1,1)	$S(2,2,4,4) \dashrightarrow S(2,4,5)$	$\mathbb{H}$
(2,2,2,2,2)	(10)	(3,3,1,1,1,1)	$S(3,3,3,3) \dashrightarrow S(2,3,10)$	$\mathbb{H}$
(2,2,2,2,2)	(3,3,3,1)	(7,2,1)	$S(2,3,7,14) \dashrightarrow S(2,3,14)$	$\mathbb{H}$
(2,2,2,2,2)	(3,3,3,1)	(5,4,1)	$S(3,4,5,20) \dashrightarrow S(2,3,20)$	$\mathbb{H}$
(2,2,2,2,2)	(3,3,3,1)	(5,3,2)	$S(3,6,10,15) \dashrightarrow S(2,3,30)$	$\mathbb{H}$
(2,2,2,2,2)	(3,3,3,1)	(4,3,3)	$S(3,3,4,4) \dashrightarrow S(2,3,12)$	$\mathbb{H}$
(2,2,2,2,2)	(4,4,2)	(7,1,1,1)	$S(2,7,7,7) \dashrightarrow S(2,4,7)$	$\mathbb{H}$
(2,2,2,2,2)	(4,4,2)	(6,2,1,1)	$S(2,3,6,6) \dashrightarrow S(2,4,6)$	$\mathbb{H}$
(2,2,2,2,2)	(4,4,2)	(4,2,2,2)	$S(2,2,2,2) \dashrightarrow S(2,4,4)$	$\mathbb{E}$
(2,2,2,2,2)	(4,4,1,1)	(8,1,1)	$S(4,4,8,8) \dashrightarrow S(2,4,8)$	$\mathbb{H}$
(2,2,2,2,2)	(4,4,1,1)	(6,3,1)	$S(2,4,4,6) \dashrightarrow S(2,4,6)$	$\mathbb{H}$
(2,2,2,2,2)	(4,4,1,1)	(6,2,2)	$S(3,3,4,4) \dashrightarrow S(2,4,6)$	$\mathbb{H}$
(5,5)	(4,4,1,1)	(2,2,2,2,1,1)	$S(2,2,4,4) \dashrightarrow S(2,4,5)$	$\mathbb{H}$
(3,3,3,1)	(3,3,3,1)	(4,4,1,1)	$S(3,3,4,4) \dashrightarrow S(3,3,4)$	$\mathbb{H}$
(8,2)	(3,3,3,1)	(2,2,2,2,1,1)	$S(2,2,3,4) \dashrightarrow S(2,3,8)$	$\mathbb{H}$
(9,1)	(3,3,3,1)	(2,2,2,2,1,1)	$S(2,2,3,9) \dashrightarrow S(2,3,9)$	$\mathbb{H}$
(4,4,2)	(4,4,2)	(2,2,2,2,1,1)	$S(2,2,2,2) \dashrightarrow S(2,4,4)$	$\mathbb{E}$

Table 10. Partitions of $d = 10$ giving rise to candidate orbifold covers of the form $S(\alpha,\beta,\gamma,\delta) \dashrightarrow S(p,q,r)$ and the corresponding geometries.

(**c**) corresponds to discarding non-$\mathbb{H}$ geometries, after which we get the 16 items with numbers 43 through 58 in Tables 2 and 3. $\square$

We now turn to the following:

Proposition 4.4. *The degrees d and partitions Π_1, Π_2, Π_3 relevant to Theorem 4.1 with $d \geqslant 13$ are the 64 items listed in Tables 4 and 5.*

Proof. In this case we proceed in reverse order, from $\widetilde{X} \dashrightarrow X$ to d and the partitions. Namely we first analyze which hyperbolic candidate orbifold covers $\widetilde{X} \overset{d:1}{\dashrightarrow} X$ exist with $d \geqslant 13$, and in the course of this analysis we determine for what choices of d and Π_1, Π_2, Π_3 these candidates can arise.

Let us then consider a hyperbolic candidate $\widetilde{X} \overset{d:1}{\dashrightarrow} X$ with $d \geqslant 13$, $\widetilde{X} = S(\alpha, \beta, \gamma, \delta)$ and $X = S(p, q, r)$. Let us always assume that $\alpha \leqslant \beta \leqslant \gamma \leqslant \delta$ and $p \leqslant q \leqslant r$. Since $0 < -\chi^{\mathrm{orb}}(\widetilde{X}) = 2 - \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}\right) < 2$ and $\chi^{\mathrm{orb}}(\widetilde{X}) = d \cdot \chi^{\mathrm{orb}}(X)$, we deduce that

$$0 < -\chi^{\mathrm{orb}}(X) = 1 - \left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right) < \frac{2}{13} \quad \Rightarrow \quad \frac{11}{13} < \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1.$$

Assuming $p \leqslant q \leqslant r$ it is now very easy to check that the last inequality is satisfied only for (p, q, r) as follows:

(**A**) $(2, 3, r)$ with $7 \leqslant r \leqslant 77$;

(**B**) $(2, 4, r)$ with $5 \leqslant r \leqslant 10$;

(**C**) $(2, 5, 5)$, $(2, 5, 6)$, $(3, 3, 4)$ or $(3, 3, 5)$.

We now remark that:

(**I**) Each of $\alpha, \beta, \gamma, \delta$ must be a divisor of one of p, q, r;

(**II**) $d = \frac{\chi^{\mathrm{orb}}(\widetilde{X})}{\chi^{\mathrm{orb}}(X)}$ must be an integer.

Let us now establish the following auxiliary result:

Lemma 4.5. *Let $S(\alpha, \beta, \gamma, \delta) \overset{d:1}{\dashrightarrow} S(p, q, r)$ be a hyperbolic candidate orbifold cover. Set $d_{\max}(p, q, r) = \frac{2 - \frac{4}{r}}{1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}}$. Then $d \leqslant d_{\max}(p, q, r)$.*

$\widetilde{X}$	$S(6,6,6,6)$	$S(4,6,6,6)$	$S(3,6,6,6)$	$S(4,4,6,6)$	$S(3,4,6,6)$	$S(4,4,4,6)$
d	16	15	14	14	13	13

Table 11. The relevant $\widetilde{X}$ for $X = S(2,4,6)$ and the corresponding $d = \frac{\chi^{\mathrm{orb}}(\widetilde{X})}{\chi^{\mathrm{orb}}(X)}$.

Proof. By (**I**) and the condition $p \leqslant q \leqslant r$ we have $\alpha, \beta, \gamma, \delta \leqslant r$, whence $-\chi^{\mathrm{orb}}(S(\alpha,\beta,\gamma,\delta)) = 2 - \frac{1}{\alpha} - \frac{1}{\beta} - \frac{1}{\gamma} - \frac{1}{\delta} \leqslant 2 - \frac{4}{r}$ and the conclusion follows from (2), because $-\chi^{\mathrm{orb}}(S(p,q,r)) = 1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}$. □

Getting back to the cases (**A**), (**B**), and (**C**) that we must consider, we start from the last one and note that

$$d_{\max}(2,5,5) = 12, \quad d_{\max}(2,5,6) = 10, \quad d_{\max}(3,3,4) = 12, \quad d_{\max}(3,3,5) = 9.$$

Since the assumption $d \geqslant 13$ is in force, we conclude that case (**C**) does not yield relevant candidates. Turning to case (**B**), we have

$$d_{\max}(2,4,5) = 24, \quad d_{\max}(2,4,6) = 16, \quad d_{\max}(2,4,7) = 13.\overline{3},$$
$$d_{\max}(2,4,8) = 12, \quad d_{\max}(2,4,9) = 11.2, \quad d_{\max}(2,4,10) = 10.\overline{6}.$$

Therefore, the cases $r = 8$, $r = 9$ and $r = 10$ do not yield relevant candidates. The case $r = 7$, namely $X = S(2,4,7)$, also does not, because for $\widetilde{X} = S(7,7,7,7)$ we have that $\frac{\chi^{\mathrm{orb}}(\widetilde{X})}{\chi^{\mathrm{orb}}(X)} = 13.\overline{3}$, which violates (**II**), and for the next biggest possible $-\chi^{\mathrm{orb}}(\widetilde{X})$ in view of (**I**), corresponding to $\widetilde{X} = S(4,7,7,7)$, we have $\frac{\chi^{\mathrm{orb}}(\widetilde{X})}{\chi^{\mathrm{orb}}(X)} = 12.\overline{3}$, which again violates (**II**). Let now analyze the case $X = S(2,4,6)$. Picking $\alpha, \beta, \gamma, \delta \in \{2,3,4,6\}$, as imposed by (**I**), and discarding the cases where $\frac{\chi^{\mathrm{orb}}(\widetilde{X})}{\chi^{\mathrm{orb}}(X)} \leqslant 12$, we find the possible $\widetilde{X}$'s of Table 11. To conclude with the case $X = S(2,4,6)$ we must now discuss for which $\widetilde{X}$ as in Table 11 there actually exist partitions of d inducing a candidate $\widetilde{X} \to X$. We do this in full detail to give the reader a taste of the arguments one can use to this end. In similar cases below we will omit all details, addressing to [19].

Proposition 4.6. *The only candidate covers with $X = S(2,4,6)$ and $\widetilde{X}$ as in Table 11 are those described in items* 90 *and* 103 *in Table 4.*

Proof. For $\widetilde{X} = S(6,6,6,6)$ the partition of $d = 16$ corresponding to the cone point of order 6 in X must include four 1's, which already give all four cone points of $\widetilde{X}$, so the partition must be $(6,6,1,1,1,1)$ and the other two must be

If ...	$r=7$	$r=8$	$r=9$	$r=10$	$r=11$
then $d \leqslant$	60	36	28	24	21
If ...	$r=12$	$13 \leqslant r \leqslant 14$	$r=15$	$16 \leqslant r \leqslant 18$	$19 \leqslant r \leqslant 22$
then $d \leqslant$	20	18	17	16	15
If ...	$23 \leqslant r \leqslant 30$	$31 \leqslant r \leqslant 54$	$r \geqslant 55$		
then $d \leqslant$	14	13	12		

Table 12. Upper bounds on d depending on the values of r.

$(2,\ldots,2)$ and $(4,4,4,4)$, whence item 103 in Table 4. For $\widetilde{X} = S(4,6,6,6)$, $\widetilde{X} = S(3,4,6,6)$ and $\widetilde{X} = S(4,4,4,6)$ the fact that $\widetilde{X}$ has no cone point of order 2 implies that the partition of d corresponding to the cone point of order 2 in X consists of 2's only, which is impossible because d is odd. A similar argument shows that $\widetilde{X} = S(3,6,6,6)$ does not give a candidate cover: since $\widetilde{X}$ has no cone point of order 2 or 4, the partition corresponding to the cone point of order 4 in X must consist of 4's only, which is impossible because $d = 14$ is not a multiple of 4. Finally, let us consider $\widetilde{X} = S(4,4,6,6)$; these cone orders tell us that in both partitions corresponding to the cone points of orders 4 and 6 in X there must be two 1's, but then the only possibility is $(2,\ldots,2)$, $(4,4,4,1,1)$ and $(6,6,1,1)$, which gives item 90 in Table 4. □

To conclude case (**B**) we must deal with $X = S(2,4,5)$. This is done in very much the same way as for $X = S(2,4,6)$ and leads to items 95, 104, 106, 116, 120, 127, and 132 in Tables 4 and 5, see [19].

Finally, we examine case (**A**), where $X = S(2,3,r)$ and $7 \leqslant r \leqslant 77$. We first note that for the function $d_{\max}$ introduced in Lemma 4.5 we have $d_{\max}(2,3,r) = 12 \cdot \frac{r-2}{r-6}$. Imposing $d \leqslant \lfloor d_{\max}(2,3,r) \rfloor$ we then easily get:

Lemma 4.7. *Let* $S(\alpha,\beta,\gamma,\delta) \stackrel{d:1}{\dashrightarrow} S(2,3,r)$ *be a candidate orbifold cover. Then, depending on the value of* r*, the degree* d *satisfies the upper bound described in Table 12.*

Since our aim is to list the relevant candidate covers with $d \geqslant 13$ we see in particular that we can restrict to $7 \leqslant r \leqslant 54$. This leaves however several cases to consider, to reduce which we establish the following:

If $d \equiv \ldots \pmod 6$	0	1	2	3	4	5
then $r \leqslant$	54	11	13	15	20	13

Table 13. Upper bounds on r depending on the congruence class of d modulo 6.

Lemma 4.8. *Let* $S(\alpha,\beta,\gamma,\delta) \stackrel{d:1}{\dashrightarrow} S(2,3,r)$ *be a candidate orbifold cover. Set*

$$d_{\max}^{\equiv 1\,(2)}(r) = 9\cdot\tfrac{r-2}{r-6},\quad d_{\max}^{\equiv 1\,(3)}(r) = 2\cdot\tfrac{5r-9}{r-6},$$
$$d_{\max}^{\equiv 2\,(3)}(r) = 4\cdot\tfrac{2r-3}{r-6},\quad d_{\max}^{\equiv 1\,(6)}(r) = \tfrac{7r-12}{r-6}.$$

If $d \equiv k \pmod n$ *then* $d \leqslant d_{\max}^{\equiv k\,(n)}(r)$.

Proof. If $d \equiv 1 \pmod 2$ then the partition of d corresponding to the cone point of order 2 in $S(2,3,r)$ must have at least one entry equal to 1, so $\alpha = 2$, whence $-\chi^{\mathrm{orb}}(S(\alpha,\beta,\gamma,\delta)) \leqslant 2-\frac{1}{2}-\frac{3}{r}$; therefore, $d \leqslant \frac{2-\frac{1}{2}-\frac{3}{r}}{1-\frac{1}{2}-\frac{1}{3}-\frac{1}{r}} = d_{\max}^{\equiv 1\,(2)}(r)$. The other cases are treated in a similar way. □

Lemma 4.9. *Let* $S(\alpha,\beta,\gamma,\delta) \stackrel{d:1}{\dashrightarrow} S(2,3,r)$ *be a candidate orbifold cover with* $d \geqslant 13$. *Then, depending on the congruence class of* d *modulo* 6, *the cone order* r *satisfies the upper bounds described in Table 13.*

Proof. The conclusion is obtained by direct computation after imposing the appropriate $d_{\max}^{\equiv k\,(n)}(r) \geqslant 13$. □

Combining the restrictions given by Lemmas 4.7, 4.8 and 4.9 we can now conclude our analysis of case (**A**).

Proposition 4.10. *No candidate orbifold cover* $S(\alpha,\beta,\gamma,\delta) \stackrel{d:1}{\dashrightarrow} S(2,3,r)$ *exists with* $d \geqslant 13$ *and* $r \geqslant 15$.

Proof. Suppose first that $r \geqslant 19$. Then Lemma 4.7 implies that $d \leqslant 15$, so d can attain the values 13, 14 and 15, and Lemma 4.9 implies that $r \leqslant 15$, a contradiction. For $15 \leqslant r \leqslant 18$ Lemma 4.7 implies that $d \leqslant 17$, and then Lemma 4.9 shows that either $d = 16$ or $d = r = 15$. For $d = 16$ we then get a contradiction invoking Lemma 4.8 because $d_{\max}^{\equiv 1\,(3)}(r) < 16$ for $15 \leqslant r \leqslant 18$. Similarly for $d = r = 15$ we get a contradiction because $d_{\max}^{\equiv 1\,(2)}(15) = 13 < 15$. □

Proposition 4.11. *The only candidates* $S(\alpha,\beta,\gamma,\delta)\overset{d:1}{\dashrightarrow}S(2,3,r)$ *with* $d\geqslant 13$ *and* $13\leqslant r\leqslant 14$ *are items* `100` *and* `109` *in Table 4.*

Proof. Start with $r=14$. Lemma 4.7 implies that $d\leqslant 18$, whence Lemma 4.9 implies that d can attain the values 15, 16 and 18. For $d=15$ we have $d\leqslant d_{\max}^{\equiv 1\,(2)}(14)=13.5$ and for $d=16$ we have $d\leqslant d_{\max}^{\equiv 1\,(3)}(14)=15.25$, whence a contradiction in both cases. For $d=18$ we note that the only partitions Π of $d=18$ consisting of divisors of 14 and such that $c(\Pi)\leqslant 4$ are

$$(7,7,2,2),\quad (14,2,2),\quad (14,2,1,1),\quad (14,1,1,1,1).$$

For $(7,7,2,2)$ and $(14,1,1,1,1)$ we have $c(\Pi)=4$, so the other two partitions of $d=18$ must be $(2,\ldots,2)$ and $(3,\ldots,3)$, but the total length is $d+2=20$ only for $(14,1,1,1,1)$, which gives `109`. For $(14,2,2)$ we have $c(\Pi)=2$, so the two other partitions must be $(2,\ldots,2,1,1)$ and $(3,\ldots,3)$, but then the total length is 19. For $(14,2,1,1)$ we have $c(\Pi)=3$ and no choice is possible for the two other partitions.

For $r=13$ again we have $d\leqslant 18$, and using as above the values of $d_{\max}^{\equiv k\,(n)}(13)$ we see that $d=14$, $d=15$ and $d=17$ are impossible. For $d=16$ the only partition Π of d consisting of divisors of 13 with $c(\Pi)\leqslant 4$ is $(13,1,1,1)$, which implies that the two other partitions must be $(2,\ldots,2)$ and $(3,\ldots,3,1)$, whence `100`. For $d=18$ there is no partition Π of d consisting of divisors of 13 with $c(\Pi)\leqslant 4$. □

The rest of the discussion leading to the proof of Proposition 4.4 is now quite similar to the arguments already used. For a decreasing value of r between 12 and 7 one has an increasingly complicated argument consisting of:

- A use of Lemmas 4.7 to 4.9, to exclude some values of d;
- The analysis of what partitions Π_1,Π_2,Π_3 of $2,3,r$ satisfy $\text{l.c.m.}(\Pi_1)=2$, $\text{l.c.m.}(\Pi_2)=3$, $\text{l.c.m.}(\Pi_3)=r$, $c(\Pi_1)+c(\Pi_2)+c(\Pi_3)=4$ and $\ell(\Pi_1)+\ell(\Pi_2)+\ell(\Pi_3)=d+2$; this last discussion is easier for $r=7$ and $r=11$, since Π_3 can only consist of r's and 1's.

We address the reader to [19] for a careful description of this argument, only mentioning that for $7\leqslant r\leqslant 12$ one gets exactly the 53 items in Tables 4 and 5 excluding the 9 coming from case (**B**) and the 2 already found in case (**A**). More precisely for $r=7,\ 8,\ 9,\ 10,\ 11,\ 12$ one gets respectively $13,\ 22,\ 5,\ 7,\ 1,\ 5$ candidates, for a total of 53. □

Combining Propositions 4.3 and 4.4 we obtain the conclusion of the proof of Theorem 4.1. □

Proof of 4.2. We use the same notation as above. Since $\widetilde{\Sigma}$ is now the torus T and Σ is the sphere S, the Riemann-Hurwitz formula (1) reads

$$\ell(\Pi_1)+\ell(\Pi_2)+\ell(\Pi_3)=d. \tag{5}$$

Therefore, we need to enumerate the degrees d, and the partitions Π_1,Π_2,Π_3 satisfying (5) and

$$c(\Pi_1)+c(\Pi_2)+c(\Pi_3)=1, \tag{6}$$

because in this case in the associated candidate orbifold cover $\widetilde{X} \dashrightarrow X$ we automatically have that $\widetilde{X}=T(\alpha)$ with $\alpha>1$ is hyperbolic, so is X.

Relations (5) and (6) imply that for any given d we must find divisors $p>1$ and $q>1$ of d, an integer $r>1$ and a divisor $r'\neq r$ of r such that $d-r'$ is a multiple of r and $\frac{d}{p}+\frac{d}{q}+\frac{d-r'}{r}+1=d$. This is very easily done for $d\leqslant 17$ and leads to the first 18 items in Table 6. (We note in passing that in Table 6 the partitions Π_1,Π_2,Π_3 defining a candidate cover are rearranged for increasing l.c.m.) As an only example, we present the argument for $d=12$, addressing the reader to [19] for the other degrees up to 17.

So, for $d=12$, we note that the pairs (r,r') with $r'\neq r$ a divisor of r and $12-r'$ a multiple of r are the following ones:

$$(8,4),\quad (9,3),\quad (10,2),\quad (11,1).$$

For each of them we have $\frac{12-r'}{r}=1$, so we must now find two divisors p and q of 12 such that $\frac{12}{p}+\frac{12}{q}=10$, and one readily sees that up to permutation the only choice is $p=2$ and $q=3$. We conclude that for $d=12$ there are 4 relevant candidate covers, listed as items `159` to `162` in Table 6.

Turning to the case $d\geqslant 18$, consider a candidate orbifold cover $\widetilde{X}\overset{d:1}{\dashrightarrow}X$ with hyperbolic $\widetilde{X}=T(\alpha)$ and $X=S(p,q,r)$. Since $0<-\chi^{\rm orb}(\widetilde{X})=1-\frac{1}{\alpha}<1$ and $\chi^{\rm orb}(\widetilde{X})=d\cdot\chi^{\rm orb}(X)$, we readily deduce that $\frac{17}{18}<\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$, which implies that either $p=2$, $q=4$ and $r=5$ or $p=2$, $q=3$ and $7\leqslant r\leqslant 8$. For $p=2$, $q=4$ and $r=5$ one must have $\alpha\in\{2,4,5\}$ and correspondingly $d\in\{10,15,16\}$, which contradicts $d\geq 18$. For $p=2$, $q=3$ and $r=7$ the partitions of d giving rise to the candidate must have one of the following forms:

$$(2,\ldots,2,1),(3,\ldots,3),(7,\ldots,7),$$
$$(2,\ldots,2),(3,\ldots,3,1),(7,\ldots,7),$$
$$(2,\ldots,2),(3,\ldots,3),(7,\ldots,7,1).$$

Correspondingly, (5) translates into

$$\begin{array}{lcl} \frac{d-1}{2}+1+\frac{d}{3}+\frac{d}{7}=d & \Rightarrow & d=21 \\ \frac{d}{2}+\frac{d-1}{3}+1+\frac{d}{7}=d & \Rightarrow & d=28 \\ \frac{d}{2}+\frac{d}{3}+\frac{d-1}{7}+1=d & \Rightarrow & d=36 \end{array}$$

and we get the candidates 166 to 168 in Table 6. For $r=8$ one carries out a similar analysis, this time with five different triples of partitions (because 8 has three divisors smaller than itself, while 2 and 3 have one), and one finds as the only new candidate item 165 in Table 6. Note that for the partitions $(2,\ldots,2),(3,\ldots,3),(8,\ldots,8,1)$, imposing $\frac{d}{2}+\frac{d}{3}+\frac{d-1}{8}+1=d$, one finds $d=21$, but this does not give a candidate, since $\frac{d}{2}$ and $\frac{d-1}{8}$ are not integers. See the details in [19]. □

5 Overview of the techniques used to prove realizability and exceptionality

In this section we briefly present the methods using which we have proved realizability or exceptionality for each of the 168 candidate covers in Theorems 3.1 and 3.2.

Dessins d'enfant (DE) This is a classical technique, introduced by Grothendieck in [7] for studying algebraic maps between Riemann surfaces, which proves a powerful tool both to exhibit realizability and to show exceptionality of candidate covers. We introduce Grothendieck's dessins in their original form, that is valid only for covers of the sphere with three branching points, but we mention that the method was generalized in [20] to the case of more branching points.

To begin recall that a bipartite graph is a finite 1-complex whose vertex set is split as $V_1 \sqcup V_2$ and each edge has one endpoint in V_1 and one in V_2. We now give the following:

Definition 5.1. *A* dessins d'enfant *on a surface* $\widetilde{\Sigma}$ *is a* bipartite *graph* $D \subset \widetilde{\Sigma}$ *such that* $\widetilde{\Sigma}\backslash D$ *consists of open discs. The* length *of one of these discs* *is the number of edges of D along which its boundary passes, counted with multiplicity.*

The connection between dessins d'enfant and branched covers comes from the next result (see [20] for a proof):

Proposition 5.2. *The realizations of a candidate* $\widetilde{\Sigma} \dashrightarrow \dashrightarrow \overset{d:1}{\underset{(d_{11},\ldots,d_{1m_1}),\ldots,(d_{31},\ldots,d_{3m_3})}{\dashrightarrow}} \dashrightarrow \dashrightarrow S$ *correspond to the dessins d'enfant $D \subset \widetilde{\Sigma}$ with set of vertices $V_1 \sqcup V_2$ such that for $i = 1, 2$ the vertices in V_i have valences $(d_{ij})_{j=1}^{m_i}$, and the discs in $\widetilde{\Sigma} \setminus D$ have lengths $(2d_{3j})_{j=1}^{m_3}$.*

Graph moves (GM) This method will be used below only to prove exceptionality. Its description here is rather generic, and in a sense obvious, but in several practical cases we can indeed make the method work.

Proposition 5.3. *Let c and c' be partial dessins d'enfant. Let $S \overset{d:1}{\underset{\Pi}{\dashrightarrow}} S$ be a candidate surface branched cover. Suppose that:*

(1) While trying to construct a dessin d'enfant realizing $S \overset{d:1}{\underset{\Pi}{\dashrightarrow}} S$ one is forced to insert a portion c;

(2) Any completion of c to a dessin d'enfant realizing $S \overset{d:1}{\underset{\Pi}{\dashrightarrow}} S$, if any, could also be used to complete c' and would give a to a dessin d'enfant realizing another candidate $S \overset{d':1}{\underset{\Pi'}{\dashrightarrow}} S$.

If $S \overset{d':1}{\underset{\Pi'}{\dashrightarrow}} S$ is exceptional then $S \overset{d:1}{\underset{\Pi}{\dashrightarrow}} S$ also is.

The key point of this result is that one can establish condition (2) by examining c, c' and $S \overset{d:1}{\underset{\Pi}{\dashrightarrow}} S$ only, without searching the completions of c realizing $S \overset{d:1}{\underset{\Pi}{\dashrightarrow}} S$. We also mention here the remarkable fact that in Proposition 6.3 we will apply the GM method with c' more complicated than c, namely with d' greater than d.

Very even data (VED) and block decompositions (BD) If $d = d' + d''$, we will say that a partition Π of d *refines* the partition (d', d'') if Π splits as $\Pi' \sqcup \Pi''$ with Π' a partition of d' and Π'' a partition of d''. The next result established in [20] will be used to show exceptionality of several candidates:

Proposition 5.4. *Consider a candidate surface cover* $\widetilde{\Sigma} \xrightarrow[\Pi_1,\Pi_2,\Pi_3]{d:1} \Sigma$ *with* d *and each element of* Π_i *for* $i = 1, 2$ *being even. If the candidate is realizable then* Π_3 *must refine the partition* $(d/2, d/2)$.

The next result was also shown in [20] and it is based on the same idea that under certain divisibility assumptions a surface branched cover can be expressed as the composition of two other ones:

Proposition 5.5. *Consider a candidate surface cover* $\widetilde{\Sigma} \xrightarrow[\Pi_1,\Pi_2,\Pi_3]{d:1} \Sigma$ *with* d *and each entry of* Π_i *for* $i = 1, 2$ *being divisible by some* k*. If the candidate is realizable then each entry of* Π_3 *is less than or equal to* d/k.

We mention that some more specific realizability criteria were provided in [20] for even d and $\Pi_1 = (2, \ldots, 2)$ and either $\Pi_2 = (5, 3, 2, \ldots, 2, 1)$, or $\Pi_2 = (3, 3, 2, \ldots, 2)$ or $\Pi_2 = (3, 2, \ldots, 2, 1)$. In an earlier version of [20] the technique leading to Propositions 5.4 and 5.5 was also generalized to a certain theory of *block decompositions*, which allows in particular to prove exceptionality of several candidate surface covers in degree 12. Since these candidates can be alternatively and quite easily discussed using DE, we will refrain from restating these specific results here.

Geometric gluings (GG) The main idea of [18] was to use the geometry of 2-orbifolds to analyze candidate surface covers. But this was actually done only in the spherical and Euclidean case, while for the 11 hyperbolic candidates corresponding to orbifold covers between triangular orbifolds the technique of DE was only used. In this paper for the first time we actually apply the geometry of hyperbolic orbifolds to discuss realizability of candidate surface branched covers. The statement we give here, just like Proposition 5.3, is a rather straight-forward one, but we can actually employ it in several concrete examples, to show both realizability and exceptionality:

Proposition 5.6. *Consider a candidate surface branched cover* $\widetilde{\Sigma} \xrightarrow[\Pi_1,\Pi_2,\Pi_3]{d:1} S$ *with associated candidate orbifold cover* $\widetilde{X} \dashrightarrow S(p, q, r)$ *of hyperbolic type.*

Let D be a fundamental domain for $S(p,q,r)$ obtained by mirroring the hyperbolic triangle $T(p,q,r)$ with inner angles $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$ in one of its edges. Then the candidate is realizable if an only if one can realize $\widetilde{X}$ by gluing d copies of D along orientation-reversing hyperbolic isometries, in such a way that, upon mapping each copy of D to the original D, the resulting orbifold cover $\widetilde{X} \to S(p,q,r)$ matches the covering instructions given by the original cover.

Monodromy representation (MR) We recall here a very classical viewpoint dating back to [8] (see also [20]), based on the remark that a realization of a candidate branched cover $\widetilde{\Sigma} \xrightarrow[(d_{11},\ldots,d_{1m_1}),\ldots,(d_{n1},\ldots,d_{nm_n})]{d:1} \Sigma$ corresponds to the choice of its monodromy, which is a (suitable) representation of the fundamental group of the n-punctured sphere into the symmetric group $\mathfrak{S}_d$. More precisely, one has that that a realization of the given candidate cover corresponds to the choice of permutations $\sigma_1, \ldots, \sigma_n \in \mathfrak{S}_d$ such that:

- σ_i has cycles of lengths $(d_{ij})_{j=1}^{m_i}$;
- the product $\sigma_1 \cdots \sigma_n$ is the identity;
- $\langle \sigma_1, \ldots, \sigma_n \rangle < \mathfrak{S}_d$ acts transitively on $\{1, \ldots, d\}$.

For $n = 3$, to apply this method in practice, one should fix a permutation σ_1 with cycle lengths $(d_{1j})_{j=1}^{m_1}$, let σ_2 vary in the conjugacy class of permutations of cycle lengths $(d_{2j})_{j=1}^{m_2}$ and check whether $\langle \sigma_1, \sigma_2 \rangle$ is transitive and $\sigma_1 \cdot \sigma_2$ has cycle lengths $(d_{3j})_{j=3}^{m_1}$. When the degree d is high this can be computationally quite demanding, but a C++ code [13] written by Maurizio Monge, building also on some formulae proved in [25], refines this approach for the special case of permutations of the form relevant to Theorem 3.2. The program employs the correspondence between Young diagrams and representations of the symmetric group, automatically generating the p-core diagrams (which are very easy for the particular permutations involved) to find all the relevant characters. This, together with some computational tricks again specific to the special permutations of Table 6, allows the program to establish within a handful of seconds the realizability of any candidate of the appropriate type up to degree 200.

6 Realizability and exceptionality of the relevant candidate covers

In this section we discuss the realizability of all the 168 candidates described in Theorems 4.1 and 4.2, thereby completing the proof of Theorems 3.1 and 3.2. We begin with the following:

Proposition 6.1. *The* 117 *candidate surface covers described in Tables 1 to 5 and indicated there to be realizable are indeed realizable.*

Proof. For each of the 117 candidates we have been able to draw a dessin d'enfant proving realizability. To avoid showing all of them, we will present here only one for each degree up to 12, one for each candidate having a prime degree greater than 12, and some for the largest degrees in the tables. See [19] for all other realizable candidates. In all our dessins, with notation as in Tables 1 to 5, we will associate white vertices to the entries of partition Π_1 and black vertices to those in Π_3, so the regions of the complement of the dessin will correspond to the entries of Π_2.

For degree up to 12 the examples we have chosen to show correspond to items 3, 6, 15, 24, 33, 45, 60, and 62. The dessins proving that they are all realizable are provided in Figure 1.

For degree greater than 12 the candidate covers with prime degree d in Tables 1 to 5 are items 83, 84, 85, 86, 105, 106, 117, 137, and 142. The dessins proving that they are all realizable are provided in Figure 2.

For each of the candidates 143, 144 and 145 a dessin d'enfant showing its realizability is shown in Figure 3. Even if we have one, we do not show a dessin for candidate 146 because a proof of its realizability will be given below in Proposition 6.4 using GG. □

Let us turn to the following:

Proposition 6.2. *The* 26 *candidate surface covers described in Tables 1 to 5 and indicated there to be exceptional are indeed exceptional.*

Proof. For degree up to 20 we could actually merely refer to the computer-generated census of Zheng [25], but we prefer to give theoretical proofs. To begin, we note that the VED criterion of Proposition 5.4 shows exceptionality of candidates 25, 31, 49, 50, 69, 97, 98, 116 and 126. Candidates 35 and 36 are exceptional due to Proposition 5.5, while 23 and 28 due to

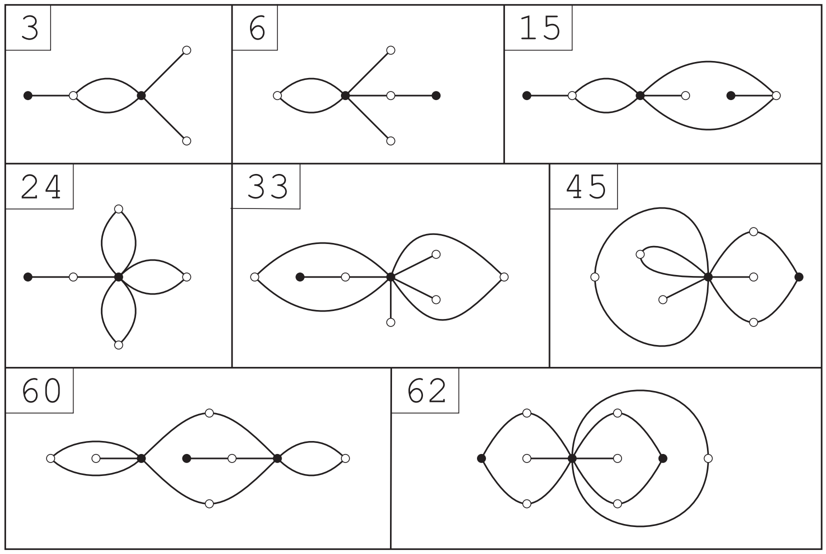

Figure 1. Realization via DE of a sample of candidate covers for degree up to 12.

Propositions 1.3 and 1.2 of [20], respectively. The BD criterion of [20, Section 5] alluded to after Proposition 5.5 implies that candidates 72-78 are exceptional. For degrees up to 20 this leaves out only candidates 41, 57, 79, 113, 114 and 115 for which we prove exceptionality using DE in Figure 4. Here we always associate white vertices to the entries of Π_1 and black ones to those in Π_2, and we use their valences and the lengths of some of the complementary regions to construct forced portions of dessin d'enfant in which one sees offending lengths of the complementary regions and/or one finds it impossible to get a connected dessin with the prescribed lengths of the complementary regions.

We are left to prove exceptionality of candidates 122, 125 and 131, in degrees 21, 22 and 24 respectively. We start with 125, for which we use DE again. Let us try to construct a dessin with white vertices corresponding to $\Pi_1 = (2, \ldots, 2)$ and black vertices corresponding to $\Pi_3 = (8, 8, 4, 1, 1)$. The proof that this is actually impossible is contained in Figure 5. In part (a) we show that neither of the black 1-valent vertices can be joined to the 4-valent one (otherwise a region of length at least 4 would arise), so both are joined to an 8-valent one, and in part (b) we show that the two 1-valent black vertices are joined to different 8-valent black vertices (for the same reason). Then in

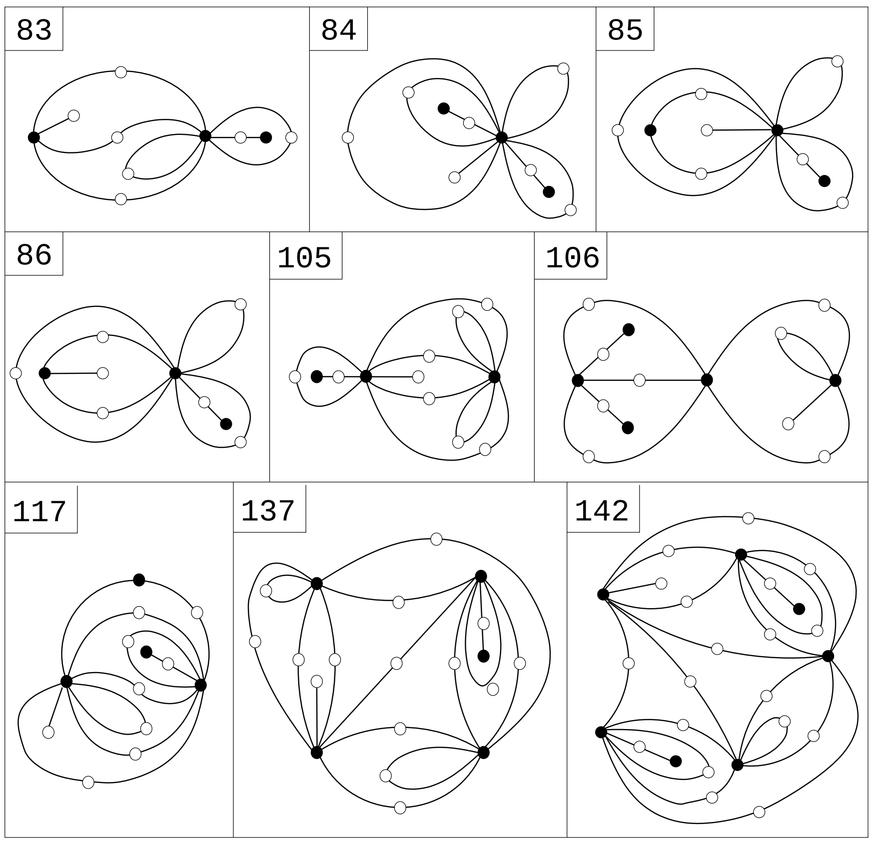

Figure 2. Realization via DE of the candidate covers with prime degree larger than 12.

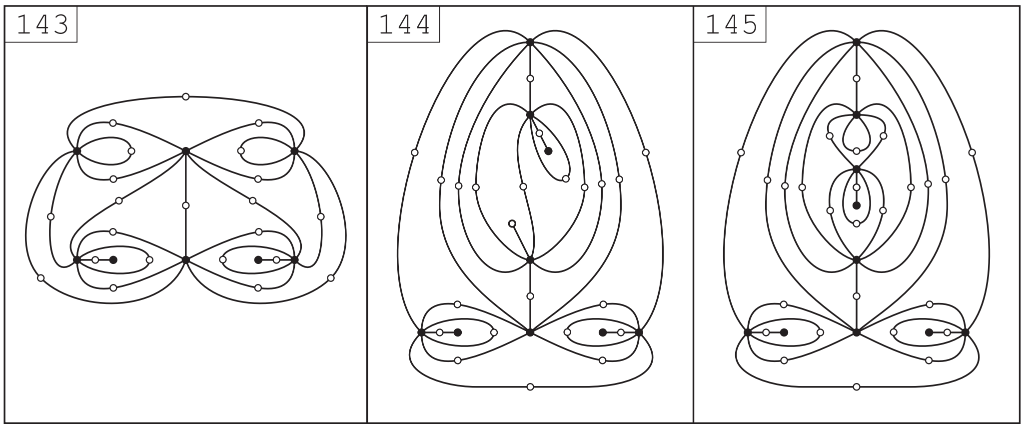

Figure 3. Realization via DE of the candidate covers with degrees 44, 45 and 52.

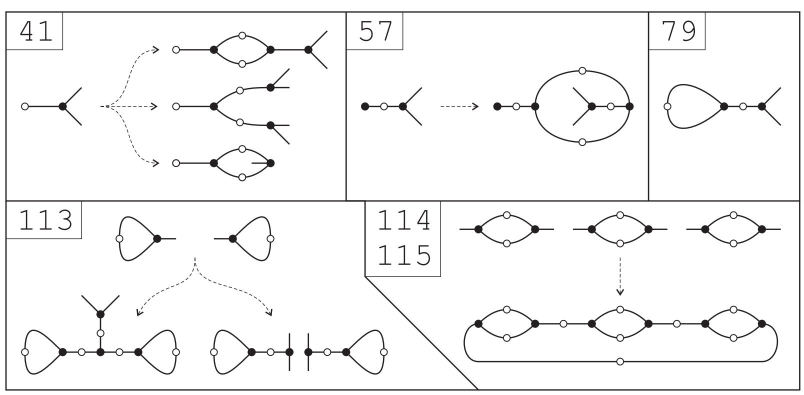

Figure 4. Exceptionality proofs via DE.

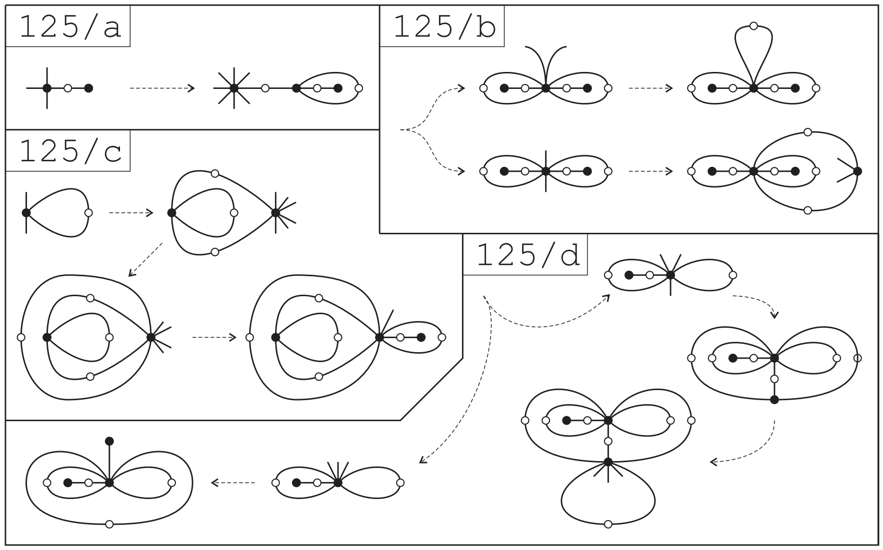

Figure 5. Exceptionality of 125 via DE.

part (c) we show that for the region incident to only one black vertex, this cannot be the 4-valent one; note in particular that we use part (b) in the last passage. This implies that there is an 8-valent black vertex joined to a 1-valent one and incident to a region of length 2, and in part (d) we show that this is impossible, once again by contradicting the fact that all regions but one should have length 3.

To deal with 122 and 131 we will use the new techniques GM and GG introduced in Section 5, but we prefer first to provide alternative proofs of exceptionality and realizability in lower degree using these techniques, to allow the reader to familiarize with them. We begin with the following:

Proposition 6.3. *The candidates* 28*,* 41 *and* 122 *can be shown to be exceptional using the* GM *technique.*

Proof. Recall that for 28 the partitions are $\Pi_1 = (2,2,2,2)$, $\Pi_2 = (3,3,1,1)$ and $\Pi_3 = (5,3)$, and let us try to construct a dessin with white and black vertices associated to Π_1 and Π_3 respectively. Since there are two regions of length 2, at least one of them has a 5-valent black vertex, and we apply the move shown in Figure 6. A dessin realizing 28 would then give one realizing the candidate $S \xrightarrow[(2,\ldots,2),(3,3,3,1),(6,3,1)]{10:1} S$, which is exceptional by [18, Theorem 3.6].

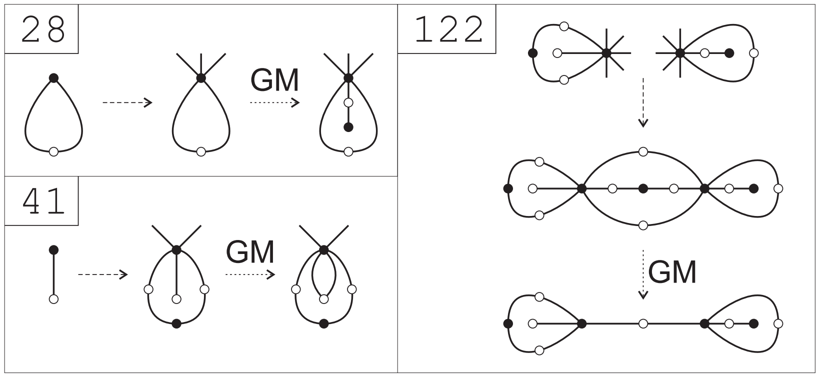

Figure 6. Exceptionality proofs via GM.

For `41` we proceed similarly, assigning white vertices to the partition $(2,2,2,2,1)$ and black ones to $(5,2,2)$, leaving the partition $(3,3,3)$ for the regions. The 1-valent white vertex must be joined to a 5-valent black one as in Figure 6, so we apply the move shown, which proves that if `41` is realizable then $S \xrightarrow[(2,\ldots,2),(3,3,3,1),(6,2,2)]{10:1} S$ also is, which is false by the VED criterion.

Turning to `122` we recall that $d = 21$, $\Pi_1 = (2,\ldots,2,1)$, $\Pi_2 = (3,\ldots,3)$ and $\Pi_3 = (8,8,2,2,1)$. Again we use white for Π_1 and black for Π_3. Two configurations as at the top of Figure 6 must exist, and the only case in which the two edges emanating from a valence-2 black vertex end on the same black vertex occurs in the first of these configurations; therefore, a configuration as that to which we apply the move in Figure 6 occurs. Note that at each valence-8 black vertex two emanating germs of edges are missing, because we apply the move regardless of their position. After the move we get the candidate $S \xrightarrow[(2,\ldots,2,1),(3,\ldots,3),(6,6,2,1)]{15:1} S$, which is exceptional by [18, Theorem 3.6], and the proof is complete. □

We now turn to the GG technique, that we first employ to provide alternative proofs of three already shown realizability results, and of one such result not explicitly given above. One more such realizability result will be proved below in Proposition 6.6.

Proposition 6.4. *The candidates* `1`*,* `10`*,* `13`*, and* `132` *can be shown to be realizable using the* GG *technique.*

Proof. For the first three candidates we show a realization in Figure 7, as we

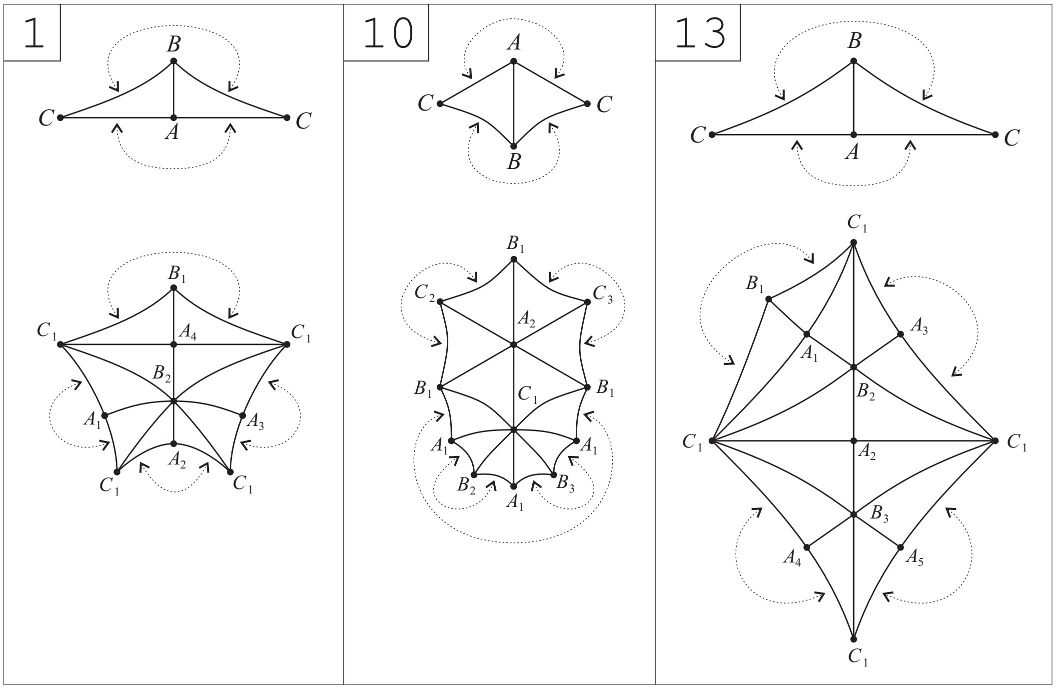

Figure 7. Realizability proofs via GG.

now explain. For 1 we should have a cover $S(2,2,2,4) \overset{5:1}{\dashrightarrow} S(2,4,5)$ with covering instructions $(2,2,2) \dashrightarrow 2$ and $4 \dashrightarrow 4$. In the figure we show $S(2,4,5)$ as a gluing of two copies of the hyperbolic triangle $T(2,4,5)$ with inner angles $\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{5}$. The cone points of orders $2,4,5$ are respectively A, B, C. And in the same figure we show $S(2,2,2,4)$ as a gluing of 10 copies of $T(2,4,5)$. This induces a covering $S(2,2,2,4) \xrightarrow{5:1} S(2,4,5)$ with each A_i, B_i, C_i mapped respectively to A, B, C. Since the cone points of $S(2,2,2,4)$ are A_1, A_2, A_3 of order 2 and B_1 of order 4, the required covering instructions are realized. Note that A_4 and B_2 are obviously non-singular, and so is C_1, because there are 10 angles $\frac{\pi}{5}$ incident to it.

The argument for 10 is similar. We must realize $S(4,4,4,4) \overset{6:1}{\dashrightarrow} S(3,4,4)$ with the instructions $(4,4) \dashrightarrow 4$ and $(4,4) \dashrightarrow 4$ (which are automatic in this case). In Figure 7 the cone points of $S(3,4,4)$ are A, B, C of orders $3,4,4$ respectively, and those of $S(4,4,4,4)$ are B_2, B_3, C_2, C_3 all of order 4, so the cover is as desired.

For 13 we must realize $S(2,2,2,3) \overset{7:1}{\dashrightarrow} S(2,3,7)$ with, of course, $(2,2,2) \rightarrow$

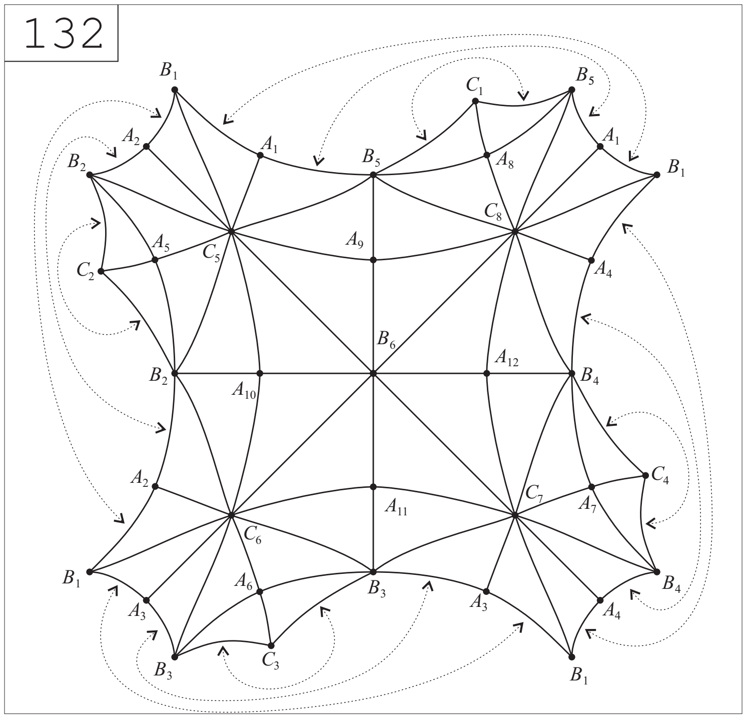

Figure 8. Realizability of candidate `132` via `GG`.

2 and $3 \to 3$. Here the cone points are A, B, C of orders $2, 3, 7$ in $S(2, 3, 7)$, and A_3, A_4, A_5 of order 2 and B_1 of order 3 in $S(2, 2, 2, 3)$.

We now turn to `132`, which is treated in Figure 8. The cover to be realized is $S(5, 5, 5, 5) \stackrel{24:1}{\dashrightarrow} S(2, 4, 5)$ with (of course) instructions $(5, 5, 5, 5) \dashrightarrow 5$. For $S(2, 4, 5)$ the fundamental domain is the same used for `1`, and the reader can check that Figure 8 contains 48 copies of $T(2, 4, 5)$ giving $S(5, 5, 5, 5)$ with cone points at C_1, C_2, C_3, C_4. $\square$

Proposition 6.5. *The candidates* `23` *and* `131` *can be shown to be exceptional using the* `GG` *technique.*

Proof. To `23` we associate $S(2, 3, 3, 6) \stackrel{8:1}{\dashrightarrow} S(2, 4, 6)$ with $(2, 3, 3, 6) \dashrightarrow 6$. We show the fundamental domain D of $S(2, 4, 6)$ in Figure 9, with cone points of order $2, 4, 6$ at A, B, C. Since, in a realization of the candidate, B is covered by two non-singular points (and four singular ones), of the 8 copies of D there must be 4 around some B_1 and the other four around some B_2.

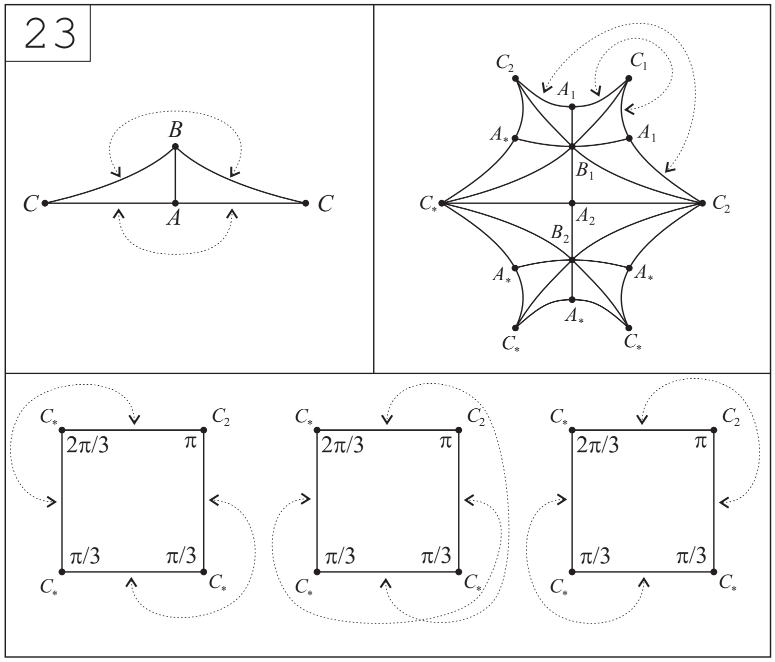

Figure 9. Exceptionality of candidate 23 via GG.

Now A is also covered by non-singular points, which implies that no edge A_*C_* is glued to the adjacent A_*C_*. In other words, each (straight) segment $C_*A_*C_*$ can be regarded as a single edge, and gets glued to another such segment. Since $S(2,3,3,6)$ is connected, the two blocks of 4 copies of D already described are glued together, forming a single block of 8 copies of D with 6 free edges of type $C_*A_*C_*$. But, to get a cone point of order 6, at some top or bottom C_*, say C_1, we must have a gluing as shown in the figure. We now abandon hyperbolically correct pictures and use combinatorial ones, but we keep track of geometry. To do this we note that the block of 8 copies of D, after performing the gluing shown, becomes a "quadrangle" (because we can ignore the A_*'s) with inner angles $\pi, \frac{2}{3}\pi, \frac{\pi}{3}, \frac{\pi}{3}$ (and an inner cone point of order 6, not shown in the picture). As shown in Figure 9, there are now three ways to pair the edges of this quadrangle by orientation-reversing maps. Since we have already realized a cone point at C_1 of angle $\frac{2\pi}{6}$, these gluings give rise to a hyperbolic cone surface as follows:

- based on the sphere, with cone angles $\frac{2\pi}{6}, \frac{2\pi}{6}, \frac{2\pi}{3}, \frac{4\pi}{3}$;

- based on the torus with cone angles $\frac{2\pi}{6}, \frac{7\pi}{3}$;
- based on the sphere with cone angles $\frac{2\pi}{6}, \frac{2\pi}{6}, \frac{2\pi}{2}, \frac{2\pi}{2}$.

Neither of these is the desired $S(2,3,3,6)$, and the exceptionality of 23 is proved. We note however that if one disregards geometry the previous gluings do give rise of realizations of candidate branched covers of degree 8, but not of the desired one. Namely:

- $S \xrightarrow[(2,2,2,2),(4,2,1,1),(4,4)]{8:1} S$, with the associated Euclidean $S(2,4,4) \dashrightarrow S(2,4,4)$;
- $T \xrightarrow[(2,2,2,2),(7,1),(4,4)]{8:1} S$, with associated $T(7) \dashrightarrow S(2,4,7)$;
- $S \xrightarrow[(2,2,2,2),(3,3,1,1),(4,4)]{8:1} S$ with the associated spherical $S(3,3) \dashrightarrow S(2,3,4)$.

Turning to 131, the associated candidate is $S(3,9,9,9) \xrightarrow{24:1} S(2,3,9)$ with instructions $(3,9,9,9) \dashrightarrow 9$. We show in Figure 10 its fundamental domain D, with cone orders $2,3,9$ at A,B,C. As for 23, using the fact that A must be covered by non-singular points, we see we can forget A, and redraw D as a combinatorial triangle, but keeping track of its geometry by writing the angles at the vertices, with $\alpha = \frac{\pi}{9}$. Since C is covered by two non-singular points, one of order 3 and three of order 9, the 24 copies of D can be grouped in two groups of 9 copies giving the blocks n_1, n_2 with nine edges, 3 copies giving blocks o_1, o_2, o_3 with one edge, and 3 copies giving a block t with three edges. Note that at this stage all vertices should cover B and they all have angle $6\alpha = \frac{2}{3}\pi$. We should now assemble the blocks so that at each glued vertex the total angle is $18\alpha = 2\pi$, because all points covering B are non-singular. This implies that gluing the edge of some o_i to another edge e of some block forces the two edges of the block adjacent to e to be glued together. Therefore, no o_i is glued to t, and (up to change of notation) o_1 and o_2 are glued to n_1, while o_3 is glued to n_2. This gives two new blocks h and t', with angles as shown, that we must now use with t. The edge of t' whose ends have angle 12α must be glued to an edge whose ends have angle 6α, and two more gluings are then forced. This implies that the said edge of t' is not glued to t, and there are two ways up to symmetry to glue it to an edge of h. One of them gives a triangle with angles $6\alpha, 6\alpha, 24\alpha$, and the other one a triangle with angles $6\alpha, 12\alpha, 18\alpha$, which easily implies that the process cannot be carried to the end. $\square$

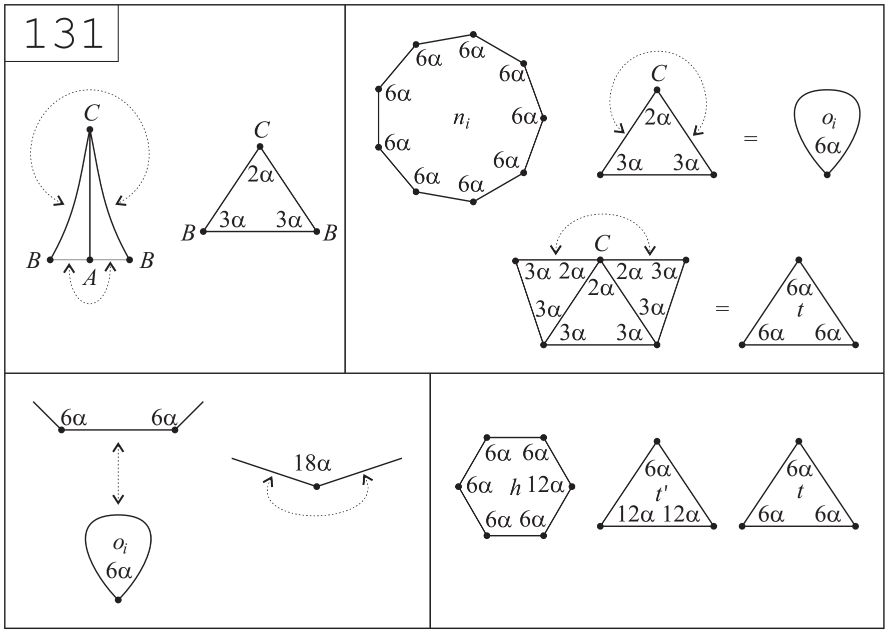

Figure 10. Exceptionality of 131 via GG.

This conclues the proof of Proposition 6.2. $\square$

As already announced we now show via GG that our highest degree candidate is realizable, which was not explicitly done within the proof of Proposition 6.1.

Proposition 6.6. *The candidate* 146 *can be shown to be realizable using the* GG *technique.*

Proof. We must realize $S(7,7,7,7) \stackrel{60:1}{\dashrightarrow} S(2,3,7)$ with $(7,7,7,7) \to 7$. If A, B, C are the cone points of orders $2,3,7$, the fact that A is covered by non-singular points implies as above that we can ignore it, then we must use 60 copies of a triangle with vertices B, B, C and angles $\frac{\pi}{3}, \frac{\pi}{3}, \frac{2}{7}\pi$. Taking into account the way C is covered we then get blocks h_i for $i = 1, \ldots, 8$ and o_i for $i = 1,2,3,4$, as shown in Figure 11, with $\alpha = \frac{\pi}{3}$, that we must assemble creating non-singular points only, that is, having angle $6\alpha = 2\pi$. Gluing o_i to h_{i+4} we get the blocks q_i for $i = 1,2,3,4$, and we still have h_i for $i = 1,2,3,4$. Now we glue the two edges of q_i separated by the vertex with angle 4α to an edge of h_i, getting the blocks h'_i for $i = 1,2,3,4$. We glue them in pairs as

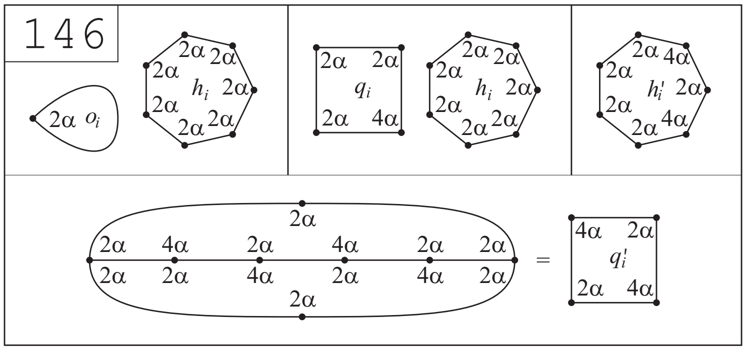

Figure 11. Realizability of 146 via GG.

illustrated, getting two blocks q_i' for $i = 1, 2$, that we can now glue together to finish the process. ☐

The analysis of candidates with associated $S(\alpha, \beta, \gamma, \delta) \dashrightarrow S(p, q, r)$ is complete, so we turn to the case $T(\alpha) \dashrightarrow S(p, q, r)$. We begin with:

Proposition 6.7. *The* 17 *candidate surface covers described in Table 6 and indicated there to be realizable are indeed realizable.*

Proof. Again we simply exhibit one dessin d'enfant for each relevant candidate, which is done in Figures 12 and 13. In all our dessins we associate white vertices to the entries of partition Π_1 and black vertices to those in Π_2, so the regions of the complement correspond to the entries of Π_3. ☐

To conclude the proof of Theorem 3.2 one would now need to show exceptionality of candidates 152, 163, 164, 166 and 168. We have actually done this using the MR criterion and the code [13], as explained in Section 5. We will however show here the same fact in a geometric fashion for the smallest and for the largest candidates:

Proposition 6.8. *Candidates* 152 *and* 168 *can be shown to be exceptional using the* GG *technique.*

Proof. For 152 we should realize $T(2) \overset{6:1}{\dashrightarrow} S(3, 3, 4)$ with (of course) $2 \to 4$. Taking the cone orders $3, 3, 4$ at A, B, C we should then assemble 12 copies of a triangle with vertices A, B, C getting, after the gluing, vertices A_1, A_2, B_1, B_2 each adjacent to 6 triangles. Taking into account this property for A_1, A_2, B_1 we get the pattern first shown in Figure 14, and taking into

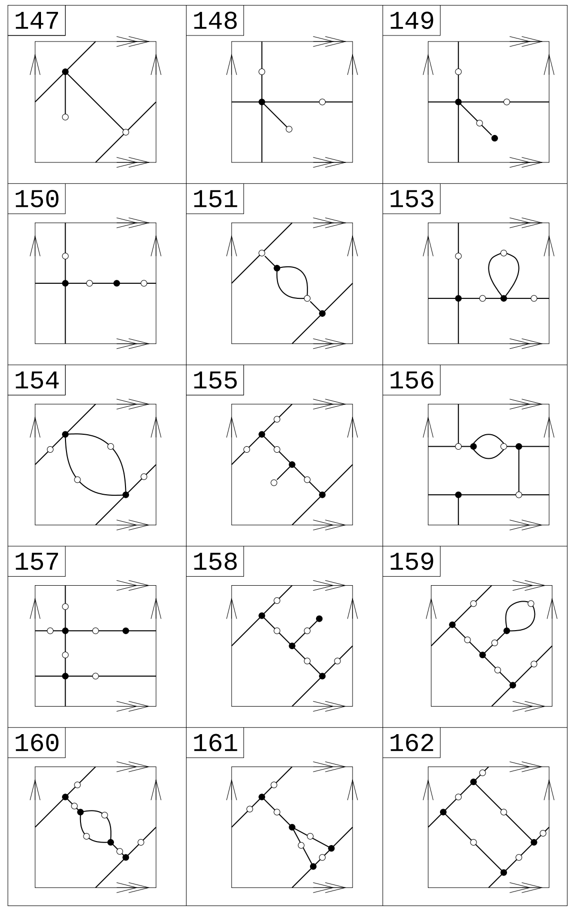

Figure 12. Dessins d'enfant realizing the candidate covers with degree up to 12.

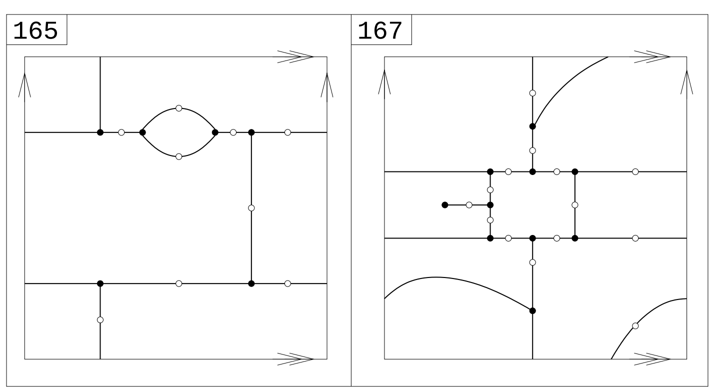

Figure 13. Dessins d'enfant realizing the candidate covers with degree greater than 12

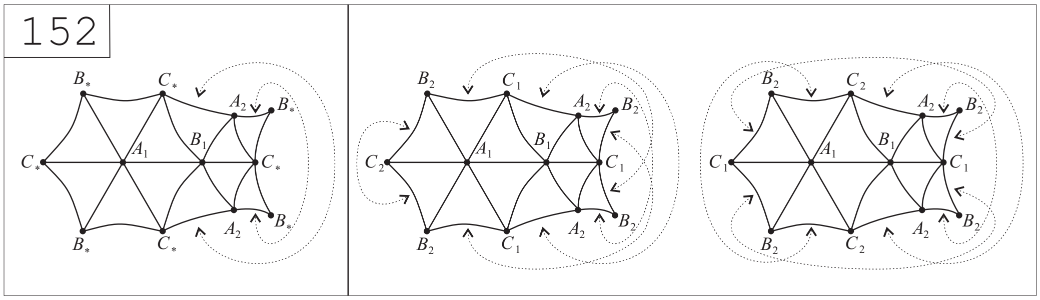

Figure 14. Exceptionality of `152` via `GG`.

account also B_2 we get the two possible patterns also shown in the figure. These patterns do not give orbifolds, because in the first one the cone angle at C_1 is $\frac{5\pi}{2}$, and in the second one the cone angles at C_1, C_2 are $\frac{3\pi}{2}$. Note however that combinatorially they give realization of candidate `151` and of the Euclidean $T \overset{6:1}{\dashrightarrow} S(3,3,3)$.

Turning to `168`, we confine ourselves to a sketch, because the complete argument is long. We should realize $T(7)$ gluing 72 copies of $T(2,3,7)$. Using the fact that 2 is covered by smooth points and 7 is covered by 5 smooth points and one of order 7, we see that the triangles must get assembled into 5 heptagons and one monogon, all with angles 2α, with $\alpha = \frac{\pi}{3}$. For short, let us write n instead of $n\alpha$. We should now glue these blocks getting angle 6 at all vertices. In particular when after a partial gluing an angle of 6 is reached, an extra gluing of edges is forced. Gluing the monogon to one heptagon we see the latter gets replaced by square with angles $2,2,2,4$. This square cannot glue to itself, so another heptagon gets replaced by one with angles $4,2,4,2,2,2,2$ (in this order). Checking various possibilities one sees that an edge incident to an angle 4 of this heptagon cannot be glued to an edge of the same heptagon, so, from the 3 original heptagons left, one gets replaced by a decagon with angles $4,4,2,2,2,4,2,2,2,2$. The edge of the decagon with angle 4 at both ends cannot be glued to another edge of the decagon, so one of the two remaining original heptagons gets replaced by an 11-gon with angles $2,4,2,2,2,4,2,2,2,4,2$. At least one edge of the heptagon gets glued to an edge of this 11-gon, and there are 6 possibilities up to symmetry. Looking at them we reduce to a single block, that is either a 14-gon with angles

$$2,2,4,2,2,4,2,2,4,2,2,4,2,2 \quad \text{or} \quad 2,2,4,2,2,4,2,2,4,2,2,2,4,2$$

or a 16-gon with angles

$$4,2,2,2,2,2,4,4,2,2,2,4,2,2,4,2 \quad \text{or} \quad 4,2,2,2,2,2,4,4,2,2,4,2,2,2,4,2.$$

And with some patience one sees that from one such block it is impossible to get a torus imposing angle 6 at all vertices. □

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