On the Enhancement of High-Gain Observers for State Estimation of Nonlinear Systems

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***Abstract*— An enhanced high-gain observer is proposed to estimate the state variables of dynamic systems with Lipschitz nonlinearities. Such an observer has a more general structure as compared with the standard high-gain observer, which can be regarded as a particular case of enhanced high-gain observer because of a special choice of the design parameters. The more general structure allows for additional degrees of freedom in the selection of the observer parameters, which however entails some difficulties in the design. To overcome such difficulties, a convenient design procedure is presented that is based on the use of the Young inequality and linear matrix inequalities. Numerical results are reported to evaluate the effectiveness of the proposed observer and its related design tools as compared with the high-gain observer.**

I. Introduction

The high-gain observer is by far the most popular estimation method used in nonlinear control. First results on what will be called high-gain observer are reported in [1], but for a complete overview on the original approach the reader is referred to [2]. The high-gain observer is based on the idea to dominate the effect of uncertainty or nonlinear terms in the dynamics of the estimation error through the selection of a sufficiently large gain. Unfortunately, the higher the gain, the larger the peaking in the transient, which may cause the destabilization of the control loop if the high-gain observer is used in cascade with a feedback regulator [3].

To reduce the peaking various solutions have been proposed. The so-called extended high-gain observer is presented in [4], Moreover, a lot of gain adaptation methods have been proposed to account for the presence of disturbances acting on the system. For example, in [5] a tuning mechanism based on the solution of a Riccati equation is adopted; a switching-gain tuning is proposed in [6]; in [7], [8] moving-horizon schemes are suggested to set the gain; [9] relies on the use of a nonlinear adaptation law.

The selection of a high gain stems also from the need to account for the nonlinearities in the error dynamics, which are usually modelled as Lipschitz functions. In [10], the gain adaptation allows one to account for the unknown Lipschitz constant. Resetting rules are proposed in [11]. The use of a time-varying gain is addressed in [12], [13], where a Lyapunov functional is used for the purpose of the stability analysis of the estimation error instead of the classical quadratic Lyapunov function. A novel state observer with a nested-saturation strucure is proposed in [14] (see also [15]).

In this paper, we will present a new observer structure as compared with the standard high-gain observer reported in the literature for triangular systems having Lipschitz nonlinearities. Such a more general structure comprises the standard one as a particular case, and let the designer to exploit additional degree of freedom at the prize of some difficulty. Indeed, the stability analysis based on a quadratic Lyapunov function provides some more complex conditions to deal. However, such issues can be overcome by using the Young inequality in a convenient manner in such a way to turn the problem into a set of bilinear matrix inequalities (BMIs). The use of gridding technique allows one to reduce the design problem to an iterative search with the satisfaction of more tractable linear matrix inequalities (LMIs) [16].

The paper is organized as follows. In Section II, we describe the basic assumptions on the system and the proposed enhanced high-gain observer. The stability analysis of the resulting estimation error is presented in Section III. In Section IV, we deal with its design. The results obtained by the proposed estimator as compared with high-gain observer are shown in Section V. The conclusions are drawn in Section VI.

The following notations will be used throughout this paper. For a real square matrix P, $P > 0$ ($P \geq 0$) means that it is symmetric positive definite (semidefinite). Given two symmetric real matrices P and Q, $P < Q$ means that the matrix $Q - P$ is positive definite. The minimum and maximum eigenvalues of $P > 0$ are denoted by $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$, respectively. If $P > 0$, $\|P\| = \lambda_{\max}(P)$.

II. System and Observer Assumptions

Let us consider dynamic systems described by

$$\begin{cases} \dot{x} = A\,x + f(x,t) \\ y = C\,x \end{cases}, \; t \geq 0 \qquad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector and $y(t) \in \mathbb{R}$ is a scalar measurement; $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^n$, and the function $f : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ are defined as follows:

$$A := \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \quad C := [1\ 0\ \dots\ 0],$$

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$$f(x,t) := \begin{bmatrix} f_1(x_1,t) \\ f_2(x_1,x_2,t) \\ \vdots \\ f_{n-1}(x_1,x_2,\dots,x_{n-1},t) \\ f_n(x_1,x_2,\dots,x_n,t) \end{bmatrix} .$$

To estimate $x(t)$, we consider the full-order state observer

$$\dot{\hat{x}} = A\hat{x} + f(\hat{x},t) + G(\gamma,K)\ (y - C\hat{x}) \ \ , \ t \geq 0 \qquad (2)$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the estimate of $x(t)$ at time t and

$$G(\gamma,K) := \begin{bmatrix} \gamma_1\, k_1 \\ \gamma_2\, k_2 \\ \vdots \\ \gamma_n\, k_n \end{bmatrix}$$

where γ_i is the i-th component of $\gamma \in \mathbb{R}^n$ and $K := [k_1\ \ k_2\ \ \dots\ \ k_n]^\top$ with $k_i \in \mathbb{R}$, $i = 1,2,\dots,n$, to be suitably chosen. Clearly, the observer (2) has just the same structure of the high-gain observer but with more many possibilities of choosing the design parameters of the gain. In a few words, (2) with all positive γ_i such that $\gamma_2 = \gamma_1^2$, $\gamma_3 = \gamma_1^3$, $\dots, \gamma_n = \gamma_1^n$ becomes quite a standard high-gain observer. Here we will present an approach to the construction of (2) that takes advantage of the increased number of parameters to be tuned in such a way to improve the performances.

Assumption 1: The function f is continuous and there exist $L \in \mathbb{R}^n_{\geq 0}$ such that, for all $x, w \in \mathbb{R}^n$ and $t \geq 0$,

$$\big| f_i(x_1 + w_1, x_2 + w_2, \dots, x_i + w_i, t) - f_i(x_1, x_2, \dots, x_i, t) \big| \leq \sum_{j=1}^{i} L_j\, |w_j| \ . \qquad (3)$$

Note that Assumption 1 ensures the existence of the solutions of both (1) and (2) (see [13] for details).

Instead of studying the stability of the estimation error that genuinely descends from (1) and (2) (i.e., $\hat{e} := x - \hat{x}$), we perform a change of variables $\hat{e} = T(\gamma)\, e$, $e \in \mathbb{R}^n$ with

$$T(\gamma) = \mathrm{diag}\,(\gamma_1, \gamma_2, \dots, \gamma_n)$$

and study the stability in e. From (1) and (2) we obtain the error dynamics

$$\dot{\hat{e}}(t) = (A - G(\gamma,K)C)\ \hat{e}(t) + f(x(t),t) - f\,(x(t) - \hat{e}(t), t)\ . \qquad (4)$$

Clearly, the error dynamics in $\hat{e}(t)$ is stable if and only if so it is also for the error dynamics in $e(t)$.

In the following, we will focus on novel stability conditions that are more general with respect to the state-of-art. Toward this end, we need a technical result that is just the generalization of [12, Lemma 1].

Lemma 1: If $\gamma_1 > 0$ and

$$\gamma_1 \leq \gamma_2 \leq \dots \leq \gamma_{n-1} \leq \gamma_n \qquad (5)$$

there exists $k_f > 0$ such that

$$\Big\| T(\gamma)^{-1} \left(f(x(t),t) - f(x(t) - T(\gamma)\, e(t), t) \right) \Big\| \leq k_f\, \|e(t)\| \qquad (6)$$

for all $t \geq 0$, and k_f does not depend on γ and t.

Proof. Consider the various components of the l.h.s. of (6):

$$\big\| T(\gamma)^{-1}\, (f(x,t) - f(x - T(\gamma)\, e, t)) \big\| = \left\| \begin{bmatrix} \dfrac{1}{T_{11}(\gamma)} \ \big(f_1(x_1,t) - f_1(x_1 - T_{11}(\gamma)e_1, t)\,\big) \\ \dfrac{1}{T_{22}(\gamma)} \ \big(f_2(x_1,x_2,t) \\ \quad -f_2(x_1 - T_{11}(\gamma)e_1, x_2 - T_{22}(\gamma)e_2, t)\,\big) \\ \vdots \\ \dfrac{1}{T_{nn}(\gamma)} \ \big(f_n(x_1,x_2,\dots,x_n,t) \\ \quad -f_n(x_1 - T_{11}(\gamma)e_1, x_2 - T_{22}(\gamma)e_2, \\ \quad \dots, x_n - T_{nn}(\gamma)e_n, t)\,\big) \end{bmatrix} \right\| .$$

As to the first one, using Assumption 1 it follows

$$\left| \frac{1}{T_{11}(\gamma)} \big(f_1(x_1,t) - f_1(x_1 - T_{11}(\gamma)e_1, t)\right| \leq \frac{L_1}{\gamma_1} |x_1 - (x_1 - \gamma_1 e_1)| = L_1 e_1 \qquad (7)$$

as $\gamma_1 > 0$. Following the same steps based on the use of Assumption 1, we obtain

$$\begin{aligned} &\left| \frac{1}{T_{ii}(\gamma)} \ \big(f_i(x_1,x_2,\dots,x_i,t) - f_i(x_1 - T_{11}(\gamma)e_1, x_2 \right. \\ &\left. - \ T_{22}(\gamma)e_2, \dots, x_i - T_{ii}(\gamma)e_i, t)\big) \right| \leq L_1 \frac{T_{11}(\gamma)}{T_{ii}(\gamma)} |e_1| \\ &+ L_2 \frac{T_{22}(\gamma)}{T_{ii}(\gamma)} |e_2| + \dots + L_{n-1} \frac{T_{i-1\,i-1}(\gamma)}{T_{ii}(\gamma)} |e_{i-1}| \\ &+ L_i |e_i| = \sum_{k=1}^{i-1} \frac{\gamma_k}{\gamma_i}\, L_k |e_k| + L_i |e_i| \leq \sum_{k=1}^{i} L_k |e_k| \end{aligned} \qquad (8)$$

for $i = 2,3,\dots,n$, where the last inequality stems from (5). Using the inequality

$$\begin{aligned} \left(\sum_{k=1}^{i} L_k |e_k| \right)^2 &\leq i \left(\max_{i=1,2,\dots,n} L_i \right)^2 \sum_{k=1}^{i} e_k^2 \\ &\leq i \left(\max_{i=1,2,\dots,n} L_i \right)^2 \sum_{k=1}^{n} e_k^2 \end{aligned}$$

on the squared r.h.s. of (8), we obtain (6) by choosing, for example,

$$k_f := \sqrt{\frac{n(n+1)}{2}} \max_{i=1,2,\dots,n} L_i\, .$$

□

Based on the aforesaid, from (4) we obtain:

$$\begin{aligned} \dot{e}(t) &= T(\gamma)^{-1}\ (A - G(\gamma,K)\, C)\ T(\gamma)\, e(t) \\ &+ T(\gamma)^{-1} \Big(f(x(t),t) - f\,(x(t) - T(\gamma)\, e(t), t) \Big) . \end{aligned}$$

Because of the particular observer structure, the previous equation can be written as follows:

$$\dot{e}(t) = \gamma_1 \left(A - KC + \Omega(\gamma)\right) e(t) + T(\gamma)^{-1} \Big(f(x(t), t) - f(x(t) - T(\gamma)\, e(t), t)\Big) \tag{9}$$

where

$$\Omega(\gamma) := \begin{bmatrix} 0 & z_1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & z_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & z_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

where $z_i = \gamma_{i+1}/(\gamma_1\gamma_i) - 1$, $i = 1, 2, \ldots, n-1$.

In the next section, we will address the design problem for the proposed observer.

III. Stability of the Estimation Error

To study the stability of (9), we will use the Lyapunov function $V(e) = e^\top P e$ with $P \in \mathbb{R}^{n\times n}$ symmetric positive definite. The time derivative of V is

$$\dot{V}(e) = \gamma_1 e^\top \Big((A - KC + \Omega(\gamma))^\top P + \gamma P(A - KC + \Omega(\gamma))\Big) e + 2e^\top P T(\gamma)^{-1} \Big(f(x,t) - f(x - T(\gamma)\, e, t)\Big) \tag{10}$$

Using the Schartz inequality and Assumption 1, the second term in (10) can be bounded as follows:

$$\begin{aligned} &2e^\top P\, T(\gamma)^{-1} \left(f(x,t) - f(x - T(\gamma)\, e, t)\right) \\ &\leq 2 \left| e^\top P\, T(\gamma)^{-1} \left(f(x,t) - f(x - T(\gamma)\, e, t)\right)\right| \\ &\leq 2k_f\, \lambda_{\max}(P)\, \|e\|^2 . \end{aligned}$$

If $\gamma \in \mathbb{R}^n_{>0}$ is such that (5) hold

$$(A - KC + \Omega(\gamma))^\top P + P(A - KC + \Omega(\gamma)) + \lambda I < 0$$

we obtain

$$\dot{V}(e) \leq -\gamma_1 \lambda \|e\|^2 + 2k_f \lambda_{\max}(P) \|e\|^2$$

and hence $\dot{V}(e)$ turns out to be negative definite for $e \neq 0$ if, in addition, we impose $\gamma_1 > 2k_f\lambda_{\max}(P)/\lambda$.

The aforesaid can be summarized as follows.

Theorem 1: If there exist $P > 0$, $\lambda > 0$, K, and γ such that

$$(A - KC + \Omega(\gamma))^\top P + P(A - KC + \Omega(\gamma)) + \lambda I < 0 \tag{11}$$

$$\gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_{n-1} \leq \gamma_n \tag{12}$$

$$\gamma_1 > \frac{2k_f \lambda_{\max}(P)}{\lambda} \tag{13}$$

then the estimation error exhibited by observer (2) in performing estimation for system (1) is asymptotically stable. □

The satisfaction of the stability conditions stated in the above theorem is difficult for various reasons. First, in general there exists the coupling of (11) and (12) except when $z_i = 0$ for all $i = 1, 2, \ldots, n-1$, which corresponds to the "classical" high-gain solution. Here, we need to explicitly account for such a coupling. Second, we have to impose the constraints (12) so as to enforce the use of the inequality (6), which is crucial to prove stability. Third, also (13) is required to ensure stability. Nevertheless, we will provide an efficient method to construct the proposed *enhanced high-gain observer* by overcoming the coupling thanks to the use of the parameters z_i instead of γ_i. The values of γ_i will be computed in such a way to satisfy (13). Toward this end, note that

$$\Omega(\gamma) = A_1 Z A_2$$

where $Z := \mathrm{diag}(z_1, \ldots, z_{n-1}) \in \mathbb{R}^{(n-1)\times(n-1)}$,

$$A_1 := \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & 1 \\ 0 & \ldots & 0 & 0 \end{bmatrix} \in \mathbb{R}^{n\times(n-1)} ,$$

$$A_2 := \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \end{bmatrix} \in \mathbb{R}^{(n-1)\times n} .$$

In practice, A_1 and A_2 can be obtained from A by erasing the first column and last row, respectively. Moreover, the Young's inequality will be exploited according to the following formulation.

Lemma 2: Let E and F be two real rectangular matrices. Then, the following inequality holds:

$$E^\top F + F^\top E \leq \frac{1}{2} \left(E + SF\right)^\top S^{-1} \left(E + SF\right)$$

for any symmetric $S > 0$ of appropriate dimension.

Proof: The proof follows from the standard Young's relation and it is omitted for the space limitation. □

The significance of Lemma 2 does not lie in its proof, which is trivial: the point to retain is that only the half quantity of $E^\top F + F^\top E$ is upper bounded by the Young's relation. This technique plays an important role on the observer design problem we address in this paper.

Based on the aforesaid, we are able to state the following result.

Theorem 2: If there exist $P > 0$, $\lambda > 0$, Y, $S > 0$ diagonal, and $W \leq 0$ diagonal such that

$$\begin{bmatrix} A^\top P + PA - C^\top Y^\top - YC + \lambda I & \star \\ A_1^\top P + W A_2 & -2\,S \end{bmatrix} < 0 \tag{14}$$

$$W > -S \tag{15}$$

and γ is chosen such that

$$\gamma_1 \geq \frac{1}{\lambda_{\min}(Z) + 1} \tag{16}$$

$$\gamma_1 > \frac{2k_f \lambda_{\max}(P)}{\lambda} \tag{17}$$

where $Z = S^{-1}W$, the estimation error given by observer (2) having gain $G(\gamma, K)$ with $K = P^{-1}Y$ and

$$\gamma_i = \gamma_1^i \prod_{k=1}^{i-1} (z_k + 1)\,,\ i = 2, \ldots, n \tag{18}$$

is asymptotically stable.

Proof. Based on Theorem 1, we need to account for (11), (12), and (13).

First, let us consider (11), which can be rewritten as

$$A^\top P + PA - C^\top Y^\top - YC + \lambda I + \Omega^\top P + P\,\Omega < 0$$

since $Y = KP$. Moreover, since

$$\begin{aligned}\Omega^\top P + P\Omega &= (A_1^\top P)^\top (ZA_2) + (ZA_2)^\top (A_1^\top P)\\ &\le \frac{1}{2}\left(A_1^\top P + SZA_2\right)^\top S^{-1}\left(A_1^\top P + SZA_2\right)\end{aligned}$$

thanks to Lemma 2, it is straightforward to get (14) as a sufficient condition for (11) to hold by using the Schur lemma.

Second, (18) stems from the definition of z_i. Using (18), it follows that (12) holds if

$$\gamma_1 \ge \frac{1}{z_i + 1} \tag{19}$$

with

$$z_i \in (-1, 0] \tag{20}$$

for all $i = 1, 2, \ldots, n-1$. The constraints (20) can be imposed by using (15) since S and W are both diagonal as well as positive definite and negative semidefinite, respectively. Of course, (19) is equivalent to

$$\gamma_1 \ge \frac{1}{\min\limits_i (z_i) + 1} \tag{21}$$

or (16) for the sake of brevity. Finally, we need (13), which is just reported as (17).

■

The LMI-based stability conditions presented so far are well-suited for the purpose of design, which will be addressed in the next section.

IV. Observer Design Based on Convex Optimization

This section is devoted to present some guidelines for the design of the proposed enhanced high-gain observer. Clearly, to get a convenient observer gain $G(\gamma, K)$ so as to reduce the effect of measurement noise but with a sufficiently fast transient, we need to select the design parameters in a careful way. Unfortunately, such an optimization is not convex, but a suitable procedure will be presented to construct observers with a tradeoff between transient response and steady state behavior in the presence of disturbances on the output.

Theorem 2 provides the stability conditions to be satisfied. Among them, (14) and (15) are LMIs that can be easily treated, whereas (16) and (17) deserve some attention. The goal is that of finding γ_1 and z_i, $i = 1, 2, \ldots, n-1$ and then γ_i, $i = 2, 3, \ldots, n$ by using (18).

First of all, let us consider (17). Thanks to homogeneity, we may choose $\gamma_1 = 1/\lambda$ under the constraint

$$P < \frac{1}{2k_f} I \tag{22}$$

since the above LMI ensures that $\lambda_{\max}(P) < 1/(2k_f)$ and hence

$$\frac{2k_f \lambda_{\max}(P)}{\lambda} < \frac{1}{\lambda} = \gamma_1\,. \tag{23}$$

Second, (16) can be taken into account together with (23) by choosing

$$\gamma_1 = \max\left(\frac{1}{\lambda}, \frac{1}{\lambda_{\min}(Z) + 1}\right).$$

Clearly, the maximization of λ is well-suited to reducing γ_1. In so doing, we need also to maximize the z_i variables since otherwise there may be no reduction of γ_1. A large λ ensures a fast response in the transient by means of a gain K and hence $G(\gamma, K)$ that may become too big as compared with the measurement noise at steady state. Thus, a compromise between how large the norm of K and λ may be taken has to be pursued. As is well-known, a possible strategy consists in exploiting homogeneity by imposing the LMIs

$$P > I$$

and

$$\begin{bmatrix} \alpha & Y^\top \\ Y & \alpha I \end{bmatrix} > 0\,. \tag{24}$$

From the former we get $P^{-1} < I$, whereas the latter is equivalent to

$$\|Y\|^2 = Y^\top Y \le \alpha^2\,.$$

and hence, since

$$\|K\| = \|P^{-1}Y\| \le \|P^{-1}\|\,\|Y\| \le \|Y\|\,,$$

the minimization of $\|K\|$ follows from the minimization of α. Unfortunately, the condition $P > I$ is not compatible with (22) in general. To overcome such a difficulty, we may introduce the new LMI

$$P > \beta I \tag{25}$$

where $\beta > 0$. From (25), we have $P^{-1} < (1/\beta)I$ and, following the same reasoning, we can minimize $\|K\|$ by minimizing of α after fixing β and if (24) holds since

$$\|K\| = \|P^{-1}Y\| \le \|P^{-1}\|\,\|Y\| \le \frac{\|Y\|}{\beta}\,.$$

The selection of β can be done by using some gridding technique, which will be also well-suited to maximizing the z_i variables later on. Gridding methods allows one to treat problems that are essentially bilinear but, after keeping some parameters constant, they can be solved by using LMIs [17].

Concerning the maximization of the z_i variables, we have to overcome the difficulty that they are not available as naive LMI variables. Nevertheless, since they are the components along the diagonal of the matrix $S^{-1}W$ we may succeed in such a maximization by introducing the diagonal matrix $R =$

$\text{diag}(r_1, r_2, \ldots, r_{n-1})$ with $r_i \in [0, 1)$, $i = 1, 2, \ldots, n-1$, such that

$$W \leq -SR \tag{26}$$

where each r_i has to regarded as lower bound on z_i and thus it is well-suited to being maximized to maximize z_i. The effect of the maximization on z_i is successful only if we obtain a feasible, non negative value of r_i. Since the problem is still composed of LMIs if the r_i variables are fixed, we may apply gridding also for dealing with such variables, likewise proposed to account for β .

Summing up, the design will be performed by gridding β on $(0, 1/(2k_f)$ and $r = (r_1, r_2, \ldots, r_{n-1})$ on $[0, 1)^{n-1}$ and minimizing the objective function $\alpha - c\lambda$ with a convenient choice of $c > 0$ in the inner loop and $G(\gamma, K)$ in the outer loop as follows:

Design procedure

Input: A, C, A_1, A_2, k_f, c, N

Output: G^*

1: Generate a sampling sequence $\{(\tilde{\beta}(i), \tilde{r}(i))\}$, $i = 1, 2, \ldots, N$ belonging to $(0, 1/(2k_f)) \times [0, 1)^{n-1}$
2: *best_norm* $\leftarrow +\infty$
3: `for` $i = 1$ `to` N `do`
4: solve the optimization problem $\min(\alpha - c\lambda)$ w.r.t. $\alpha > 0$, $P > 0$, $\lambda > 0$, Y, $S > 0$ diagonal, and $W \leq 0$ diagonal subject to (14), (15), (22), (24), (25), (26) with $\beta = \tilde{\beta}(i)$, $R = \text{diag}(\tilde{r}(i))$
5: $K \leftarrow P^{-1}Y$
6: $Z \leftarrow S^{-1}W$
7: $\gamma_1 \leftarrow \max\left(\frac{1}{\lambda}, \frac{1}{\lambda_{\min}(Z) + 1}\right)$
8: $\gamma_i \leftarrow \gamma_1^i \prod_{k=1}^{i-1} (z_k + 1)$, $i = 2, \ldots, n$
9: $G(\gamma, K) \leftarrow T(\gamma)K$
10: `if` $\big(\|G(\gamma, K)\| <$ *best_norm*$\big)$ `then`
11: *best_norm* $\leftarrow \|G(\gamma, K)\|$
12: $G^* \leftarrow G(\gamma, K)$
13: `endif`
14: `endfor`

with G^* as the final result of the procedure.

Note that, if we perform the minimization at step "4:" for different values of c, we force the optimization toward a larger K and a smaller γ_1 by increasing c.

Next section will be devoted to the simulation results to evaluate the effectiveness of the proposed observer as compared with the standard high-gain observer.

V. Simulation Results

This section is devoted to some numerical comparisons between the high-gain observer and the enhanced high-gain observer, which will be denoted by HGO and EHGO, respectively. Let us consider the third-order system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = k_f \sin x_3 \\ y = x_1 \end{cases} \tag{27}$$

where $x(t) = (x_1(t), x_2(t), x_3(t))$ is the state vector. Thus, (27) can be written as

$$\dot{x} = A\,x + f(x)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \; C = [\,1\;0\;0\,],$$

and

$$f(x) = [\,0 \quad 0 \quad k_f \sin x_3\,]^\top .$$

We applied the proposed design procedure using Yalmip [18] to find the gain of the EHGO with different choice of k_f. The same procedure "mutatis mutandis" was used to find the gains of the HGO. Table I allows for a comparison of the results by using percentage reduction of the gain norms, i.e.,

$$\Delta_G = \frac{\|G(\gamma_{\text{HG}}, K_{\text{HG}})\| - \|G(\gamma_{\text{EHG}}, K_{\text{EHG}})\|}{\|G(\gamma_{\text{HG}}, K_{\text{HG}})\|}$$

where γ_{HG}, K_{HG} and γ_{EHG}, K_{EHG} stand for the gains of the HGO and EHGO, respectively. Specifically, the results reported in Table I are obtained by using linearly spaced gridding on $[0, 1)$ for the r_i variables, whereas logarithmic spacing on $(0, 1/(2k_f))$ was chosen for β. The gridding was made of $N = 1000$ points.

Fig. 1 shows the result of a simulation run with additive, uniform random noises in the range $[-1, 1]$ on the measure and on the last dynamic equation of (27), initial state equal to $(5\,,\,5\,,\,5)$ and initial estimated state (for both HGO and EHGO) equal to $(-5\,,\,-5\,,\,-5)$.

Table I and Fig. 1 are helpful to illustrate the effectiveness of the proposed EHGO as compared with the HGO. For small values of k_f, there is no difference between HGO and EHGO. By contrast, for large values of k_f the results obtained by the EHGO are much better in terms of transient behavior of the estimates of the inner state variables as well as at regime, where the smaller gain of the EHGO makes the estimation less sensible to the measurement noise.

Remark 1: It is worth to note that the classical HGO cannot be better than the EHGO since the HGO stability conditions result from a particular choice of the design matrices according to Theorem 2, namely $W = 0$, or $R = 0$ in the design procedure. In such a case, Theorem 2 provides the standard conditions of stability for the HGO, i.e.,

$$A^\top P + PA - C^\top Y^\top - YC + \lambda I < 0$$
$$\gamma_1 > \frac{2k_f \lambda_{\max}(P)}{\lambda}$$

with, as usual, $K_{\text{HG}} = P^{-1}Y$ and $\gamma_i = \gamma_1^i$, $i = 2, \ldots, n$. Table I confirms the aforesaid in the case $k_f = 0.1$, for which no appreciable difference in the final result is shown between HGO and EHGO.

TABLE I

RESULTS OF THE DESIGN FOR DIFFERENT VALUES OF k_f AND c

c	k_f	HGO				EHGO						Δ_G
		γ_{HG}	K_{HG}			γ_{EHG}: γ_1	γ_2	γ_3	K_{EHG}			
1	0.1	2.34	1.06	0.86	0.29	2.34	5.48	12.81	1.06	0.86	0.29	0%
	1	23.45	1.06	0.86	0.29	14.82	219.75	977.29	7.46	5.95	1.87	41%
	10	234.51	1.06	0.86	0.29	148.24	21975.16	977274.59	7.46	5.95	1.87	51%
10	0.1	1.00	2.58	4.60	2.11	1.00	1.00	1.00	2.58	4.60	2.11	0%
	1	5.74	2.58	4.60	2.11	5.82	33.81	117.98	2.45	4.56	2.16	30%
	10	57.36	2.58	4.60	2.11	58.15	3381.46	117979.65	2.45	4.56	2.16	36%
100	0.1	1.00	2.58	4.88	2.21	1.00	1.00	1.00	2.58	4.88	2.21	0%
	1	5.71	2.58	4.88	2.20	5.78	33.37	115.66	2.40	4.88	2.23	31%
	10	57.14	2.58	4.88	2.21	57.77	3336.98	115659.61	2.40	4.88	2.23	37%
1000	0.1	1.00	2.58	4.91	2.22	1.00	1.00	1.00	2.58	4.91	2.22	0
	1	5.71	2.58	4.91	2.21	5.78	33.37	115.65	2.40	4.91	2.24	31%
	10	57.14	2.58	4.91	2.22	57.76	3336.72	115645.85	2.40	4.91	2.24	37%

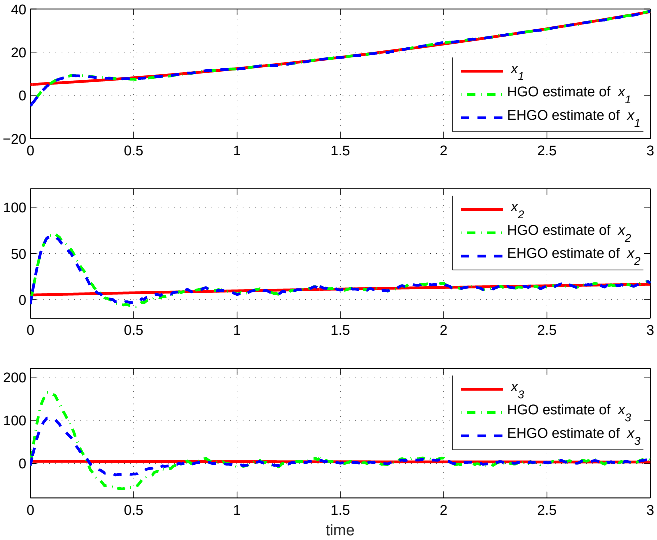

Fig. 1. Simulation run for a system instance with $k_f = 1$, HGO and EHGO designed with $c = 10$.

VI. CONCLUSION

As evolution of the high-gain observer, we have presented an enhanced high-gain observer together with an effective design method. Such an observer can be designed in such a way to ensure stability in a noise-free setting with a smaller gain and hence a more reduced sensitivity to the measurement noise as compared with the standard high-gain observer. The complications arisen in the design have been successfully addressed, as shown by means of the numerical results we have reported.

As a future work, we will investigate the combination of the enhanced high-gain observer with the increasing-gain observer [12], [13] with the goal to simultaneously guarantee good performances in the transient and at steady state.

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