Continuous transition between traveling mass and traveling oscillator using mixed variables

Antonio Cazzani[a,*], Nils Wagner[b], Peter Ruge[a,c], Flavio Stochino[a,d]

[a] *University of Cagliari, DICAAR — Dept. of Civil and Environmental Engineering and Architecture, via Marengo, 2, I-09123 Cagliari, Italy*
[b] *Intes GmbH, Schulze-Delitzsch-Straße 16, D-70565 Stuttgart, Germany*
[c] *Technische Universität Dresden, Institut Statik und Dynamik der Tragwerke, Schumann-Straße 10, D-01062 Dresden, Germany*
[d] *University of Sassari, DADU — Dept. of Architecture, Design and Urban Planning, Asilo Sella, via Garibaldi, 35, I-07041 Alghero (SS), Italy*

Abstract

The interaction between cars or trains and bridges has been often described by means of a simplified model consisting of a beam loaded by a traveling mass, or by a traveling oscillator.
Among others, two aspects are essential when dealing with masses traveling along flexible vibrating supports: (*i*) a complete relative kinematics; and (*ii*) a continuous transition between a traveling mass, rigidly coupled, and a traveling oscillator, elastically coupled with the support.
The kinematics is governed by normal and tangential components —with respect to the curved trajectory— of the acceleration. However in literature these parts are oriented with reference to the undeformed beam configuration. This model is improved here by a nonlinear second-order enriched contribution.
The transition between a traveling oscillator and a traveling mass is governed by the stiffness k of the elastic or viscoelastic coupling which, in the latter case (i.e. rigid coupling), has to tend towards infinity.
However, very large stiffness values cause high frequencies and significant problems are mentioned in literature in order to establish numerically stable

*Corresponding author. Tel: +39–070–6755420; Fax: +39–070–6755418.
Email addresses: antonio.cazzani@unica.it (Antonio Cazzani), nils.wagner@intes.de (Nils Wagner), peter.ruge@tu-dresden.de (Peter Ruge), fstochino@uniss.it (Flavio Stochino)

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables"*International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017

and reliable results and in order to realize a continuous evolution between absolute and relative formulations.
By using mixed state variables, generalized displacements and coupling forces, the contribution from the stiffness changes from k to its inverse $1/k$, the coupling force itself becomes a member of the solution-space and the problems, which have been mentioned in literature, disappear. As a matter of fact, the coupling force can also take into account a viscoelastic contribution; moreover, a larger number of traveling oscillators can be considered, too.
Finally, for a periodic sequence of moving oscillators the dynamic stability is treated in the time-domain along several periods, as well as in the spectral domain, by using Floquet's theorem.

Keywords: Mixed variables, Traveling mass-traveling oscillator, Curved beam, Floquet's theorem, Consistent kinematics, Dynamic stability.

1 Introduction

The body of literature devoted to traveling oscillators is large. State-of-the-art overviews are available from Ouyang [1] and Au *et al.* [2]. The classical treatment concentrates on finding the critical constant velocity which leads to a continuous increase of deformations if a sequence of masses crosses the beam/bridge.

Well-known studies on this dynamic stability problem have been presented by Bolotin [3], Fryba [4], Luongo [5], [6], [7], [8] and Piccardo and his coworkers [9], [10], [11], [12], [13], [14].

In order to avoid dynamic instability, the use of piezoelectric actuators can be effective: some applications to beams and plates are shown in [15], [16], [17], [18].

A group of recently published papers presents a discussion whether absolute or relative formulations should be used [19]; deals with the equivalence of the moving mass and moving oscillator problems [20], [21]; and gives attention to the dynamic stability if a sequence of oscillators crosses the supporting structure, which has been already deformed by the foregoing oscillators [22].

Another group of papers [23], [24] deals with more sophisticated models for both, bridge and vehicles, which are represented as an assembly of rigid bodies, springs and dampers. Sometimes, in addition, the bridge is modeled as a continuous in-plane curved beam or the wheel-rail interaction is of special concern: see, for instance [25], [26], [27], [28], [29], [30].

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables"*International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017

Plates and beams on generalized foundations subjected to moving loads are treated in several papers like [31], [32], [33] including tensionless foundations and special models for finite and infinite soil domains using Boundary Elements, Infinite elements or the Scaled Boundary Finite Element Method.

The analysis and control of bridges with traveling masses has been studied, too, during earthquakes: several papers are listed in the above-mentioned review [1].

Finally, local and global changes of the supporting structure due to abrupt changes in the bridge-railway interface or due to separation and impact-reattachment have been treated in [34], [35], [36].

The first aspect of this paper concerns the introduction of mixed state variables, generalized displacements and coupling forces, for the description of the traveling oscillator. In doing so, a continuous transition between a traveling mass and a traveling oscillator can be established without incurring numerical problems due to high frequencies caused by stiffness coefficients tending towards infinity. The key idea behind this new approach is to change from the stiffness k to its inverse $1/k$; in doing so, the coupling force automatically appears as an additional member of the solution-space.

However, to prepare a common basis for the description of the deformations and the total acceleration of the oscillator, Section 2 describes in a detailed way the model of a circle-like beam in a horizontal plane. Thus, the straight beam case is rigorously recovered when the curvature radius R tends toward infinity; practically, it is included in the presented model when this radius is significantly increased.

In Section 3 the kinematic model of the traveling mass is presented and discussed. Section 4 is devoted to the second innovative aspect of this paper: a consistent representation of the normal and tangential acceleration with respect to the curved trajectory of the traveling mass or oscillator. These accelerations are part of the second total derivative $\mathrm{d}^2\mathbf{x}/\mathrm{d}t^2$ of the position vector $\mathbf{x}$ of the traveling coupling point: up to now, the classical formulation shows that the normal acceleration is applied along the vertical direction (which is a first-order approximation), and not along the normal to the supporting curve, which is bent by strains, and constitutes the actual trajectory traveled by the coupling point. Here, a nonlinear second-order theory for the description of the position vector is introduced, which results in a normal acceleration correctly oriented along the normal vector of the deformed beam axis. For the sake of simplicity, the procedure is applied here only in the case of an initially straight beam.

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables"*International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017

In order to restrict the amount of variables and to concentrate on the benefits from the mixed formulation, the shear deformations due to shear-forces are neglected, and the system of partial differential equations in space and time is solved in Section 5 by means of a semi-analytical approach using the sinusoidal modal space of the curved beam. Thus, the beam is assumed to be a simply supported one with zero vertical displacements and torsional rotations at both boundaries.

The resulting time-variant ordinary differential equations are presented in Section 6 and then solved in Section 7 by means of a linear interpolation of all relevant quantities. The evaluation of Floquet's theorem is prepared in Section 8, while in Section 9, some typical results are presented and discussed. Finally, in Section 10, some conclusions are drawn.

2 Formulation of the curved beam model

The curved beam/bridge is assumed to have a circle-like shape in the horizontal plane, spanned by the (fixed) Cartesian unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and perpendicular to the vertical unit vector $\mathbf{e}_3$.

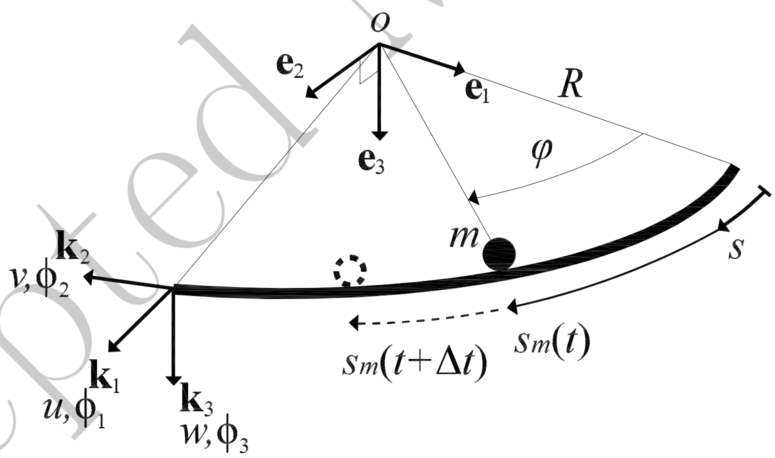


Figure 1: Circle-like beam lying in a horizontal plane.

2.1 Basic definitions

The coordinates and displacements of the beam points are defined locally by the following variables, clearly depicted in Figures 1–2:

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables"*International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017

- R: radius, referred to the curved beam axis, i.e. to the cross-section centroid, G.
- r: radial position of a point, P, measured from the center of curvature O.
- φ: angular position of a point, measured from the center of curvature O.
- $s = R\varphi$: coordinate in circumferential direction along the curved beam axis.
- ξ: coordinate in the cross-section of the beam measured in radial direction from the centroid, G; the corresponding radial position r is: $r = R + \xi$.
- z: coordinate in the cross-section of the beam measured in vertical direction from the centroid.
- u: displacement in radial direction, defined by the local axis $\mathbf{k}_1$.
- v: displacement in tangential direction, defined by the local axis $\mathbf{k}_2$.
- w: displacement in vertical direction, defined by local/global vertical axis $\mathbf{k}_3$.
- ϕ_1: rotation around the local radial axis $\mathbf{k}_1$.
- ϕ_2: rotation around the local tangential axis $\mathbf{k}_2$.
- ϕ_3: rotation around the local/global vertical axis $\mathbf{k}_3 = \mathbf{e}_3$.

2.2 Kinematics and strains

Using the cylindrical coordinates r, φ, z, for any point of the beam the position vector $\mathbf{x}$ can be described with respect to the fixed initial Cartesian base $\mathbf{E} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$:

$$\mathbf{x} = r\cos\varphi\,\mathbf{e}_1 + r\sin\varphi\,\mathbf{e}_2 + z\,\mathbf{e}_3 \tag{1}$$

Adopting the comma notation for derivatives, (i.e. $\mathbf{x}_{,r} = \dfrac{\partial \mathbf{x}}{\partial r}$, etc.), the total differential, d$\mathbf{x}$ of this position vector:

$$\mathrm{d}\mathbf{x} = \mathbf{x}_{,r}\,\mathrm{d}r + \mathbf{x}_{,\varphi}\,\mathrm{d}\varphi + \mathbf{x}_{,z}\,\mathrm{d}z \tag{2}$$

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables"*International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017

defines the unit vectors $\mathbf{k}_j$, with $j = 1, 2, 3$, of the local base $\mathbf{K}$:

$$\begin{aligned} \mathbf{k}_1 =& \mathbf{x},_r = \cos\varphi\, \mathbf{e}_1 + \sin\varphi\, \mathbf{e}_2, \\ \mathbf{k}_2 =& \frac{\mathbf{x},_\varphi}{r} = -\sin\varphi\, \mathbf{e}_1 + \cos\varphi\, \mathbf{e}_2, \\ \mathbf{k}_3 =& \mathbf{x},_z = \mathbf{e}_3. \end{aligned} \tag{3}$$

It should be noticed that, with the definitions of eq. (3), such local base consists again of three mutually orthogonal unit vectors.

On the other hand, the total differential d$\mathbf{x}$ of the above-mentioned position vector can also be expressed by means of the fixed base, $\mathbf{E}$, as:

$$\mathrm{d}\mathbf{x} = \mathbf{e}_1 \mathrm{d}x + \mathbf{e}_2 \mathrm{d}y + \mathbf{e}_3 \mathrm{d}z. \tag{4}$$

So, the components of d$\mathbf{x}$ which are different, when referred to the fixed base, $\mathbf{E}$, and to the local one, $\mathbf{K}$, can be compared:

$$\begin{aligned} \mathrm{d}x =& \cos\varphi \mathrm{d}r - \sin\varphi r \mathrm{d}\varphi \\ \mathrm{d}y =& \sin\varphi \mathrm{d}r + \cos\varphi r \mathrm{d}\varphi. \end{aligned}$$

Similarly, the displacement vector $\mathbf{u}$ may be described with reference to the global and to the local base:

$$\mathbf{u} = u_x\, \mathbf{e}_1 + u_y\, \mathbf{e}_2 + u_z\, \mathbf{e}_3 = u\, \mathbf{k}_1 + v\, \mathbf{k}_2 + w\, \mathbf{k}_3, \tag{5}$$

so that the following equalities hold:

$$\begin{aligned} u_x =& u\cos\varphi - v\sin\varphi \\ u_y =& u\sin\varphi + v\cos\varphi \\ u_z =& w. \end{aligned} \tag{6}$$

The total differential of the displacement, d$\mathbf{u}$ can be deduced in a way similar to eq. (2):

$$\mathrm{d}\mathbf{u} = \mathbf{u},_r\, \mathrm{d}r + \mathbf{u},_\varphi\, \mathrm{d}\varphi + \mathbf{u},_z\, \mathrm{d}z \tag{7}$$

It follows then, taking into account that, by eq. (3), $\mathbf{k}_1$ and $\mathbf{k}_2$ depend only on the angular coordinate φ:

$$\begin{aligned} \mathbf{u},_r &= (u\mathbf{k}_1 + v\mathbf{k}_2 + w\mathbf{k}_3),_r = (u,_r\, \mathbf{k}_1 + v,_r\, \mathbf{k}_2 + w,_r\, \mathbf{k}_3) \\ \mathbf{u},_\varphi &= (u\mathbf{k}_1 + v\mathbf{k}_2 + w\mathbf{k}_3),_\varphi = [(u,_\varphi - v)\mathbf{k}_1 + (v,_\varphi + u)\mathbf{k}_2 + w,_\varphi\, \mathbf{k}_3] \\ \mathbf{u},_z &= (u\mathbf{k}_1 + v\mathbf{k}_2 + w\mathbf{k}_3),_z = (u,_z\, \mathbf{k}_1 + v,_z\, \mathbf{k}_2 + w,_z\, \mathbf{k}_3), \end{aligned} \tag{8}$$

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables"*International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017

so that the complete expression of d**u** reads as:

$$\begin{aligned} \mathrm{d}\mathbf{u} =& (u_{,r}\,\mathbf{k}_1 + v_{,r}\,\mathbf{k}_2 + w_{,z}\,\mathbf{k}_3)\mathrm{d}r + \\ & (\frac{u_{,\varphi} - v}{r}\mathbf{k}_1 + \frac{v_{,\varphi} + u}{r}\mathbf{k}_2 + \frac{w_{,\varphi}}{r}\mathbf{k}_3) r\mathrm{d}\varphi + \\ & (u_{,z}\,\mathbf{k}_1 + v_{,z}\,\mathbf{k}_2 + w_{,z}\,\mathbf{k}_3)\mathrm{d}z. \end{aligned} \tag{9}$$

Therefore, by eq. (9) the total differential of the displacement **u** may be expressed in matrix form as a function of the coordinate increment d**x** and of the displacement gradient, $\boldsymbol{\nabla}\mathbf{u}$; indeed:

$$\mathrm{d}\mathbf{u} = \boldsymbol{\nabla}\mathbf{u}\,\mathrm{d}\mathbf{x}, \tag{10}$$

where d**u**, d**x** and $\boldsymbol{\nabla}\mathbf{u}$ have these matrix form:

$$\mathrm{d}\mathbf{u} = \begin{bmatrix} \mathrm{d}u \\ \mathrm{d}v \\ \mathrm{d}w \end{bmatrix}, \quad \mathrm{d}\mathbf{x} = \begin{bmatrix} \mathrm{d}r \\ r\mathrm{d}\varphi \\ \mathrm{d}z \end{bmatrix}; \quad \boldsymbol{\nabla}\mathbf{u} = \begin{bmatrix} u_{,r} & \dfrac{u_{,\varphi} - v}{r} & u_{,z} \\ v_{,r} & \dfrac{v_{,\varphi} + u}{r} & v_{,z} \\ w_{,r} & \dfrac{w_{,\varphi}}{r} & w_{,z} \end{bmatrix}. \tag{11}$$

The components $(\nabla u)_{ij}$ of the displacement gradient $\boldsymbol{\nabla}\mathbf{u}$ allow deducing, with the usual procedure, the corresponding components of the infinitesimal strain tensor:

$$\varepsilon_{ij} = \varepsilon_{ji} = \frac{(\nabla u)_{ij} + (\nabla u)_{ji}}{2}. \tag{12}$$

The result is:

$$\begin{aligned} &\varepsilon_{11} = u_{,r} && \varepsilon_{12} = \frac{1}{2}\left(v_{,r} + \frac{u_{,\varphi} - v}{r}\right) \\ &\varepsilon_{22} = \frac{v_{,\varphi} + u}{r} && \varepsilon_{13} = \frac{1}{2}\left(u_{,z} + w_{,r}\right) \\ &\varepsilon_{33} = w_{,z} && \varepsilon_{23} = \frac{1}{2}\left(v_{,z} + \frac{w_{,\varphi}}{r}\right). \end{aligned} \tag{13}$$

With reference to Figure 2, let now $\mathbf{x}_G$ denote the position of the centroid, G, of the cross-section, whose coordinates are given by the triple $(R, \varphi, 0)$; and

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables"*International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017

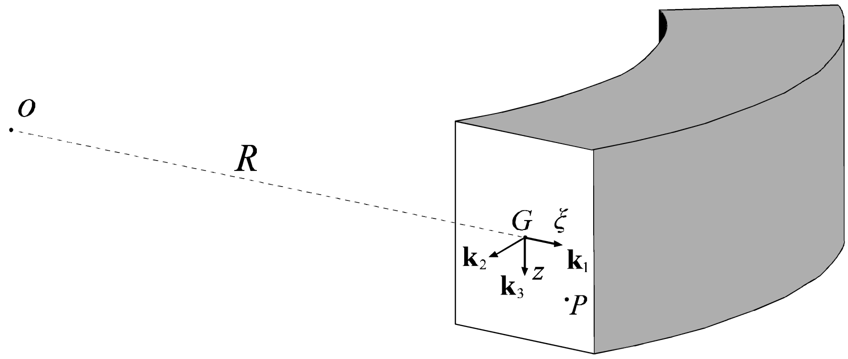


Figure 2: Cross-section and local coordinates of the circle-like beam. O denotes the center of the beam axis, G the centroid and P a generic point of the cross-section

$\mathbf{x}_P$ that of any other point P which is not coinciding with the centroid; its coordinates are then expressed by the following triple: ($r_P = R + \xi$, $\varphi_P = \varphi$, $z_P = z$).

If u, v, w identify, from now on, the components of the displacement vector $\mathbf{u}$ of the centerline of the beam, (i.e. for any cross section those referred to the centroid) the displacements of any point P, $\mathbf{u}_P$, whose components are denoted respectively by u_P, v_P w_P are linked to those of G by the usual kinematic constraint which apply to beam-like solids, viz. :

$$\mathbf{u}_P = \mathbf{u} + \boldsymbol{\phi} \times (\mathbf{x}_P - \mathbf{x}_G), \tag{14}$$

where $\boldsymbol{\phi}$ is the infinitesimal rotation vector which defines the rotation of the cross-section and has components ϕ_1, ϕ_2, ϕ_3 along the directions defined by the local base $\mathbf{K}$, see Figure 2.

Taking into account that $r = R + \xi$, since R is constant for a circle-like beam, it is also $\mathrm{d}r = \mathrm{d}\xi$, and $\partial(.)/\partial r = \partial(.)/\partial \xi$. Then the component form of eq. (14) is:

$$\begin{aligned} u_P &= u + z\phi_2 \\ v_P &= v - z\phi_1 + \xi\phi_3 \\ w_P &= w - \xi\phi_2. \end{aligned} \tag{15}$$

The infinitesimal length of a fibre lying outside the centerline of the beam is $\mathrm{d}s_P = (R + \xi)\mathrm{d}\varphi$. Since the displacement $\mathbf{u}$ and the rotation $\boldsymbol{\phi}$ are only unknown functions of the angular coordinate φ or (which is the same) of the

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables"*International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017

arc-length $s = R\varphi$, so that $\mathrm{d}s = R\,\mathrm{d}\varphi$ and hence $\partial(.)/\partial s = \partial(.)/R\partial\varphi$, it follows, by eq. (13), that $\varepsilon_{11} = 0$; $\varepsilon_{33} = 0$; $\varepsilon_{13} = 0$ and the only non-zero strain components are:

$$\begin{aligned}
\varepsilon_{22} =& \frac{v_{,s} + \dfrac{u}{R}}{1 + \dfrac{\xi}{R}} + \frac{z\left(\dfrac{\phi_2}{R} - \phi_{1,s}\right)}{1 + \dfrac{\xi}{R}} + \frac{\xi\phi_{3,s}}{1 + \dfrac{\xi}{R}}\\
\gamma_{12} =& 2\varepsilon_{12} = \frac{u_{,s} - \dfrac{v}{R} + \phi_3}{1 + \dfrac{\xi}{R}} + \frac{z\left(\dfrac{\phi_1}{R} + \phi_{2,s}\right)}{1 + \dfrac{\xi}{R}} \qquad (16)\\
\gamma_{23} =& 2\varepsilon_{23} = \frac{w_{,s} - \phi_1}{1 + \dfrac{\xi}{R}} - \frac{\xi\left(\dfrac{\phi_1}{R} + \phi_{2,s}\right)}{1 + \dfrac{\xi}{R}}.
\end{aligned}$$

In eq. (16) warping effects have been neglected.

Moreover, if the radius of curvature R of the beam axis is large compared to the cross section width, then $\xi/R \ll 1$ and the corresponding terms, appearing in the denominators of eq. (16), can be disregarded. This will be tacitly assumed in the sequel.

However, in cases when R becomes comparable with the cross-section beam width, Winkler's theory [37] for strongly curved beams applies: some examples and analytical solutions are discussed in [38].

2.3 Potential and kinetic energy

The potential energy $\mathcal{E}$ and the kinetic energy $\mathcal{T}$ of the beam are simply:

$$2\mathcal{E} = \int G(\gamma_{12}^2 + \gamma_{23}^2)\mathrm{d}s\,\mathrm{d}A + \int E\varepsilon_{22}^2\,\mathrm{d}s\,\mathrm{d}A, \qquad (17)$$

$$\begin{aligned}
2\mathcal{T} =& \int \rho(\dot{u}_P^2 + \dot{v}_P^2 + \dot{w}_P^2)\,\mathrm{d}s\,\mathrm{d}A \qquad (18)\\
=& \rho A\int(\dot{u}^2 + \dot{v}^2 + \dot{w}^2)\,\mathrm{d}s + \rho I_1\int\dot{\phi}_1^2\,\mathrm{d}s + \rho I_3\int\dot{\phi}_3^2\,\mathrm{d}s + \rho I_P\int\dot{\phi}_2^2\,\mathrm{d}s,
\end{aligned}$$

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables"*International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017

where G and E are the shear and the Young's modulus of the beam, ρ is its density, which is constant since the beam is assumed to be homogeneous and to have a constant cross-section, whose area is A, $I_1 = \int z^2 \mathrm{d}A$ and $I_3 = \int \xi^2 \mathrm{d}A$ are the cross section moments of inertia about the $\mathbf{k}_1$ and $\mathbf{k}_3$ directions and $I_P = (I_1 + I_3)$ is the corresponding cross-section polar moment of inertia.

Once these energies are known, Hamilton's principle can be applied in order to get the governing Partial Differential Equations (PDEs); alternatively, variationally-based numerical approaches can be used by assuming some shape-functions for modeling the unknown variables $u(s)$, $v(s)$, $w(s)$, $\phi_1(s)$, $\phi_2(s)$, $\phi_3(s)$.

2.4 Simplifying assumptions

The first term which appears in the elastic potential, $2\mathcal{E}$, in eq. (17), namely

$$2\mathcal{E}_G = \int G(\gamma_{12}^2 + \gamma_{23}^2)\,\mathrm{d}s\,\mathrm{d}A$$

accounts for the effects of shear deformation on strain energy; it consists of two contributions, one of them is due to shear forces (with a shear stiffness given by GA_T, A_T being the reduced shear area of the cross-section), the other is due to torsion (whose stiffness is GI_P).

Once strains components, see eqs. (16), are substituted into eq. (17) it results:

$$2\mathcal{E}_G = GA_T \int (\phi_1 - w,_s)^2 \mathrm{d}s + GA_T \int (\phi_3 + u,_s - \frac{v}{R})^2\,\mathrm{d}s + GI_P \int (\phi_{2,s} + \frac{\phi_1}{R})^2 \mathrm{d}s.$$

The Euler-Bernoulli beam model is characterized by zero strains due to shear forces:

$$GA_T \int (\phi_1 - w,_s)^2 \mathrm{d}s = 0 \;\rightarrow\; \phi_1 = w,s$$

$$GA_T \int (\phi_3 + u,_s - \frac{v}{R})^2 \mathrm{d}s = 0 \;\rightarrow\; \phi_3 = -u,_s + \frac{v}{R}\;;$$

this allows eliminating ϕ_1 and ϕ_3 from the state variables.

Moreover, here only the out-of-plane dynamics will be treated, which means that only state variables ϕ_2 and w (which are decoupled from the

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables" *International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017

pair u, v of in-plane displacements) need to be introduced. Then, eqs. (16) become:

$$\varepsilon_{22} = z\left(\frac{\phi_2}{R} - w_{,ss}\right), \quad \gamma_{12} = z\left(\frac{w_{,s}}{R} + \phi_{2,s}\right), \quad \gamma_{23} = -\xi\left(\frac{w_{,s}}{R} + \phi_{2,s}\right). \tag{19}$$

The corresponding out-of-plane potential energy for strain-less shear forces, containing the vertical displacement w and the torsional rotation ϕ_2, is:

$$2\mathcal{E} = EI_1 \int (\frac{\phi_2}{R} - w_{,ss})^2 \mathrm{d}s + GI_P \int (\phi_{2,s} + \frac{w_{,s}}{R})^2 \mathrm{d}s. \tag{20}$$

By applying the same reduction of variables to the kinetic energy, see eq. (18), but retaining the rotational inertia, it results:

$$2\mathcal{T} = \rho A \int \dot{w}^2 \mathrm{d}s + \rho I_P \int \dot{\phi}_2^2 \mathrm{d}s. \tag{21}$$

2.5 Equations of motion

By applying Hamilton's principle to the total energy of the beam, given by the sum of $\mathcal{E}$, eq. (20), and $\mathcal{T}$, eq. (21), the equations of motion of the beam model are established:

$$EI_1(w_{,ssss} - \frac{\phi_{2,ss}}{R}) - \frac{GI_P}{R}(\frac{w_{,ss}}{R} + \phi_{2,ss}) + \rho A w_{,tt} = 0 \tag{22}$$

$$\frac{EI_1}{R}(\frac{\phi_2}{R} - w_{,ss}) - GI_P(\frac{w_{,ss}}{R} + \phi_{2,ss}) + \rho I_P \phi_{2,tt} = 0 \tag{23}$$

They correspond to the extension to curved beams of the so-called Rayleigh beam model, as defined by Pilkey [39, p. 537].

3 Kinematics of a traveling mass

3.1 General case: traveling mass along a plane-curved beam

A point-mass traveling along the curved axis (i.e. the center-line) of the circular beam is now considered. As it is shown in Figure 1, the variable position of the mass is measured by an arc-length coordinate $s(t) = R\varphi(t)$. The local base $\mathbf{K}$, to which the displacements and rotations of the beam are referred, is connected to the mass, and it is therefore moving along the curved beam axis, in such a way that $\mathbf{k}_1$ and $\mathbf{k}_2$ are always oriented, respectively,

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables"*International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017

according to the radial and the tangential directions. It is assumed that the traveling speed c of the mass, defined as:

$$\frac{\mathrm{d}s}{\mathrm{d}t} = R\frac{\mathrm{d}\phi}{\mathrm{d}t} = R\omega = c = \text{const}, \tag{24}$$

so that also the angular velocity ω is constant. Since the traveling speed of base $\mathbf{K}$ is c, it is also $\varphi = \omega t$, $\omega = c/R$. The position vector $\mathbf{x}_r$ to any point of the curved beam axis in the undeformed configuration of the beam (or, which is the same, if the mass trajectory is considered a rigid curve) is time-dependent; particular it results: $\mathbf{x}_r = R\,\mathbf{k}_1$, with $\mathbf{k}_1$ expressed by eq. $(3)_1$ in terms of the global base $\mathbf{E}$. The total derivatives with respect to time of the position vector $\mathbf{x}_r$ come out to be oriented along the $\mathbf{k}_1$ and $\mathbf{k}_2$ unit vectors; indeed it is simply:

$$\frac{\mathrm{d}\mathbf{x}_r}{\mathrm{d}t} = R\omega\,\mathbf{k}_2, \qquad \frac{\mathrm{d}^2\mathbf{x}_r}{\mathrm{d}t^2} = -R\omega^2\mathbf{k}_1, \tag{25}$$

again with $\mathbf{k}_2$ given by eq. $(3)_2$. In the deformed configuration of the beam, the position vector $\mathbf{x}$ to *any* point of its axis can be thought of as the superposition of a displacement $\mathbf{u}$ (measured again in the local basis $\mathbf{K}$) to the undeformed configuration:.

$$\mathbf{x} = \mathbf{x}_r + \mathbf{u} = \mathbf{x}_r + (u\mathbf{k}_1 + v\mathbf{k}_2 + w\mathbf{k}_3)\,, \tag{26}$$

where $\mathbf{k}_3$ is defined in eq. $(3)_3$

In evaluating the first derivative of $\mathbf{x}$ with respect to time, namely the velocity vector, one has to consider the time-dependency of the basis $\mathbf{K}$ and the dependency of the state-variables with respect to the position of the traveling mass: indeed, $u = u(s,t)$, $v = v(s,t)$, $w = w(s,t)$. The velocity vector is then:

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\,t} = \frac{\mathrm{d}\mathbf{x}_r}{\mathrm{d}\,t} + \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\,t}, \tag{27}$$

where:

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{u}_{,t} + \mathbf{u}_{,s}\, s_{,t} = \mathbf{u}_{,t} + c\,\mathbf{u}_{,s}\,. \tag{28}$$

Finally, it results:

$$\begin{aligned} \mathbf{u}_{,t} &= (u_{,t} - \omega v)\,\mathbf{k}_1 + (v_{,t} + \omega u)\,\mathbf{k}_2 + w_{,t}\,\mathbf{k}_3, \\ \mathbf{u}_{,s} &= \left(u_{,s} - \frac{v}{R}\right)\mathbf{k}_1 + \left(v_{,s} + \frac{u}{R}\right)\mathbf{k}_2 + w_{,s}\,\mathbf{k}_3. \end{aligned} \tag{29}$$

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables"*International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017

So, the velocity vector $\mathbf{v}$ of the traveling mass can be explicitly computed:

$$\mathbf{v} = R\omega\,\mathbf{k}_2 + \mathbf{u},_t + c\mathbf{u},_s\,. \tag{30}$$

With a similar procedure, the second total time-derivative of the position vector $\mathbf{x}$ gives the acceleration of the traveling mass, $\mathbf{a}$:

$$\mathbf{a} = \frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}\,t^2} = \frac{\mathrm{d}^2\mathbf{x}_r}{\mathrm{d}\,t^2} + \frac{\mathrm{d}^2\mathbf{u}}{\mathrm{d}\,t^2}. \tag{31}$$

The total second time-derivative of $\mathbf{u}$ can be written in this compact form:

$$\frac{\mathrm{d}^2\mathbf{u}}{\mathrm{d}t^2} = \mathbf{u},_{tt} + 2c\,\mathbf{u},_{ts} + c^2\mathbf{u},_{ss}\,. \tag{32}$$

The explicit expressions of $\mathbf{u},_{tt}$, $\mathbf{u},_{ts}$, $\mathbf{u},_{ss}$ are:

$$\begin{aligned}
\mathbf{u},_{tt} &= \left(u,_{tt} - 2\omega v,_t - u\omega^2\right)\mathbf{k}_1 + \left(v,_{tt} + 2\omega u,_t - v\omega^2\right)\mathbf{k}_2 + w,_{tt}\,\mathbf{k}_3,\\
\mathbf{u},_{ts} &= \left(u,_{ts} - \omega v,_s - \frac{v,_t + \omega u}{R}\right)\mathbf{k}_1 + \left(v,_{ts} + \omega u,_s + \frac{u,_t - \omega v}{R}\right)\mathbf{k}_2 + w,_{ts}\,\mathbf{k}_3,\\
\mathbf{u},_{ss} &= \left(u,_{ss} - 2\frac{v,_s}{R} - \frac{u}{R^2}\right)\mathbf{k}_1 + \left(v,_{ss} + 2\frac{u,_s}{R} - \frac{v}{R^2}\right)\mathbf{k}_2 + w,_{ss}\,\mathbf{k}_3.
\end{aligned} \tag{33}$$

By using eqs. (33) the acceleration vector $\mathbf{a}$ of the traveling mass, formulated with respect to the local base $\mathbf{K}$ results:

$$\mathbf{a} = -R\omega^2\mathbf{k}_1 + \mathbf{u},_{tt} + 2c\,\mathbf{u},_{ts} + c^2\mathbf{u},_{ss}\,. \tag{34}$$

It should be noticed that the expression of acceleration contains Coriolis-type terms, indicated by the typical factor $2c = 2\omega R$; moreover, for the acceleration vector expressed in the local base:

$$\mathbf{a} = a_r\mathbf{k}_1 + a_\varphi r\mathbf{k}_2 + a_z\mathbf{k}_3 \tag{35}$$

the component acting in the vertical direction (namely that spanned by $\mathbf{k}_3$) contains only parts related to the vertical displacement w of the beam:

$$a_z = w,_{tt} + 2cw,_{st} + c^2 w,_{ss}\,. \tag{36}$$

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables"*International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017

3.2 Special case: traveling mass along a straight beam

With a similar procedure one can deduce the acceleration for the out-of-plane dynamics in the case of a mass traveling on an originally straight beam. With reference to Figure 6 (where there is a slight change of the notation adopted up to here: the horizontal direction is denoted by x and η describes the arc-length abscissa, previously denoted by s) it is indeed:

$$\frac{\mathrm{d}^2 w}{\mathrm{d}t^2} = w_{,tt} + 2cw_{,\eta t} + c^2 w_{,\eta\eta} + aw_{,\eta}\,, \tag{37}$$

where $c = \mathrm{d}\eta/\mathrm{d}t$, $a = \mathrm{d}^2\eta/\mathrm{d}t^2$. It should be remarked that eq. (37) has been deduced in a more general framework, where the traveling speed of the mass, *c might be not constant*; hence the presence of the last term in the right-hand side. On the other hand, if $c = \text{const}$ then $a = 0$, and the same expression provided by eq. (36) is recovered.

4 Second-order nonlinear kinematics of a traveling mass

With reference to a mass traveling along a straight beam, as described at the end of Section 3, the vertical acceleration can be typically split in two parts, which are identified by their geometric orientation:

- $c^2\, w_{,\eta\eta}$, with $c = \eta_{,t}$: *normal* acceleration, which is related to the squared value of the speed of the traveling, mass/oscillator contact point;
- $a\, w_{,\eta}$, with $a = \eta_{,tt}$: *tangential* acceleration, which depends on the magnitude of acceleration of the traveling mass/oscillator contact point.

However, the second total derivative of the position vector $\mathbf{x}$ of the coupling point of the traveling mass/oscillator does not result truly aligned along the *normal* direction, as some simple computations show.

The problem is a plane one; its description is simplified if an horizontal x-axis and a vertical, downwards pointing z-axis are assumed, as in Figure 3; for the sake of conciseness, vectors are here represented component-wise and stored in suitable column matrices.

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables"*International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017

It results, then for the position, velocity and acceleration vectors corresponding to the traveling mass/oscillator contact point:

$$\begin{aligned} \mathbf{x} &= \begin{bmatrix} \eta(t) \\ w(\eta,t) \end{bmatrix}, \\ \mathbf{v} &= \mathrm{d}\mathbf{x}/\mathrm{d}\,t = \begin{bmatrix} 0 \\ w_{,t} \end{bmatrix} + \eta_{,t} \begin{bmatrix} 1 \\ w_{,\eta} \end{bmatrix}, \\ \mathbf{a} &= \mathrm{d}^2\mathbf{x}/\mathrm{d}t^2 = w_{,\eta\eta}\,(\eta_{,t})^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \eta_{,tt} \begin{bmatrix} 1 \\ w_{,\eta} \end{bmatrix} + 2\eta_{,t} \begin{bmatrix} 0 \\ w_{,\eta t} \end{bmatrix} + \begin{bmatrix} \eta_{,tt} \\ w_{,tt} \end{bmatrix}. \end{aligned} \tag{38}$$

If Figure 3 is considered, it is clear that the so-called *normal* acceleration

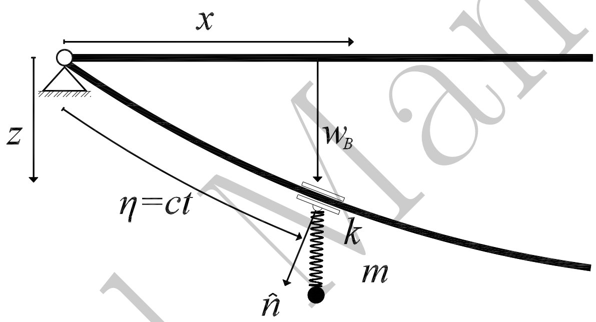


Figure 3: Absolute description of the motion of an elastically coupled mass traveling along a beam, standard model: spring elongation occurs in the vertical direction, and not along that of the unit normal $\hat{\mathbf{n}}$ to the deformed beam axis.

defined by eq. $(38)_3$ is not directed along the unit normal $\hat{\mathbf{n}}$ to the deformed beam, at the coupling point between the traveling mass/oscillator and the beam, whose expression is:

$$\hat{\mathbf{n}} = \frac{1}{\sqrt{1 + w_{,\eta}{}^2}} \begin{bmatrix} -w_{,\eta} \\ 1 \end{bmatrix}, \tag{39}$$

but in the vertical direction:

$$\mathbf{e}_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{40}$$

However, a careful look at the deformed situation of the supporting structure, see Figure 4, shows a difference $\Delta u(x,t)$ between the position occupied by

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables"*International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017

the coupling point, if the beam were rigid, x, and the actual position of the coupling point on the deformed beam. If the reasonable hypothesis of inextensibility is introduced, so that $x = \eta$, then this difference is given by:

$$\Delta u(x,t) = -\int_0^x \frac{1}{2} w_{,\eta}{}^2 \mathrm{d}\eta. \tag{41}$$

Thus, the position vector $\mathbf{x}$ of the coupling point has to be enriched by this difference. When this is done, it results that the expressions of the position, velocity and acceleration vectors corresponding to the traveling mass/oscillator contact point are no more given by eqs. (38), but are instead:

$$\begin{aligned}
\mathbf{x} &= \begin{bmatrix} \eta(t) - \int_0^x \frac{1}{2} w_{,\eta}^2 \mathrm{d}\eta \\ w(\eta,t) \end{bmatrix}, \\
\mathbf{v} &= \begin{bmatrix} 0 \\ w_{,t} \end{bmatrix} + \eta_{,t} \begin{bmatrix} 1 - \frac{1}{2} w_{,\eta}^2 \\ w_{,\eta} \end{bmatrix}, \\
\mathbf{a} &= (\eta_{,t})^2 \, w_{,\eta\eta} \begin{bmatrix} -w_{,\eta} \\ 1 \end{bmatrix} + \eta_{,tt} \begin{bmatrix} 1 - \frac{1}{2} w_{,\eta}{}^2 \\ w_{,\eta} \end{bmatrix} + \eta_{,t} \, w_{,\eta t} \begin{bmatrix} -w_{,\eta} \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ w_{,tt} \end{bmatrix}.
\end{aligned} \tag{42}$$

For this enhanced kinematic model, eq. (42), the normal acceleration is indeed oriented along the direction identified by the unit normal $\hat{\mathbf{n}}$, eq. (39).

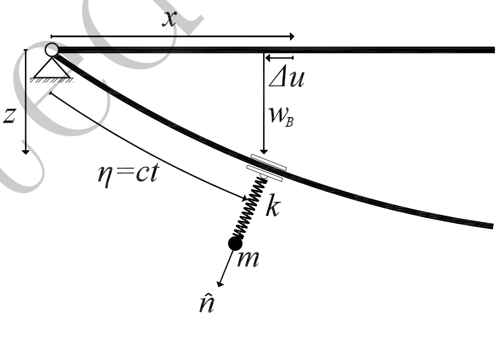


Figure 4: Absolute description of the motion of an elastically coupled mass traveling along a beam, enhanced kinematic model: spring elongation occurs in the direction of the unit normal $\hat{\mathbf{n}}$ to the deformed beam axis.

For a constant velocity of the traveling mass, $\eta_{,t} = c$, $\eta_{,tt} = 0$, the acceleration along the vertical (z) direction:

$$a_z = c^2 w_{,\eta\eta} + 2c w_{,\eta t} + w_{,tt} \tag{43}$$

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables" *International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017

is the same as for the standard, not enriched formulation; however, the acceleration along the horizontal (x) direction

$$a_x = -cw_{,\eta}\,(cw_{,\eta\eta} + w_{,\eta t}\,) \tag{44}$$

does not vanish as it happens for the standard formulation.

Including the difference Δu in the analysis produces another improvement concerning that part, $\mathbf{v}_c$, of the velocity vector which is related to the (imposed) driving velocity $\eta_{,t} = c$.

Indeed, for the standard, not enriched kinematic model it results:

$$\mathbf{v}_c = c\begin{bmatrix} 1 \\ w_{,\eta} \end{bmatrix}; \quad |\mathbf{v}_c| = c\sqrt{1 + {w_{,\eta}}^2} \approx c(1 + \frac{1}{2}{w_{,\eta}}^2), \tag{45}$$

while for the enriched one it follows:

$$\mathbf{v}_c = c\begin{bmatrix} 1 - \frac{1}{2}{w_{,\eta}}^2 \\ w_{,\eta} \end{bmatrix}; \quad |\mathbf{v}_c| = c\sqrt{1 + \frac{1}{4}{w_{,\eta}}^4} \approx c(1 + \frac{1}{8}{w_{,\eta}}^4). \tag{46}$$

Thus the discrepancy between $|\mathbf{v}_c|$ and the imposed traveling velocity c is much less for the enriched version, provided that the slope of the deformed beam axis is conveniently small: $w_{,\eta} \ll 1$.

It should be noticed that in almost all papers dealing with a traveling oscillator along a beam the displacement of the oscillator is always assumed to be in a vertical direction, as it is depicted in Figure 3. A more precise analysis, however, should take the oscillator as a pendulum with an additional rotational degree-of-freedom (DOF) β, as it is shown in Figure 5, resulting in a 2 DOFs variable length pendulum, see [40]. This model, of course, reduces to the traditional one when the constraint $\beta = 0$ is introduced.

5 Semi-analytical approach for the dynamics of a beam with a traveling oscillator

The dynamics of the system composed by the beam and the traveling oscillator is described by this system of partial differential equations:

$$\begin{aligned} &EI_1(w_{,ssss} - \frac{\phi_{2,ss}}{R}) - \frac{GI_P}{R}(\frac{w_{,ss}}{R} + \phi_{2,ss}) + \rho A w_{,tt} = F\delta(s - ct), \\ &\frac{EI_1}{R}(\frac{\phi_2}{R} - w_{,ss}) - GI_P(\frac{w_{,ss}}{R} + \phi_{2,ss}) + \rho I_P \phi_{2,tt} = 0, \end{aligned} \tag{47}$$

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables"*International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017

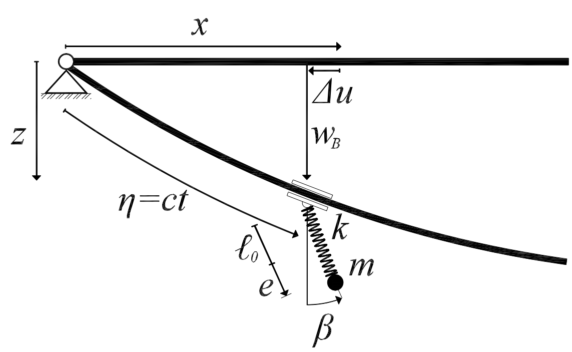


Figure 5: Absolute description of the motion of an elastically coupled mass traveling along a beam, enhanced 2 DOFs kinematic model: the direction of spring elongation e forms an angle β with the vertical direction.

where F is the coupling force in the interface between supporting beam and traveling mass/oscillator. This forcing term, which applies only to eq. $(47)_1$, the one governing the w displacement-component of the beam, is a Dirac's δ-function acting in the vertical direction; ct is the instantaneous position occupied by the traveling oscillator, driven with a constant speed c along the curved, circular beam axis.

The system of PDEs eq. (47) is solved by a semi-analytical approach for a *simply supported* circle-like beam, having a developed length (i.e. measured along the curved axis), L, with the following Boundary Conditions (BCs):

$$@s = 0: \quad w = 0\,, \quad \phi_2 = 0\,; \tag{48}$$

$$@s = L: \quad w = 0\,, \quad \phi_2 = 0\,. \tag{49}$$

5.1 Discretization in the space-domain

The equations of motion can be solved by a discretization procedure in the space-domain: for this purpose sin-like shape functions are chosen in order to satisfy for any value of time t the above-mentioned BCs (48)–(49):

$$w(s,t) = \sum_{j=1}^{N} W_j(t) \sin(j\frac{\pi s}{L}), \tag{50}$$

$$\phi_2(s,t) = \sum_{j=1}^{N} T_j(t) \sin(j\frac{\pi s}{L}). \tag{51}$$

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables"*International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017

In eqs. (50)–(51) $W_j(t)$ and $T_j(t)$ represent still unknown time-dependent amplitude coefficients. Solution is sought by substituting these shape functions, eqs. (50)–(51), inside the equations of motions and then projecting all the resulting terms against the orthogonal functions space spanned by $\sin(j\pi s/L)$ $(j = 1, \ldots, N)$, which correspond to the homogeneous solutions of the curved beam in the space-domain.

This procedure can be thought of as a Fourier-series expansion of the unknown variables: based on the particular symmetry of the problem, it should be noticed that only sin-type terms are involved.

Taking into account that these functions are orthogonal, i.e. :

$$\begin{aligned} \int_0^L \sin(j\frac{\pi s}{L})\sin(k\frac{\pi s}{L})\mathrm{d}s &= 0 \quad \forall\, k{\neq}j \\ \int_0^L \sin(j\frac{\pi s}{L})\sin(k\frac{\pi s}{L})\mathrm{d}s &= \frac{L}{2} \quad \text{for}\, k = j \end{aligned} \tag{52}$$

the procedure produces a set of ODE's in the unknown variables $W_j(t)$ and $T_j(t)$; more precisely, it results (for $j = 1, \ldots, N$):

$$\begin{aligned} b_{11j}W_j(t) + b_{12j}T_j(t) + \rho A\ddot{W}_j(t) &= \frac{2F}{L}\sin(j\alpha\, t), \\ b_{21j}W_j(t) + b_{22j}T_j(t) + \rho I_P\ddot{T}_j(t) &= 0, \end{aligned} \tag{53}$$

where these short-hand definitions have been adopted:

$$\begin{aligned} b_{11j} &= EI(\frac{j\pi}{L})^4 + \frac{GI_P}{R^2}(\frac{j\pi}{L})^2, \\ b_{12j} &= \frac{GI_P + EI}{R}(\frac{j\pi}{L})^2 = b_{21j}, \\ b_{22j} &= GI_P(\frac{j\pi}{L})^2 + \frac{EI}{R^2}. \end{aligned} \tag{54}$$

The r.h.s. in eq. $(53)_1$ descends from the integration properties of Dirac's delta function; indeed:

$$\int_0^L F\sin(j\frac{\pi s}{L})\delta(s - ct)\mathrm{d}s = F\sin(j\frac{\pi c\, t}{L}) = F\sin(j\alpha\, t),$$

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables"*International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017

where $\alpha = \pi c/L$.

5.2 Modeling of the coupling force between beam and traveling mass/oscillator

The equation of motion for the traveling oscillator contains the coupling force:

$$F = f_k + f_d \tag{55}$$

due to the elastic (f_k) and to the viscous (f_d) contributions, so that the equation of motion of the oscillator, whose mass is m, by Newton's law is:

$$m\ddot{w}_M = mg - f_k - f_d \quad \text{with } f_k = k(w_M - w_B), \text{ and } f_d = d(\dot{w}_M - \dot{w}_B) \tag{56}$$

where gravity acceleration is denoted by g. In eq. (56) both the *absolute* vertical displacement $w_B = z_B$ of the supporting beam and the *absolute* vertical deflection $w_M = z_M - \ell_0$ of the oscillator's mass appear, according to Figure 6; moreover, ℓ_0 describes the stress-less length of the spring.

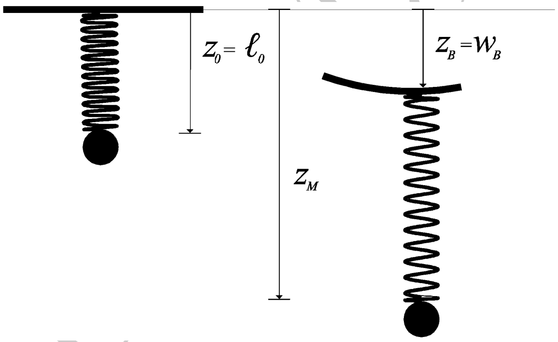


Figure 6: Absolute description of the motion of an elastically coupled mass traveling along a beam, standard model: spring elongation $e = z_M - z_B - \ell_0$ is constrained to be in the vertical direction.

The viscous force f_d can be written equivalently in this way: $f_d = \dfrac{d}{k}\dot{f}_k$; then the viscoelastic coupling force, eq. (55) becomes:

$$F = f_k + \frac{d}{k}\dot{f}_k, \tag{57}$$

and the oscillator's equation of motion, eq. (56) results in:

$$m\ddot{w}_M = mg - f_k - \frac{d}{k}\dot{f}_k. \tag{58}$$

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables"*International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017

Then, the second derivative of the *absolute* vertical displacement of the mass, w_M, can be described by $\ddot{w}_M = \ddot{w}_B + \ddot{f}_k/k$ and thus, the w_M degree-of freedom (DOF) can be eliminated by switching to the new state variable, f_k.

Finally, the equation of motion of the mass-spring-dashpot oscillator is this:

$$m\ddot{w}_B + \frac{m}{k}\ddot{f}_k + \frac{d}{k}\dot{f}_k + \frac{k}{k}f_k = mg. \tag{59}$$

Thus, the whole system characterized by an oscillator traveling along a flexible beam/bridge is described by mixed variables: displacements for the beam and forces for the oscillator.

The second total derivative of the displacement w_B with respect to time at the interface between beam and traveling oscillator can then be taken from eq. (37).

$$\frac{\mathrm{d}^2 w}{\mathrm{d}t^2} = w_{,tt} + 2cw_{,\eta t} + c^2 w_{,\eta\eta} + aw_{,\eta}, \tag{60}$$

with c and a denoting, respectively, the driving speed and acceleration of the traveling contact point.

6 Semi-discretized, time-variant equation of motion for a curved beam

For $j = 1$ up to N, the projections of all terms of the given PDEs — see eqs. (47) — against the domain of orthogonal functions $\sin(j\pi s/L)$ give as a result a system of ODEs (whose size depends on the number N of the considered Fourier series harmonics) which are characterized by time-dependent nonsymmetric matrices:

$$\mathbf{M}(t)\ddot{\mathbf{q}} + \mathbf{D}(t)\dot{\mathbf{q}} + \mathbf{C}(t)\mathbf{q} = \mathbf{r}, \tag{61}$$

where $\{\mathbf{M}(t), \mathbf{D}(t), \mathbf{C}(t)\} \in \mathbb{R}^{(2N+1)\times(2N+1)}$ and $\{\mathbf{q}, \mathbf{r}\} \in \mathbb{R}^{2N+1}$. In eq. (61) a standard notation is adopted: $\mathbf{M}$, $\mathbf{D}$ and $\mathbf{C}$ denote, respectively, the inertia, damping and stiffness matrices, while $\mathbf{q}$, $\mathbf{r}$ are column-matrices storing componentwise the unknown displacements and the assigned external forces.

Before giving explicitly the structure of matrices appearing in eq. (61) it is useful introducing some new definitions, namely $\mu = \rho A$, which is the density per unit length of the beam; m, d, k which denote, as before, the mass, viscous damping and elastic stiffness of the moving oscillator; and a new variable, $\alpha = \pi c/L$, having the dimension of the reciprocal of time: it

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables"*International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017

should be remembered that c is the traveling speed of the oscillator, while L the developed length of the beam. It follows, hence:

$$\mathbf{M}(t) = \begin{bmatrix} \mu \mathbf{I}_{NN} & \mathbf{0}_{NN} & \mathbf{0}_N \\ \mathbf{0}_{NN} & \rho I_P \mathbf{I}_{NN} & \mathbf{0}_N \\ \mathbf{m}_{31}^T & \mathbf{0}_N^T & \frac{m}{k} \end{bmatrix}, \tag{62}$$

$$\mathbf{m}_{31}^T = m \begin{bmatrix} \sin(\alpha t) & \cdots & \sin(N\alpha t) \end{bmatrix},$$

$$\mathbf{D}(t) = \begin{bmatrix} \mathbf{0}_{NN} & \mathbf{0}_{NN} & \mathbf{d}_{13} \\ \mathbf{0}_{NN} & \mathbf{0}_{NN} & \mathbf{0}_N \\ \mathbf{d}_{31}^T & \mathbf{0}_N^T & \frac{d}{k} \end{bmatrix}, \tag{63}$$

$$\mathbf{d}_{13}^T = -\frac{2d}{Lk} \begin{bmatrix} \sin(\alpha t) & \cdots & \sin(N\alpha t) \end{bmatrix},$$
$$\mathbf{d}_{31}^T = 2m\alpha \begin{bmatrix} \cos(\alpha t) & \cdots & N\cos(N\alpha t) \end{bmatrix},$$

$$\mathbf{C}(t) = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} & \mathbf{c}_{13} \\ \mathbf{C}_{12}^T & \mathbf{C}_{22} & \mathbf{0}_N \\ \mathbf{c}_{31}^T & \mathbf{0}_N^T & 1 = \frac{k}{k} \end{bmatrix}, \tag{64}$$

$$\mathbf{C}_{11} = \operatorname{diag}\left[EI\left(\frac{\pi}{L}\right)^4 + \frac{GI_P}{R^2}\left(\frac{\pi}{L}\right)^2, \quad \cdots \quad, EI\left(\frac{N\pi}{L}\right)^4 + \frac{GI_P}{R^2}\left(\frac{N\pi}{a}\right)^2 \right],$$
$$\mathbf{C}_{22} = \operatorname{diag}\left[GI_P\left(\frac{\pi}{L}\right)^2 + \frac{EI}{R^2}, \quad \cdots \quad, GI_P\left(\frac{N\pi}{L}\right)^2 + \frac{EI}{R^2} \right],$$
$$\mathbf{C}_{12} = \left(\frac{GI_P + EI}{R}\right) \operatorname{diag}\left[\left(\frac{\pi}{L}\right)^2, \quad \cdots \quad, \left(\frac{N\pi}{L}\right)^2 \right],$$

$$\mathbf{c}_{31}^T = -m\alpha^2 \begin{bmatrix} \sin(\alpha t) & \cdots & N^2\sin(N\alpha t) \end{bmatrix},$$
$$\mathbf{c}_{13}^T = -\frac{2}{L} \begin{bmatrix} \sin(\alpha t) & \cdots & \sin(N\alpha t) \end{bmatrix},$$

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables" *International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017

Matrices $\mathbf{I}_{NN}$, $\mathbf{0}_{NN}$ and $\mathbf{0}_N$ appearing in eqs. (62)–(64) are respectively a square identity matrix of order N, a square null matrix of order N and a null column matrix having N rows. Moreover, it results:

$$\mathbf{q}^T(t) = \begin{bmatrix} W_1 & \cdots & W_N & T_1 & \cdots & T_N & F \end{bmatrix}, \tag{65}$$

$$\mathbf{r}^T = \begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & mg \end{bmatrix}. \tag{66}$$

The corresponding matrices for a straight beam traveled by an oscillator will be deduced in Appendix A.

7 Time discretization of the equation of motion

The final equation of motion is, in matrix form, a second-order Ordinary Differential Equation (ODE), see eq. (61):

$$\mathbf{M}(t)\ddot{\mathbf{q}} + \mathbf{D}(t)\dot{\mathbf{q}} + \mathbf{C}(t)\mathbf{q} = \mathbf{r}(t), \tag{67}$$

where the time-dependent forcing term $\mathbf{r}(t)$ can be expressed as:

$$\mathbf{r}(t) = mg \left[\mathbf{0}_N^T \mathbf{0}_N^T 1\right]^T = mg\, \mathbf{e}_{2N+1}. \tag{68}$$

In eq. (68) $\mathbf{e}_{2N+1}$ is a column matrix consisting of $2N+1$ rows: the first $2N$ entries are filled with zeros, while the last one stores a unit value. The equation of motion eq. (67) can be rewritten as a system of first-order ODEs, with additional state variables $\mathbf{v}$:

$$\mathbf{M}(t)\dot{\mathbf{v}} + \mathbf{D}(t)\dot{\mathbf{q}} + \mathbf{K}(t)\mathbf{q} = \mathbf{r}(t), \tag{69}$$

$$\dot{\mathbf{q}} = \mathbf{v}(t). \tag{70}$$

Time integration is performed along time-steps of equal length h using a linear interpolation (corresponding to a Newmark-like algorithm), for the

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables"*International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017

unknown state variables $\mathbf{q}$, $\mathbf{v}$ and for the given matrices $\mathbf{M}$, $\mathbf{D}$, $\mathbf{C}$ as well:

$$\begin{aligned}
\mathbf{M}(t) &= \left(1 - \frac{t}{h}\right)\mathbf{M}_0 + \frac{t}{h}\mathbf{M}_1,\\
\mathbf{D}(t) &= \left(1 - \frac{t}{h}\right)\mathbf{D}_0 + \frac{t}{h}\mathbf{D}_1,\\
\mathbf{C}(t) &= \left(1 - \frac{t}{h}\right)\mathbf{C}_0 + \frac{t}{h}\mathbf{C}_1,\\
\mathbf{q}(t) &= \left(1 - \frac{t}{h}\right)\mathbf{q}_0 + \frac{t}{h}\mathbf{q}_1,\\
\mathbf{v}(t) &= \left(1 - \frac{t}{h}\right)\mathbf{v}_0 + \frac{t}{h}\mathbf{v}_1,
\end{aligned} \tag{71}$$

where $\mathbf{M}_0 = \mathbf{M}|_{t=t_0}$, $\mathbf{M}_1 = \mathbf{M}|_{t=t_1=t_0+h}$, and so on.

Numerical integration ends up with a corresponding difference equation:

$$\begin{aligned}
\frac{1}{2}\left[\mathbf{M}_0 + \mathbf{M}_1\right](\mathbf{v}_1 - \mathbf{v}_0) + \frac{1}{2}\left[\mathbf{D}_0 + \mathbf{D}_1\right](\mathbf{q}_1 - \mathbf{q}_0) + &\\
\frac{h}{6}\left[\mathbf{C}_0 + 2\mathbf{C}_1\right]\mathbf{q}_1 + \frac{h}{6}\left[2\mathbf{C}_0 + \mathbf{C}_1\right]\mathbf{q}_0 &= mg\,\mathbf{e}_{2N+1},\\
\mathbf{q}_1 - \mathbf{q}_0 &= \frac{h}{2}(\mathbf{v}_0 + \mathbf{v}_1)\,.
\end{aligned} \tag{72}$$

Here, the initial values $\mathbf{q}_0$, $\mathbf{v}_0$ are known by initial conditions, while $\mathbf{q}_1$, $\mathbf{v}_1$ are the unknowns. Column matrix $\mathbf{v}_1$ can be taken explicitly from the last eq. $(72)_2$

$$\mathbf{v}_1 = \frac{2}{h}(\mathbf{q}_1 - \mathbf{q}_0) - \mathbf{v}_0,$$

and eliminated in the other difference equation, eq. $(72)_1$:

$$\begin{aligned}
\frac{1}{h}\left[\mathbf{M}_0 + \mathbf{M}_1 + \frac{h}{2}(\mathbf{D}_0 + \mathbf{D}_1) + \frac{h^2}{6}(\mathbf{C}_0 + 2\mathbf{C}_1)\right]\mathbf{q}_1 &=\\
\frac{1}{h}\left[\mathbf{M}_0 + \mathbf{M}_1 + \frac{h}{2}(\mathbf{D}_0 + \mathbf{D}_1) - \frac{h^2}{6}(2\mathbf{C}_0 + \mathbf{C}_1)\right]\mathbf{q}_0 &+\\
\left[\mathbf{M}_0 + \mathbf{M}_1\right]\mathbf{v}_0 + mg\,\mathbf{e}_{2N+1}&.
\end{aligned} \tag{73}$$

This algebraic system has to be solved for an amount of $n = T/h$ time-steps where $T = L/c$ is the period that the oscillator needs to cross the

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables"*International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017

beam/bridge (whose length, measured along the arc-length abscissa s, is L) at a constant speed, c.

A special situation concerns a sequence of traveling oscillators having a fixed distance L, namely equal to the developed length of the beam. Then the displacements of the beam may increase continuously from one crossing to the next, such that instability occurs. The stability of such parameter-excited problems (here such parameter is the velocity c) can be studied, too, by Floquet's theorem.

8 Stability analysis by Floquet's theorem

The starting point for treating the dynamical stability by means of Floquet's theorem is constituted by the two homogeneous difference equations (72), which are put together into one single system of double order, starting with the first time-step with $j = 0$, $k = 1$ and ending with $j = n-1$, $k = n$:

$$\mathbf{A}_{j,k}\mathbf{z}_k = \mathbf{B}_{j,k}\mathbf{z}_j, \quad \mathbf{z}_k = \mathbf{T}_{j,k}\mathbf{z}_j, \quad \mathbf{T}_{j,k} = \mathbf{A}_{j,k}^{-1}\mathbf{B}_{j,k}, \tag{74}$$

where matrices used in eq. (74) have the following structure:

$$\mathbf{A}_{j,k} = \begin{bmatrix} \mathbf{I} & -\frac{h}{2}\mathbf{I} \\ \mathbf{D}_{j,k} + \frac{h}{6}(\mathbf{C}_j + 2\mathbf{C}_k) & \mathbf{M}_{j,k} \end{bmatrix}, \tag{75}$$

$$\mathbf{B}_{j,k} = \begin{bmatrix} \mathbf{I} & \frac{h}{2}\mathbf{I} \\ \mathbf{D}_{j,k} - \frac{h}{6}(2\mathbf{C}_j + \mathbf{C}_k)\mathbf{C}_{j,k} & \mathbf{M}_{j,k} \end{bmatrix}. \tag{76}$$

In eqs. (75)–(76) these short-hand notations have been used:

$$\mathbf{M}_{j,k} = \frac{1}{2}(\mathbf{M}_j + \mathbf{M}_k), \quad \mathbf{D}_{j,k} = \frac{1}{2}(\mathbf{D}_j + \mathbf{D}_k). \tag{77}$$

Thus the local transition from $\mathbf{z}_j$ to $\mathbf{z}_k = \mathbf{z}_{j+1}$ can be described by means of the local transition matrix $\mathbf{T}_{j,k}$, which is calculated by column-wise solving the linear algebraic system $\mathbf{A}_{j,k}\mathbf{T}_{j,k} = \mathbf{B}_{j,k}$.

However, the stability analysis by Floquet's theorem requires computing the relation

$$\mathbf{z}|_{t=T} = \mathbf{T}_{0,n}\ \mathbf{z}|_{t=0} \tag{78}$$

along the whole typical period of the dynamic system.

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables"*International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017

The global transition matrix $\mathbf{T}_{0,n}$ entering in eq. (78) is built by the product of all transition matrices for each time-step,

$$\mathbf{T}_{0,n} = \mathbf{T}_{n-1,n} \times \mathbf{T}_{n-2,n-1} \times \cdots \times \mathbf{T}_{1,2} \times \mathbf{T}_{0,1}, \tag{79}$$

and the local counting with time increments h is replaced by a global counting along a period $T = nh$, which generates a global difference equation:

$$\mathbf{z}_1 = \mathbf{T}_{0,T}\mathbf{z}_0. \tag{80}$$

This equation is solved by assuming:

$$\mathbf{z}_1 = \lambda^1 \mathbf{z}_0 \quad \text{and thus} \quad \mathbf{z}_k = \lambda^k \mathbf{z}_0, \tag{81}$$

with λ descending from the corresponding eigenvalue-problem:

$$\mathbf{T}_{0,T}\mathbf{x} = \lambda \mathbf{x}. \tag{82}$$

Typically, the eigenvalues λ are complex numbers, which can be written alternatively in the following forms, with $a \in \mathbb{R}$, $b \in \mathbb{R}$:

$$\lambda = a + \mathrm{i}b = \rho^\star \exp(\mathrm{i}\Phi); \quad \rho^\star = \sqrt{a^2 + b^2}; \quad \tan\Phi = \frac{b}{a}. \tag{83}$$

Hence it follows from eq. (81) that

$$\mathbf{z}_k = \rho^{\star k} \exp(\mathrm{i}k\Phi)\mathbf{z}_0. \tag{84}$$

Consequently, the quantities $\mathbf{z}_k$ remain bounded and the solution is stable only if the magnitude (or modulus) $\rho^\star$ of any eigenvalue λ does not exceed the value 1. Otherwise, for any $\rho^\star > 1$ the system is unstable.

It is useful to notice that only the eigenvalue with largest modulus $\rho^\star$ is needed in order to check the stability of the system.

On the other hand, it should be emphasized, even though this property has not been used in this context, that it is the eigenvector $\mathbf{x}^\star$ related to the eigenvalue with $\rho^\star > 1$ which contains the information about the critical mode leading to instability.

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables"*International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017

9 Numerical examples

The numerical examples presented in this section are conceived to outline these main issues:

1. To describe a continuous transition between traveling mass and traveling oscillator using mixed variables;
2. To detect dynamic instability in the time domain by means of a sequence of traveling oscillators and in the spectral domain by means of Floquet's theorem;
3. To evaluate in a quantitative way the non-linear second-order effects produced by the enhanced approximation of the traveling mass kinematics described in Section 4.

9.1 Sequence of traveling oscillators moving along a curved beam

The same data provided by Yang *et al.* [30] have been adopted here:

- Radius of the centerline: $R = 45.840$ m,
- Mass of traveling oscillator: $m = 29869.5$ kg,
- Mass density of the bridge: $\rho = 2400$ kg/m^3,
- Young's modulus of the bridge: $E = 3.22 \cdot 10^{10}$ N/m^2,
- Poisson's ratio: $\nu = 0.2$,
- Cross-section area of the beam: $A = 5.0 \cdot 1.8 = 9.0$ m^2,
- Cross-section moment of inertia about the $\mathbf{k}_1$ axis: $I_1 = 5.0 \cdot 1.8^3/12$ m^4,
- Cross-section polar moment of inertia: $I_P = I_1 + 1.8 \cdot 5.0^3/12$ m^4,
- Aperture angle of the beam: $\theta = \pi/6$
- Beam length measured along the arc-length: $L = R \cdot \theta = 24.00$ m;

All the presented results have been computed by considering a number $N = 3$ of harmonic contributions: the size of the problem, in terms of DOFs is $2 \cdot (2N + 1) = 14$.

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables"*International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017

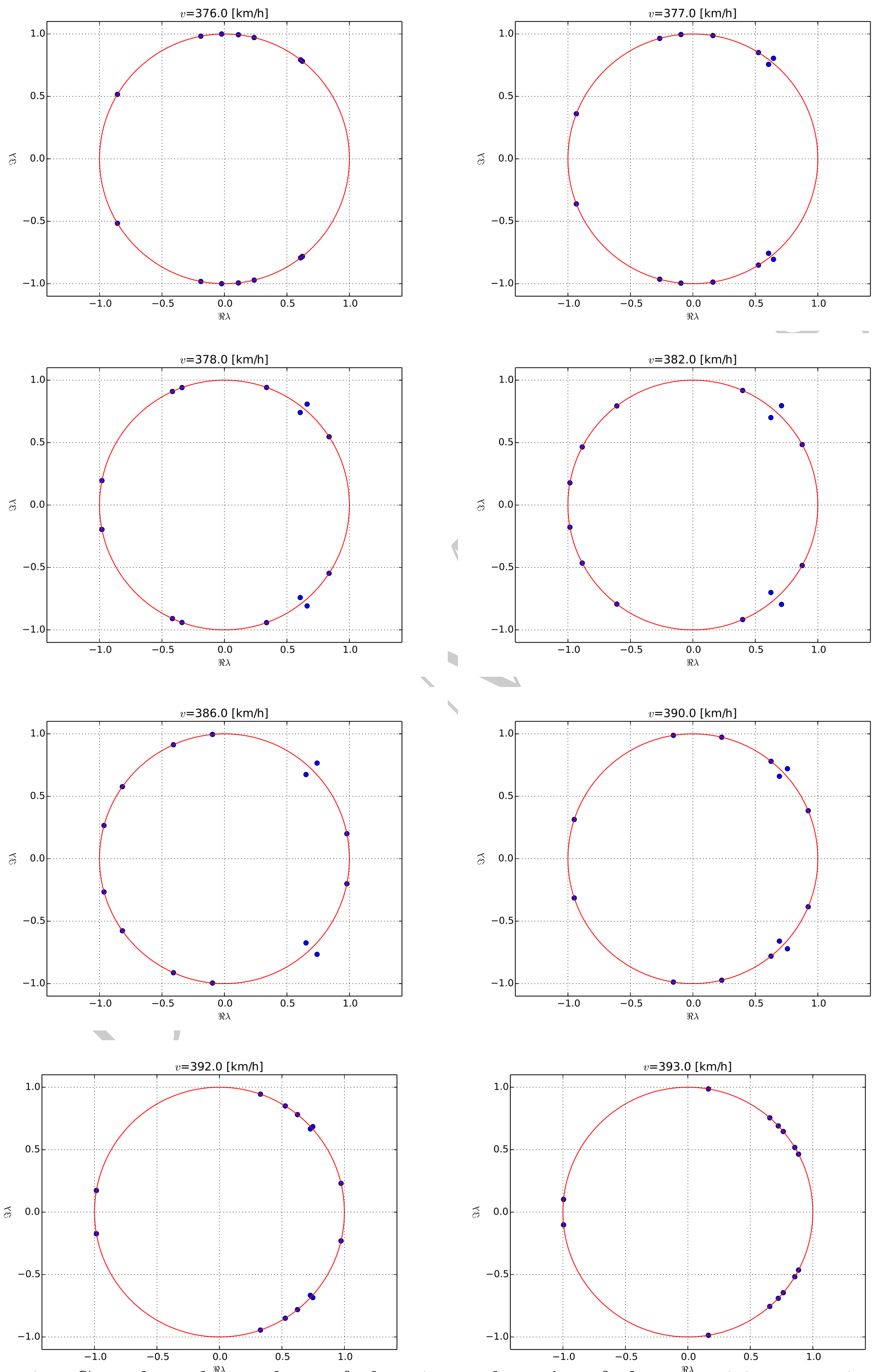
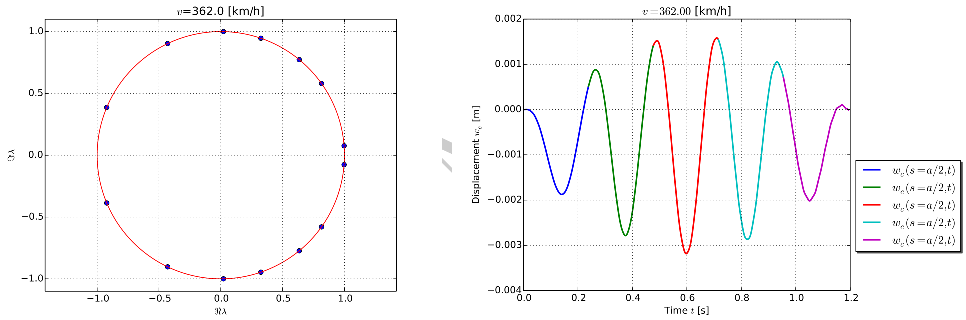


Figure 7: Complex plane plots of the eigenvalues λ_i of the transition matrix for the traveling mass elastically coupled with a plane-curved beam, as a function of the mass speed.

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables" *International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017

The development of the dynamical instability according to the constant velocity of the traveling oscillator is shown in Figure 7 in the spectral domain by means of the eigenvalues λ of the transition matrix in the complex plane. The stable situation is characterized by eigenvalues running around the unit circle. However, in between the values from about $c = 377$ km/h up to about $c = 392$ km/h eigenvalues escape outwards from the unit circle and that indicates instability.

For a velocity $c = 362$ km/h (i.e. outside the critical interval) the displacement of the oscillator along time, as well as the position of the eigenvalues of the corresponding transition matrix in the complex plane, are shown in Figure 8, for a sequence of five masses crossing the bridge. A continuous amplification from one crossing to the following one is not visible.



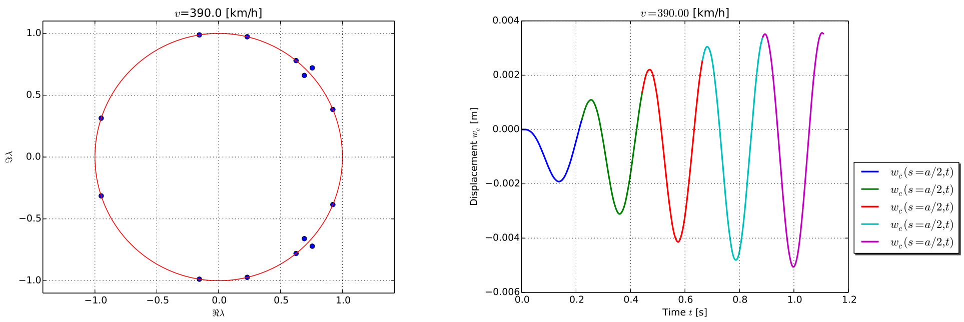
(a) (b)

Figure 8: Complex plane plots of the eigenvalues λ_i of the transition matrix for the traveling mass elastically coupled with a plane-curved beam, for a mass speed $c = 362$ km/h (a) and corresponding displacement history of the 5 equally spaced oscillators as a function of time (b).

The same situation, but now for a critical velocity $c = 390$ km/h is demonstrated in Figure 9, where a continuous increase of the displacements occurs. Obviously, the results in the spectral domain and in the time domain correspond to each other.

The development of a characteristic quantity, for example the coupling force between the oscillator's traveling mass and the supporting beam, is shown along the coupling stiffness in Figure 10. There the largest coupling force F_{max} during the passing of only one oscillator along the beam is shown

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables"*International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017



(a) (b)

Figure 9: Complex plane plots of the eigenvalues λ_i of the transition matrix for the traveling mass elastically coupled with a plane-curved beam, for a mass speed $c = 390$ km/h (a) and corresponding displacement history of the 5 equally spaced oscillators as a function of time (b).

in relation to the static force $F_0 = mg$, for a mass speed $c = 360$ km/h (i.e 100 m/s).

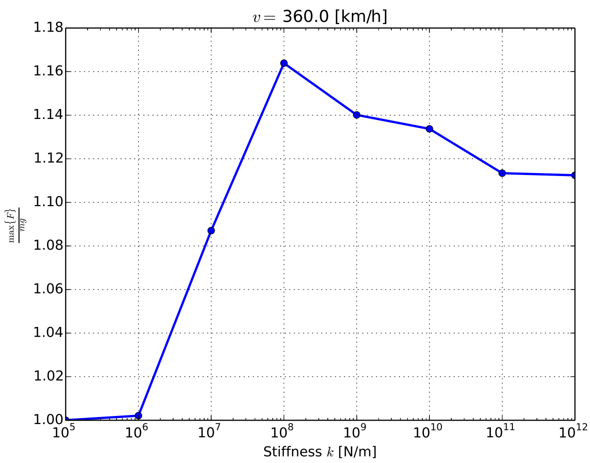


Figure 10: Coupling force between the oscillator's traveling mass and the supporting beam, for a mass speed $c = 360$ km/h, for several values of the elastic stiffness parameter k.

Obviously, the normalized coupling force tends towards the situation of a traveling mass, rigidly connected with the supporting beam in normal di-

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables" *International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017

rection and the force increases as stiffness increases. Before asymptotically reaching the rigid situation there happens a local amplification.

Finally, the center point displacement w_B of the supporting beam is presented in Figure 11 as a function of the time and of the coupling stiffness k [N/m] during the crossing of one oscillator from $t = 0$ (entering the bridge) until $t = 0.6$ s (leaving the bridge) according to the relation $L = cT$, this time with $c = 144$ km/h (i.e. 40 m/s), $L = 24$ m and thus $T = 0.6$ s. All curves, including those for high stiffness values exhibit a smooth behavior and the maximum center point displacement occurs after the oscillator has passed the center point.

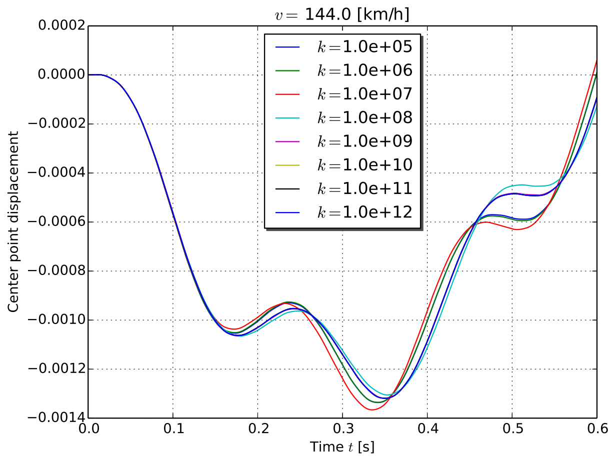


Figure 11: Center displacement w_B of the supporting beam, for a mass speed c = 144 km/h, as a function of the elastic stiffness parameter k.

9.2 *Single mass traveling along a straight beam: second-order nonlinear kinematics*

The numerical influence of the nonlinear second-order correction, eq. (41):

$$\Delta u(x,t) = -\int_0^x \frac{1}{2} {w_{,\eta}}^2 \mathrm{d}\eta$$

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables"*International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017

in eq. (42) is shown *a posteriori* for a beam which is nearly a straight one by adopting a radius $R = 1000$ m of the centerline[1]; furthermore, by assuming a very stiff spring, $k = 4.0 \cdot 10^{12}$ N/m, the traveling oscillator becomes so rigid that the case of a traveling mass is approached. All other data are the same already used for the first example above.

By following the semi-discretization procedure described in Section 5, it is possible to show that for an initially straight beam, the vertical and horizontal components of the traveling mass, eqs. (43) and (44), assume these expressions:

$$a_z(t) = -c^2 \sum_{j=1}^{N} (j\frac{\pi}{L})^2 W_j(t) \sin(j\frac{\pi c t}{L}) + 2c \sum_{j=1}^{N} (j\frac{\pi}{L}) \dot{W}_j(t) \cos(j\frac{\pi c t}{L}) + \sum_{j=1}^{N} \ddot{W}_j(t) \sin(j\frac{\pi c t}{L}),$$

and

$$a_x(t) = -c \left[\sum_{j=1}^{N} (j\frac{\pi}{L}) W_j(t) \cos(j\frac{\pi c t}{L}) \right] \times \left[\sum_{j=1}^{N} (j\frac{\pi}{L}) \dot{W}_j(t) \cos(j\frac{\pi c t}{L}) - c \sum_{j=1}^{N} (j\frac{\pi}{L})^2 W_j(t) \sin(j\frac{\pi c t}{L}) \right].$$

These values, for $c = 390$ km/h are computed along one period:

$$T = \frac{\pi R}{6c} = (\frac{\pi}{6}) \frac{1000 \cdot 3.6}{390} = 4.83 \text{ s}$$

and the results are compared in Figure 12.

As it can be seen from Figure 12 the horizontal acceleration a_x plays a marginal role compared with the vertical acceleration a_z. For the given values, indeed, it comes out that a_z is more than 1000 times larger than a_x. However, for different data and for high performance systems the horizontal component a_x of acceleration can become important and thus, further work should be carried out on this issue.

[1]The same code developed for the curved beam has been used in this case, too; therefore, a large, but finite, value of R was selected, for approximating a truly straight beam

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables"*International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017

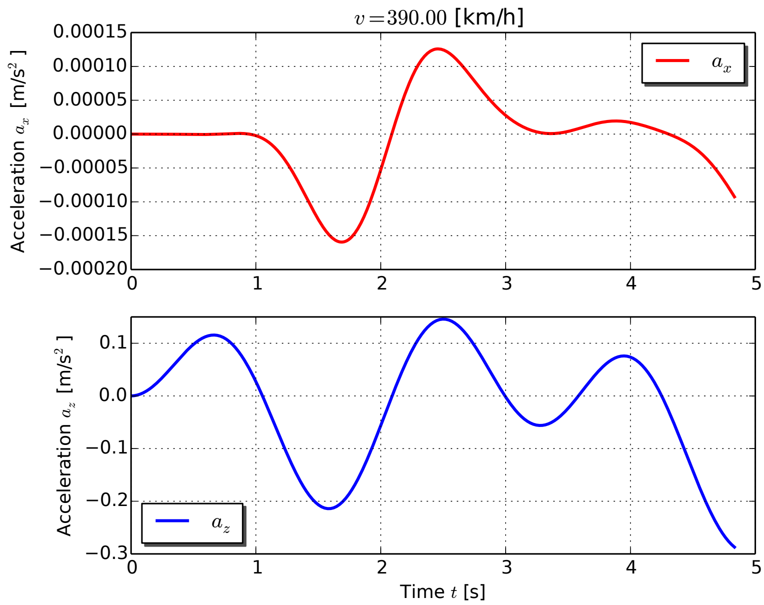


Figure 12: Comparison of the horizontal, a_x, and vertical, a_z, components of acceleration for the enhanced kinematic model of a single mass traveling along a straight beam, for a mass speed $c = 390$ km/h.

10 Conclusions

This paper provides a common kinematics for a mass traveling along a circle-like curved beam as well as for the elastic strains. In order to switch from a traveling mass to a traveling oscillator, papers which are available in literature use either the absolute displacement w_M of the traveling oscillator or the relative displacement $\Delta w = w_M - w_B$ between supporting beam and the oscillator's mass. Here, instead, the coupling force F is introduced, which allows a continuous transition between a traveling oscillator and a traveling mass, to be thought of as a material point rigidly connected to the supporting beam. Even for a coupling stiffness k tending towards infinity and thus modeling a traveling mass, the numerical treatment remains stable due to the inverse appearing like $1/k$ in the equations of motion and as a byproduct, the coupling force appears directly in the solution space. Thus this concept of mixed variables presents a numerically robust tool in order to deal with dynamical problems involving almost rigid connections and can be used for a much larger variety of problems with traveling oscillators including control, earthquake, separation and reattachment. Further research in this direction

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables"*International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017

is on the way.

Concerning the treatment of the dynamical stability of bridges excited periodically by a sequence of traveling oscillators, two different approaches in the time-domain and in the spectral-domain have been elaborated and the results clearly correspond to each other. Thus further studies can use any of these two methods in order to analyze the dynamic stability of systems with a sequence of traveling oscillators.

In addition, an enriched kinematics for a traveling mass running along a straight elastic beam has been presented which shows the so-called normal acceleration component of the whole total second derivative of the position vector of the traveling mass is indeed correctly oriented along the normal direction referred to the supporting flexible beam. This nonlinear enrichment containing a second-order contribution to the horizontal displacement and thus to the corresponding acceleration has been evaluated *a posteriori* for one example and has been found to be marginal compared with the vertical one. However, there is a need of further investigation about this important issue.

Possible extensions of the theory developed in this paper could involve the following topics:

- dynamic and stability analyses by nonlibear methods, including perturbation techniques: [41], [42], [43], [44], [45], [46], [47];
- inverse problems and system identification, see [48], [49], [50];
- applications to micro-structured or higher-gradient materials, as in [51], [52], [53], [54], [55], [56];
- applications to bidimensional structures, like membranes, plates and shells, see [57], [58], [59], [60], [61], [62];
- applications to contact/impact problems or to fractured/damaged structures, see, for instance: [63], [64], [65], [66];
- applications to complex materials: composite, plastic, poro/two phases-elastic, or nonlinear elastic ones, like, for instance, those described in [67], [68], [69], [70], [71], [72], [73].

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables"*International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017

Appendix A Time-variant system for a straight beam

For a straight beam, by assuming $R \to \infty$, and, as a consequence of the decoupling between flexural and torsional response, $\phi_2 = 0\, \forall\, t, s$, matrices $\mathbf{M}$, $\mathbf{D}$, $\mathbf{C}$ corresponding to those which appear in eq. (61) can be easily deduced starting from the more general formulation which holds for the curved beam.

In the case of the straight beam eq. (61) has again this expression:

$$\mathbf{M}(t)\ddot{\mathbf{q}} + \mathbf{D}(t)\dot{\mathbf{q}} + \mathbf{C}(t)\mathbf{q} = \mathbf{r}, \tag{A.1}$$

but this time, due to the flexural-torsional decoupling, matrix dimensions are different: indeed $\{\mathbf{M}(t), \mathbf{D}(t), \mathbf{C}(t)\} \in \mathbb{R}^{(N+1)\times(N+1)}$ and $\{\mathbf{q}(t)\}, \{\mathbf{q}(t)\} \in \mathbb{R}^{N+1}$.

Looking inside the structure of these matrices, and adopting the same notation already used in Section 5, it results, :

$$\begin{aligned}
\mathbf{M}(t) &= \begin{bmatrix} \mu\mathbf{I}_{NN} & \mathbf{0}_N \\ \mathbf{m}_{21}^T & \dfrac{m}{k} \end{bmatrix}, \\
\mathbf{m}_{21}^T &= m \begin{bmatrix} \sin(\alpha t) & \cdots & \sin(N\alpha t) \end{bmatrix},
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
\mathbf{D}(t) &= \begin{bmatrix} \mathbf{0}_{NN} & \mathbf{d}_{12} \\ \mathbf{d}_{21}^T & \dfrac{d}{k} \end{bmatrix}, \\
\mathbf{d}_{12}^T &= -\frac{2d}{Lk} \begin{bmatrix} \sin(\alpha t) & \cdots & \sin(N\alpha t) \end{bmatrix}, \\
\mathbf{d}_{21}^T &= 2m\alpha \begin{bmatrix} \cos(\alpha t) & \cdots & N\cos(N\alpha t) \end{bmatrix},
\end{aligned} \tag{A.3}$$

$$\begin{aligned}
\mathbf{C}(t) &= \begin{bmatrix} \mathbf{C}_{11} & \mathbf{c}_{12} \\ \mathbf{c}_{21}^T & 1 = \dfrac{k}{k} \end{bmatrix}, \\
\mathbf{c}_{21}^T &= -m\alpha^2 \begin{bmatrix} \sin(\alpha t) & \cdots & N^2\sin(N\alpha t) \end{bmatrix}, \\
\mathbf{c}_{12}^T &= -\frac{2}{L} \begin{bmatrix} \sin(\alpha t) & \cdots & \sin(N\alpha t) \end{bmatrix}, \\
\mathbf{C}_{11} &= \mathrm{diag}\left[EI\left(\frac{\pi}{L}\right)^4, \quad \cdots \quad, EI\left(\frac{N\pi}{L}\right)^4 \right],
\end{aligned} \tag{A.4}$$

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables"*International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017

Again, matrices $\mathbf{I}_{NN}$, $\mathbf{0}_{NN}$ and $\mathbf{0}_N$ appearing in eqs. (A.2)–(A.4) are respectively a square identity matrix of order N, a square null matrix of order N and a null column matrix having N rows. Finally, it results:

$$\mathbf{q}^T(t) = \begin{bmatrix} W_1 & \cdots & W_N & F \end{bmatrix}, \tag{A.5}$$

$$\mathbf{r}^T = \begin{bmatrix} 0 & \cdots & 0 & mg \end{bmatrix}. \tag{A.6}$$

Acknowledgement

The financial support of MIUR, the Italian Ministry of Education, University and Research, under grant PRIN 2010–2011 (project 2010MBJK5B—*Dynamic, Stability and Control of Flexible Structures*) is gratefully acknowledged.

References

[1] H. Ouyang, Moving-load dynamic problems: A tutorial (with a brief overview), Mechanical Systems and Signal Processing 25 (2011) 2039–2060.

[2] F. Au, Y. Cheng, Y. Cheung, Vibration analysis of bridges under moving vehicles and trains: an overview, Progress in Structural Engineering and Materials 3 (2001) 299–304.

[3] V. Bolotin, The dynamic stability of elastic systems, Holden-Day, San Francisco, 1964.

[4] L. Fryba, Vibration of solids and structures under moving loads, 3rd ed., Thomas Telford, London, 1999.

[5] F. Romeo, A. Luongo, Invariant representation of propagation properties for bi-coupled periodic structures, Journal of Sound and Vibration 257 (2002) 869–886.

[6] A. Luongo, G. Piccardo, A continuous approach to the aeroelastic stability of suspended cables in 1:2 internal resonance, Journal of Vibration and Control 14 (2008) 135–157.

[7] A. Luongo, D. Zulli, Parametric, external and self-excitation of a tower under turbulent wind flow, Journal of Sound and Vibration 330 (2011) 3057–3069.

[8] A. Luongo, D. Zulli, Dynamic instability of inclined cables under combined wind flow and support motion, Nonlinear Dynamics 67 (2012) 71–87.

[9] G. Piccardo, F. Tubino, Dynamic response of Euler-Bernoulli beams to resonant harmonic moving loads, Structural Engineering and Mechanics 44 (2012) 681–704.

[10] M. Ferretti, G. Piccardo, Dynamic modeling of taut strings carrying a traveling mass, Continuum Mechanics and Thermodynamics 25 (2013) 469–488.

[11] G. Piccardo, F. Tubino, Parametric resonance of flexible footbridges under crowd-induced lateral excitation, Journal of Sound and Vibration 311 (2008) 353–371.

[12] G. Piccardo, F. Tubino, Simplified procedures for vibration serviceability analysis of footbridges subjected to realistic walking loads, Computers & Structures 87 (2009) 890–903.

[13] G. Piccardo, F. Tubino, Equivalent spectral model and maximum dynamic response for the serviceability analysis of footbridges, Engineering Structures 40 (2012) 445–456.

[14] F. Tubino, G. Piccardo, Tuned mass damper optimization for the mitigation of human-induced vibrations of pedestrian bridges, Meccanica 50 (2015) 809–824.

[15] M. Porfiri, F. dell'Isola, F. Mascioli, Circuit analog of a beam and its application to multimodal vibration damping, using piezoelectric transducers, International Journal of Circuit Theory and Applications 32 (2004) 167–198.

[16] M. Porfiri, F. dell'Isola, E. Santini, Modeling and design of passive electric networks interconnecting piezoelectric transducers for distributed vibration control, International Journal of Applied Electromagnetics and Mechanics 21 (2005) 69–87.

[17] S. Alessandroni, U. Andreaus, F. dell'Isola, M. Porfiri, A passive electric controller for multimodal vibrations of thin plates, Computers and Structures 83 (2005) 1236–1250.

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables"*International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017

[18] H. Shen, J. Qiu, H. Ji, K. Zhu, M. Balsi, I. Giorgio, F. dell'Isola, A low-power circuit for piezoelectric vibration control by synchronized switching on voltage sources, Sensors and Actuators, A: Physical 161 (2010) 245–255.

[19] G. Muscolino, A. Palmeri, A. Sofi, Absolute versus relative formulations of the moving oscillator problem, International Journal of Solids and Structures 46 (2009) 1085–1094.

[20] A. Pesterev, L. Bergman, C.-A. Tan, T.-C. Tsao, B. Yang, On asymptotics of the solution of the moving oscillator problem, Journal of Sound and Vibration 260 (2003) 519–536.

[21] A. Pesterev, B. Yang, L. Bergman, C. Tan, Revisiting the moving force problem, Journal of Sound and Vibration 261 (2003) 75–91.

[22] A. V. Pesterev, B. Yang, L. Bergman, C.-A. Tan, Response of elastic continuum carrying multiple moving oscillators, ASCE Journal of Engineering Mechanics 127 (2001) 260–265.

[23] E. Dimitrakopoulos, Q. Zeng, A three-dimensional dynamic analysis scheme for the interaction between trains and curved railway bridges, Computers & Structures 149 (2015) 43–60.

[24] V. Mermertas, Dynamic interaction between the vehicle and simply supported curved bridge deck, Computer Methods in Applied Mechanics and Engineering 162 (1998) 125–131.

[25] J. Genin, E. Ting, Z. Vafa, Curved bridge response to a moving vehicle, Journal of Sound and Vibration 81 (1982) 469–475.

[26] C. Huang, Y. Tseng, C. Hung, An accurate solution for the responses of circular curved beams subjected to a moving load, International Journal for Numerical Methods in Engineering 48 (2000) 1723–1740.

[27] F. Javid, E. Esmailzadeh, D. Younesian, An investigation into the vehicle-curved bridge dynamic interaction, International Journal of Automotive Engineering 1 (2011) 235–243.

[28] M. Rostam, F. Javid, E. Esmailzadeh, D. Younesian, Vibration suppression of curved beams traversed by off-center moving loads, Journal of Sound and Vibration 352 (2015) 1–15.

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables"*International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017

[29] N. Wagner, P. Ruge, Kinetische Stabilität von Kreisbogenträgern unter einer Folge wandernder Einzelmassen (Dynamic stability of circle-like curved beams due to a chain of moving masses), ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik 77 (1997) S357–S358.

[30] Y.-B. Yang, C.-M. Wu, J.-D. Yau, Dynamic response of a horizontally curved beam subjected to vertical and horizontal moving loads, Journal of Sound and Vibration 242 (2001) 519–537.

[31] E. Sapountzakis, A. Kampitsis, Nonlinear response of shear deformable beams on tensionless nonlinear viscoelastic foundation under moving loads, Journal of Sound and Vibration 330 (2011) 5410–5426.

[32] J.-S. Chen, Y.-K. Chen, Steady state and stability of a beam on a damped tensionless foundation under a moving load, International Journal of Non-Linear Mechanics 46 (2011) 180–185.

[33] P. Castro Jorge, F. Simões, A. Pinto da Costa, Dynamics of beams on non-uniform nonlinear foundations subjected to moving loads, Computers & Structures 148 (2015) 26–34.

[34] K. K. Ang, J. Dai, Response analysis of high-speed rail system accounting for abrupt change of foundation stiffness, Journal of Sound and Vibration 332 (2013) 2954–2970.

[35] L. Baeza, H. Ouyang, Dynamics of a truss structure and its moving-oscillator exciter with separation and impact-reattachment, Proceedings of the Royal Society A: Mathematical, Physical & Engineering Sciences 464 (2008) 2517–2533.

[36] D. Stăncioiu, H. Ouyang, J. Mottershead, Vibration of a beam excited by a moving oscillator considering separation and reattachment, Journal of Sound and Vibration 310 (2008) 1128–1140.

[37] E. Winkler, Formänderung und Festigkeit gekrümmter Körper, insbesondere der Ringe (Deformation and resistance of curved bodies, in particular the rings), Der Civilingenieur 4 (1858) 231–248.

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables"*International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017

[38] A. Cazzani, M. Malagù, E. Turco, F. Stochino, Constitutive models for strongly curved beams in the frame of isogeometric analysis, Mathematics and Mechanics of Solids (2015) 1–28. DOI:10.1177/1081286515577043.

[39] W. D. Pilkey, Formulas for Stress, Strain, and Structural Matrices, 2nd ed., Wiley, 2004.

[40] A. O. Belyakov, A. P. Seyranian, A. Luongo, Dynamics of the pendulum with periodically varying length, Physica D: Nonlinear Phenomena 238 (2009) 1589–1597.

[41] A. Luongo, Mode localization by structural imperfections in one-dimensional continuous systems, Journal of Sound and Vibration 155 (1992) 249–271.

[42] A. Luongo, A. Paolone, G. Piccardo, Postcritical behavior of cables undergoing two simultaneous galloping modes, Meccanica 33 (1998) 229–242.

[43] A. Luongo, A. Paolone, Multiple scale analysis for divergence-hopf bifurcation of imperfect symmetric systems, Journal of Sound and Vibration 218 (1998) 527–539.

[44] A. Luongo, Mode localization in dynamics and buckling of linear imperfect continuous structures, Nonlinear Dynamics 25 (2001) 133–156.

[45] A. Di Egidio, A. Luongo, F. Vestroni, A non-linear model for the dynamics of open cross-section thin-walled beams Part I: formulation, International Journal of Non-Linear Mechanics 38 (2003) 1067–1081.

[46] A. Paolone, M. Vasta, A. Luongo, Flexural-torsional bifurcations of a cantilever beam under potential and circulatory forces I: Non-linear model and stability analysis, International Journal of Non-Linear Mechanics 41 (2006) 586–594.

[47] F. Vestroni, A. Luongo, A. Paolone, A perturbation method for evaluating nonlinear normal modes of a piecewise linear two-degrees-of-freedom system, Nonlinear Dynamics 54 (2008) 379–393.

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables"*International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017

[48] E. Turco, Load distribution modelling for pin-jointed trusses by an inverse approach, Computer Methods in Applied Mechanics and Engineering 165 (1998) 291–306.

[49] E. Turco, A strategy to identify exciting forces acting on structures, International Journal for Numerical Methods in Engineering 64 (2005) 1483–1508.

[50] F. Presta, C. Hendy, E. Turco, Numerical validation of simplified theories for design rules of transverse stiffened plate girders, Structural Engineer 86 (2008) 37–46.

[51] F. dell'Isola, L. Rosa, C. Woźniak, Dynamics of solids with micro periodic nonconnected fluid inclusions, Archive of Applied Mechanics 67 (1997) 215–228.

[52] F. dell'Isola, L. Rosa, C. Woźniak, A micro-structured continuum modelling compacting fluid-saturated grounds: The effects of pore-size scale parameter, Acta Mechanica 127 (1998) 165–182.

[53] L. Placidi, S. H. Faria, K. Hutter, On the role of grain growth, recrystallization and polygonization in a continuum theory for anisotropic ice sheets, Annals of Glaciology 39 (2004) 49–52.

[54] L. Placidi, K. Hutter, An anisotropic flow law for incompressible polycrystalline materials, Zeitschrift für Angewandte Mathematik und Physik (ZAMP) 57 (2005) 160–181.

[55] F. dell'Isola, P. Seppecher, A. Madeo, How contact interactions may depend on the shape of cauchy cuts in N-th gradient continua: approach "à la d'Alembert", Zeitschrift für Angewandte Mathematik und Physik (ZAMP) 63 (2012) 1119–1141.

[56] L. Placidi, G. Rosi, I. Giorgio, A. Madeo, Reflection and transmission of plane waves at surfaces carrying material properties and embedded in second-gradient materials, Mathematics and Mechanics of Solids (2013) 1–24. DOI:10.1177/1081286512474016.

[57] A. Cazzani, C. Lovadina, On some mixed finite element methods for plane membrane problems, Computational Mechanics 20 (1997) 560–572.

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables"*International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017

[58] W. Pietraszkiewicz, V. Eremeyev, V. Konopińska, Extended non-linear relations of elastic shells undergoing phase transitions, ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik 87 (2007) 150–159.

[59] H. Altenbach, V. Eremeyev, Analysis of the viscoelastic behavior of plates made of functionally graded materials, ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik 88 (2008) 332–341.

[60] H. Altenbach, V. A. Eremeyev, N. F. Morozov, Linear theory of shells taking into account surface stresses, Doklady Physics 54 (2009) 531–535.

[61] H. Altenbach, V. A. Eremeev, N. F. Morozov, On equations of the linear theory of shells with surface stresses taken into account, Mechanics of Solids 45 (2010) 331–342.

[62] H. Altenbach, V. A. Eremeyev, N. F. Morozov, Surface viscoelasticity and effective properties of thin-walled structures at the nanoscale, International Journal of Engineering Science 59 (2012) 83–89.

[63] M. Cuomo, G. Ventura, Complementary energy approach to contact problems based on consistent augmented Lagrangian formulation, Mathematical and Computer Modelling 28 (1998) 185–204.

[64] U. Andreaus, L. Placidi, G. Rega, Soft impact dynamics of a cantilever beam: equivalent SDOF model versus infinite-dimensional system, Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science 225 (2011) 2444–2456.

[65] L. Greco, M. Cuomo, On the force density method for slack cable nets, International Journal of Solids and Structures 49 (2012) 1526–1540.

[66] L. Contrafatto, M. Cuomo, F. Fazio, An enriched finite element for crack opening and rebar slip in reinforced concrete members, International Journal of Fracture 178 (2012) 33–50.

[67] F. dell'Isola, R. Batra, Saint-Venant's problem for porous linear elastic materials, Journal of Elasticity 47 (1997) 73–81.

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables"*International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017

[68] G. Sciarra, F. dell'Isola, K. Hutter, A solid-fluid mixture model allowing for solid dilatation under external pressure, Continuum Mechanics and Thermodynamics 13 (2001) 287–306.

[69] V. A. Eremeev, A. B. Freidin, L. L. Sharipova, Nonuniqueness and stability in problems of equilibrium of elastic two-phase bodies, Doklady Physics 48 (2003) 359–363.

[70] S. Federico, A. Grillo, S. Imatani, G. Giaquinta, W. Herzog, An energetic approach to the analysis of anisotropic hyperelastic materials, International Journal of Engineering Science 46 (2008) 164–181.

[71] S. Federico, Volumetric-distortional decomposition of deformation and elasticity tensor, Mathematics and Mechanics of Solids 15 (2010) 672–690.

[72] M. Assidi, J.-F. Ganghoffer, Composites with auxetic inclusions showing both an auxetic behavior and enhancement of their mechanical properties, Composite Structures 94 (2012) 2373–2382.

[73] A. Cheviakov, J.-F. Ganghoffer, R. Rahouadj, Finite strain plasticity models revealed by symmetries and integrating factors: The case of Dafalias spin model, International Journal of Plasticity 44 (2013) 47–67.

Please cite this document as: A. Cazzani, N. Wagner, P. Ruge and F. Stochino "Continuous transition between traveling mass and traveling oscillator using mixed variables"*International Journal of Non-Linear Mechanics*, first published on August 24, 2015, DOI:10.1016/j.ijnonlinmec.2015.06.017