NONSYMMETRIC ELLIPTIC OPERATORS WITH WENTZELL BOUNDARY CONDITIONS IN GENERAL DOMAINS

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Abstract. *We study nonsymmetric second order elliptic operators with Wentzell boundary conditions in general domains with sufficiently smooth boundary. The ambient space is a space of L^p- type, $1 \leq p \leq \infty$. We prove the existence of analytic quasicontractive (C_0)-semigroups generated by the closures of such operators, for any $1 < p < \infty$. Moreover, we extend a previous result concerning the continuous dependence of these semigroups on the coefficients of the boundary condition. We also specify precisely the domains of the generators explicitly in the case of bounded domains and $1 < p < \infty$, when all the ingredients of the problem, including the boundary of the domain, the coefficients, and the initial condition, are of class C^∞.*

1. Introduction

In the recent years, after the paper [10], much attention was devoted to the study of symmetric elliptic operators of the type

$$M_0 u = \sum_{i,j=1}^{N} \partial_i (a_{ij}(x) \partial_j u)$$

with Wentzell boundary condition

$$(WBC)_0 \quad M_0 u + \beta \partial_\nu^{\mathcal{A}} u + \gamma u = 0 \quad on \quad \partial\Omega,$$

as generators of analytic semigroups on spaces of L^p type, or on $C(\overline{\Omega})$, defined in a bounded domain Ω of $\mathbf{R}^N$ with sufficiently smooth boundary $\partial\Omega$, see e.g. [2], [3], [8], [12], [16]. Here $\mathcal{A}(\cdot)$ is the $N \times N$ matrix $(a_{ij}(\cdot))$, $\partial_\nu^{\mathcal{A}} u = (\mathcal{A}\nabla u) \cdot \nu$ is the conormal derivative of u with respect to $\mathcal{A}$, $0 < \beta$ and γ are real-valued and all these functions are sufficiently regular. In [11] we discovered that generation results can be also obtained for some classes of nonsymmetric operators of the type

$$Mu = \sum_{i,j=1}^{N} \partial_i (a_{ij}(x) \partial_j u) + \sum_{i=1}^{N} c_i \partial_i u + ru$$

having the coefficients c_j, r sufficiently regular in Ω, with boundary condition

$$Mu + \beta \partial_\nu^{\mathcal{A}} u + \gamma u = 0 \quad on \quad \partial\Omega.$$

On the other hand, in [1] we dealt with the more general Wentzell boundary condition

$$(WBC) \quad Mu + \beta \partial_\nu^{\mathcal{A}} u + \gamma u - q\beta L_\partial u = 0 \quad on \quad \partial\Omega,$$

where Ω is a general domain, $0 < \beta$, $\frac{1}{\beta}$ and γ are bounded, $q \in [0, \infty)$ and L_∂ is a suitable generalization of the Laplace-Beltrami operator. Thus, a natural question arises: Can existence and analyticity results be stated in a more general setting concerning the domain (possibly unbounded domain) and, at the same time, the operator (nonsymmetric operator)? Here we give a positive answer under suitable assumptions on the operator and the boundary condition. Indeed, we prove existence and analyticity of the (C_0) semigroup generated by the closure of the nonsymmetric elliptic operator M equipped with Wentzell boundary condition (WBC) on the space $X_p(\Omega)$, $1 < p < \infty$. For the definition of the spaces $X_p(\Omega)$, as well as the general notation and assumptions, we refer to Section 2. The main results are proved in Section 3. In this framework, the continuous dependence on the coefficients of the boundary condition holds, as an extension of [4, Theorem 3.1] and [1, Section 2]. See Section 4. Finally, in the case of bounded domains, under suitable additional assumptions, an explicit representation of the domain of the generator is given, for $1 < p < \infty$. See Section 5.

2. Notation and Main Assumptions

In the following Ω will be a domain of $\mathbf{R}^N$ having its nonempty boundary $\partial\Omega$ consisting of sufficiently smooth $(N-1)$ dimensional manifolds. *Sufficiently smooth* means that the divergence theorem can be used in Ω, Stokes' theorem can be used on $\partial\Omega$, and the usual trace theorems for Sobolev classes hold. The assumption that $\partial\Omega$ is of class $C^{2+\delta}$ for some $\delta > 0$ is more than enough.

In addition, let us assume that

$(\mathbf{A_1})$ *$\mathcal{A}(x) = (a_{ij}(x))$, $i, j = 1, ..., N$ is an $N \times N$ real Hermitian matrix function on $\overline{\Omega}$ such that $a_{ij} \in C^{1+\varepsilon}(\overline{\Omega}, \mathbf{R}) \cap L^\infty(\Omega, \mathbf{R})$, for some $\varepsilon > 0$, for all i, j and there exist $0 < \alpha_0 \le \alpha_1 < \infty$ such that*

$$\alpha_0 |\xi|^2 \le \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \le \alpha_1 |\xi|^2, \quad x \in \overline{\Omega}, \ \xi \in \mathbf{R}^N, \ \xi = (\xi_1, .., \xi_N). \tag{2.1}$$

$(\mathbf{A_2})$ *$c = (c_i)_{1 \le i \le N} \in C^{1+\varepsilon}(\overline{\Omega}, \mathbf{R}^N) \cap L^\infty(\Omega, \mathbf{R}^N)$, for some $\varepsilon > 0$, $r \in C^\varepsilon(\overline{\Omega}, \mathbf{R}) \cap L^\infty(\Omega, \mathbf{R})$ for some $\varepsilon > 0$.*

$(\mathbf{A_3})$ *$\mathcal{B}(x) = (b_{ij}(x))$, $i, j = 1, ..., N-1$ is an $(N-1) \times (N-1)$ real Hermitian matrix function on $\partial\Omega$ such that $b_{ij} \in C^{1+\varepsilon}(\partial\Omega, \mathbf{R}) \cap L^\infty(\partial\Omega, \mathbf{R})$ for some $\varepsilon > 0$, for all i, j, and*

$$\alpha_0 |\xi|^2 \le \sum_{i,j=1}^{N-1} b_{ij}(x)\xi_i\xi_j \le \alpha_1 |\xi|^2, \quad x \in \partial\Omega, \ \xi \in \mathbf{R}^{N-1}, \ \xi = (\xi_1, .., \xi_{N-1}). \tag{2.2}$$

($\mathbf{A_4}$) $\beta, \gamma \in C^{1+\varepsilon}(\partial\Omega, \mathbf{R}) \cap L^\infty(\partial\Omega, \mathbf{R})$, *for some* $\varepsilon > 0$, $q \in [0, \infty)$, *and there exists* $\delta > 0$ *such that*

$$0 < \delta < \beta(x) < \frac{1}{\delta}, \ |\gamma(x)| < \frac{1}{\delta}$$

for all $x \in \partial\Omega$.

We associate with $\mathcal{A}$, c, r the formal differential operator M defined by

$$Mu = \sum_{i,j=1}^{N} \partial_i(a_{ij}(x)\partial_j u) + \sum_{i=1}^{N} c_i \partial_i u + ru, \quad x \in \Omega \tag{2.3}$$

and we denote by M_0 the 'homogeneous symmetric version' of M corresponding to replacing each of c_i, r by zero, i.e.,

$$M_0 u = \sum_{i,j=1}^{N} \partial_i(a_{ij}(x)\partial_j u).$$

We associate with $\mathcal{B}$ the operator L_∂, given by

$$L_\partial u = \nabla_\tau \cdot (\mathcal{B}(x)\nabla_\tau u), \qquad x \in \partial\Omega. \tag{2.4}$$

Here ∇_τ is the tangential gradient on $\partial\Omega$. The operator L_∂ becomes the Laplace-Beltrami operator Δ_{LB} when $\mathcal{B} = I$, the identity matrix.

We consider the boundary conditions of Wentzell-type as follows,

$$(WBC)_0 \quad M_0 u + \beta\partial_\nu^{\mathcal{A}} u + \gamma u - q\beta L_\partial u + q\tilde{a} \cdot \nabla_\tau u + \tilde{r}u = 0 \quad \textit{on} \quad \partial\Omega,$$

and

$$(WBC) \quad Mu + \beta\partial_\nu^{\mathcal{A}} u + \gamma u - q\beta L_\partial u + q\tilde{a} \cdot \nabla_\tau u + \tilde{r}u = 0 \quad \textit{on} \quad \partial\Omega,$$

where ν is the unit outer normal on $\partial\Omega$ and $\partial_\nu^{\mathcal{A}} u = (\mathcal{A}\nabla u) \cdot \nu$ is the conormal derivative of u with respect to $\mathcal{A}$, $\tilde{r} \in L^\infty(\partial\Omega, \mathbf{R})$, and

$$\tilde{a} \in (W^{1,\infty}(\partial\Omega, \mathbf{R}))^N. \tag{2.5}$$

($\mathbf{A_5}$) *The initial value problem*

$$\frac{\partial w}{\partial t} = \tilde{a} \cdot \nabla_\tau w, \quad w(0, x) = h(x), \quad x \in \partial\Omega, \quad t \geq 0$$

is governed by a generalized translation semigroup $\mathcal{S} = (S(t))_{t \geq 0}$, $w(t, x) = (S(t)h)(x) = h(\delta(x, t))$ *for a suitable* $\delta : \partial\Omega \times \mathbf{R}_+ \to \partial\Omega$ *which can be obtained using the theory* *of characteristics. We assume that $\mathcal{S}$ is a (C_0) semigroup on $Y_p := L^p(\partial\Omega, \frac{dS}{\beta})$ for each $1 < p < \infty$. It follows that*

$$\|S(t)\| \leq K_p e^{\omega_p t}$$

for all $t \geq 0$ and some constants K_p, ω_p.

If $\tilde{a}$ has compact support, then $\mathcal{S}$ is of class (C_0) on each Y_p. If, say, $\Omega = \Omega_1 \setminus \Omega_2$, where Ω_1 is a half space,

$$\Omega_1 = \{x = (x_1, \cdots, x_N) \in \mathbf{R}^N : x_N > 0\}$$

and Ω_2 is a ball whose closure is in the interior of Ω_1, then

$$\partial\Omega_1 = \{x = (x_1, \cdots, x_N) \in \mathbf{R}^N : x_N = 0\}$$

(which can be viewed as $\mathbf{R}^{N-1}$), and $\partial\Omega_2$ is bounded. Then, for any $\tilde{a}$ satisfying (2.5), $\mathcal{S}$ is of class (C_0) on Y_p for all $1 < p < \infty$. But when the part of $\partial\Omega$ *near infinity* is curved, it is not clear that $\mathcal{S}$ will be of class (C_0) on Y_p, thus we assume this is as in $(\mathbf{A_5})$.

The purpose of $(\mathbf{A_5})$ is to allow the boundary conditions $(WBC)_0$ and (WBC) to be nonsymmetric, due to the presence of the first order terms involving $\tilde{a}$. Condition $(\mathbf{A_5})$ allows for these terms to determine a Kato perturbation of the basic symmetric operator. But the full semigroup governed by M and its boundary conditions need not be quasicontractive. Therefore we delay the use of $(\mathbf{A_5})$ until Theorem 3.5.

Following [10] and [8], let us introduce some notation and spaces. We identify every $u \in C(\overline{\Omega})$ with $U = (u|_\Omega, u|_{\partial\Omega})$ and define $X_p(\Omega)$, or simply X_p, to be the completion of $\{u \in C(\overline{\Omega}) : \|u\|_{X_p} < \infty\}$ where the norm $||\cdot||_{X_p}$ is given by

$$||U||_{X_p} := \left(\int_\Omega |u|^p \, dx + \int_{\partial\Omega} |u|^p \frac{dS}{\beta} \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \tag{2.7}$$

or, for $p = \infty$,

$$X_\infty(\Omega) := C(\overline{\Omega})$$

if Ω is bounded, or

$$X_\infty(\Omega) := C_0(\overline{\Omega})$$

if Ω is unbounded, where $C_0(\overline{\Omega})$ is the space of all continuous functions on $\overline{\Omega}$ vanishing at infinity. In any case, $X_\infty(\Omega)$, or briefly X_∞, is equipped with the sup norm

$$||U||_{X_\infty} := ||u||_\infty,$$

where $||u||_\infty = \sup_{x \in \overline{\Omega}} |u(x)|$. In general, a member of X_p is $H = (f, g)$, where $f \in L^p(\Omega)$, $g \in L^p(\partial\Omega, \frac{dS}{\beta})$. Here $\frac{dS}{\beta}$ denotes the natural surface measure dS on $\partial\Omega$ with weight $\frac{1}{\beta}$. Note that, for $p < \infty$, f may not have a trace on $\partial\Omega$, and even if f does, this trace needs not equal g. For $p = 2$, X_2 is a Hilbert space equipped with the inner product

$$< H_1, H_2 >_{X_2} := < f_1, f_2 >_{L^2(\Omega)} + < g_1, g_2 >_{L^2(\partial\Omega, \frac{dS}{\beta})},$$

$$H_i = (f_i, g_i) \in X_2, \quad i = 1, 2.$$

Remark 2.1 Let $\varepsilon \geq 0$ and define

$$\begin{aligned} C_c^{2+\varepsilon}(\overline{\Omega}) :=& \{h \in C^{2+\varepsilon}(\overline{\Omega}) : \text{There is an}\, R = R_h > 0 \\ & \text{such that}\, h(x) = 0 \text{ for } x \in \Omega,\ |x| > R\}. \end{aligned}$$

Observe that for $\varepsilon \geq 0$, the subspace $C_c^{2+\varepsilon}(\overline{\Omega})$ is dense in X_p, $1 \leq p \leq \infty$. Note that if $\partial\Omega$ is bounded, then $h \in C_c^{2+\varepsilon}(\overline{\Omega})$ need not vanish anywhere on $\partial\Omega$.

3. Nonsymmetric Operators in General Domains

The following theorem is proved in detail in [10], [8, Section 3] in the case of bounded domains, provided that in [8] we replace the Laplace-Beltrami operator by L_∂. The extension to general domains in the X_2 case is done in [1]. Combining the arguments of these papers proves the more general and more comprehensive statement of it. We point out that the analyticity holds in X_p, for $1 < p < \infty$, by the Stein interpolation theorem, as discussed in [8].

Theorem 3.1. *Suppose that $r = 0$, $c_i = 0$, for any $1 \leq i \leq N$, $\tilde{a} = \mathbf{0}, \tilde{r} = 0$ and the assumptions $(\mathbf{A_1})$-$(\mathbf{A_4})$ hold. For any $1 \leq p \leq \infty$, denote by $M_{0,p}$ the realization of M_0 in X_p with domain*

$$D(M_{0,p}) = \{U = (u|_\Omega, u|_{\partial\Omega}) \in X_p \,:\, u \in C_c^2(\overline{\Omega}),\, (WBC)_0 \quad \textit{holds}\}.$$

Then, for any $1 \leq p \leq \infty$, the closure G_p of $M_{0,p}$ is quasi-m-dissipative on X_p. Moreover, the semigroup generated by G_p is analytic for any $1 < p < \infty$.

Remark 3.2. *(i)* According to [8, Theorem 3.2], if Ω is bounded and $q = 0$, the analyticity of the semigroup holds on X_1 and X_∞ provided $\partial\Omega$ and all the coefficients are of class C^∞.
(ii) If Ω is bounded and $q > 0$, the analyticity of the semigroup holds on X_1 and X_∞ by the results in [13].

In the following, we shall consider an unbounded domain Ω which satisfies the assumptions stated at the beginning of Section 2. For this kind of Ω, by [1, Section 2] and similar arguments as in [8], analogous results as in Theorem 3.1 hold, provided that the assumptions $(\mathbf{A_1})$-$(\mathbf{A_4})$ remain true.

Now, let us focus on the main results of the paper concerning the nonsymmetric operator M in general domains as above. We will need some preliminary results.

Lemma 3.3. *According to the notation introduced in Section 2, we have*
(i) $X_1 \cap X_\infty$ is dense in X_p for any $p \in [1, \infty]$.
(ii) Let $(\Lambda, \Sigma, \lambda)$ be a σ-finite measure space and denote $L^p = L^p(\Lambda, \Sigma, \lambda)$. If

$$f \in \bigcap_{1<p\leq\infty} L^p \quad \textit{and} \quad \sup_{1<p\leq\infty} \|f\|_p \leq k < \infty, \tag{3.1}$$

then

$$f \in L^1 \quad \textit{and} \quad \|f\|_{L^1} \leq k.$$

Proof. The statement *(i)* follows from the definition of the spaces X_p, $1 \le p \le \infty$. In order to prove *(ii),* let $(\Lambda_n)_{n\ge 1}$ be an increasing sequence in Σ such that for each $n \ge 1$,

$$\lambda(\Lambda_n) < \infty \quad and \quad \bigcup_{n\ge 1} \Lambda_n = \Omega.$$

Let f satisfy (3.1). Let us define $f_n := f\chi_{\Lambda_n}$, where χ_{Λ_n} is the characteristic function of Λ_n, $n \ge 1$. Since

$$|f_n| \le \|f\|_\infty$$

the sequence $|f_n|^{1+\frac{1}{m}} \to |f_n|$ a.e. as $m \to \infty$ and $|f_n|^{1+\frac{1}{m}} \le \max\{\|f\|_\infty^2, \|f\|_\infty\}$ on Λ_n for any $m \ge 1$. By the dominated convergence theorem,

$$f_n \in L^1 \quad and \quad \|f_n\|_1 = \lim_{p\to 1^+} \|f_n\|_p \le k.$$

Thus it follows that $f_n \to f$ a.e., $f \in L^1(\Lambda)$ and $\|f\|_1 \le k$.

Theorem 3.4. *Assume that* $\tilde{a} = 0_{\mathbf{R}^N}, \tilde{r} = 0$ *and that* $(\mathbf{A_1})$-$(\mathbf{A_4})$ *hold. Let* $1 \le p \le \infty$ *and* M_p *be the realization of the operator* M *in* X_p *with domain*

$$D(M_p) = \{U = (u|_\Omega, u|_{\partial\Omega}) \in X_p \, : \, u \in C_c^2(\overline{\Omega}), \, (WBC) \ holds\}.$$

Then, for $1 \le p \le \infty$*, the operator* $M_p - \omega I$ *is dissipative on* X_p *for some* $\omega \in \mathbf{R}$*, and the closure* N_p *of* M_p *is quasi-m-dissipative on* X_p*.*

Proof. Let us consider the case $p = 2$. Let $U, V \in D(M_2)$, $U = (u|_\Omega, u|_{\partial\Omega})$, $V = (v|_\Omega, v|_{\partial\Omega})$ and compute

$$\begin{aligned}
< M_2U, V >_{X_2} &= \int_\Omega (Mu)\overline{v}\, dx + \int_{\partial\Omega} (Mu)\overline{v}\, \frac{dS}{\beta} \\
&= -\sum_{i,j=1}^N \int_\Omega a_{ij}\partial_j u \partial_i \overline{v}\, dx - \int_{\partial\Omega} \gamma u \overline{v}\, \frac{dS}{\beta} \\
&\quad + \int_{\partial\Omega} q(L_\theta u)\overline{v}\, dS + \int_\Omega (\sum_{i=1}^N c_i \partial_i u + ru)\overline{v}\, dx \\
&= -\sum_{i,j=1}^N \int_\Omega a_{ij}\partial_j u \partial_i \overline{v}\, dx - \int_{\partial\Omega} \gamma u \overline{v}\, \frac{dS}{\beta} \\
&\quad - \int_{\partial\Omega} q(\mathcal{B}\nabla_\tau u)\cdot \nabla_\tau \overline{v} dS + \int_\Omega (\sum_{i=1}^N c_i \partial_i u + ru)\overline{v}\, dx,
\end{aligned}$$

by the divergence theorem, the boundary condition and Stokes' theorem on $\partial\Omega$. Moreover, we use the fact that each u in the domain is in $C_c^2(\overline{\Omega})$.

In particular, for all $U = (u|_\Omega, u|_{\partial\Omega}) \in D(M_2)$ we deduce that

$$\begin{aligned}
Re < M_2U, U >_{X_2} \le &- \alpha_0 \int_\Omega |\nabla u|^2\, dx \\
&- \int_{\partial\Omega} \gamma |u|^2 \frac{dS}{\beta} - \alpha_0 \int_{\partial\Omega} q|\nabla_\tau u|^2\, dS \\
&+ Re \int_\Omega (c\cdot \nabla u)\overline{u}\, dx + \int_\Omega r|u|^2\, dx.
\end{aligned} \tag{3.2}$$

Now, for any $\varepsilon > 0$,

$$Re \int_\Omega (c \cdot \nabla u)\overline{u}\, dx \leq \|c\| \|\nabla u\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \\ \leq \|c\| \left(\varepsilon \|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{4\varepsilon} \|u\|^2_{L^2(\Omega)} \right),$$

where $\|c\| = \sup_{1 \leq i \leq N} \|c_i\|_\infty$.

Consequently

$$Re < M_2 U, U >_{X_2} \leq (-\alpha_0 + \varepsilon \|c\|) \|\nabla u\|^2_{L^2(\Omega)} \\ + \|\gamma_-\|_\infty \|u\|^2_{L^2(\partial\Omega, \frac{dS}{\beta})} + \frac{\|c\|}{4\varepsilon} \|u\|^2_{L^2(\Omega)} \\ + \|r_+\|_\infty \|u\|^2_{L^2(\Omega)},$$

where γ_- (resp. r_+) denotes the negative part of γ (resp. the positive part of r).

Assume $\varepsilon > 0$ is such that

$$\varepsilon \|c\| \leq \frac{\alpha_0}{2},$$

and define

$$k = \|\gamma_-\|_\infty + \|r_+\|_\infty + \frac{\|c\|}{4\varepsilon}.$$

It follows that

$$Re < M_2 U, U >_{X_2} \leq -\frac{\alpha_0}{2} \|\nabla u\|^2_{L^2(\Omega)} + k \|u\|^2_{X_2}.$$

Hence M_2 is quasi-dissipative on X_2 and thus its closure N_2 is also quasi-dissipative. In order to prove the range condition for N_2, we consider $Re\,\lambda > \|c\| + \|r_+\|_\infty + \|\gamma_-\|_\infty$. We must solve the equation

$$\lambda U - N_2 U = H, \tag{3.3}$$

for all H in a dense subspace of X_2. Let us take $h : \overline{\Omega} \to \mathbf{C}$, h sufficiently smooth on Ω such that $H = (h|_\Omega, h|_{\partial\Omega}) \in X_1 \cap X_\infty$. For each $U \in D(M_2)$ let us evaluate the inner product of both hand sides of (3.3) by $V \in D(M_2)$. By using similar arguments as above we find that

$$\int_\Omega (\mathcal{A}\nabla u) \cdot \nabla \overline{v}\, dx + \int_\Omega \lambda u \overline{v}\, dx + \int_{\partial\Omega} [\frac{(\gamma + \lambda)}{\beta} u \overline{v} + q(\mathcal{B}\nabla_\tau u) \cdot \nabla_\tau \overline{v}]\, dS \\ + \int_\Omega (\sum_{i=1}^N c_i \partial_i u + ru)\overline{v}\, dx = \int_\Omega h\overline{v}\, dx + \int_{\partial\Omega} h\overline{v}\, \frac{dS}{\beta}. \tag{3.4}$$

Define the sesquilinear form $B_\lambda(U, V)$ as the left hand side of (3.4) and define as $C(V)$ the right hand side. For $q \geq 0$, let us introduce $\mathcal{V}_q$ as

$$\mathcal{V}_0 : = H^1(\Omega) \qquad \text{for} \quad q = 0,$$

$$\mathcal{V}_q : = \{u \in \mathcal{V}_0 : \, u|_{\partial\Omega} \in H^1(\partial\Omega, \frac{dS}{\beta})\} \quad \text{for} \quad q > 0.$$

The norm defined by

$$\|V\|_{\mathcal{V}_q}^2 := \|v\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\partial\Omega,\frac{dS}{\beta})}^2 + \|\sqrt{q}\nabla_\tau v\|_{L^2(\partial\Omega,dS)}^2$$

for $V = (v|_\Omega, v|_{\partial\Omega})$, makes $\mathcal{V}_q$ a Hilbert space continuously embedded into X_2. Then for $q \geq 0$, according to our previous calculations, $B_\lambda(\cdot,\cdot)$ is a sesquilinear form on $\mathcal{V}_q$ which is bounded and coercive and $C(\cdot)$ is a bounded linear functional on $\mathcal{V}_q$. Let H be such that $h \in C_c^{2+\varepsilon}(\overline{\Omega})$. Hence the Lax-Milgram Lemma (see e.g. [15, Theorem 6, p.57]) implies that there exists a unique weak solution U of (3.3) for all H such that $h \in \mathcal{V}_q$. If $h \in C^{2+\varepsilon}(\overline{\Omega})$, then a standard elliptic regularity argument shows that $u \in C^{2+\varepsilon}(\overline{\Omega})$ and satisfies the elliptic equation a.e. Furthermore, since $u|_{\partial\Omega} \in H^1(\partial\Omega, \frac{dS}{\beta})$ (when $q > 0$), it is possible to apply the divergence theorem obtaining that $N_2 U \in X_2$. Thus the assertion follows for $p = 2$.

Now let us show that M_p is quasidissipative on X_p, for all $p \in [1,\infty)$. We start with the case $p > 2$. Let $JU := |U|^{p-2}\overline{U}\chi_{\{U\neq 0\}}$ be the duality map of X_p (modulo a positive constant multiple which depends on $||U||_{X_p}$, for $U \neq 0$). Take $0 \neq U = (u|_\Omega, u|_{\partial\Omega}) \in D(M_p)$. We have

$$\begin{aligned}
< M_pU, JU > &= \int_\Omega M_p u(|u|^{p-2}\overline{u})\chi_{\{u\neq 0\}}\, dx + \int_{\partial\Omega} M_p u(|u|^{p-2}\overline{u})\chi_{\{u\neq 0\}}\frac{dS}{\beta} \\
&= \int_\Omega M_p u(|u|^{p-2}\overline{u})\chi_{\{u\neq 0\}}\, dx - \int_{\partial\Omega} \beta\partial_\nu^{\mathcal{A}} u|u|^{p-2}\overline{u}\chi_{\{u\neq 0\}}\frac{dS}{\beta} \\
&\quad - \int_{\partial\Omega} \gamma u|u|^{p-2}\overline{u}\chi_{\{u\neq 0\}}\frac{dS}{\beta} + \int_{\partial\Omega} q\mathcal{B}(\nabla_\tau u)\cdot(\nabla_\tau(|u|^{p-2}\overline{u})\chi_{\{u\neq 0\}}\, dS \\
&= - \sum_{i,j=1}^N \int_\Omega a_{ij}\partial_j u\partial_i(|u|^{p-2}\overline{u})\chi_{\{u\neq 0\}}\, dx \\
&\quad + \int_\Omega \sum_{I=1}^N c_i\partial_i u|u|^{p-2}\overline{u}\chi_{\{u\neq 0\}}dx + \int_\Omega ru|u|^{p-2}\overline{u}\chi_{\{u\neq 0\}}dx \\
&\quad - \int_{\partial\Omega} q\mathcal{B}(\nabla_\tau u)\cdot(\nabla_\tau(|u|^{p-2}\overline{u})\chi_{\{u\neq 0\}}\, dS - \int_{\partial\Omega}\gamma|u|^p\frac{dS}{\beta}
\end{aligned}$$

by the divergence theorem, the boundary condition and Stokes' theorem. Observe that, if we call

$$Z := \sum_{i,j=1}^N \int_\Omega a_{ij}\partial_j u\partial_i(|u|^{p-2}\overline{u})\chi_{\{u\neq 0\}}\, dx,$$

then $Re\, Z \geq 0$. Indeed

$$\begin{aligned}
Z &= \int_\Omega \sum_{i,j=1}^N a_{ij}\partial_j u\partial_i(\overline{u})(|u|^{p-2})\chi_{\{u\neq 0\}}\, dx \\
&\quad + \int_\Omega \sum_{i,j=1}^N a_{ij}\partial_j u\overline{u}(p-2)|u|^{p-4}Re\,(u\partial_i\overline{u})\chi_{\{u\neq 0\}}\, dx,
\end{aligned}$$

since

$$\begin{aligned}
\partial_j|u|^q = \partial_j\left[(|u|^2)^{\frac{q}{2}}\right] &= \frac{q}{2}|u|^{2(\frac{q}{2}-1)}\left[u\partial_j\overline{u} + (\partial_j u)\overline{u}\right] \\
&= q|u|^{q-2}\, Re\,(u\partial_j\overline{u}),
\end{aligned}$$

where $|u|^2 = u\overline{u} \neq 0$. From

$$Re\,(\overline{u}\partial_i u)\,Re\,(u\partial_i \overline{u}) = [Re\,(u\partial_i \overline{u})]^2 \geq 0$$

and the positive definiteness of $\mathcal{A}(x)$, it follows that $Re\,Z \geq 0$. Arguing in a similar way, one can also deduce that

$$Re \int_{\partial\Omega} q\mathcal{B}(\nabla_\tau u) \cdot (\nabla_\tau(|u|^{p-2}\overline{u})\chi_{\{u\neq 0\}}\, dS \geq 0.$$

Thus, taking into account that all c_i and r_+ are essentially bounded, we deduce that

$$Re\, < M_p U - (\|c\| + \|r_+\|_\infty + \|\gamma_-\|_\infty)U, JU > \leq 0.$$

Hence M_p is quasidissipative for $p > 2$. The same result holds for $p \in [1,2)$ according, for instance, to the proof of [7, Lemma 4.4.3]. Hence, for any $p \in [1,\infty)$, M_p is quasi-dissipative. Now, M_p quasi-dissipative implies

$$\|(\lambda - M_p)^{-1}\| \leq \frac{1}{Re\,\lambda - (\|c\| + \|r_+\|_\infty + \|\gamma_-\|_\infty)} \tag{3.5}$$

for $Re\,\lambda > \|c\| + \|r_+\|_\infty + \|\gamma_-\|_\infty$. Hence the range of $\lambda - N_p$ is closed, where $N_p = \overline{M_p}$. Let us show that it is dense, too. Let $H \in X_p$ for $p \in (1,\infty)$. We can argue for any $H = (h|_\Omega, h|_{\partial\Omega})$, where $h \in C_c^{2+\varepsilon}(\overline{\Omega})$, $\varepsilon > 0$, sufficiently small. Then, by the previous case, there exists $U \in D(N_2), U = (u|_\Omega, u|_{\partial\Omega})$ such that

$$(\lambda - N_2)U = H$$

for $Re\,\lambda$ sufficiently large and

$$\nabla \cdot \mathcal{A}\nabla u + c \cdot \nabla u + ru = h.$$

By elliptic regularity for the Wentzell problem in bounded domains (see [8]), we have that there exists $u \in D(N_p)$ such that

$$\|u\|_{W^{2,p}(\Omega)} \leq K_{p,R}\|h\|_{L^p(\Omega)}$$

if $h \in C_c^{2+\varepsilon}(\overline{\Omega})$. Since $C_c^{2+\varepsilon}(\overline{\Omega})$ is dense on each X_p, then N_p is quasi-m-dissipative on X_p, $1 < p < \infty$. In the case $p = 1$, let $h \in C_c^{2+\varepsilon}(\overline{\Omega})$ and λ sufficiently large, say $\lambda > \omega$. Then, for any $p \in (1,\infty)$ there exists a unique $U \in D(N_p)$ such that

$$\lambda U - N_p U = h$$

for any $p \in (1,\infty)$. Moreover,

$$\|U\|_{X_p} \leq \frac{1}{\lambda - \omega}\|h\|_{X_p} \leq \frac{1}{\lambda - \omega}\, \max\{\|h\|_{X_p}, \|h\|_\infty\} := k.$$

By Lemma 3.1, we deduce that $U \in X_1$ and $\|U\|_{X_1} \leq k$. Hence N_1, the closure of M_1, is quasi-m-dissipative on X_1. Finally, let M_∞ be the realization of M in X_∞ with domain $D(M_\infty)$, obtained by replacing X_p by X_∞ in the definition of $D(M_p)$ and let N_∞ be the closure of M_∞ on X_∞. Then, as a consequence of (3.5), as $p \to \infty$ we deduce that M_∞, and hence, N_∞, is quasidissipative on X_∞. Moreover, since the range condition for each N_p is essentially p-independent, N_∞ is quasi-m-dissipative on X_∞. $\square$

Theorem 3.5. *Suppose that the assumptions* $(\mathbf{A}_1)$-$(\mathbf{A}_5)$ *hold. For any* $1 < p < \infty$, *let* $\widehat{M_p}$ *be the realization of the operator* M *in* X_p *with domain* $D(M_p)$ *as in Theorem 3.4.We use* $\widehat{M_p}$ *and not* M_p *because at this time we include the tangential gradient term. Then, the closure* $\hat{N}_p$ *of* $\hat{M}_p$ *generates an analytic semigroup on* X_p.

We shall show that $\hat{N}_p$ is a Kato perturbation of the operator N_p in Theorem 3.4 in X_p, for $1 < p < \infty$. The proof will require an extension of the Kallman-Rota inequality, which we present now.

Lemma 3.6. *Let* G *generate a uniformly bounded (by* $k > 0$*)* (C_0) *semigroup* $\mathcal{W} = (W(t))_{t \geq 0}$ *on a Banach space* E. *Then, for all* $f \in D(G^2)$ *we have*

$$\|Gf\|^2 \leq 2k(k+1)\|G^2 f\| \, \|f\|. \tag{3.6}$$

Proof. The classical Kallman-Rota inequality is for the case of $k = 1$, in which case the constant in (3.6) is 4. Let $f \in D(G^2), t > 0$. By Taylor's formula,

$$\begin{aligned} W(t)f - f = & \int_0^t \frac{d}{ds} W(s)f \, ds = \int_0^t GW(s)f \, ds \\ = & \int_0^t G \left[f + \int_0^s \frac{d}{dr} W(r)Gf \, dr \right] ds \\ = & tGf + \int_0^t \left(\int_r^t W(r)G^2 f \, ds \right) dr \\ = & tGf + \int_0^t (t-r)W(r)G^2 f \, dr, \end{aligned}$$

whence

$$tGf = (W(t)f - f) - \int_0^t sW(t-s)G^2 f \, ds.$$

It follows that

$$\begin{aligned} t\|Gf\| \leq & (k+1)\|f\| + \int_0^t sk\|G^2 f\| \, ds \\ = & (k+1)\|f\| + \frac{t^2}{2} k\|G^2 f\|. \end{aligned} \tag{3.7}$$

If $G^2 f = 0$, letting $t \to \infty$ shows $Gf = 0$ and there is nothing to prove. If $G^2 f \neq 0$, let us consider the polynomial in t,

$$P(t) = \frac{t^2}{2} k\|G^2 f\| - t\|Gf\| + (k+1)\|f\|.$$

Due to (3.7) we have that $P(t) \geq 0$ for all $t \in \mathbf{R}$. Hence

$$\|Gf\|^2 - 2k(k+1)\|G^2 f\|\|f\| \leq 0$$

and (3.6) follows.

Remark 3.7.

As a consequence of the previous Lemma, for any $\varepsilon > 0$,

$$\|Gf\| \leq \varepsilon \|G^2 f\| + \frac{2k(k+1)}{\varepsilon}\|f\|.$$

Thus if B is any closable operator with $D(G) \subset D(B)$, then (by the closed graph theorem) B is a Kato perturbaion of G^2 (see [14, Corollary 6.9]). Now, let us return to the proof of Theorem 3.5.

Proof of Theorem 3.5. We can interpret $\hat{M}_p$ as the operator matrix

$$\begin{pmatrix} M_{0,p} & 0 \\ -\beta\frac{\partial^{\mathcal{A}}}{\partial\nu} & -\gamma + q\beta L_\partial + q\tilde{a}\nabla_\tau + \tilde{r} \end{pmatrix} + \begin{pmatrix} P_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & P_2 \end{pmatrix},$$

where P_1 (resp. P_2) represents the lower order terms of the operator $\hat{M}_p$ acting on Ω (resp. $\partial\Omega$). Now $M_{0,p}$ includes the terms with $\tilde{a}$ and $\tilde{r}$ described in ($\mathbf{A}_5$).

Observe that, by [1, Section 2], the closure G_2 of $M_{0,2}$ generates an analytic semigroup which is analytic in the sector

$$\Sigma(\theta) = \{z \in \mathbf{C} : \, Re z > 0, |arg\, z| < \theta\}$$

with $\theta = \frac{\pi}{2}$. Similar arguments as in [10, 8] allow us to obtain that G_p generates a semigroup analytic in the sector $\Sigma(\theta_p) = \Sigma(\theta_{p'})$ where for $2 \leq p < \infty, \theta_p = \frac{\pi}{p}$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Alternatively, Neuberger's theorem could be used here (cf. [14]).

When $q = 0$, P_2 is a bounded operator and then $P = \begin{pmatrix} P_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & P_2 \end{pmatrix}$ is a Kato perturbation of $M_{0,p}$ and it follows that $\hat{N}_p$ generates a (C_0) semigroup analytic in $\Sigma(\theta_p), 1 < p < \infty$. When $q > 0$ and $\tilde{a}$ is nonzero, the argument following the proof of Lemma 3.4 shows that P is a Kato perturbation of $M_{0,p}$, but the quasidissipativity of $\hat{N}_p$ need not be valid in this case. Still, $\hat{N}_p$ generates a (C_0) semigroup analytic in $\Sigma(\theta_p)$, $1 < p < \infty$. The theorem is now proved. ☐

4. Continuous Dependence

As a consequence of the Trotter-Neveu-Kato approximation theorem (see e.g. [14, Theorem 7.3]) and according to the results in [4] and [1], one can easily deduce the following theorem.

Theorem 4.1. *Let $\mathbf{N}_0 = \{0, 1, 2, ..\}$, and Ω, $M_k, L_k, \mathcal{A}_k, \mathcal{B}_k$ satisfy the assumptions ($\mathbf{A}_1$)-($\mathbf{A}_4$) and assume that in (WBC) the functions $\tilde{a}, \tilde{r}$ vanish. Then the solution of our problem associated with M_p and (WBC) depends continuously on (β, γ, q) in X_p in the following sense. Let $\beta_k, \gamma_k \in C^1(\partial\Omega)$ be real for $k \in \mathbf{N}_0 = \{0, 1, 2, ..\}$ and suppose that*

$$sup\{|\gamma_k(x)| + \frac{1}{|\gamma_k(x)|} : \, k \in \mathbf{N}_0, \, x \in \partial\Omega\} < \infty,$$

$$sup\{\beta_k(x) + \frac{1}{\beta_k(x)} : k \in \mathbf{N}_0,\, x \in \partial\Omega\} < \infty,$$
$$\beta_k \to \beta_0, \gamma_k \to \gamma_0$$

uniformly as $k \to \infty$ with $\beta_k(x) > 0$ for all k and x. Let $q_k \in (0,\infty)$ for $k \in \mathbf{N}_0$ or $q_k = 0$ for $k \in \mathbf{N}_0$ with $q_k \to q_0$ and

$$X_{kp} = L^p(\Omega, dx) \oplus L^p(\partial\Omega, \frac{dS}{\beta_k}), \quad 1 \le p \le \infty,$$

Let N_{kp} be the corresponding N_p for each $k \in \mathbf{N}_0, p \in [1,\infty]$. Note that X_{kp} and X_{0p} are equal as sets and have uniformly equivalent norms. Let $\mathcal{T}_k = (T_k(t))_{t\ge 0}$, $k \in \mathbf{N}_0$, be the semigroup generated by N_{kp} on X_{kp}.

Then, for all $1 \le p \le \infty$, for any $f \in X_{0p}$, $T_k(t)f \to T_0(t)f$ for all $t \ge 0$, uniformly for t in bounded intervals.

5. Domain Characterization

Let Ω be a bounded domain and assume that $\tilde{a}, \tilde{r}$ vanish and $\partial\Omega$, all the coefficients (a_{ij}), (b_{ij}), c_i, r, β, γ and the initial function $u(0,x) = f(x)$ are all of class C^∞, either on $\overline{\Omega}$ and on $\partial\Omega$. In the symmetric case, we showed under these hypotheses that

$$D(G_2) = H^2(\Omega) \quad \text{if} \quad q = 0$$

and

$$D(G_2) = \{u \in H^2(\Omega) : tr(u) \in H^2(\partial\Omega)\} \quad \text{if} \quad q > 0.$$

We proved this in [4] for $p = 2$ and the symmetric case because we were focussing on hyperbolic problems such as the wave and telegraph equations. The proof was based on the theory of (uniformly) elliptic boundary value problems, developed by Agmon, Douglis, Nirenberg, Lions and others, and extended by Triebel and others for $1 < p < \infty$. This is explained in detail in H. Triebel's book [18]. The context was that of a bounded domain, with the boundary and all functions appearing in the problem being of class C^∞ on their maximal domains. The proof was based on pseudodifferential operator theory, which requires everything to be C^∞.

But now we are considering parabolic problems in the L^p context. The theory in Triebel's book works for Ω bounded, $1 < p < \infty$ and M nonsymmetric (but still uniformly elliptic) and everything being C^∞. Thus with much work on the considerable technical details, we mimicked the proof in [4] and obtained the following result.

Theorem 5.1. *Let Ω be a bounded domain and suppose $\tilde{a}, \tilde{r}$ vanish, $\partial\Omega$ and all the coefficients (a_{ij}), (b_{ij}), c_i, r, β, γ are of class C^∞ on $\overline{\Omega}$ and on $\partial\Omega$.*

Then for $1 < p < \infty$,

$$D(N_p) = W^{2,p}(\Omega) \quad \textit{if} \quad q = 0$$

and

$$D(N_p) = \{u \in W^{2,p}(\Omega) : tr(u) \in W^{2,p}(\partial\Omega)\} \quad \textit{if} \quad q > 0.$$

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