Merging relations: a way to compact Petri nets' behaviors uniformly[★]

Giovanni Casu and G. Michele Pinna

Dipartimento di Matematica e Informatica, Università di Cagliari, Cagliari, Italy
giovanni.casu@unica.it, gmpinna@unica.it

Abstract. Compacting Petri nets behaviors means to develop a more *succinct* representation of all the possible executions of a net, still giving the capability to *reason* on properties fulfilled by the computations of the net. To do so suitable equivalences on alternative executions have to be engineered. We introduce a general notion of *merging relation* covering the existing approaches to compact behaviors and we discuss how to enforce that the more succinct net is an unravel net, namely a net where dependencies can be identified (almost) synctactically.

1 Introduction

The non sequential behavior of a Petri net [1, 2] can be described in many ways, *e.g* using *traces* [3], but probably the most popular and used one is the notion of *unfolding* [4, 5]. The unfolding of a net N is particularly relevant as it allows to record conflicts and dependencies among the activities modeled with N, and the possibility of finding a finite representation of it (the prefix), has given profitability to the notion, otherwise confined to the purely theoretical modeling realm [6, 7]. However the size of a finite unfolding, even of the prefix, can be too large, hence manageable only with big efforts. Several approaches to reduce it have been proposed, based on the idea of identifying suitable conflicting conditions of the unfolding. In the case of *merged process* [8] the criterion is that the conditions must be equally labeled and have the same token occurrence (*i.e.* they represent the same token, in the collective token philosophy of [9]) whereas in the case of *trellis processes* [10] the criterion is the distance of the equally labeled conditions from the initial condition of its component (measuring the *time*). Once conditions have been identified, isomorphic *futures* are identified as well.

The identification of conflicting conditions seems to be a good starting point for compacting nets' behaviors. We pursue this idea further, casting it in a general framework. We first define a notion of incompatibility among places of the behavior that it is not based on the syntax, as in causal nets, but on the semantics; and then we introduce an equivalence relation on places with some minimal requirements: it should respect the notion of incompatibility and it should respect the labeling (which is, in the case of the non sequential behavior of a net, the folding morphism). The incompatibility relation should capture the

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idea that, assuming that a place is identified with a resource, two resources are never *used* together in a computation. This equivalence relation is the basis for compacting behaviours that, following the classic motto, are nets [11, 12].

Some of the definable equivalences may give a rather compact net, where executions have to be recovered with some efforts, and causality and conflicts may be deduced only after all the executions have been exploited. To ease the task of finding causality and conflict we propose the notion of *unravel* net which allows to capture easily and syntactically dependencies in each computation, whereas conflict is deduced in a semantic way. Indeed, unravel nets are such that each execution gives an acyclic net, hence causality is easily traceable. Furthermore we believe that this notion is robust enough for representing the non sequential behaviors of nets as it is closely related to a brand of event structures, namely *bundle event structure* [13]. Steps towards this direction have already been done in [14], and here we present this attempt in the more general framework. Our approach is rather flexible. The merging relation may be induced by a *measure* on the places of the net to be compacted and, depending on the kind of measure adopted, the result is an unravel net or there are ways to enrich the net in such a way that the behaviors are preserved and reflected, and the result of the compaction is still an unravel net. Thus the problem of finding the proper merging relation is moved to the search of the useful measure on the places of the net. Clearly a measure which is injective gives the trivial merging relation.

In Sec. 2 we introduce unravel nets and relate them to bundle event structures, then, in Sec. 3, we propose our general framework, which we show adequate in Sec. 4 by casting in it the classic approaches to behaviors compaction. In Sec. 5 we discuss how to ensure that the result of the compaction is still an unravel net.

2 Nets and Bundle Event Structures

With $\mathbb{N}$ we denote the set of natural numbers. Let A be a set, a *multiset* of A is a function $m : A \to \mathbb{N}$. The set of multisets of A is denoted by μA. The usual operations on multisets, like multiset union $+$ or multiset difference $-$, are used. We write $m \subseteq m'$ if $m(a) \leq m'(a)$ for all $a \in A$. If $m \in \mu A$, we denote with $[\![m]\!]$ the multiset defined as $[\![m]\!](a) = 1$ if $m(a) > 0$ and $[\![m]\!](a) = 0$ otherwise; and we use $\mathit{supp}(m)$ as the denotation of the set $\{a \in A \mid m(a) \geq 1\}$. Finally, when a multiset m of A is a set, *i.e.* $m = [\![m]\!]$ (hence $m = \mathit{supp}(m)$), we write $a \in m$ to denote that $m(a) \neq 0$, and often confuse the multiset m with $\mathit{supp}(m)$.

A *Petri net* is a 4-tuple $N = \langle S, T, F, \mathsf{m}\rangle$, where S is a set of *places* and T is a set of *transitions* (with $S \cap T = \emptyset$), $F \subseteq (S \times T) \cup (T \times S)$ is the *flow* relation, and $\mathsf{m} \in \mu S$ is called the *initial marking*. Petri nets are depicted as usual.

Given a net $N = \langle S, T, F, \mathsf{m}\rangle$ and $x \in S \cup T$, we define the following multisets: ${}^{\bullet}x = F(-, x)$ and $x^{\bullet} = F(x, -)$. If $x \in S$ then ${}^{\bullet}x$ ($x^{\bullet}$) is in μT and if $x \in T$ then ${}^{\bullet}x$ ($x^{\bullet}$) is in μS A transition $t \in T$ is enabled at a marking $m \in \mu S$, denoted with $m\,[t\rangle$, whenever ${}^{\bullet}t \subseteq m$. A transition t enabled at a marking m can *fire* and its firing produces the marking $m' = m - {}^{\bullet}t + t^{\bullet}$. The firing of a transitions t at a marking m is denoted with $m\,[t\rangle\, m'$. We assume that each transition t of a net N is such that ${}^\bullet t \neq \emptyset$ (which means that no transition may fire *spontaneously*). Given a generic marking m (not necessarily equal to the initial one), the *firing sequence* (fs) starting at m of the net $N = \langle S, T, F, \mathsf{m}\rangle$, is defined as usually: (a) m is a fs, and (b) if $m\,[t_1\rangle\, m_1 \cdots\; m_{n-1}\,[t_n\rangle\, m_n$ is a fs and $m_n\,[t\rangle\, m'$ then also $m\,[t_1\rangle\, m_1 \cdots m_{n-1}\,[t_n\rangle\, m_n\,[t\rangle\, m'$ is a fs. The set of firing sequences of a net N starting at a marking m is denoted with $\mathbf{\Sigma}_m^N$ and it is ranged over by σ. Given fs $\sigma = m\,[t_1\rangle\, \sigma'\,[t_n\rangle\, m_n$, with $start(\sigma)$ we denote the marking m, with $lead(\sigma)$ the marking m_n and with $tail(\sigma)$ the fs $\sigma'\,[t_n\rangle\, m_n$. Given a net $N = \langle S, T, F, \mathsf{m}\rangle$, a marking m is *reachable* iff there exists a fs $\sigma \in \mathbf{\Sigma}_{\mathsf{m}}^N$ such that $lead(\sigma)$ is m; the set of reachable markings of N is $\mathcal{M}_N = \bigcup_{\sigma\in\mathbf{\Sigma}_{\mathsf{m}}^N} lead(\sigma)$. Given a fs $\sigma = m\,[t_1\rangle\, m_1 \cdots m_{n-1}\,[t_n\rangle\, m'$, with $X_\sigma = \sum_{i=1}^n \{t_i\}$ we denote the multiset of transitions associated to this fs. We call this multiset a *state* of the net. The set of states of a Petri net is then $\mathsf{St}(N) = \{X_\sigma \in \mu T \mid \sigma \in \mathbf{\Sigma}_{\mathsf{m}}^N\}$.

Given a set of markings M, with $\mathsf{P}(M)$ we denote the set of places that are marked at some marking in M, namely $\{s \in S \mid \exists m \in M.\ m(s) > 0\}$, and given a fs σ, $\mathsf{M}(\sigma)$ are the markings associated to the fs σ, where $\mathsf{M}(\sigma)$ is $\mathsf{M}(\sigma) = \{m\}$ if $\sigma = m$ and $\mathsf{M}(\sigma) = \{start(\sigma)\} \cup \mathsf{M}(tail(\sigma))$ otherwise.

A net $N = \langle S, T, F, \mathsf{m}\rangle$ is said *safe* if each marking $m \in \mathcal{M}_N$ is such that $m = [\![m]\!]$. In this paper we consider safe nets $N = \langle S, T, F, \mathsf{m}\rangle$ where each transition can be fired, *i.e.* $\forall t \in T\ \exists m \in \mathcal{M}_N.\ m\,[t\rangle$, and each place is marked in a computation, *i.e.* $\forall s \in S\ \exists m \in \mathcal{M}_N.\ m(s) = 1$. A subnet of a net is a net obtained restricting places and transitions, and correspondingly also the relation F and the initial marking. Let $N = \langle S, T, F, \mathsf{m}\rangle$ be a Petri net and let $T' \subseteq T$. Then the subnet generated by T' is the net $N|_{T'} = \langle S', T', F', \mathsf{m}'\rangle$, where $S' = \bigcup_{t\in T'} \left([\![{}^\bullet t]\!] \cup [\![t^\bullet]\!]\right)\ \cup\ supp(\mathsf{m})$, F' is the restriction of F to S' and T', and m' is the obvious restriction of m to places in S'. Analogously we can restrict to a subset of places. Let $S' \subseteq S$, then the subnet generated by S' is the net $N|_{S'} = \langle S', T', F', \mathsf{m}'\rangle$, where $T' = \bigcup_{s\in S'} \left([\![{}^\bullet s]\!] \cup [\![s^\bullet]\!]\right)$, F' is the restriction of F to S' and T', and m' is the restriction of m to places in S'.

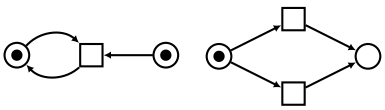
A net $N = \langle S, T, F, \mathsf{m}\rangle$ is said to be *acyclic with respect to a subset of places* S' whenever, given $N|_{S'} = \langle S', T', F', \mathsf{m}'\rangle$, the transitive and reflexive closure of $[\![F]\!]$ is a partial order on $S' \cup T'$.

An *1-occurrence net* $O = \langle S, T, F, \mathsf{m}\rangle$ is a Petri net where each state is a set, i.e. $\forall X \in \mathsf{St}(O)$ it holds that $X = [\![X]\!]$. The notion of occurrence net we use here is the one called 1-occurrence net in [12] and the intuition behind it is the following: regardless how tokens are produced or consumed, an occurrence net *guarantees* that each transition can *occur* only once.

The notion of occurrence net is a *semantical* one, whereas the one of *causal net* is more syntax oriented. Given a net N, we define $x <_N y$ iff $(x, y) \in F$, and $\leq_N$ is the transitive and reflexive closure of this relation. For denoting places and transitions of a causal net we use B and E (see [4]) and call them conditions and events respectively. A causal net is acyclic, when the whole set of conditions is considered, and equipped with a *conflict* relation. Thus, a *causal net* $C = \langle B, E, F, \mathsf{m}\rangle$ is a safe net satisfying the following restrictions: (1) $\forall b \in [\![\mathsf{m}]\!]$, ${}^\bullet b = \emptyset$, (2) $\forall b \in B.\ \exists b' \in [\![\mathsf{m}]\!]$ such that $b' \leq_C b$, (3) $\forall b \in B.\ {}^\bullet b$ is either empty or a singleton, (4) for all $e \in E$ the set $\{e' \in E \mid e' \leq_C e\}$ is finite, and (5) $\#$ is an irreflexive and symmetric relation defined as follows: (5.a) $e \#_r e'$ iff $e, e' \in E$, $e \neq e'$ and ${}^\bullet e \cap {}^\bullet e' \neq \emptyset$, (5.b) $x \# x'$ iff $\exists y, y' \in E$ such that $y \#_r y'$ and $y \leq_C x$ and $y' \leq_C x'$. The intuition behind this notion is the following: each condition b represents the occurrence of a token, which is produced by the *unique* event in ${}^\bullet b$, unless b belongs to the initial marking, and it is used by only one transition (hence if $e, e' \in b^\bullet$, then $e \# e'$). On causal nets it is natural to define a notion of *causality* among elements of the net: we say that x is *causally dependent* from y iff $y \leq_C x$. Given a causal net $C = \langle B, E, F, \mathsf{m}\rangle$, if $\forall b \in B$ it holds that $b^\bullet$ is at most a singleton, we say that it is a *conflict-free* causal net (the relation $\#$ is empty). Observe that each causal net $C = \langle B, E, F, \mathsf{m}\rangle$ is also an 1-occurrence net. Causal nets capture dependencies (and conflicts) whereas 1-occurrence nets capture the unique occurrence property of each transition. We define a net which will turn out to be, so to say, in between 1-occurrence and causal nets. Like in 1-occurrence nets we assure that each transition happens just once, and we are still able to retrieve dependencies among the firings of transitions, though in a more *semantical* way.

Definition 1. *An* unravel net $R = \langle B, E, F, \mathsf{m}\rangle$ *is a safe occurrence net such that for each state* $X \in \mathsf{St}(R)$ *the net* $R|_{[\![X]\!]}$ *is a conflict-free causal net.*

It is straightforward to observe that if $C = \langle B, E, F, \mathsf{m}\rangle$ is a causal net then it is an unravel net as well. The contrary does not hold (see the nets on the side). The net in (b) is an unravel net which is not a causal net, whereas the net (a) is a safe 1-occurrence net which is not an unravel one. We state some simple facts on unravel nets. First, as we consider nets where each transition can be executed at some marking, if $s \in \mathsf{m}$ then ${}^\bullet s = \emptyset$. Furthermore if two different transitions t and t' are such that ${}^\bullet t \cap {}^\bullet t' \neq \emptyset$ or $t^\bullet \cap t'^\bullet \neq \emptyset$, then the two transitions cannot appear in the same state of the net.



(a) (b)

Unravel nets are closely related to *bundle event structures* [13]. In this brand of event structure causality among events is represented by pairs (X, e), the *bundles*, where X is a non empty set of events and e an event. The meaning of a bundle (X, e) is that if e happens then one (and only one) event of X has to have happened before (events in X are pairwise conflicting). An event e can be caused by several bundles, and for each bundle an event in it should have happened.

Definition 2. *A* bundle event structure *is a triple* $\beta = (\mathsf{E}, \mapsto, \#)$*, where (a)* E *is a set of events, (b)* $\#$ *is an irreflexive and symmetric binary relation on* E *(the* conflict *relation), (c)* $\mapsto \subseteq \mathbf{2}^{\mathsf{E}}_{fin} \times \mathsf{E}$ *is the* enabling *relation such that if* $X \mapsto \mathsf{e}$ *then for all* $\mathsf{e}_1, \mathsf{e}_2 \in X$. $\mathsf{e}_1 \neq \mathsf{e}_2$ *implies* $\mathsf{e}_1 \# \mathsf{e}_2$*, and (d) for each* $\mathsf{e} \in \mathsf{E}$ *the set* $\{X \subseteq \mathsf{E} \mid X \mapsto \mathsf{e}\}$ *is finite.*

The configurations of a BES are defined as usual. Let $\beta = (\mathsf{E}, \mapsto, \#)$ be a BES and $X \subseteq \mathsf{E}$ be a set of events. Then X is a *configuration* or β iff (a) it is *conflict* *free*, *i.e.* $\forall \mathsf{e}, \mathsf{e}' \in X.\ \neg(\mathsf{e} \ \#\mathsf{e}')$, and (b) there exists a linearization $\{\mathsf{e}_1, \dots, \mathsf{e}_n, \dots\}$ of the events in X such that $\forall i \in \mathbb{N}$ and for all bundles $X_i \mapsto \mathsf{e}_i$ it holds that $X_i \cap \{\mathsf{e}_1, \dots, \mathsf{e}_{i-1}\} \neq \emptyset$. The set of configurations of a BES β is denoted with $\mathsf{Conf}(\beta)$.

Given an unravel net $R = \langle B, E, F, \mathsf{m}\rangle$, the associated BES is the triple $\mathcal{E}_{\text{BES}}(R) = (E, \mapsto, \#)$ where $\mapsto$ is defined taking, for each $b \in {}^{\bullet}e$, the set of events ${}^{\bullet}b$ (thus ${}^{\bullet}b \mapsto e$) and the conflict relation $\#$ is defined as $e \ \# \ e'$ iff $\forall X \in \mathsf{St}(R).\ \{e, e'\} \not\subseteq [\![X]\!]$. The configurations of the BES associated to an unravel net are precisely the states of the unravel net: $\mathsf{Conf}(\mathcal{E}_{\text{BES}}(R)) = \mathsf{St}(R)$. Furthermore, given a BES $\beta = (\mathsf{E}, \mapsto, \#)$, we can associate an unravel net $\mathcal{N}(\beta) = \langle B, \mathsf{E}, \mathrm{F}, \mathsf{m}\rangle$ where $B = \{(\mathsf{e}, i) \mid \mathsf{e} \in E\} \cup \{(\mathsf{e}, \mathsf{e}') \mid \mathsf{e} \ \# \ \mathsf{e}'\} \cup \ \{(Y, \mathsf{e}) \mid Y \mapsto \mathsf{e}\}$, $(s, \mathsf{e}) \in \mathrm{F}$ if $s = (\mathsf{e}, i)$ or $s = (\mathsf{e}, \mathsf{e}')$ or $s = (\mathsf{e}', \mathsf{e})$ or $s = (Y, \mathsf{e})$; and $(\mathsf{e}, s) \in \mathrm{F}$ if ($s = (Y, \mathsf{e}')$ and $\mathsf{e} \in Y$) or $s = (\mathsf{e}, o)$, and $\mathsf{m} = \{(\mathsf{e}, i) \mid \mathsf{e} \in E\} \cup \{\{\mathsf{e}, \mathsf{e}' \mid \mathsf{e} \ \# \ \mathsf{e}'\}$ and, as before $\mathsf{Conf}(\beta) = \mathsf{St}(\mathcal{N}(\beta))$. Thus among unravel nets and BES there is a similar relationship as the one we have among causal nets and *prime event structure*.

We introduce now labeled nets. Let Λ be a set of labels, a *labeled* net N is the pair (N, l), where $N = \langle S, T, F, \mathsf{m}\rangle$ is a Petri net and $l\colon S \cup T \to \Lambda$ a total mapping such that $l(T) \cap l(S) = \emptyset$. Given a labeled net $\mathsf{N} = (N, l)$ and a fs $\sigma = m\,[t_1\rangle\, m_1\,[t_2\rangle\, m_2 \cdots m_{n-1}\,[t_n\rangle\, m_n$, with $run(\sigma)$ we denote the word on $l(T)^*$ defined as $l(t_1 t_2 \cdots t_n)$, and we call it *trace*. To the fs $\sigma = m$ the empty trace is associated, *i.e.* $run(\sigma) = \epsilon$. The length of a fs σ is the length of $run(\sigma)$.

Let $\mathsf{N} = (N, l)$ be a labeled net where $N = \langle S, T, F, \mathsf{m}\rangle$ and let $t, t' \in T$ be two transitions. We say that t and t' are *identifiable* whenever ${}^{\bullet}t = {}^{\bullet}t'$, $t^{\bullet} = t'^{\bullet}$ and $l(t) = l(t')$. Thus on T it is possible to define an equivalence relation $\simeq$ such that $t \simeq t'$ iff t and t' are identifiable. The set of transitions can be quotiented through this equivalence relation obtaining the set $\{[t]_{\simeq} \mid t \in T\}$. Observe that the transitions in $[t]_{\simeq}$ are pairwise conflicting.

Definition 3. *Let $(N, l) = (\langle S, T, F, \mathsf{m}\rangle, l)$ be a labeled net and let $\simeq$ be the equivalence relation induced by transitions identifiability. Then we can construct the labeled net $\widehat{\mathsf{N}} = (\widehat{N}, \hat{l})$ where $\widehat{N}$ is the Petri net $\langle S, \widehat{T}, \widehat{F}, \mathsf{m}\rangle$ with $\widehat{T} = \{[t]_{\simeq} \mid t \in T\}$, $\widehat{F}(s, [t]_{\simeq}) = F(s, t)$ and $\widehat{F}([t]_{\simeq}, s) = F(t, s)$, and $\hat{l}$ is the mapping defined as $\hat{l}(s) = l(s)$ and $\hat{l}([t]_{\simeq}) = l(t)$.*

Let $\sigma \in \mathbf{\Sigma}^{N}_{\mathsf{m}}$, then $\widehat{\sigma}$ is constructed as follows: $\widehat{\sigma} = \mathsf{m}$ if $\sigma = m$ and $\widehat{\sigma} = \widehat{\sigma'}\ [[t]_{\simeq}\rangle\, m$ if $\sigma = \sigma'\ [t\rangle\, m$. The firing sequences of N and of $\widehat{N}$ are clearly related, as the following proposition shows. Observe that $run(\widehat{\sigma}) = l([t_1]_{\simeq} \cdots [t_n]_{\simeq}) = run(\sigma)$.

Proposition 1. *Let $(N, l) = (\langle S, T, F, \mathsf{m}\rangle, l)$ be a labeled net, and let $\simeq$ be the equivalence relation induced by transitions identifiability. Let $\sigma \in \mathbf{\Sigma}^{N}_{\mathsf{m}}$ be a fs, then $\widehat{\sigma} \in \mathbf{\Sigma}^{\widehat{N}}_{\mathsf{m}}$.*

3 Merging relation

In this section we propose a simple and general framework to compact labeled nets. We first introduce a *semantical* notion of incompatibility on places, capturing the idea that, if a place is akin to a *resource*, two resources are incompatible if they never appear in the same computation, even at different stages. Then we show that, given a suitable equivalence relation, related to places incompatibility, a more succinct version of the net we started with can be obtained, whose behaviors are still related to the original one.

Given a net $N = \langle S, T, F, \mathsf{m} \rangle$, we say that two different places $s, s' \in S$ are *incompatible* iff for each firing sequence $\sigma \in \mathbf{\Sigma}^N_{\mathsf{m}}$ it holds that $\{s, s'\} \not\subseteq \mathsf{P}(\mathsf{M}(\sigma))$, and we denote it with $s \bowtie s'$ (observe that this notion is quite similar to the one of conflict we introduced on unravel nets). Clearly $\bowtie$ is a symmetric and irreflexive relation and if $s \in [\![m]\!]$ and $s \bowtie s'$ then $\forall \sigma \in \mathbf{\Sigma}^N_m.\ s' \not\in \mathsf{P}(\mathsf{M}(\sigma))$.

Example 1. Consider the labeled net $\mathsf{N} = (N, l)$ in Fig. 2(a), with initial marking $\mathsf{m} = \{c_0\}$. The relation $\bowtie$ contains the pairs (c_i, c_j) such that $i, j > 0$ and if i is odd then j is even and vice versa as well. Thus $c_4 \bowtie c_7$ and $c_9 \bowtie c_6$ but $c_6 \not\bowtie c_4$.

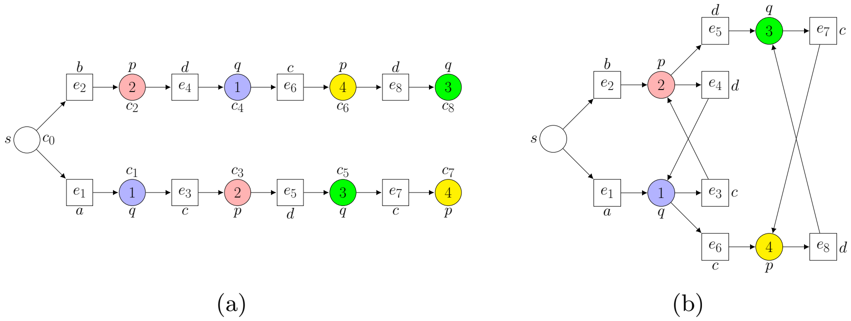
We introduce now the notion of *merging relation*.

Definition 4. *Let* $\mathsf{N} = (N, l)$ *be a labeled net where* $N = \langle S, T, F, \mathsf{m} \rangle$*, let* $\circledast\ \subseteq \bowtie$*, and let* $\sim$ *be an equivalence relation such that* $s \sim s' \iff (s \circledast s' \vee s = s') \wedge l(s) = l(s')$*. Then* $\sim$ *is a* merging relation *for* N.

A merging relation is any equivalence relation respecting labeling and incompatibility (better, a relation included into the one of incompatibility). Observe that the identity on places is a trivial merging relation. Furthermore if s is initially marked then $[s]_\sim = \{s\}$.

The *merging* relation is used to *compact* the net. Similarly to what is done in [8], we first merge places by identifying equivalent ones, thus the merged places will be $S' = \{[s]_\sim \mid s \in S\}$. Then, when needed, we may identify also transitions.

Definition 5. *Let* $\mathsf{N} = (N, l)$ *be a labeled unravel net where* $N = \langle S, T, F, \mathsf{m} \rangle$*, and let* $\sim$ *be a merging relation. Then we construct the labeled net* $\widetilde{\mathsf{N}} = (\widetilde{N}, \widetilde{l})$*, where* $\widetilde{N}$ *is the Petri net* $\langle \widetilde{S}, T, \widetilde{F}, \widetilde{\mathsf{m}} \rangle$ *defined as* $\widetilde{S} = \{[s]_\sim \mid s \in S\}$*,* $\widetilde{F}([s]_\sim, t) = F(s, t)$*,* $\widetilde{F}(t, [s]_\sim) = F(t, s)$ *and* $\widetilde{\mathsf{m}}([s]_\sim) = \sum_{s \in [s]_\sim} \mathsf{m}(s)$*, and* $\widetilde{l}$ *is the labeling mapping defined as* $\widetilde{l}([s]_\sim) = l(s)$ *and* $\widetilde{l}(t) = l(t)$.



(a) (b)

Fig. 2: A labeled net N (a) and its compact representation (b).

The flow relation is well defined, as $\forall t \in T.\ |[\![{}^\bullet t]\!] \cap [s]_\sim| \leq 1$ and $\forall t \in T.\ |[\![t^\bullet]\!] \cap [s]_\sim| \leq 1$ as well, and the same for the initial marking, as the equivalence class of each place in the initial marking contains just that place.

Example 2. Consider the net of the Ex. 1. A suitable merging relation can be $c_1 \sim c_4$, $c_2 \sim c_3$, $c_6 \sim c_7$ and $c_5 \sim c_8$ and the result of the merging of these places is the net in Fig. 2(b). Another merging relations could be $c_1 \sim' c_8$, $c_2 \sim' c_3$, $c_6 \sim' c_7$ and $c_5 \sim' c_4$, or simply $c_1 \sim'' c_4$.

The construction can be lifted to the reachable markings and firing sequence. Let $m \in \mathcal{M}_N$, then $\widetilde{m} \in \mu\widetilde{S}$ is defined as $\widetilde{m}([s]_\sim) = \sum_{s \in [s]_\sim} m(s)$. Observe that, as places in $[s]_\sim$ are in conflict, at most one may contain tokens. Consider then $\sigma \in \mathbf{\Sigma}^N_{\mathsf{m}}$, then $\widetilde{\sigma}$ is obtained as follows: if $\sigma = \mathsf{m}$ then $\widetilde{\sigma} = \widetilde{\mathsf{m}}$, if $\sigma = \sigma' \, [t\rangle \, m$ then $\widetilde{\sigma} = \widetilde{\sigma'} \, [t\rangle \, \widetilde{m}$. The following proposition points out the obvious relation among the firing sequences of both nets.

Proposition 2. *Let $\mathsf{N} = (N, l)$ be a labeled net, let $\sim$ be a merging relation and $(\widetilde{N}, \widetilde{l})$ be the labeled net of Def. 5. Then $\forall \sigma \in \mathbf{\Sigma}^N_{\mathsf{m}}\ \exists \sigma' \in \mathbf{\Sigma}^{\widetilde{N}}_{\widetilde{\mathsf{m}}}.\ \widetilde{\sigma} = \sigma'$.*

By merging places it may happen that two equally labeled transitions have the same preset and the same postset. Hence the equivalence relation $\simeq$ induced by transition identifiability may be non trivial, *i.e.* different from the identity. We can then apply the construction of Def. 3.

Proposition 3. *Let $\mathsf{N} = (N, l)$ be a labeled net and let $\sim$ be a merging relation. Let $(\overline{N}, \overline{l})$ be the labeled net obtained applying first the construction of Def. 5 and then the one of Def. 3. Then $\forall \sigma \in \mathbf{\Sigma}^N_{\mathsf{m}}\ \exists \sigma' \in \mathbf{\Sigma}^{\overline{N}}_{\overline{\mathsf{m}}}.\ run(\sigma) = run(\sigma')$.*

By merging places and transitions we do not lose any behavior, but the obtained net may have more behaviors with respect to the one we started with, as the net in Fig. 2(b) shows.

4 Compacting causal behaviors of safe nets

The framework we have devised in the previous section can be applied to any kind of labeled net representing the behavior of a Petri net. Here we test it by focussing on the causal behaviors of *safe* nets, represented as branching processes.

Given a Petri net $N = \langle S, T, F, \mathsf{m} \rangle$, a *branching process* of N (see [4] and [5]) is a labeled causal net $\mathsf{C} = (C, p)$ with $C = \langle B, E, F, \mathsf{m}_0 \rangle$ and $p \colon B \cup E \to S \cup T$ is such that (a) $p(B) = S$, $p(E) = T$, (b) the initial markings m_0 and m are bijectively related and (c) there are bijections between $p({}^\bullet e)$ and ${}^\bullet p(e)$, and between $p(e^\bullet)$ and $p(e)^\bullet$. Such kind of labeling mapping is called *folding* morphism. We apply our framework to merged processes (see [8] for details). The idea behind merged processes is the following: the conditions representing the *same* occurrence of a token in the original net can be identified. The consequence of this identification is that all the computations producing the same set of *tokens* can be considered as equivalent with respect to their possible futures, *i.e.* all these computations share equivalent futures that can be identified. A conflict relation is easily and structurally identifiable on causal nets, without resorting to the semantics, thus we do need some more information to identify conditions in the causal net. Let $\mathsf{C} = (C, p)$ be a branching process of N. Given a condition $b \in B$, the *occurrence depth* of b is $|\{b' \in B \mid b' \leq_C b \textit{ and } p(b) = p(b')\}|$ and it is denoted with $\mathsf{tok}(b)$. We are now ready to introduce the merging relation for merged processes: two conditions b and b' are equivalent if (a) they are in conflict, (b) they have the same labeling, and (c) they have the same occurrence depth. The fact that our unifying framework covers the notion of merged process is summarized by the following theorem.

Theorem 1. *Let $\mathsf{C} = (C, p)$ be branching process of N, and let $\sim_{tok}$ be the merging relation defined as follows: $b \sim_{tok} b'$ iff $b \natural b'$, $p(b) = p(b')$ and $\mathsf{tok}(b) = \mathsf{tok}(b')$, where $\natural$ is the reflexive closure of the conflict relation $\#$ of the causal net C restricted to conditions only. Then $\overline{\mathsf{C}} = (\overline{C}, \overline{p})$ is a merged process of N.*

Example 3. Consider the net N in Fig. 3 (a). A branching process for N is the labeled net $\mathsf{C} = (C, p)$ in Fig. 3 (b) and the merging relation is $c_4 \sim c_7$ (both conditions have token occurrence equal to 1 and they are labeled with p_4) and $c_5 \sim c_6$ (both conditions have token occurrence equal to 1 and they are labeled with p_5). The result of the identification of these conditions gives the net $\overline{\mathsf{C}} = (\overline{C}, \overline{p})$ (c), depicted in Fig. 3. Observe that the last step is irrelevant in this case as no transitions have to be identified. The net C is acyclic whereas the $\overline{C}$ is not. Furthermore $\overline{C}$ is not an unravel net.

Instead of the token occurrence, the *time* can be taken into account. We identify conditions when they have been produced at the same time, still provided that they bear the same label. To formalize this idea, on which the notion of *trellis* process is based (see [10]), we do need to guarantee that the proper time can be identified for each condition. To do so we resort to *multi-clock* nets. A multi-clock net is a safe net $N = \langle S, T, F, \mathsf{m} \rangle$ equipped with a *partition* mapping $\nu \colon S \to [\![\mathsf{m}]\!]$ such that $\nu([\![\mathsf{m}]\!])$ is the identity and $\forall s, s' \in [\![\mathsf{m}]\!]\ \nu^{-1}(s) \cap \nu^{-1}(s') \neq \emptyset \ \Rightarrow \ s = s'$. Thus a net can be partitioned in a number of components (which are identified

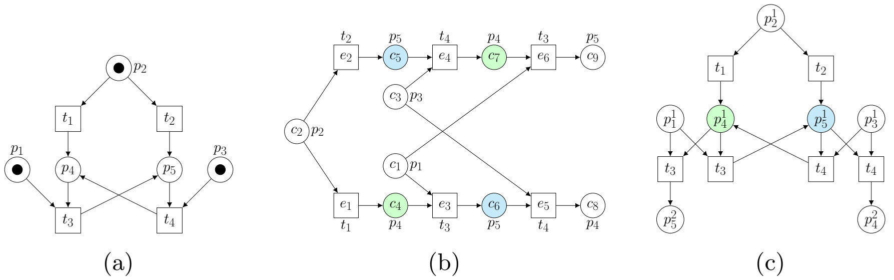

p_2
t_1
t_2
p_1
p_3
p_4
p_5
t_3
t_4
t_2 e_2 p_5 c_5 t_4 e_4 p_4 c_7 t_3 e_6 p_5 c_9
c_3 p_3
c_2 p_2
c_1 p_1
e_1 t_1 c_4 p_4 e_3 t_3 c_6 p_5 e_5 t_4 c_8 p_4
p_2^1
t_1 t_2
p_1^1 p_4^1 p_5^1 p_3^1
t_3 t_3 t_4 t_4
p_5^2 p_4^2
(a)
(b)
(c)

Fig. 3: A labeled net N (a), its branching process (b) and its merged process (c).

using the partition mapping) which synchronize on some common transitions, and each component is a finite state automata which has just a place initially marked. A *trellis* process of the multi-clock net $N = \langle S, T, F, \mathsf{m}\rangle$ with ν as partition mapping, is a labeled net $\mathsf{R} = (R, p)$, where (a) $R = \langle B, E, F', \mathsf{m}'\rangle$ is an unravel net, (b) p is a folding morphism, and (c) R is a multi-clock net under the partition mapping ν_R defined as $\nu_R(b) = \nu(p(b))$; furthermore for each $s \in [\![\mathsf{m}]\!]$ any subnet $R|_{[p^{-1}(s)]_\nu}$ is acyclic, where $[p^{-1}(s)]_{\nu_R} = \{b \in B \mid \nu_R(b) = \nu(s)\}$. For further details on trellises and multi-clock nets we refer to [10].

Let $\mathsf{C} = (C, p)$ be a branching process of the multi-clock N, with ν as partition mapping. Given a condition b of C, the *height* of b, denoted with $\mathsf{height}(b)$, is $|\{b' \in B \mid b' \leq_C b \textit{ and } \nu(p(b)) = \nu(p(b'))\}|$, where B are the conditions of C. The height of a condition is well defined in a casual net which is a multi-clock as well, as in the case of branching processes arising from multi-clock nets.

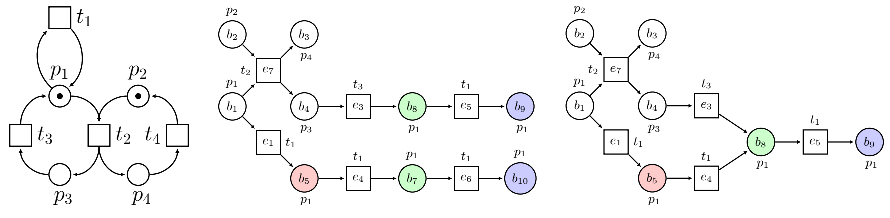
Theorem 2. *Let $\mathsf{C} = (C, p)$ be branching process of N, and let $\sim_{tr}$ be the merging relation defined as follows: $b \sim_{tr} b'$ iff $b \sharp b'$, $p(b) = p(b')$ and $\mathsf{height}(b) = \mathsf{height}(b')$, where $\sharp$ is the reflexive closure of the conflict relation $\#$ of the causal net C restricted to conditions only. Then $\overline{\mathsf{C}} = (\overline{C}, \overline{p})$ is a trellis process of N.*

Example 4. Consider the net N in Fig. 4(a). It is a multi-clock net considering $\nu(p_3) = p_1$ and $\nu(p_4) = p_2$. The equivalence induced by the height of the conditions in the branching process in Fig. 4(b) is $b_7 \sim b_8$ and $b_9 \sim b_{10}$, whereas $b_5 \not\sim b_8$ as they have a different height though they have the same label. The merging can be applied and the result, which is depicted in Fig. 4(c), is the trellis process associated to this specific branching process. This is an unravel net.

We conclude this section observing that the merging relation is induced assigning to each place a *measure*, and stipulating that two equally labeled and incompatible places are *equivalent* whenever they have the same measure.

5 Enriching to obtain unravel nets

When compacting behaviors we usually start from a labeled unravel net, but the produced net is not necessarily an unravel one, and in the compaction we may



(a) (b) (c)

Fig. 4: A multi-clock net (a) one of its branching processes (b) and the corresponding trellis process (c).

lose the tight correspondence among nets and event structures. In fact cycles may be introduced when merging places and transitions, and in some case the cycles may be executable (thus the associated event structure would have a configuration where dependencies are not a partial order).

The possibility of obtaining an unravel net is connected with the measure, associated to places, the merging relation is based upon. Consider a labeled unravel net $\mathsf{R} = (R, l)$, with $R = \langle S, T, F, \mathsf{m}\rangle$, and a *locality* mapping $loc : S \to \mathsf{Loc}$, where Loc is a finite set of *localities*, and take any *measure* $\delta : S \to \mathbb{N}$ such that $\delta(s) = \delta(s')$ implies $s \bowtie s'$. We say that δ is *strictly increasing* if for each state $X \in \mathsf{St}(R)$, and each pair of places s, s' of the net $R|_{[\![X]\!]}$, it holds that $s \leq s' \ \wedge \ loc(s) = loc(s') \ \Rightarrow \ \delta(s) < \delta(s')$.

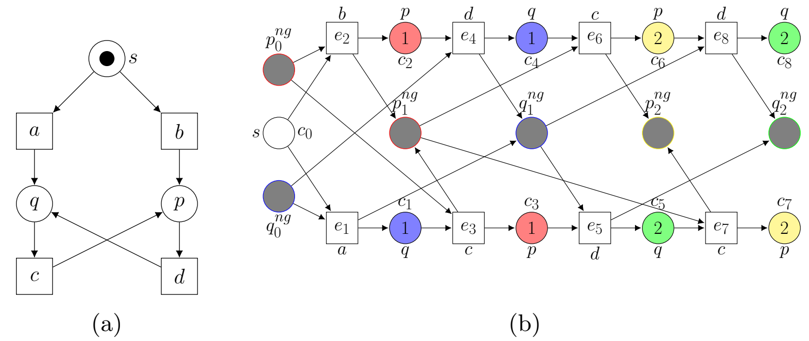
We can prove the following theorem, stating that the property of being an unravel net is preserved when strictly increasing measure are considered.

Theorem 3. *Let $\mathsf{R} = (R, l)$ where $R = \langle S, T, F, \mathsf{m}\rangle$ be a labeled unravel net and $loc : S \to \mathsf{Loc}$ a locality mapping, let $\delta : S \to \mathbb{N}$ be* strictly increasing *and let $\sim$ be the equivalence relation induced by δ,* i.e. *$s \sim s'$ iff $(\delta(s) = \delta(s')$ and $l(s) = l(s'))$ or $s = s'$. Then the resulting compact labeled net $\overline{\mathsf{R}} = (\overline{R}, \bar{l})$ is a labeled unravel net.*

The measure defined for trellis processes is a strictly increasing one, the locality mapping being the one induced by the partition mapping. Hence we can consider Th. 2 as a special case of the Th. 3 above. When the measure is not of this kind we may still obtain an unravel net, but sometimes at the price of *enriching* it in order to forbid certain unwanted executions in the compact version.

Given a labeled unravel net $\mathsf{R} = (R, l)$ with $R = \langle S, T, F, \mathsf{m}\rangle$, we say that a measure $\delta : S \to \mathbb{N}$ is *homogeneous* iff for each $X \in \mathsf{St}(R)$ and each subset of places $\hat{S}$ of $R|_{[\![X]\!]} = \langle S', [\![X]\!], F', \mathsf{m}\rangle$ such that there exists a label a such that $l^{-1}(\mathsf{a}) \cap S' = \hat{S}$, it holds that $\hat{S}'$ can be totally ordered with respect to the reflexive and transitive closure of F', $\delta(\hat{S}') = \{1, \ldots, |\hat{S}'|\}$ and $s \leq s' \ \Rightarrow \ \delta(s) \leq \delta(s')$. An homogeneous measure on causal nets is the token count of merged process. Again, as done before, we may introduce an equivalence relation which is based on an homogeneous measure δ by stipulating that $s \sim s'$ iff either $s = s'$ or $(l(s) = l(s') \ \wedge \ \delta(s) = \delta(s'))$. As the measure is an homogenous one, we do not have to require that the two places are incompatible as it is implied by the definition of the measure itself. When compacting using this merging relation the result may be not an unravel net. However this net may be turned into an unravel one without losing behaviors of the original net by adding some places which solely purpose is to forbid unwanted executions.

Take an unravel net $\mathsf{R} = (\langle S, T, F, \mathsf{m}\rangle, l)$ and an homogeneous measure δ on S. We can add to the net $R = \langle S, T, F, \mathsf{m}\rangle$ a set of places $S_{ng} = \{(l(s), \delta(s), ng) \mid s \in S\} \cup \{(l(s), 0, ng) \mid s \in S \setminus [\![\mathsf{m}]\!]\}$, and connect them to the transitions in T as follows: $F_{ng}((l(s), n, ng), t) = 1$ whenever $\exists s' \in t^{\bullet}.\ l(s) = l(s')$ and $\delta(s') = n + 1$, and $F_{ng}(t, (l(s), n, ng)) = 1$ whenever $\exists s' \in t^{\bullet}.\ l(s) = l(s')$ and $\delta(s') = n$; finally the places $(l(s), 0, ng)$ are initially marked as well as $(l(s), 1, ng)$ if $b \in \mathsf{m}$ (and are the multiset m_{ng}). We call these places *no-gap* as in the case that the δ is



(a) (b)

Fig. 5: A labeled net N (a) and one of its enriched branching process $\mathsf{Ng}(C)$ (b).

precisely the token count and $\mathsf{R} = (R, l)$ is a branching process of a safe net N, they assure that the tokens in a place of the original net N are *produced* in the proper sequence.

The net obtained adding these new places S_{ng}, namely $\mathsf{Ng}(R) = \langle S \cup S_{ng}, T, F \cup F_{ng}, \mathsf{m} + \mathsf{m}_{ng}\rangle$, is an unravel net such that to each fs σ of $\mathsf{Ng}(R)$ a fs σ' of R corresponds and they are such that $run(\sigma) = run(\sigma')$ but also the *vice versa* holds, thus to each fs $\hat{\sigma} \in \boldsymbol{\Sigma}^{R}_{\mathsf{m}}$ a fs $\hat{\sigma}' \in \boldsymbol{\Sigma}^{\mathsf{Ng}(R)}_{\mathsf{m}+\mathsf{m}_{ng}}$ corresponds such that $run(\hat{\sigma}) = run(\sigma)'$, hence both unravel nets have exactly the same states, which means that this enriching does not change the behaviors of the net.

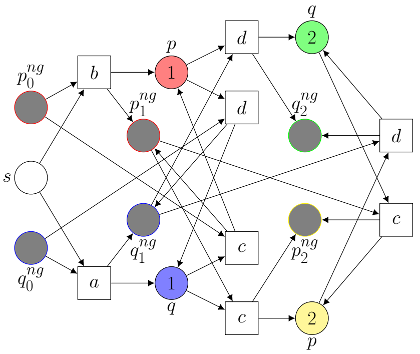


Fig. 6: The resulting unravel net of the compaction of $\mathsf{Ng}(C)$ in Fig. 5.

Theorem 4. *Let $\mathsf{R} = (R, l)$ be a labeled unravel net, where $R = \langle S, T, F, \mathsf{m}\rangle$. Let $\mathsf{Ng}(\mathsf{R}) = (\mathsf{Ng}(R), \mathsf{Ng}(l))$ where $\mathsf{Ng}(l)(s) = l(s)$ if $s \in S$ and $l((s, n, ng)) = l(n)$ obtained with respect to an homogeneous measure δ. Let $\sim$ be the equivalence relation induced by this measure on the places in S. Then $\overline{\mathsf{Ng}(\mathsf{R})} = (\overline{\mathsf{Ng}(R)}, \overline{\mathsf{Ng}(l)})$ is an unravel net.*

As a corollary of this theorem we have the following.

Corollary 1. *Let $\mathsf{C} = (C, p)$ be a branching process of the safe net N, where $C = \langle B, E, F, \mathsf{m}\rangle$. Let $\mathsf{Ng}(\mathsf{C}) = (\mathsf{Ng}(C), \mathsf{Ng}(p))$ be the unravel net obtained applying Ng to C and p and consider the equivalence relation $\sim$ induced by $\sim_{tok}$, where $\sim_{tok}$ is defined on conditions in B. $\overline{\mathsf{Ng}(\mathsf{C})} = (\overline{\mathsf{Ng}(C)}, \overline{\mathsf{Ng}(p)})$ in a labeled unravel net and furthermore $\mathsf{Ng}(\mathsf{C})|_{B_{\sim_{tok}}}$ is a merging process of N, where $B_{\sim_{tok}}$ are the merged resource conditions.*

Example 5. The causal net in Fig. 2(a), with the appropriate labeling, is a branching process of the net in Fig. 5(a). The net in Fig. 5(b) is the result of enriching this causal net with no-gap conditions (the black ones). For instance, the no-gap condition p_1^{ng} is $(p, 1, ng)$ and the transitions (events) putting a token in it are e_2 and e_3, whereas the events e_6 and e_7 consume the token in it. Clearly $e_2 \# e_3$ and $e_6 \# e_7$ The conditions with the same color have the same token count (the number depicted in the conditions) and are equally labeled. The result of its compaction is the unravel net in Fig. 6.

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