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A new phenomenon in the critical exponent for structurally damped semi-linear evolution equations

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Abstract

In this paper, we find the critical exponent for global small data solutions to the Cauchy problem in $\mathbb{R}^n$, for dissipative evolution equations with power nonlinearities $|u|^p$ or $|u_t|^p$,

$$u_{tt} + (-\Delta)^\delta u_t + (-\Delta)^\sigma u = \begin{cases} |u|^p, \\ |u_t|^p. \end{cases}$$

Here $\sigma, \delta \in \mathbb{N} \setminus \{0\}$, with $2\delta \leq \sigma$. We show that the critical exponent for each of the two nonlinearities is related to each of the two possible asymptotic profiles of the linear part of the equation, which are described by the diffusion equations:

$$v_t + (-\Delta)^{\sigma-\delta} v = 0,$$
$$w_t + (-\Delta)^\delta w = 0.$$

The nonexistence of global solutions in the critical and subcritical cases is proved by using the test function method (under suitable sign assumptions on the initial data), and lifespan estimates are obtained. By assuming small initial data in Sobolev spaces, we prove the existence of global solutions in the supercritical case, up to some maximum space dimension $\bar{n}$, and we derive L^q estimates for the solution, for $q \in (1, \infty)$. For $\sigma = 2\delta$, the result holds in any space dimension $n \geq 1$. The existence result also remains valid if σ and/or δ are fractional.

Keywords: semi-linear evolution equations, critical exponent, global small data solutions, structural damping, test function method
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1. Introduction

In this paper, we look for the critical exponents for global small data solutions to

$$\begin{cases} u_{tt} + (-\Delta)^\delta u_t + (-\Delta)^\sigma u = |u|^p, & t \geq 0, x \in \mathbb{R}^n, \\ (u, u_t)(0, x) = (u_0, u_1)(x), \end{cases} \tag{1}$$

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with $\sigma \in \mathbb{N} \setminus \{0\}$, $\delta \in \mathbb{N}$, and to

$$\begin{cases} u_{tt} + (-\Delta)^\delta u_t + (-\Delta)^\sigma u = |u_t|^p, & t \geq 0, x \in \mathbb{R}^n, \\ (u, u_t)(0, x) = (u_0, u_1)(x), \end{cases} \tag{2}$$

with $\sigma, \delta \in \mathbb{N} \setminus \{0\}$. When $2\delta \leq \sigma$, we prove that these critical exponents are, respectively,

$$p_0 := 1 + \frac{2\sigma}{(n - 2\delta)_+}, \tag{3}$$

$$p_1 := 1 + \frac{2\delta}{n}. \tag{4}$$

By critical exponent we mean that suitable global small data solutions exist in the supercritical case, whereas global solutions cannot exist, under suitable sign assumption on the data, in the critical and subcritical case. The term $(-\Delta)^\delta u_t$ represents a damping term. When $\delta > 0$, the damping is sometimes said to be *structural* (or strong). In the case $n \leq 2\delta$, the notation $p_0 = \infty$ in (3) (see Notation 4 in Section 1.3) denotes that the nonexistence result holds for (1) for any $p > 1$.

Exponents (3) and (4) are easily found by homogeneity arguments when $2\delta \leq \sigma$, whereas the same arguments lead to the exponents $1 + 2\sigma/(n - \sigma)$ and $1 + \sigma/n$, respectively, for (1) and (2), when $2\delta \geq \sigma$. Indeed, by using a quite standard test function method, we prove that global, weak, solutions cannot exist, under suitable sign assumption on the data, for critical and subcritical powers, in all these cases (Theorems 1 and 2), and we prove some lifespan estimates for the local solutions.

On the other hand, it is well-known that existence of global small data solutions may not be proved in the whole supercritical range, for some partial differential equations, as a counterpart of a nonexistence result related to homogeneity arguments. For instance, the critical exponent $1 + \sqrt{2}$ for the existence of global small data solutions to the semilinear wave equation $u_{tt} - \Delta u = |u|^p$ in space dimension $n = 3$ (see [27]) is strictly greater than the critical exponent 2 found by homogeneity arguments [29] (see [19, 20], and the reference therein, for the existence exponent in higher space dimension).

By the converse, in 2001, G. Todorova and B. Yordanov [46] proved global existence of small data solution for the semilinear damped wave equation ($\sigma = 1$ and $\delta = 0$ in (1)),

$$\begin{cases} u_{tt} - \Delta u + u_t = |u|^p, & t \geq 0, x \in \mathbb{R}^n, \\ (u, u_t)(0, x) = (u_0, u_1)(x), \end{cases} \tag{5}$$

in the supercritical range $p > 1+2/n$, by assuming small data in weighted energy space. Here $1+2/n$ is Fujita exponent, obtained by homogeneity arguments (see, in particular, [47]). By only assuming data in Sobolev spaces, the existence result was proved in space dimension $n = 1, 2$ in [25], by using energy methods, and in space dimension $n \leq 5$ in [38], by using $L^r - L^q$ estimates, $1 \leq r \leq q \leq \infty$.

Indeed, the main difference with respect to the wave equation with no dissipation, is that the damping term u_t in (5) produces the *diffusion phenomenon*. This effect modifies the asymptotic profile of the solution to the corresponding linear problem so that it can be described by the solution to a heat equation with suitable initial data (see [23] and, later, [22, 30, 40]).

We mention that, recently, the first author, together with S. Lucente and M. Reissig [12, 14], studied a wave equation with time-dependent dissipation, for which the existence exponent coincides with Fujita exponent in space dimension $n = 1, 2$, and it is larger than this latter in (odd) space dimension $n \geq 3$.

Having in mind that the asymptotic profile of solutions to the linear part of the equation influences the critical exponent for the problem with power nonlinearity, we consider the linear evolution equation related to models (1) and (2):

$$\begin{cases} u_{tt} + (-\Delta)^\delta u_t + (-\Delta)^\sigma u = 0, & t \geq 0, x \in \mathbb{R}^n, \\ (u, u_t)(0, x) = (u_0, u_1)(x). \end{cases} \tag{6}$$

For any $\delta \geq 0$, the equation in (6) is a dissipative σ-evolution equation (it is called, sometimes, 2σ-th order damped wave equation, or with other names). In particular, its energy (see Notation 2 in Section 1.3)

$$E(t) = \frac{1}{2}\|u_t(t,\cdot)\|_{L^2}^2 + \frac{1}{2}\||D|^\sigma u(t,\cdot)\|_{L^2}^2$$

is non-increasing, due to:

$$E'(t) = -\||D|^\delta u_t(t,\cdot)\|_{L^2}^2.$$

Moreover, if $2\delta \in [0,\sigma)$ (*effective* damping, according to the classification introduced in [9]), the solution to (6) may be written as the sum of two terms, whose asymptotic profiles as $t \to \infty$ are described by the solutions to the two diffusion problems [7, 28]:

$$\begin{cases} v_t + (-\Delta)^{\sigma-\delta} v = 0, \\ v(0,x) = f(u_0,u_1,\delta), \end{cases} \tag{7}$$

$$\begin{cases} w_t + (-\Delta)^{\delta} w = 0, \\ w(0,x) = g(u_0,u_1,\sigma,\delta), \end{cases} \tag{8}$$

for suitable f, g. In the limit case $\delta = 0$, the asymptotic profile of u is always described by the solution to (7) (since $w(t,x) = e^{-t}w(0,x)$). Therefore, in view of our previous discussion, when $2\delta \in [0,\sigma)$, it is a natural expectation that global existence of small data solutions to (1) holds in the supercritical range $p > p_0$ (as proved in [46] for $\delta = 0$ and $\sigma = 1$). The second, natural, question that arises is if global existence of small data solutions to (2) holds in the supercritical range $p > p_1$. In particular, this second question is meaningful only if $\delta > 0$ (since $p_1 = 1$ for $\delta = 0$).

The main goal of our paper is to give a positive answer to these questions.

The answer is positive in the easier, limit, case $2\delta = \sigma$ (see [42]), even if the solutions to (7) and (8) no longer describe the asymptotic profile of the solution to (6). On the other hand, in the case $2\delta > \sigma$, the asymptotic profile of the solution is completely different, in particular, the wave structure appears and oscillations come into play (the case $\sigma = \delta = 1$ has been studied into details by R. Ikehata [24]), consistently with the classification introduced in [9]. For this reason, if $2\delta > \sigma$ we cannot expect, in general, an existence result in the whole supercritical range $p > 1 + 2\sigma/(n-\sigma)_+$ in (1) or $p > 1 + \sigma/n$ in (2), as it happened for the semilinear wave equation. Moreover, when $\delta > \sigma$, completely different phenomena appear in the linear equation, with respect to the case $\delta \in [0,\sigma]$ (see Section 9).

In order to prove our existence result for $2\delta \in [0,\sigma)$, we derive suitable, sharp, decay estimates for the linear evolution problem (6). The sharpness of the estimates for (6) is guaranteed by the *diffusion phenomenon*, i.e. by the sharpness of the estimates for the solutions to (7) and (8). These linear estimates can be easily extended, in the limit case $2\delta = \sigma$.

Our existence result remains valid in the supercritical case, even for non integer values of σ, δ, if we denote by $(-\Delta)^b = |D|^{2b}$, the fractional Laplacian operator defined by its action $|D|^{2b} f = \mathfrak{F}^{-1}(|\xi|^{2b}\hat{f})$, where $\mathfrak{F}$ is the Fourier transform with respect to the space variable, and $\hat{f} = \mathfrak{F}f$ (see Notation 2 in Section 1.3), and we replace the usual Sobolev spaces $W^{m,q}$ with Bessel potential spaces $H^{m,q}$, in our statements, when m is not an integer. However, if the exponents σ, δ are non-integer, then the test function method cannot be applied, in general, to prove the non existence counterpart of the existence result.

In the case of structurally damped waves with power nonlinearity $|u|^p$, i.e. $\sigma = 1$ and $\delta \in (0,1]$ in (1), the first global existence result has been obtained by the first author and M. Reissig [15] in low space dimension, by using energy estimates. For $2\delta \in (0,1]$, the existence critical exponent was $p_0 = 1 + 2/(n-2\delta)_+$, whereas for $2\delta \in (1,2]$, the existence of global small data solutions was proved only for $p > 1 + (1+2\delta)/(n-\sigma)_+$. In particular, in the case $\sigma = \delta = 1$, there appeared a gap between the exponent $1 + 3/(n-1)$ and the nonexistence exponent $1 + 2/(n-1)$ found by homogeneity arguments and test function method.

By taking advantage of some linear $L^r - L^q$ estimates, $1 \leq r \leq q \leq \infty$, obtained in [39], the result for $2\delta \in (0,1]$ was later extended by the authors [8] to higher space dimensions, up to some maximum dimension, monotonously tending to ∞ as $2\delta \to 1$.

The difficulty in dealing with higher space dimension is related to the loss of regularity appearing when one deals with $L^q - L^q$ estimates, with $q \in (1,2)$. Indeed, these estimates come into play in a natural way to deal with power nonlinearities $|u|^p$, when $p \in (1,2)$, and the critical exponent eventually becomes smaller than 2 in high space dimension n (for instance, Fujita exponent $1 + 2/n$ is smaller than 2 in space dimension $n \geq 3$).

The loss of regularity for $L^q - L^q$ estimates, with $q \in (1,2)$, is related to the wave structure of the equation at high frequencies, as studied into details in [38] for (5). However, the presence of the structural damping in (6), when $\delta > 0$, generates a smoothing effect on the solution, which does not appear for the classical damping u_t. This smoothing effect allows us to recover the additional regularity by using estimates which are singular at $t = 0$. The singularity order is proportional to $n(\sigma - 2\delta)/2\delta$, and it vanishes at $\sigma = 2\delta$. This effect explains, roughly speaking, the possibility to employ these estimates in higher space dimensions when $\sigma/(2\delta)$ tends to 1.

We conclude this overview, citing a few results about damped evolution operators, with $\sigma > 1$. The linear damped plate equation without rotational inertia corresponds to take $\sigma = 2$ in (6). On the other hand, linear estimates for the damped plate equation with rotational inertia $-\Delta u_{tt}$, for which a regularity-loss type decay appears, has been investigated in [1, 3, 5, 45].

A general result for linear evolution equations, which includes model (6), has been recently obtained in [4]. This result can be easily applied to study global existence of small data solutions for the problem with power nonlinearity $|u|^p$, but in absence of more general $L^r - L^q$ estimates, global existence for all supercritical exponents $p > p_0 = 1 + 2\sigma/(n - 2\delta)_+$ could only be proved, following the ideas in [25], in low space dimension.

For several existence/nonexistence and blow-up results for higher order nonlinear equations, not only of parabolic type, but also of hyperbolic type, and for dispersion and Schrödinger equations, we address the reader to the book of V.A. Galaktionov, E.L. Mitidieri and S.I. Pohozaev [18]. In particular, higher order diffusion equations (see later, (10)) are considered in Section 2.8 in [18] (see also [17]).

Finally, we mention that the first author and E. Jannelli recently found an explicit way to construct a dissipative term for any linear higher order hyperbolic equation and to obtain the long time decay estimates [11] for the solution of the related Cauchy problem, providing the basic tool to investigate small data global solutions in presence of a power nonlinearity.

1.1. Motivation for this paper

The main motivation for this paper is related to the "shape" of the two critical exponents for problems (1) and (2). It is well-know that critical exponents for the existence of small data global solutions are related to the decay rate of the solution in suitable space, which are fixed to deal with the nonlinearity. The linear equation (6) has two possible types of decay rates, each one related to one of the two diffusion problems in (7)-(8), in particular, to its scaling properties (see Section 2.3). According to which norm is considered for the solution, one or the other profile appears.

In this paper, we show that the critical exponent p_0 in (3) for problem (1) is related to the scaling properties of the operator in (7) with nonlinearity $|I_{2\delta}v|^p$, namely,

$$v_t + (-\Delta)^{\sigma-\delta} v = |I_{2\delta}v|^p. \tag{9}$$

Here $I_{2\delta}$ stands for the Riesz potential, roughly speaking, the inverse of $(-\Delta)^\delta$ (see later, Notation 2 in Section 1.3). Equation (9) is formally obtained by deleting u_{tt} in (1) and setting $v = (-\Delta)^\delta u$. The well-known Riesz potential mapping properties motivate why no existence result holds if $n \leq 2\delta$.

On the other hand, the critical exponent p_1 in (4) for problem (2) is related to the scaling of the operator in (8) with nonlinearity $|w|^p$, namely,

$$w_t + (-\Delta)^\delta w = |w|^p. \tag{10}$$

This latter equation is formally obtained by deleting $(-\Delta)^\sigma u$ in (2) and setting $w = u_t$.

This interesting phenomenon only appears in the case of structural damping, i.e. if $\delta > 0$ (since (10) reduces to an ordinary differential equation if $\delta = 0$).

The second motivation for this paper is to obtain sharp $L^r - L^q$ estimates, with $1 < r \leq q < \infty$, for the linear problem, and optimal estimates for the L^q norm of the solution and its time derivative, with $q \in (1,\infty)$, for the nonlinear problem. In the special case $\sigma = 2\delta$, this task can be easily accomplished, also for $1 \leq r \leq q \leq \infty$ (see [6] for $\sigma = 1$ and fractional $\delta = 1/2$). On the other hand, several difficulties appear if $2\delta < \sigma$. In particular, apart from the special case $\sigma = 1$ which we consider in Section 8 (for which $L^1 - L^1$ and $L^\infty - L^\infty$ linear estimates are available), we rely on a multiplier theorem (Theorem 10). In particular, we show how the smoothing effect created by the structural damping allows to use linear estimates with a singularity at $t = 0$ of order lesser than 1, when dealing with the nonlinear problem.

As an additional benefit, thanks to the obtained $L^r - L^q$ estimates in the case of non integer powers σ and/or δ, the maximum space dimension for which we may state our global existence result for any supercritical power, tends to ∞ as $\sigma - 2\delta \to 0$, allowing us to avoid the restriction on the space dimension, coming from to the usual Sobolev embeddings $H^k \subset L^\infty$, $n < 2k$ (see Remark 2.4).

1.2. Two sample models

Before going into details of our result, we present two sample models, to which our result applies, setting $\delta = 1$ and $\sigma = 2, 3$, in (1) and (2).

Example 1.1. Let us consider the semilinear plate equation with strong damping:

$$u_{tt} + \Delta^2 u - \Delta u_t = \begin{cases} |u|^p, \\ |u_t|^p. \end{cases}$$

The critical exponent is $1 + 4/(n-2)_+$ for the power nonlinearity $|u|^p$, and $1 + 2/n$ for the power nonlinearity $|u_t|^p$.

Assuming u_1 sufficiently small in $L^1 \cap L^\infty$, and u_0, together with its first and second derivatives, sufficiently small in $L^1 \cap L^\infty$, we find global existence of small data solutions to (1), for any space dimension $n \geq 3$ and power $p > 1 + 4/(n-2)$, and to (2), for any space dimension $n \geq 1$ and power $p > 1 + 2/n$, and we derive L^q estimates for the solution, for any $q \in [1,\infty]$ (Theorem 5). Global solutions cannot exist, under a suitable sign assumption on u_1, in space dimension $n = 1, 2$ for any $p > 1$, and in space dimension $n \geq 3$, for any $p \in (1, 4/(n-2)]$ (Theorem 1), and in space dimension $n \geq 1$, for any $p \in (1, 1 + 2/n]$ (Theorem 2), respectively.

Example 1.2. Let us consider:

$$u_{tt} - \Delta^3 u - \Delta u_t = \begin{cases} |u|^p, \\ |u_t|^p, \end{cases} \tag{11}$$

in space dimension $n = 3$. The critical exponent is 7 for the power nonlinearity $|u|^p$, and $5/3$ for the power nonlinearity $|u_t|^p$. These exponents correspond to the critical exponents of the problems

$$v_t + \Delta^2 v = |I_2 v|^p,$$
$$w_t - \Delta w = |w|^p,$$

respectively (see (9) and (10)). Assuming u_0 and u_1 sufficiently small in $L^1 \cap L^\infty$, together with their derivatives, up to the order 5 and 2, respectively, we find global existence of small data solutions to (1) and (2), in the supercritical ranges $p > 7$ and $p > 5/3$, respectively, and we derive L^q estimates of the solution for any $q \in (1,\infty)$ (Theorems 6 and 7). Additional details are given in Section 2.4.

Global solutions cannot exist, under a suitable sign assumption on u_1, for any $p \in (1, 7]$ (Theorem 1) and for any $p \in (1, 5/3]$ (Theorem 2), respectively.

1.3. Notation

Through this paper, we use the following.

Notation 1. Let $f, g : \Omega \to \mathbb{R}$ be two functions. We use the notation $f \approx g$ if there exist two constants $C_1, C_2 > 0$ such that $C_1 g(y) \leq f(y) \leq C_2 g(y)$ for all $y \in \Omega$. If the inequalities hold in a neighborhood of some $\bar{y} \in \Omega$, we use the notation $f \sim g$ (as $y \to \bar{y}$).

If the inequality is one-sided, namely, if $f(y) \leq Cg(y)$ (resp. $f(y) \geq Cg(y)$) for all $y \in \Omega$, then we write $f \lesssim g$ (resp. $f \gtrsim g$).

Notation 2. We denote $\hat{f} = \mathfrak{F}f$, the Fourier transform of a function f with respect to the x variable. For $b \geq 0$, we denote by $|D|^b f = \mathfrak{F}^{-1}(|\xi|^b \hat{f})$, the possibly fractional Laplace operator, and by $I_b f = \mathfrak{F}^{-1}(|\xi|^{-b}\hat{f})$, the Riesz potential operator.

Notation 3. Let χ_0, χ_1 be $C_c^\infty(\mathbb{R}^n)$ cut-off nonnegative functions satisfying

$$\chi_0 + \chi_1 = 1, \qquad \operatorname{supp}\chi_0 \subset \{|\xi| \leq 1/2\}, \qquad \text{and} \qquad \operatorname{supp}\chi_1 \subset \{|\xi| \geq 1/4\}.$$

In particular, it follows that $\chi_0 = 1$ in $\{|\xi| \leq 1/4\}$ and $\chi_1 = 1$ in $\{|\xi| \geq 1/2\}$. To localize a function g at low and high frequencies, we denote $g_{\chi_j} = \mathfrak{F}^{-1}(\chi_j \widehat{g})$, $j = 0, 1$.

Notation 4. By $[\cdot] : \mathbb{R} \to \mathbb{N}$, we denote the floor function:

$$[x] = \max\{n \in \mathbb{N} : n \leq x\}.$$

By $(x)_+$ we denote the positive part of $x \in \mathbb{R}$, i.e. $(x)_+ = \max\{x, 0\}$. As usual, we set $1/(x)_+ = \infty$, when $x \leq 0$.

Notation 5. For any $q \in [1, \infty]$, and $m \in \mathbb{N}$, we denote by $W^{m,p} = \{\partial_x^\alpha u \in L^p,\ |\alpha| \leq m\}$ the usual Sobolev space of order m, with $W^{0,p} = L^p$. For $s \in [0, +\infty)$, we denote by H^s the Bessel potential space:

$$H^s = \left\{(1 - |D|^2)^{\frac{s}{2}} f \in L^2\right\}.$$

We recall that $H^s = W^{s,2}$, for $s \in \mathbb{N}$.

Notation 6. For any $q \in [1, \infty]$, we denote by q' its Hölder conjugate, i.e. $q' = q/(q-1)$.

Notation 7. For any $p \in [1, \infty)$, by L^p_{loc} we denote the space of distributions, whose restrictions on any compact subset, are in L^p.

1.4. Scheme of the paper

For the ease of reading, we summarize the various arguments treated in each Section:

- in Section 2.1, we state our nonexistence results and lifespan estimates (Thorems 1, 2, 3 and 4);
- in Section 2.2, we state our existence result in the special, easier, case $2\delta = \sigma$ (Theorem 5);
- in Section 2.3, we discuss the linear estimates, related to the asymptotic profile for (7) or (8);
- in Section 2.4, we state our existence results in the case $2\delta < \sigma$ (Theorems 6, 7, 8 and 9);
- in Section 2.5, we discuss how the critical exponents change if the initial data are not in L^1;
- in Section 2.6, we give some comments about the case of classical damping, i.e. for $\delta = 0$, in (1);
- in Section 2.7, we discuss some extension of our result to more general power-type nonlinearities;

- in Section 3, we prove the non existence results and the lifespan estimates;
- in Section 4, we state and prove the linear low frequencies and high frequencies estimates, which we will use to deal with the nonlinear problem;
- in Section 5, we give a detailed proof of Theorem 5;
- in Section 6, we prove our main results (Theorems 6 and 7), following the steps introduced in Section 5;
- in Section 7, we sketch how to modify the proof in Section 6 to prove Theorems 8 and 9;
- in Section 8, we discuss which improvements are possible in the special case $\sigma = 1$;
- in Section 9, we prove a local existence result for problems (1) and (2) in the energy space.

2. Main result

In the following, we first state the nonexistence result in the subcritical and critical cases (Section 2.1), then we look for global solutions in the supercritical case, in different spaces (all of them, in particular, include the energy space), according to five different cases.

In Section 2.2, we consider the special, easier, case $2\delta = \sigma$, looking for energy solutions to (1) which also verify $u \in L^1 \cap L^\infty$, and for $W^{\sigma,1} \cap W^{\sigma,\infty}$ solutions to (1), which also verify $u_t \in L^1 \cap L^\infty$. For this model, we can prove global existence of small data solutions in any space dimension $n \geq 1$, and derive estimates for the solution in L^q, $1 \leq q \leq \infty$.

In Section 2.4, we consider the case $2\delta < \sigma$, but we replace the $L^1 \cap L^\infty$ additional regularity with an $L^\eta \cap L^{\bar{q}}$ additional regularity, for any small η and large $\bar{q}$. We derive estimates for the solution in L^q, with $q \in [\eta, \bar{q}]$. In the second part of Section 2.4, we only ask additional $L^{\min\{2,p\}}$ regularity to the energy solutions, but we consider higher order energies. Consequently, we derive estimates for the solution in L^q, only for $\min\{2, p\} \leq q \leq 2p$, where the upper bound is due to the use of Gagliardo-Nirenberg inequality.

2.1. *The critical exponent via test function method*

We first give a definition of weak solution to (1) and (2).

Definition 1. Let $p > 1$. We say that $u \in L^p_{\mathrm{loc}}([0,\infty)\times\mathbb{R}^n)$ is a global weak solution to (1), or that $u \in L^1_{\mathrm{loc}}([0,\infty)\times\mathbb{R}^n)$ with $u_t \in L^p_{\mathrm{loc}}([0,\infty)\times\mathbb{R}^n)$, is a global weak solution to (2), if, for any test function $F \in C_c^\infty([0,\infty)\times\mathbb{R}^n)$, it holds:

$$I = \int_0^\infty \int_{\mathbb{R}^n} u(t,x)(F_{tt}(t,x) - (-\Delta)^\delta F_t(t,x) + (-\Delta)^\sigma F(t,x))\,dxdt \\ - \int_{\mathbb{R}^n} u_1(x)\,F(0,x)\,dx + \int_{\mathbb{R}^n} u_0(x)\,(F_t(0,x) - (-\Delta)^\delta F(0,x))\,dx, \quad (12)$$

where

$$I = \int_0^\infty \int_{\mathbb{R}^n} |u(t,x)|^p F(t,x)\,dxdt, \quad \text{or} \quad I = \int_0^\infty \int_{\mathbb{R}^n} |u_t(t,x)|^p F(t,x)\,dxdt,$$

respectively.

Let $T > 0$. We say that $u \in L^p_{\mathrm{loc}}([0,T]\times\mathbb{R}^n)$ is a local weak solution to (1), or that $u \in L^1_{\mathrm{loc}}([0,T]\times\mathbb{R}^n)$ with $u_t \in L^p_{\mathrm{loc}}([0,T]\times\mathbb{R}^n)$, is a local weak solution to (2), if (12) is verified for the test functions as above, under the additional assumption that $\operatorname{supp} F \subset [0,T]\times\mathbb{R}^n$.

Integrating by parts, classical solutions to (1) and (2), are also weak solutions, according to Definition 1.

Remark 2.1. Several results of local existence of solutions to (1) and (2) may be given. For the sake of brevity, we only state a local existence result for energy solutions, postponing the statement and its proof to Section 9. As a consequence of this result, local solutions to (1) and (2), in the weak sense of Definition 1, exist, assuming data in a suitable space. We prove the result for powers p which may belong either to the subcritical ranges in Theorems 1 and 2, or to the supercritical ranges in Theorems 8 and 9.

Theorem 1. *Let $\delta \in \mathbb{N}$, $\sigma \in \mathbb{N} \setminus \{0\}$, and assume that $u_0 = 0$, whereas $u_1 \in L^1$ verifies*

$$\int_{\mathbb{R}^n} u_1(x)\,dx > 0. \tag{13}$$

Then there exists no global weak solution to (1)*:*

- *for any $p > 1$ if $n \leq \min\{2\delta, \sigma\}$;*
- *for any*
$$p \in \left(1, 1 + \frac{2\sigma}{n - \min\{2\delta, \sigma\}}\right],$$
if $n > \min\{2\delta, \sigma\}$.

Moreover, for any fixed $g \in L^1$, verifying (13)*, there exists $C > 0$ such that, for any subcritical value of p, the maximal existence time T of the local solution satisfies*

$$T \leq C\varepsilon^{-\frac{\kappa}{2\sigma p' - (n+\kappa)}}, \qquad \kappa = 2\sigma - \min\{2\delta, \sigma\}, \tag{14}$$

where we set $u_1 = \varepsilon g$, $\varepsilon \in (0, 1)$, as initial data. Here $p' = p/(p-1)$.

Theorem 2. *Let $\delta, \sigma \in \mathbb{N} \setminus \{0\}$, and assume that $u_0 = 0$, whereas $u_1 \in L^1$ verifies* (13)*. Then there exists no global weak solution to* (2) *for any*

$$p \in \left(1, 1 + \frac{\min\{2\delta, \sigma\}}{n}\right].$$

Moreover, for any fixed $g \in L^1$, verifying (13)*, there exists $C > 0$ such that, for any subcritical value of p, the maximal existence time T of the local solution satisfies*

$$T \leq C\varepsilon^{-\frac{\kappa}{\kappa p' - (n+\kappa)}}, \qquad \kappa = \min\{2\delta, \sigma\}, \tag{15}$$

where we set $u_1 = \varepsilon g$, $\varepsilon \in (0, 1)$, as initial data. Here $p' = p/(p-1)$.

Into determinate the critical exponents in Theorems 1 and 2, the assumption that the initial data belong to L^1 plays an essential role. If the L^1 assumption is dropped, the critical exponents change accordingly. This effect is well-known for the dissipative wave equation [26]. Dropping the assumption of initial data in L^1 may also influence the nonexistence result, as the following two results show.

Theorem 3. *Let $\delta \in \mathbb{N}$, $\sigma \in \mathbb{N} \setminus \{0\}$, and assume that $u_0 = 0$, whereas $u_1 \in L^1_{\text{loc}}$ verifies*

$$u_1(x) \geq \varepsilon(1 + |x|)^{-\mu}, \qquad \text{for some } \varepsilon \in (0, 1) \text{ and } \mu < n. \tag{16}$$

Then there exists no global weak solution to (1)*:*

- *for any $p > 1$ if $\mu \leq \min\{2\delta, \sigma\}$;*
- *for any*
$$p \in \left(1, 1 + \frac{2\sigma}{\mu - \min\{2\delta, \sigma\}}\right),$$
if $\mu > \min\{2\delta, \sigma\}$.

Moreover, there exists $C > 0$, independent of ε, such that, for any subcritical value of p, the maximal existence time T of the local solution satisfies

$$T \leq C\varepsilon^{-\frac{\kappa}{2\sigma p'-(\mu+\kappa)}}, \qquad \kappa = 2\sigma - \min\{2\delta, \sigma\}. \tag{17}$$

Theorem 4. *Let $\delta, \sigma \in \mathbb{N} \setminus \{0\}$, and assume that $u_0 = 0$, whereas $u_1 \in L^1_{\mathrm{loc}}$ verifies* (16). *Then there exists no global weak solution to* (2) *for any*

$$p \in \left(1, 1 + \frac{\min\{2\delta, \sigma\}}{\mu}\right).$$

Moreover, there exists $C > 0$, independent of ε, such that, for any subcritical value of p, the maximal existence time T of the local solution satisfies

$$T \leq C\varepsilon^{-\frac{\kappa}{\kappa p'-(\mu+\kappa)}}, \qquad \kappa = \min\{2\delta, \sigma\}. \tag{18}$$

In other words, the parameter μ in (16) in Theorems 3 and 4, plays the role once played by the space dimension n in Theorems 1 and 2. Assumption (16) on the initial data is inspired by [35].

We will prove Theorems 1, 2, 3 and 4, using the test function method. A deep description of the test function method can be found in [33], see also [31, 32, 34, 35].

2.2. Global existence in the special case $2\delta = \sigma$

For the ease of reading, we first discuss the easiest case $\sigma = 2\delta$.

Theorem 5. *Let $n \geq 1$, $2\delta = \sigma$, and $p > p_0 = 1 + 2\sigma/(n-\sigma)_+$ in* (1)*, or $p > p_1 = 1 + \sigma/n$, in* (2).

Then there exists a sufficiently small $\varepsilon > 0$ such that for any data

$$(u_0, u_1) \in \mathcal{A} \times \mathcal{B} := (W^{\sigma,1} \cap W^{\sigma,\infty}) \times (L^1 \cap L^\infty), \qquad \|u_0\|_{\mathcal{A}} + \|u_1\|_{\mathcal{B}} \leq \varepsilon, \tag{19}$$

there exists a global solution

$$u \in C([0,\infty), W^{\sigma,1} \cap W^{\sigma,\infty}) \cap C^1([0,\infty), L^1 \cap L^\infty)$$

to (1) *or* (2)*. Also, for any $q \in [1,\infty]$, the solution to* (1) *or* (2) *satisfies the decay estimate*

$$\|(|D|^\sigma u, u_t)(t,\cdot)\|_{L^q} \lesssim (1+t)^{-\frac{n}{\sigma}\left(1-\frac{1}{q}\right)}(\|u_0\|_{\mathcal{A}} + \|u_1\|_{\mathcal{B}}). \tag{20}$$

Moreover, the solution verifies the estimate

$$\|u(t,\cdot)\|_{L^q} \lesssim (1+t)^{1-\frac{n}{\sigma}\left(1-\frac{1}{q}\right)}(\|u_0\|_{\mathcal{A}} + \|u_1\|_{\mathcal{B}}). \tag{21}$$

Estimates (20) and (21) coincide with the estimates obtained for the corresponding linear problem (6).

Global existence of small data solutions in some supercritical range $(p_j, \bar{p}]$, in the case $2\delta = \sigma$ has been recently proved in low space dimension in [42]. Theorem 5 extends this result to any supercritical power $p > p_j$, in any space dimension $n \geq 1$.

2.3. Asymptotic linear estimates

We will derive estimates with no loss of information, with respect to the linear problem, also for $2\delta \in [0,\sigma)$ in (1) and $2\delta \in (0,\sigma)$ in (2). However, when $2\delta \in (0,\sigma)$, the asymptotic profile for $\|u(t,\cdot)\|_{L^q}$ and $\|u_t(t,\cdot)\|_{L^q}$, where u is the solution to the linear problem (6), changes whether condition

$$n\left(1 - \frac{1}{q}\right) - 2\delta \geq 0, \tag{22}$$

holds or not (see [7, 28] and, later, Proposition 4.1). Condition (22) may be written in a more compact form as $n \geq 2\delta q'$, where q' is the Hölder conjugate of q.

More precisely, under suitable assumption on the data, in particular for $u_1 \in L^1$, with $\hat{u}_1(0) \neq 0$ (i.e. with nonzero integral), the solution to (6) verifies

$$\||D|^b \partial_t^k u(t,\cdot)\|_{L^q} \sim \||D|^b I_{2\delta} \partial_t^k v(t,\cdot)\|_{L^q} + \||D|^b I_{2\delta} \partial_t^k w(t,\cdot)\|_{L^q} \sim t^{-\frac{1}{2(\sigma-\delta)}\left\{n\left(1-\frac{1}{q}\right)+b-2\delta\right\}-k} + t^{-\frac{1}{2\delta}\left\{n\left(1-\frac{1}{q}\right)+b-2\delta\right\}-k}, \tag{23}$$

as $t \to \infty$, where v and w are the solution to (7) and (8), with suitable data $v(0,\cdot), w(0,\cdot) \in L^1$. We remark that the term in brackets in (23) is the same in both profiles, for v and w, since the only difference between equations (7) and (8), is the power of the Laplacian.

One may see that, for $b = \sigma$ and $k = 0$, the term in the brackets in (23) is nonnegative, due to $\sigma \geq 2\delta$. Therefore, $\||D|^\sigma u(t,\cdot)\|_{L^q} \sim t^{-\gamma_q}$ as $t \to \infty$, where γ_q is the nonnegative exponent defined as

$$\gamma_q := \frac{1}{2(\sigma-\delta)}\left\{n\left(1-\frac{1}{q}\right)+\sigma-2\delta\right\}. \tag{24}$$

On the other hand, if we set $b = 0$, then the sign of the term in the brackets in (23) changes whether (22) holds or not. Therefore, $\|\partial_t^k u(t,\cdot)\|_{L^q} \sim t^{-\beta_q-k}$, as $t \to \infty$, where

$$\beta_q := \begin{cases} \dfrac{1}{2(\sigma-\delta)}\left\{n\left(1-\dfrac{1}{q}\right)-2\delta\right\} & \text{if } n \geq 2\delta q', \\ \dfrac{n}{2\delta}\left(1-\dfrac{1}{q}\right)-1 & \text{if } n \leq 2\delta q'. \end{cases} \tag{25}$$

Since these estimates for the solution to (6) are optimal, they are also optimal for global small data solutions to the semilinear problems (1) and (2).

Remark 2.2. The fact that the asymptotic profile of $\|\partial_t^k u(t,\cdot)\|_{L^q}$ changes whether (22) holds, or not, has an interesting consequence on the critical exponent for problems (1) and (2).

- The critical exponent p_0 in (3) for (1) is the solution to $p\beta_p = 1$. In particular, it follows that (22) holds for $q = p_0$, so that the decay rate $t^{-\beta_{p_0}}$ is the same obtained for $\|I_{2\delta} v(t,\cdot)\|_{L^{p_0}}$ in (7). Indeed, p_0 is related to the scaling of (9). We also remark that, if $n \leq 2\delta$, then $\|u(t,\cdot)\|_{L^q}$ decays for no $q \in (1,\infty)$, and this explains why no existence result holds.

- The critical exponent p_1 in (4) for (2) is the solution to $p(\beta_p + 1) = 1$. In particular, it follows that (22) does not hold for $q = p_1$, so that the decay rate $t^{-\beta_{p_1}-1}$ is the same obtained for $\|I_{2\delta} w_t(t,\cdot)\|_{L^{p_1}}$ in (8). Indeed, p_1 is related to the scaling of (10).

Estimates (23) are of special interest when $q = 2$, therefore we also set:

$$\theta_b := \begin{cases} \dfrac{n+2b-4\delta}{4(\sigma-\delta)} & \text{if } n+2b \geq 4\delta, \\ \dfrac{n+2b}{4\delta}-1 & \text{if } n+2b < 4\delta, \end{cases} \tag{26}$$

so that $\||D|^b \partial_t^k u(t,\cdot)\|_{L^2} \sim t^{-\theta_b-k}$, as $t \to \infty$.

As it is well-known, estimates on H^b basis are very useful to manage power nonlinearities, due to the (fractional) homogeneous Sobolev embedding (equivalently, by the mapping properties of the Riesz potential):

$$\|f\|_{L^q} \lesssim \||D|^{n\left(\frac{1}{2}-\frac{1}{q}\right)} f\|_{L^2}, \qquad \forall q \in [2,\infty). \tag{27}$$

In particular, it is clear that

$$\beta_q = \theta_b \iff b = n\left(\frac{1}{2}-\frac{1}{q}\right), \quad \forall q \in [2,\infty), \tag{28}$$

where β_q and θ_b are as in (25) and (26).

2.4. Global existence for $2\delta < \sigma$

Due to the lack of $L^1 - L^1$ linear estimates for the general case $\sigma \neq 1$ (the special case $\sigma = 1$ is discussed in Section 8), we cannot look for solutions with additional $L^1 \cap L^\infty$ regularity for u, and/or for u_t, as we did in Theorem 5. Therefore, we will look for solutions which verify $u \in C([0,\infty), L^\eta \cap L^{\bar q})$, if we consider (1), and $u_t \in C([0,\infty), L^\eta \cap L^{\bar q})$ if we consider (2), for some $\eta \in (1, \min\{2, p\}]$, and for sufficiently large $\bar q < \infty$.

Theorem 6. *Let $2\delta \in [0,\sigma)$, assume that the space dimension $n > 2\delta$ satisfies $n \leq n_0$, where*

$$n_0 = n_0(\sigma,\delta) := \max\left\{ n \in \mathbb{N} :\ n < \frac{2(\sigma + 2\delta)}{\sigma - 2\delta} \right\}, \tag{29}$$

and let $p > p_0$ in (1)*. Then there exist a sufficiently large $M \in [2p, \infty)$ and a sufficiently small $\varepsilon > 0$ such that for any data*

$$(u_0, u_1) \in \mathcal{A} \times \mathcal{B} := (W^{\sigma+2\delta,1} \cap W^{\sigma+2\delta,\infty}) \times (W^{2\delta,1} \cap W^{2\delta,\infty}), \quad \text{with } \|u_0\|_{\mathcal{A}} + \|u_1\|_{\mathcal{B}} \leq \varepsilon, \tag{30}$$

and for any $\eta \in (1, \min\{2, p\}]$ and $\bar q \in [M, \infty)$, there exists a global solution

$$u \in C([0,\infty), L^\eta \cap H^\sigma \cap L^{\bar q}) \cap C^1([0,\infty), L^2)$$

to (1)*. Also, decay estimates*

$$\||D|^\sigma u(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\gamma_2} (\|u_0\|_{\mathcal{A}} + \|u_1\|_{\mathcal{B}}), \tag{31}$$

$$\|u_t(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\beta_2 - 1} (\|u_0\|_{\mathcal{A}} + \|u_1\|_{\mathcal{B}}), \tag{32}$$

hold, and the solution verifies the estimate

$$\|u(t,\cdot)\|_{L^q} \lesssim (1+t)^{-\beta_q} (\|u_0\|_{\mathcal{A}} + \|u_1\|_{\mathcal{B}}), \quad \forall q \in [\eta, \bar q], \tag{33}$$

where γ_q and β_q are defined in (24) *and* (25)*.*

Theorem 7. *Let $2\delta \in (0,\sigma)$, and assume that the space dimension satisfies $n \leq n_1$, where*

$$n_1 = n_1(\sigma,\delta) := \max\left\{ n \in \mathbb{N} :\ n < \frac{4\delta}{\sigma - 2\delta} \right\}, \tag{34}$$

and let $p > p_1$ in (2)*. Then there exist a sufficiently large $M \in [2p, \infty)$ and a sufficiently small $\varepsilon > 0$ such that for any data as in* (30)*, and for any $\eta \in (1, \min\{2, p\}]$ and $\bar q \in [M, \infty)$, there exists a global solution*

$$u \in C([0,\infty), W^{\sigma,\eta} \cap W^{\sigma,\bar q}) \cap C^1([0,\infty), (L^\eta \cap L^{\bar q}))$$

to (2)*. Also, the solution satisfies the decay estimates*

$$\||D|^\sigma u(t,\cdot)\|_{L^q} \lesssim (1+t)^{-\gamma_q} (\|u_0\|_{\mathcal{A}} + \|u_1\|_{\mathcal{B}}), \quad \forall q \in [\eta, \bar q], \tag{35}$$

$$\|u_t(t,\cdot)\|_{L^q} \lesssim (1+t)^{-\beta_q - 1} (\|u_0\|_{\mathcal{A}} + \|u_1\|_{\mathcal{B}}), \quad \forall q \in [\eta, \bar q], \tag{36}$$

where γ_q and β_q are defined in (24) *and* (25)*, as well as estimate* (33)*.*

We notice that $n_j(\sigma,\delta) \nearrow \infty$ as $\sigma/(2\delta) \to 1$, for both $j = 0, 1$.

Remark 2.3. In the regularity obtained for the solution in both Theorems 6 and 7 it appears a loss of regularity, with respect to the initial data. This loss of regularity is related to the employment of linear estimates on $L^q - L^q$ basis, with $q \neq 2$. Indeed, at short time and low frequencies, the solution to (6) has the same structure of the solution to the evolution equation $u_{tt} + (-\Delta)^\sigma u = 0$, without dissipative terms, for which the loss of regularity is a well-known phenomenon (see, for instance, [37, 41]). The loss of regularity in Theorems 6 and 7 could be reduced, fixing the space dimension n and the power nonlinearity p, and modifying accordingly the proofs in Section 6, but this aim is beyond the scope of this paper.

The data regularity $H^{\sigma+2\delta} \times H^{2\delta}$ is obtained by adding to the standard regularity for energy solutions $H^\sigma \times L^2$, an additional power 2δ, which comes by the use of singular estimates (see Section 4.1).

Remark 2.4. Since σ and δ are integers, the condition $2\delta < \sigma$ implies that $\sigma - 2\delta \geq 1$, and this makes the assumption on the maximum space dimension (29) more restrictive than the assumption that guarantees that $H^{\sigma+2\delta}$ embeds in L^∞, i.e. $n < 2(\sigma + 2\delta)$. They are equivalent only for $\sigma - 2\delta = 1$. The same comparison can be made between assumption (34) and the assumption that guarantees that $H^{2\delta}$ embeds in L^∞, i.e. $n < 4\delta$.

For this reason, if one is not interested in having L^η regularity of the solution, for small $\eta > 1$, the maximum space dimension for which the global existence result holds can be improved by using Sobolev embeddings, at least when $\sigma - 2\delta > 1$. This approach is employed in Theorems 8 and 9.

By the converse, if we consider fractional values of σ and/or δ, we can no longer exclude that $\sigma - 2\delta$ is in $(0, 1)$. In this latter case, assumptions (29) and (34) in Theorems 6 and 7 are less restrictive than the corresponding conditions $n < 2(\sigma + 2\delta)$ and $n < 4\delta$, related to the Sobolev embeddings $H^{\sigma+2\delta} \subset L^\infty$ and $H^{2\delta} \subset L^\infty$.

Theorem 8. *Let $2\delta \in [0, \sigma)$, $n > 2\delta$, and fix p in* (1)*, such that*

$$p_0 = 1 + \frac{2\sigma}{n - 2\delta} < p < 1 + \frac{2(\sigma + 2\delta)}{(n - 2(\sigma + 2\delta))_+}. \tag{37}$$

Define

$$\kappa := \max\left\{\frac{n}{2}\left(1 - \frac{1}{p}\right), \sigma\right\}, \tag{38}$$

and assume that the space dimension satisfies

$$n\left(\frac{1}{p} - \frac{1}{2}\right)_+ (\sigma - 2\delta) < \sigma + 2\delta. \tag{39}$$

Then there exists a sufficiently small $\varepsilon > 0$ such that for any data

$$(u_0, u_1) \in \mathcal{A} \times \mathcal{B} := (W^{\sigma+2\delta,1} \cap H^{\sigma+2\delta}) \times (W^{2\delta,1} \cap H^{2\delta}), \quad \text{with } \|u_0\|_{\mathcal{A}} + \|u_1\|_{\mathcal{B}} \leq \varepsilon, \tag{40}$$

and for $\eta := \min\{2, p\}$, there exists a global solution

$$u \in C([0, \infty), L^\eta \cap H^\kappa) \cap C^1([0, \infty), H^{\kappa-\sigma})$$

to (1)*. Also, estimates*

$$\||D|^b u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\theta_b}(\|u_0\|_{\mathcal{A}} + \|u_1\|_{\mathcal{B}}), \quad \forall b \in [0, \kappa], \tag{41}$$

$$\||D|^b u_t(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\theta_b-1}(\|u_0\|_{\mathcal{A}} + \|u_1\|_{\mathcal{B}}), \quad \forall b \in [0, \kappa - \sigma], \tag{42}$$

hold, where θ_b is defined in (26)*. Moreover, if $p < 2$, the solution verifies the estimate*

$$\|u(t, \cdot)\|_{L^q} \lesssim (1 + t)^{-\beta_q}(\|u_0\|_{\mathcal{A}} + \|u_1\|_{\mathcal{B}}), \quad \forall q \in [p, 2], \tag{43}$$

where β_q is defined in (25).

The restriction from above in (37) guarantees that $\kappa < \sigma + 2\delta$ in (38).

Theorem 9. *Let $2\delta \in (0, \sigma)$ and fix p in* (2)*, such that*

$$p_0 = 1 + \frac{2\delta}{n} < p < 1 + \frac{4\delta}{(n - 4\delta)_+}. \tag{44}$$

Define

$$\kappa := \sigma + \frac{n}{2}\left(1 - \frac{1}{p}\right), \tag{45}$$

and assume that the space dimension satisfies

$$n\left(\frac{1}{p}-\frac{1}{2}\right)_+(\sigma-2\delta)<2\delta. \tag{46}$$

Then there exists a sufficiently small $\varepsilon>0$ such that for any data as in (40)*, and for $\eta:=\min\{2,p\}$, there exists a global solution*

$$u\in C([0,\infty),W^{\sigma,\eta}\cap H^\kappa)\cap C^1([0,\infty),L^\eta\cap H^{\kappa-\sigma})$$

to (2)*. Also, estimates* (41) *and* (42) *hold. Moreover, if $p<2$, the solution verifies estimate* (43) *and*

$$\|u_t(t,\cdot)\|_{L^q}\lesssim(1+t)^{-\beta_q-1}(\|u_0\|_{\mathcal{A}}+\|u_1\|_{\mathcal{B}}),\quad \forall q\in[p,2], \tag{47}$$

where β_q is defined in (25)*.*

Remark 2.5. We notice that, for any fixed p, assumption (39) is less restrictive than assumption (29), whereas assumption (46) is less restrictive than assumption (34). Replacing $p=p_0$ in (39) and, respectively, $p=p_1$ in (46), one obtains a bound on the space dimension n, given by

$$n\left(\frac{1}{p_j}-\frac{1}{2}\right)_+<\frac{(1-j)\sigma+2\delta}{\sigma-2\delta}, \tag{48}$$

which remains valid for any exponent p sufficiently close to the critical one.

Example 2.6. Coming back to Example 1.2, the critical exponent for (11) is $1+6/(n-2)_+$ for the power nonlinearity $|u|^p$, and $1+2/n$ for the power nonlinearity $|u_t|^p$. In this paper:

- we prove global existence of small data solutions to problem (1), for any

  $$1+\frac{6}{n-2}<p<1+\frac{10}{(n-10)_+}$$

  in space dimension $3\leq n\leq 20$ (Theorem 8); in space dimension $3\leq n\leq 9$, we derive L^q estimates of the solution for any $q\in(1,\infty)$ (Theorem 6);
- we prove global existence of small data solutions to problem (2), for any

  $$1+\frac{2}{n}<p<1+\frac{4}{(n-4)_+}$$

  in space dimension $1\leq n\leq 6$ (Theorem 9); in space dimension $n\leq 3$, we derive L^q estimates of the solution for any $q\in(1,\infty)$ (Theorem 7).

For any $n\geq 1$, global solutions cannot exist, under a suitable sign assumption on u_1, in the critical and subcritical cases (Theorems 1 and 2).

2.5. Critical exponents for initial data not in L^1

As we previously discussed, Theorems 3 and 4 do not contradict the existence results for $p>p_0$ and for $p>p_1$, proved in Sections 2.2 and 2.4, since (16) implies that $u_1\notin L^1$, more precisely, if $\mu\in[0,n)$ then $u_1\notin L^{\frac{n}{\mu}}$.

Indeed, it is known that the critical exponent for semilinear problems like (1) and (2) changes if the L^1 assumption on the initial data is replaced by an L^m assumption, for $m\in(1,2]$ (see, for instance, [26]). More precisely, assuming small initial data in L^m, $m\in(1,2]$, and in the energy space, the critical exponent to (1) and (2), when $2\delta\leq\sigma$, is then related to the p-th power of the decay rate for the L^m-L^{mp} linear estimate for u and, respectively, u_t, that is,

$$(1+t)^{-\min\left\{\frac{n}{2m(\sigma-\delta)}(p-1)-\frac{\delta}{\sigma-\delta}p,\,\frac{n}{2m\delta}(p-1)-p\right\}},$$

$$(1+t)^{-\min\left\{\frac{n}{2m(\sigma-\delta)}(p-1)-\frac{\delta}{\sigma-\delta}p+p,\,\frac{n}{2m\delta}(p-1)\right\}}.$$

We can see that the role played by the space dimension n when data are small in L^1, is now played by the parameter n/m. In other words, the critical exponents are (see Remark 2.2):

$$p_0(n/m) = 1 + \frac{2m\sigma}{n-2m\delta}, \qquad p_1(n/m) = 1 + \frac{2m\delta}{n}. \tag{49}$$

The global existence of small data solutions for $p > p_0(n/m)$ and $p > p_1(n/m)$, may be easily obtained with minor modifications in the statements and the proofs in this paper. In particular, it is much easier to prove the analogous of Theorems 8 and 9, when initial data are only assumed small in the energy space, namely, setting $m = 2$ in (49). Indeed, in general, one should set $\eta = \min\{2, mp\}$, so that $\eta = 2$ when $m = 2$, for any p.

Then, Theorems 3 and 4 provide the nonexistence counterpart for any $p < p_0(n/m)$ and $p < p_1(n/m)$, fixing a suitable $\mu = \mu(p) > n/m$. It only remains open the problem to prove the nonexistence result at the critical values $p = p_0(n/m)$ and $p = p_1(n/m)$.

2.6. The case of classical damping: $\delta = 0$

In the limit case $\delta = 0$, i.e. when the damping is external or frictional, there is no smoothing effect, so that the singularity employed in the high frequencies estimates do not appear. Our results for (1) remain valid but we have no result for (2) (conditions (34) and (46) are never verified for $\delta = 0$). Indeed, the asymptotic profile of the solution to (6) is always described by the solution to (7) in (23), except for the limit case of $\|\partial_t^k u(t,\cdot)\|_{L^1} \sim t^{-k}$, as $t \to \infty$.

Therefore, the main interest of this paper is when $\delta > 0$, even if the case $\delta = 0$ is included in Theorems 6 and 8.

2.7. Some generalizations of the problem

With minor modifications, it is possible to consider more regular solutions in Theorems 6 and 7, but asking more regularity influences the maximum space dimension where the estimates employed to manage the high frequencies part of the solution are not too singular at $t = 0$ (see later, Section 4.1). By assuming extra regularity for the solution in Theorems 6 and 7, it becomes possible to consider nonlinearities like $|D^\alpha u|^p$, for $\alpha \in \mathbb{N}^n$, with $a = |\alpha| \in [0, \sigma + 2\delta)$, or $|D^\beta u_t|^p$, for $\beta \in \mathbb{N}^n$, with $b = |\beta| \in [0, 2\delta)$. Also, one may consider fractional derivatives in the nonlinearities, i.e. $\||D|^a u|^p$ or $\||D|^b u_t|^p$.

The global existence of small data solutions can then be proved in the supercritical cases $p > p_0(a)$ or $p > p_1(b)$, where:

$$p_0(a) = 1 + \frac{2\sigma - a}{(n + a - 2\delta)_+}, \qquad p_1(b) = 1 + \frac{2\delta - b}{n + b}.$$

These two exponents are, respectively, related to the linear decay rates for $\||D|^a u\|_{L^p}^p$, when the asymptotic profile is described by problem (7), and $\||D|^b u_t\|_{L^p}^p$, when the asymptotic profile is described by problem (8). The exponents are consistent with the ones found in low dimension, in the case $\sigma = 2\delta$, in [42].

The maximum space dimensions corresponding to n_0 and n_1 in Theorems 6 and 7 become:

$$n_0(a) := \max\left\{ n \in \mathbb{N} : \ n < \frac{2(\sigma + 2\delta - a)}{\sigma - 2\delta} \right\},$$
$$n_1(b) := \max\left\{ n \in \mathbb{N} : \ n < \frac{2(2\delta - b)}{\sigma - 2\delta} \right\}.$$

One may proceed similarly for Theorem 5.

We omitted these results and their proofs to make the paper easier for the reader, but no substantial difficulty appears and the extension of the calculations to cover these cases is quite straight-forward. In particular, in Section 4 we derived linear estimates in a more general setting, which covers what is needed to extend the nonlinear arguments to nonlinearities like $\||D|^a u|^p$ or $\||D|^b u_t|^p$.

However, unfortunately, the test function method employed to prove the optimality of the critical exponents when σ and δ are integer, appears to be not directly extendable to deal with power nonlinearities containing spatial derivatives of u.

We also remark that it is possible to modify the linear part of the equation, putting constant positive coefficients, as in

$$u_{tt} + \mu(-\Delta)^{\delta} u_t + \nu(-\Delta)^{\sigma} u = 0,$$

with $\mu, \nu > 0$. Linear estimates remains the same, as well as the nonlinear result. The choice to fix $\mu = \nu = 1$ has been made for the sake of simplicity.

Other generalizations of the nonlinear problem, for which linear estimates obtained in Section 4 can be directly applied, include models with nonlinear memory term, like

$$\int_0^t (t-s)^{-\gamma} \, |u(s,x)|^p \, ds,$$

and systems of nonlinear coupled equations.

3. Proof of the nonexistence results

The first part of the proof of Theorem 1 can be obtained as a modified application of Theorem 4.2 in [16], to the operator

$$Lu = (L_1 + L_2)u,$$

with

$$L_1 u = u_{tt}, \qquad L_2 u = (-\Delta)^{\delta} u_t + (-\Delta)^{\sigma} u,$$

quasi-homogeneous operators of type $(4(\sigma-\delta), 2(\sigma-\delta), 1)$ and $(2\sigma, 2(\sigma-\delta), 1)$, respectively, if $2\delta \leq \sigma$, and

$$L_1 u = u_{tt} + (-\Delta)^{\sigma} u, \qquad L_2 u = (-\Delta)^{\delta} u_t,$$

quasi-homogeneous operators of type $(2\sigma, \sigma, 1)$ and $(\sigma + 2\delta, \sigma, 1)$, respectively, if $2\delta \geq \sigma$. Indeed, according to Definition 2.2 in [16], an operator $L(\partial_t, \partial_x)$ is quasi-homogeneous of type (h, d_1, d_2), if, for any $R > 0$, it holds

$$L(R^{d_1}\tau, R^{d_2}\xi) = R^h L(\tau, \xi).$$

The modification consists into consider the Cauchy problem for $Lu = |u|^p$, instead of the Lioville problem for $Lu \geq |u|^p$ (see [47]). Then, the application of Theorem 4.2 in [16] gives nonexistence of weak solutions for any $p > 1$ such that:

$$(2(\sigma-\delta) + n - \min\{4(\sigma-\delta), 2\sigma\})\, p \leq 2(\sigma-\delta) + n,$$

i.e., $(n-2\delta)(p-1) \leq 2\sigma$, if $2\delta \leq \sigma$, and for any $p > 1$ such that

$$(\sigma + n - \min\{2\sigma, \sigma + 2\delta\})\, p \leq \sigma + n,$$

i.e., $(n-\sigma)(p-1) \leq 2\sigma$, if $\sigma \geq 2\delta$. This concludes the proof of the nonexistence result. A direct proof of the non existence result in Theorem 1 can also be easily obtained. We present it for three reasons: for the ease of reading, to prove the lifespan estimate, and to give the basis of the proof of Theorem 3.

Proof (Theorem 1). We fix a nonnegative, non-increasing, test function $\varphi \in C_c^\infty([0,\infty))$ with $\varphi = 1$ in $[0,1/2]$ and $\operatorname{supp}\varphi \subset [0,1]$, and a nonnegative, radial, test function $\psi \in C_c^\infty(\mathbb{R}^n)$, such that $\psi = 1$ in the ball $B_{1/2}$, and $\operatorname{supp}\psi \subset B_1$. We also assume $\psi(x) \leq \psi(y)$ when $|x| \geq |y|$. Here B_r denotes the ball of radius r, centered at the origin. We may assume (see, for instance, [16, 33]) that

$$\varphi^{-\frac{p'}{p}}\,(|\varphi'|^{p'} + |\varphi''|^{p'}), \qquad \psi^{-\frac{p'}{p}}\,(|\Delta^\delta\psi|^{p'} + |\Delta^\sigma\psi|^{p'}), \qquad \text{are bounded.} \tag{50}$$

We remark that the assumption that δ and σ are integers plays a fundamental role here. Then, for $R \geq 1$, we define:

$$\varphi_R(t) = \varphi(R^{-\kappa}t), \quad \psi_R(x) = \psi(R^{-1}x), \tag{51}$$

for some $\kappa > 0$ which we will fix later.

Let us assume that u is a (global or local) weak solution to (1). Let $R > 0$, and also assume that $R \leq T^\kappa$, if u is a local solution in $[0,T] \times \mathbb{R}^n$. Replacing $F(t,x) = \varphi_R(t)\,\psi_R(x)$ in (12), integrating by parts, and recalling that $u_0 = 0$ and $\varphi_R(0) = 1$, we obtain

$$\int_0^\infty \int_{\mathbb{R}^n} u(\varphi_R''\psi_R - \varphi_R'(-\Delta)^\delta\psi_R + \varphi_R(-\Delta)^\sigma\psi_R)\,dxdt - \int_{\mathbb{R}^n} u_1(x)\,\psi_R(x)\,dx = I_R, \tag{52}$$

where:

$$I_R = \int_0^\infty \int_{\mathbb{R}^n} |u|^p \varphi_R\psi_R\,dxdt.$$

We may now apply Young inequality to estimate:

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^n} &|u|(|\varphi_R''|\,\psi_R + |\varphi_R'|\,|(-\Delta)^\delta\psi_R| + \varphi_R\,|(-\Delta)^\sigma\psi_R|)\,dxdt \\ &\leq \frac{1}{p}\,I_R + \frac{1}{p'}\int_0^\infty \int_{\mathbb{R}^n} (\varphi_R\psi_R)^{-\frac{p'}{p}}\,(|\varphi_R''\psi_R| + |\varphi_R'(-\Delta)^\delta\psi_R| + |\varphi_R(-\Delta)^\sigma\psi_R|)^{p'}\,dxdt,\end{aligned}$$

where $p' = p/(p-1)$. Due to

$$\begin{aligned}\varphi_R'(t) &= R^{-\kappa}(\varphi')(R^{-\kappa}t), & \varphi_R''(t) &= R^{-2\kappa}(\varphi'')(R^{-\kappa}t),\\ (-\Delta)^\delta\psi_R(x) &= R^{-2\delta}((-\Delta)^\delta\psi)(R^{-1}x), & (-\Delta)^\sigma\psi_R(x) &= R^{-2\sigma}((-\Delta)^\sigma\psi)(R^{-1}x),\end{aligned}$$

recalling (50), we may estimate

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^n} (\varphi_R\psi_R)^{-\frac{p'}{p}}\,|\varphi_R''\psi_R|^{p'}\,dxdt &\leq C\,R^{-2\kappa p'+n+\kappa},\\ \int_0^\infty \int_{\mathbb{R}^n} (\varphi_R\psi_R)^{-\frac{p'}{p}}\,|\varphi_R'(-\Delta)^\delta\psi_R|^{p'}\,dxdt &\leq C\,R^{-(\kappa+2\delta)p'+n+\kappa},\\ \int_0^\infty \int_{\mathbb{R}^n} (\varphi_R\psi_R)^{-\frac{p'}{p}}\,|\varphi_R(-\Delta)^\sigma\psi_R|^{p'}\,dxdt &\leq C\,R^{-2\sigma p'+n+\kappa}.\end{aligned}$$

We may now fix

$$\kappa = \max\{2(\sigma-\delta),\sigma\} = 2\sigma - \min\{2\delta,\sigma\},$$

so that, summarizing, we proved that

$$\frac{1}{p'}\,I_R \leq C\,R^{-2\sigma p'+n+\kappa} - \int_{\mathbb{R}^n} u_1(x)\,\psi_R(x)\,dx.$$

Assume, by contradiction, that the solution u is global. Recalling assumption (13), in the subcritical case $p < 1 + 2\sigma/(n+\kappa-2\sigma)$, it follows that $I_R < 0$, for any sufficiently large R, and this contradicts the fact that $I_R \geq 0$. The critical case $p = 1 + 2\sigma/(n + \kappa - 2\sigma)$ is treated in standard way, but we omit the details for the sake of brevity. Therefore, u cannot be a global solution.

To prove the lifespan estimate in the subcritical case, we first notice that, for any fixed $g \in L^1$, verifying (13), there exists $\bar{R} > 0$ such that

$$\int_{\mathbb{R}^n} g(x)\,\psi_R(x)\,dx \geq c > 0, \qquad \forall R \geq \bar{R}.$$

Assume that u is a local solution in $[0,T]\times\mathbb{R}^n$, with $T \geq \bar{R}^\kappa$. Then, setting $R = T^{\frac{1}{\kappa}}$ we may define I_R and, recalling $u_1 = \varepsilon g$, we obtain:

$$0 \leq \frac{1}{p'}\,I_R \leq C\,R^{-2\sigma p'+n+\kappa} - \int_{\mathbb{R}^n} u_1(x)\,\psi_R(x)\,dx \leq C\,T^{-\frac{2\sigma p'-(n+\kappa)}{\kappa}} - c\varepsilon.$$

Therefore, we derive (14), and this concludes the proof.

Proof (Theorem 2). As in the proof of Theorem 1, we fix a nonnegative, non-increasing, test function $\varphi \in C_c^\infty([0,\infty))$ with $\varphi = 1$ in $[0, 1/2]$ and $\operatorname{supp}\varphi \subset [0,1]$, and a nonnegative, radial, test function $\psi \in C_c^\infty(\mathbb{R}^n)$, such that $\psi = 1$ in the ball $B_{1/2}$, and $\operatorname{supp}\psi \subset B_1$. We also assume $\psi(x) \leq \psi(y)$ when $|x| \geq |y|$. We assume (50), and we define φ_R and ψ_R as in (51).

Let $\Phi_R \in C_c^\infty([0,\infty))$ be the test function defined by

$$\Phi_R(t) = \int_t^\infty \varphi_R(s)\,ds.$$

(Indeed, we notice that $\operatorname{supp}\Phi_R \subset [0, R^\kappa]$, since $\operatorname{supp}\varphi_R \subset [0, R^\kappa]$). In particular, $\Phi_R' = -\varphi_R$.

Let u be a (local or global) solution to (2). Let $R > 0$, and also assume that $R \leq T^\kappa$, if u is a local solution in $[0,T]\times\mathbb{R}^n$. Replacing $F(t,x) = \varphi_R(t)\,\psi_R(x)$ in (12), integrating by parts, and recalling that $u_0 = 0$ and $\varphi_R(0) = 1$, we obtain

$$\int_0^\infty \int_{\mathbb{R}^n} u_t(-\varphi_R'\psi_R + \varphi_R(-\Delta)^\delta\psi_R + \Phi_R(-\Delta)^\sigma\psi_R)\,dxdt - \int_{\mathbb{R}^n} u_1(x)\,\psi_R(x)\,dx = I_R, \tag{53}$$

where:

$$I_R = \int_0^\infty \int_{\mathbb{R}^n} |u_t|^p \varphi_R\psi_R\,dxdt.$$

We may now apply Young inequality to estimate:

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^n} &|u_t|(|\varphi_R'|\psi_R + \varphi_R|(-\Delta)^\delta\psi_R| + \Phi_R|(-\Delta)^\sigma\psi_R|)\,dxdt \\ &\leq \frac{1}{p}\,I_R + \frac{1}{p'}\int_0^\infty \int_{\mathbb{R}^n} (\varphi_R\psi_R)^{-\frac{p'}{p}}\,(|\varphi_R'\psi_R| + |\varphi_R(-\Delta)^\delta\psi_R| + \Phi_R|(-\Delta)^\sigma\psi_R|)^{p'}\,dxdt,\end{aligned}$$

where $p' = p/(p-1)$. Due to

$$\varphi_R'(t) = R^{-\kappa}(\varphi')(R^{-\kappa}t), \qquad (-\Delta)^\delta\psi_R(x) = R^{-2\delta}((-\Delta)^\delta\psi)(R^{-1}x),$$

recalling (50), we may estimate

$$\int_0^\infty \int_{\mathbb{R}^n} (\varphi_R\psi_R)^{-\frac{p'}{p}}\,|\varphi_R'\psi_R|^{p'}\,dxdt \leq C\,R^{-\kappa p'+n+\kappa}\,,$$

$$\int_0^\infty \int_{\mathbb{R}^n} (\varphi_R\psi_R)^{-\frac{p'}{p}}\,|\varphi_R(-\Delta)^\delta\psi_R|^{p'}\,dxdt \leq C\,R^{-2\delta p'+n+\kappa}\,.$$

The difference, with respect to the proof of Theorem 1, is related to the estimate of the term containing Φ. In this case, due to $\Phi_R(t) \leq \Phi_R(0) \leq R^\kappa$, and being $\Phi_R^{p'}\, \varphi_R^{-\frac{p'}{p}}$ bounded (this latter can be proved, for instance, by applying the integral mean theorem, as $\varphi_R \to 0$) one gets:

$$\int_0^\infty \int_{\mathbb{R}^n} (\varphi_R \psi_R)^{-\frac{p'}{p}}\, \Phi_R |(-\Delta)^\sigma \psi_R|^{p'}\, dxdt \leq C\, R^{-2\sigma p' + \kappa p' + n + \kappa}\,.$$

We may now fix $\kappa = \min\{2\delta, \sigma\}$, so that, summarizing, we proved that

$$\frac{1}{p'}\, I_R \leq C\, R^{-\kappa p' + n + \kappa} - \int_{\mathbb{R}^n} u_1(x)\, \psi_R(x)\, dx.$$

Assume, by contradiction, that the solution u is global. Recalling assumption (13), in the subcritical case $p < 1 + \kappa/n$, it follows that $I_R < 0$, for any sufficiently large R, and this contradicts the fact that $I_R \geq 0$. The critical case $p = 1 + \kappa/n$ is treated in standard way, but we omit the details for the sake of brevity. Therefore the solution u cannot be global.

To prove the lifespan estimate in the subcritical case, we proceed as in the proof of Theorem 1, obtaining

$$0 \leq \frac{1}{p'}\, I_R \leq C\, R^{-\kappa p' + n + \kappa} - \int_{\mathbb{R}^n} u_1(x)\, \psi_R(x)\, dx \leq C\, T^{-\frac{\kappa p' - (n+\kappa)}{\kappa}} - c\varepsilon,$$

and this concludes the proof.

Remark 3.1. In the application of the test function method to problem (1), for $2\delta < \sigma$, one may see that the part $(-\Delta)^\delta u_t + (-\Delta)^\sigma u$ of the linear equation is dominant, with respect to u_{tt}; the best scaling is given by $(t, x) \mapsto (R^{-2(\sigma-\delta)} t, R^{-1} x)$. On the other hand, when studying problem (2), for $2\delta < \sigma$, the part $u_{tt} + (-\Delta)^\delta u_t$ is dominant with respect to $(-\Delta)^\sigma u$, since the variable of the nonlinearity is u_t; the best scaling is given by $(t, x) \mapsto (R^{-2\delta} t, R^{-1} x)$. This effect is analogous to the fact that the critical exponent p_0 and p_1 are related to the asymptotic profile of problem (7) and (8), respectively.

When $2\delta > \sigma$, the dominant part is $u_{tt} + (-\Delta)^\sigma u$, for both problems (1) and (2). Indeed, if $2\delta \in (\sigma, 2\sigma]$, the asymptotic profile of the solution is described by the solution to the evolution equation $u_{tt} + (-\Delta)^\sigma u = 0$, to which the dissipative operator $e^{-t(-\Delta)^\delta}$ is applied (see [24] for the case $\sigma = \delta = 1$). The profile becomes more complicated for $\delta > \sigma$ (see Section 9).

Proof (Theorem 3). To prove Theorem 3, it is sufficient to follow the proof of Theorem 1, but replacing the estimate for the initial data with:

$$\int_{\mathbb{R}^n} u_1(x)\, \psi_R(x)\, dx \geq \varepsilon \int_{\mathbb{R}^n} (1 + |x|)^{-\mu}\, \psi_R(x)\, dx \geq c\varepsilon\, R^{n-\mu}.$$

As a consequence:

$$I_R \leq C\, R^{-2\sigma p' + n + \kappa} - c\varepsilon\, R^{n-\mu} = R^n\, (C\, R^{-2\sigma p' + \kappa} - c\varepsilon\, R^{-\mu}),$$

and the proof of both the nonexistence of global solutions, in the subcritical case, immediately follows. To prove the lifespan estimate, it is sufficient to fix $R = T^{\frac{1}{\kappa}}$, where T is the maximal existence time of a local solution. This concludes the proof.

Proof (Theorem 4). The proof of Theorem 4 is completely analogous to the proof of Theorem 3, but now we obtain

$$I_R \leq C\, R^{-\kappa p' + n + \kappa} - c\varepsilon\, R^{n-\mu} = R^n\, (C\, R^{-\kappa p' + \kappa} - c\varepsilon\, R^{-\mu}).$$

The proof follows.

4. The linear estimates

Our main tool in proving our existence results consists in the use of linear $L^{q_1}-L^q$ low-frequencies and $L^{q_2}-L^q$ high frequencies estimates for the solution to (6), with $1\leq q_j\leq q\leq\infty$, and q_1,q_2 are possibly different.

Let $2\delta\in[0,\sigma)$ and let u be the solution to (6). After applying the Fourier transform, we can write

$$\begin{cases}\widehat{u}_{tt}+|\xi|^{2\sigma}\widehat{u}+|\xi|^{2\delta}\widehat{u}_t=0\,,\\ \widehat{u}(0,\xi)=\widehat{u_0}(\xi)\,,\\ \widehat{u}_t(0,\xi)=\widehat{u_1}(\xi)\,.\end{cases}$$

The roots of the full symbol $\lambda^2+\lambda|\xi|^{2\delta}+|\xi|^{2\sigma}$ are radial and have non-positive real parts:

$$\lambda_\pm(|\xi|)=\begin{cases}\frac12\big(-1\pm\sqrt{1-4|\xi|^{2(\sigma-2\delta)}}\big)\,|\xi|^{2\delta} & \text{if } 2|\xi|^{\sigma-2\delta}<1,\\ \frac12\big(-1\pm i\,\sqrt{4|\xi|^{2(\sigma-2\delta)}-1}\big)\,|\xi|^{2\delta} & \text{if } 2|\xi|^{\sigma-2\delta}>1.\end{cases}\tag{54}$$

In particular,

$$\lambda_+\sim-|\xi|^{2(\sigma-\delta)}\,,\qquad \lambda_-\sim-|\xi|^{2\delta}\qquad \lambda_+-\lambda_-\sim|\xi|^{2\delta}\qquad\qquad \text{as } \xi\to0\tag{55}$$

$$\Re\lambda_\pm=-\frac12\,|\xi|^{2\delta}\,,\qquad \Im\lambda_\pm\sim\pm|\xi|^{\sigma}\,,\qquad\qquad \text{as } |\xi|\to\infty.\tag{56}$$

We may write

$$u(t,x)=K_0(t,|x|)\ast_{(x)}u_0(x)+K_1(t,|x|)\ast_{(x)}u_1(x)\,,$$

where

$$\widehat{K_0}(t,|\xi|)=\frac{\lambda_+e^{\lambda_-t}-\lambda_-e^{\lambda_+t}}{\lambda_+-\lambda_-},\quad \widehat{K_1}(t,|\xi|)=\frac{e^{\lambda_+t}-e^{\lambda_-t}}{\lambda_+-\lambda_-}.\tag{57}$$

We first consider low-frequencies estimates.

Proposition 4.1. *Let $2\delta\in[0,\sigma)$ and χ_0 as in Notation 3 in Section 1.3, a cut-off function localizing at low frequencies. The low frequencies part of the solution to (6) satisfies the decay estimate*

$$\|\partial_t^k|D|^bu_{\chi_0}(t,\cdot)\|_{L^q}\lesssim\sum_{j=0}^1(1+t)^{-\frac{1}{2(\sigma-\delta)}\left(n\left(\frac{1}{q_j}-\frac1q\right)+b-2j\delta\right)-k}\|u_j\|_{L^{q_j}}\,,\tag{58}$$

for any $q_0,q_1\geq1$, $q\in[\max\{q_0,q_1\},\infty]$ and $t\geq0$, $b\geq0$ and $k\in\mathbb{N}$, provided that

$$\frac{1}{q_1}-\frac1q+\frac{b-2\delta}{n}\geq0\,.\tag{59}$$

If (59) does not hold, then the solution to (6) satisfies the estimate

$$\|\partial_t^k|D|^bu_{\chi_0}(t,\cdot)\|_{L^q}\lesssim(1+t)^{-\frac{1}{2(\sigma-\delta)}\left(n\left(\frac{1}{q_0}-\frac1q\right)+b\right)-k}\|u_0\|_{L^{q_0}}+(1+t)^{-\frac{1}{2\delta}\left(n\left(\frac{1}{q_1}-\frac1q\right)+b\right)+1-k}\|u_1\|_{L^{q_1}}\,.\tag{60}$$

A special exception is given in the case $q_1=1$, $q=\infty$, $k=0$, and $2\delta-b=n$. In this case, we may prove (60) with a logarithmic power loss with respect to u_1, namely:

$$\||D|^{2\delta-n}u_{\chi_0}(t,\cdot)\|_{L^\infty}\lesssim(1+t)^{-\frac{1}{2(\sigma-\delta)}\left(2\delta-n\left(1-\frac{1}{q_0}\right)\right)}\|u_0\|_{L^{q_0}}+\log(1+t)\|u_1\|_{L^1}.$$

If $\delta=0$, then (59) is trivially verified, so that the low-frequencies decay estimates are given by (58).

Remark 4.1. Following the ideas in [7], in the special case $\sigma=1$, one may show that estimates (58) and, respectively, (60) are sharp, since $\|\partial_t^k|D|^bu_{\chi_0}(t,\cdot)\|_{L^q}$ asymptotically behaves, as $t\to\infty$, as the corresponding norm for the solution to (7) and, respectively, to (8), for a suitable choice of initial data.

We point out that estimate (60) for $k=0$ improves the corresponding one obtained in [7].

In order to prove our statement, we use a result for radial convolution kernels.

Lemma 4.1. *[Lemma 3.1 in [7]] Let $K(t,x)$ be a radial convolution kernel of the form*

$$K(t,x) *_{(x)} h = \mathfrak{F}^{-1}(f(|\xi|)e^{-g(|\xi|)t}\hat{h}),$$

with compactly supported h. Assume $g(\rho) \approx \rho^\beta$, and

$$|f^{(\kappa)}(\rho)| \lesssim \rho^{\alpha-\kappa}$$
$$|g^{(\kappa)}(\rho)| \lesssim \rho^{-\kappa} g(\rho)$$

for some $\alpha > -1$, and for any $\kappa \leq [(n+3)/2]$. Then

$$\|K(t,x) *_{(x)} h\|_{L^q} \lesssim (1+t)^{-\frac{n}{\beta}\left(\frac{1}{q_1}-\frac{1}{q}\right)-\frac{\alpha}{\beta}} \|h\|_{L^{q_1}},$$

for any $1 \leq q_1 \leq q \leq \infty$, provided that $q_1 < q$ if $\alpha = 0$ and f is not constant, and that

$$\frac{1}{q_1} - \frac{1}{q} \geq \frac{-\alpha}{n}, \qquad \text{if } \alpha \in (-1,0).$$

Lemma 4.1 allows to avoid the restrictions $q_j > 1$ and $q < \infty$, which would come by using a general multiplier theorem, like Theorem 2 in [36]. We notice that the restriction $q_1 < q$ appearing when $\alpha = 0$ can be removed, for sufficiently smooth f (namely, if $f = C + f_1(\rho)$, with $f_1^{(k)}(\rho) = o(\rho^{-k})$).

Proof (Proposition 4.1). First, assume that $b - 2\delta > -1$ or $k \geq 1$, so that we may directly apply Lemma 3.1 in [7].

Let $\xi \in \operatorname{supp} \chi_0$, i.e., $|\xi| \leq 1/2$, and fix the notation

$$\widehat{K_1^\pm}(t,|\xi|) = \pm \frac{e^{\lambda_\pm t}}{\lambda_+ - \lambda_-},$$
$$\widehat{K_0^\pm}(t,|\xi|) = \lambda_\mp \widehat{K_1^\pm}(t,|\xi|).$$

Thanks to (55), we may estimate

$$\left\| \partial_t^k |D|^b K_0^-(t,\cdot) *_{(x)} h_{\chi_0} \right\|_{L^q} \lesssim (1+t)^{-\frac{1}{2\delta}\left(n\left(\frac{1}{q_0}-\frac{1}{q}\right)+b\right)-\frac{\sigma-2\delta}{\delta}-k} \|h\|_{L^{q_0}},$$
$$\left\| \partial_t^k |D|^b K_0^+(t,\cdot) *_{(x)} h_{\chi_0} \right\|_{L^q} \lesssim (1+t)^{-\frac{1}{2(\sigma-\delta)}\left(n\left(\frac{1}{q_0}-\frac{1}{q}\right)+b\right)-k} \|h\|_{L^{q_0}},$$

for any $1 \leq q_0 \leq q \leq \infty$ and for any $k \in \mathbb{N}$ and $b \geq 0$. Since $\sigma > 2\delta$, the second decay rate is always worse than the first one. Gluing our estimates, we conclude the proof.

When we derive $L^{q_1} - L^q$ estimates for $\partial_t^k |D|^b K_1^\pm *_{(x)} h_{\chi_0}$, a restriction on (p,q) appears if $k = 0$. Though, if $k \geq 1$ or (59) holds, still using Lemma 3.1 in [7], we may estimate

$$\left\| \partial_t^k |D|^b K_1^-(t,\cdot) *_{(x)} h_{\chi_0} \right\|_{L^q} \lesssim (1+t)^{-\frac{1}{2\delta}\left(n\left(\frac{1}{q_1}-\frac{1}{q}\right)+b\right)+1-k} \|h\|_{L^{q_1}},$$
$$\left\| \partial_t^k |D|^b K_1^+(t,\cdot) *_{(x)} h_{\chi_0} \right\|_{L^q} \lesssim (1+t)^{-\frac{1}{2(\sigma-\delta)}\left(n\left(\frac{1}{q_1}-\frac{1}{q}\right)+b\right)+\frac{\delta}{\sigma-\delta}-k} \|h\|_{L^{q_1}}.$$

The second decay rate is not better than the first one if, and only if, condition (59) holds, and they are equal if the equality holds in (59). Gluing our estimates, we conclude the proof.

If $k = 0$ and (59) is violated, assuming for the sake of brevity that $u_0 = 0$, it is sufficient to use (60) with $k = 1$, in

$$\| |D|^b u_{\chi_0}(t,\cdot)\|_{L^q} \lesssim \int_0^t \|\partial_t |D|^b u_{\chi_0}(s,\cdot)\|_{L^q}\, ds \lesssim \|u_1\|_{L^{q_1}} \int_0^t (1+s)^{-\frac{1}{2\delta}\left(n\left(\frac{1}{q_1}-\frac{1}{q}\right)+b\right)} ds, \tag{61}$$

to derive the desired estimate. Indeed, the decay rate in (60) with $k = 1$ when (59) is violated is slower than $(1+t)^{-1}$, so that

$$\int_0^t (1+s)^{-\frac{1}{2\delta}\left(n\left(\frac{1}{q_1}-\frac{1}{q}\right)+b\right)}\,ds \lesssim (1+t)^{1-\frac{1}{2\delta}\left(n\left(\frac{1}{q_1}-\frac{1}{q}\right)+b\right)},$$

and this latter is an increasing power of $(1+t)$. This concludes the proof in the case $b-2\delta > -1$ or $k \geq 1$.

Now let us assume that $b - 2\delta \leq -1$ and $k = 0$. Then we cannot directly apply Lemma 3.1 in [7] to K_1. If $q < \infty$, then we define $s = 2\delta - b - 1/2 \geq 1/2$. By Riesz potential mapping properties, we get

$$\left\| |D|^b K_1^{\pm}(t,\cdot) *_{(x)} h_{\chi_0} \right\|_{L^q} \lesssim \left\| |D|^{b+s} K_1^{\pm}(t,\cdot) *_{(x)} h_{\chi_0} \right\|_{L^{q^\dagger}}, \qquad \frac{1}{q^\dagger} = \frac{1}{q} + \frac{s}{n}.$$

Then we may apply Lemma 4.1, with $\alpha = -1/2$ (in particular, we notice that $q^\dagger > 1$). We proceed similarly if $q = \infty$ and $q_1 > 1$, applying Lemma 4.1 to $|D|^{b+s}K_1^{\pm}(t,\cdot) *_{(x)} (I_s h_{\chi_0})$. If $q_1 = 1$ and $q = \infty$, the thesis follows by using standard Fourier transform property and Hölder inequality,

$$\left\| |D|^b K_1^{\pm}(t,\cdot) *_{(x)} h_{\chi_0} \right\|_{L^\infty} \lesssim \left\| |\xi|^b \hat{K}_1^{\pm}(t,\cdot) \hat{h}_{\chi_0} \right\|_{L^1} \lesssim \left\| |\xi|^b \hat{K}_1^{\pm}(t,\cdot) \right\|_{L^1} \|h\|_{L^1},$$

provided that $b + n > 2\delta$. If $b + n = 2\delta$, we cannot employ the above estimate. Tough, by integrating as in (61), we obtain:

$$\||D|^{2\delta-n} u_{\chi_0}(t,\cdot)\|_{L^\infty} \lesssim \int_0^t \|\partial_t |D|^{2\delta-n} u_{\chi_0}(s,\cdot)\|_{L^\infty} ds \lesssim \|u_1\|_{L^1} \int_0^t (1+s)^{-1} ds,$$

and this concludes the proof.

We now consider high-frequencies estimates. We recall a variant of Mikhlin-Hörmander multiplier theorem, for kernels localized at high frequencies, obtained by A. Miyachi.

Theorem 10 (Theorem 1 in [36]). *Let $q \in (1,\infty)$, $k = [n/2]+1$ (see Notation 4 in Section 1.3) and $a \geq 0$. Suppose that $m \in C^k(\mathbb{R}^n \setminus \{0\})$, $m(\xi) = 0$ if $|\xi| \leq 1$, and*

$$|\partial_\xi^\beta m(\xi)| \leq C\,|\xi|^{-na\left|\frac{1}{q}-\frac{1}{2}\right|} (A|\xi|^{a-1})^{|\beta|}, \quad |\beta| \leq k,$$

for $|\xi| \geq 1$, with some constant $A \geq 1$. Then $T_m = \mathfrak{F}^{-1}(m(t,\xi)) *_{(x)}$, defined by the action $T_m f(t,\cdot) = \mathfrak{F}^{-1}(m(t,\xi)\hat{f}(\xi))$, is continuously bounded from L^q into itself and

$$\|T_m f\|_{L^q} \leq C\, A^{n\left|\frac{1}{q}-\frac{1}{2}\right|} \|f\|_{L^q}.$$

Theorem 10 is stated only for $q \in (1,2)$ in [36], but it can be extended to $q \in (2,\infty)$ by duality arguments, whereas the extension for $q = 2$ trivially follows from Plancherel's theorem.

We are now able to prove the following.

Proposition 4.2. *Let $2\delta \in [0,\sigma)$, $q \in (1,\infty)$, and χ_1 as in Notation 3 in Section 1.3, a cut-off function localizing at high frequencies. The high frequencies part of the solution to (6) satisfies the decay estimate*

$$\|\partial_t^k |D|^b u_{\chi_1}(t,\cdot)\|_{L^q} \lesssim e^{-ct} \left(\|u_0\|_{W^{m,q}} + \|u_1\|_{W^{(m-\sigma)_+,q}}\right), \qquad \forall t \geq 0, \tag{62}$$

for some constants $c > 0$ independent on t, where $m \in \mathbb{N}$ satisfies

$$m \geq n(\sigma - 2\delta)\left|\frac{1}{q} - \frac{1}{2}\right| + k\sigma + b. \tag{63}$$

Proof. For $j = 0, 1$, let $m_j(t,\xi)$ be the multiplier associated to $\partial_t^k |D|^b \chi_1 K_j(t,|x|)$, respectively. Recalling that $|\xi| \geq 1/4$ in $\operatorname{supp}\chi_1$, for sufficiently small $c > 0$, we may estimate

$$||\xi|^{-m}\partial_\xi^\beta m_1(t,\xi)| \lesssim t^{|\beta|}\,|\xi|^{b-m+\sigma(k-1)+|\beta|(\sigma-1)}\,e^{-ct|\xi|^{2\delta}}\,e^{-ct} \lesssim |\xi|^{b-m+\sigma(k-1)+|\beta|(\sigma-2\delta-1)}\,e^{-ct},$$

where we used the boundedness of $(t|\xi|^{2\delta})^{|\beta|}e^{-t|\xi|^{2\delta}}$. Applying Theorem 10, with $a = \sigma - 2\delta$, the thesis follows for m as in (63). We proceed similarly for $m_0(t,\xi)$.

However, at high frequencies, it is also possible to use the smoothing effect produced by the structural damping to reduce the regularity required on the data, exception given for the special case $\delta = 0$. However, doing this in short time estimates, produces a singularity at $t = 0$. In particular, we have the following.

Proposition 4.3. *Let $2\delta \in (0,\sigma)$ and χ_1 as in Notation 3 in Section 1.3, a cut-off function localizing at high frequencies. The high frequencies part of the solution to (6) satisfies the singular estimate*

$$\|\partial_t^k |D|^b u_{\chi_1}(t,\cdot)\|_{L^q} \lesssim e^{-ct}\sum_{j=0}^{1} t^{-n\Xi(q_j,q)\left(\frac{\sigma-2\delta}{2\delta}\right)-\frac{1}{2\delta}\left(n\left(\frac{1}{q_j}-\frac{1}{q}\right)+b+(k-j)\sigma\right)}\|u_j\|_{L^{q_j}}, \qquad \forall t \in (0,\infty), \tag{64}$$

for $1 < q_j \leq q < \infty$, for some $c > 0$, where

$$\Xi = \begin{cases} \frac{1}{2} - \frac{1}{q_j} & \text{if } 2 \leq q_j, \\ 0 & \text{if } q_j \leq 2 \leq q, \\ \frac{1}{q} - \frac{1}{2} & \text{if } q \leq 2. \end{cases} \tag{65}$$

In order to prove Proposition 4.3, we first prove it for $q_1 = q_2 = q$.

Lemma 4.2. *Let $2\delta \in (0,\sigma)$. Then we have the singular estimate*

$$\|\partial_t^k |D|^b u_{\chi_1}(t,\cdot)\|_{L^q} \lesssim t^{-n\left|\frac{1}{q}-\frac{1}{2}\right|\left(\frac{\sigma-2\delta}{2\delta}\right)-\frac{1}{2\delta}(b+k\sigma)}e^{-ct}\left(\|u_0\|_{L^q} + t^{\frac{\sigma}{2\delta}}\|u_1\|_{L^q}\right), \qquad \forall t \in (0,\infty), \tag{66}$$

for any $q \in (1,\infty)$, $b \geq 0$, $k \in \mathbb{N}$, and for some $c > 0$.

In particular,

$$\|\partial_t^k |D|^b u_{\chi_1}(t,\cdot)\|_{L^2} \lesssim t^{-\frac{1}{2\delta}(b+k\sigma)}e^{-ct}\left(\|u_0\|_{L^2} + t^{\frac{\sigma}{2\delta}}\|u_1\|_{L^2}\right), \qquad \forall t \in (0,\infty). \tag{67}$$

Proof. For $j = 0, 1$, let $m_j(t,\xi)$ be the multipliers associated to $\partial_t^k |D|^b \chi_1 K_j(t,|x|)$, respectively. We may estimate

$$|\partial_\xi^\beta m_j(t,\xi)| \lesssim t^{|\beta|}\,|\xi|^{b+\sigma(k-j)+|\beta|(\sigma-1)}\,e^{-ct\,|\xi|^{2\delta}}e^{-ct},$$

for sufficiently small $c > 0$, thanks to $|\xi| \geq 1/4$ in $\operatorname{supp}\chi_1$. Thanks to the smoothing effect, i.e., the term $e^{-ct\,|\xi|^{2\delta}}$, we may estimate

$$t^{|\beta|}\,|\xi|^{b+\sigma(k-j)+|\beta|(\sigma-1)}\,e^{-ct\,|\xi|^{2\delta}} \lesssim t^{-\frac{b+(k-j)\sigma}{2\delta}}\left(t^{1-\frac{\sigma}{2\delta}}|\xi|^{-1}\right)^{|\beta|}.$$

Therefore, we conclude (66), for $t \in (0,1]$, by applying Theorem 10, with $A = t^{-\frac{\sigma-2\delta}{2\delta}}$. For $t \in [1,\infty)$, the proof is trivial (it is sufficient to set $A = 1$ in Theorem 10), thanks to the presence of the exponential decay e^{-ct}.

Proof (Proposition 4.3). Since the equation is linear, we may prove our estimate separately with respect to the initial datum u_0 or u_1, assuming the other one to be zero. First, let $2 \leq q_j \leq q < \infty$. We write

$$u = I_{r_j}|D|^{r_j}u, \qquad r_j = n\left(\frac{1}{q_j} - \frac{1}{q}\right).$$

By the Riesz potential mapping properties, we have

$$I_{r_j} : L^q \to L^{q_j}, \qquad \|I_{r_j} g\|_{L^q} \lesssim \|g\|_{L^{q_j}}.$$

Therefore, by applying (66) to $|D|^{r_j} u$, with initial data $(u_0, 0)$ or $(0, u_1)$, we derive

$$\|\partial_t^k |D|^b u_{\chi_1}(t,\cdot)\|_{L^q} \lesssim \|\partial_t^k |D|^{b+r_j} u_{\chi_1}(t,\cdot)\|_{L^{q_j}} \lesssim e^{-ct}\, t^{-n\left|\frac{1}{q_j}-\frac{1}{2}\right|\left(\frac{\sigma}{2\delta}-1\right)-\frac{1}{2\delta}\left(b+r_j+(k-j)\sigma\right)} \|u_j\|_{L^{q_j}}.$$

Now, let $q \le 2$. We apply (66) to u, with initial data $(I_{r_0} u_0, 0)$ or $(0, I_{r_1} u_1)$. Therefore, setting r_j as before,

$$\|\partial_t^k |D|^b u_{\chi_1}(t,\cdot)\|_{L^q} \lesssim e^{-ct}\, t^{-n\left|\frac{1}{q}-\frac{1}{2}\right|\left(\frac{\sigma}{2\delta}-1\right)-\frac{1}{2\delta}\left(b+r_j+(k-j)\sigma\right)} \|I_{r_j} u_j\|_{L^q} \lesssim e^{-ct}\, t^{-n\left|\frac{1}{q}-\frac{1}{2}\right|\left(\frac{\sigma}{2\delta}-1\right)-\frac{1}{2\delta}\left(b+r_j+(k-j)\sigma\right)} \|u_j\|_{L^{q_j}}.$$

If $q_j \le 2 \le q$, by setting r_j as before,

$$r = n\left(\frac{1}{2} - \frac{1}{q}\right),$$

and applying (66) to $|D|^r u$, with initial data $(I_{r_0 - r} u_0, 0)$ or $(0, I_{r_1 - r} u_1)$, we obtain

$$\|\partial_t^k |D|^b u_{\chi_1}(t,\cdot)\|_{L^q} \lesssim \|\partial_t^k |D|^{b+r} u_{\chi_1}(t,\cdot)\|_{L^2} \lesssim e^{-ct}\, t^{-\frac{1}{2\delta}\left(b+r_j+(k-j)\sigma\right)} \|I_{r_j} u_j\|_{L^2} \lesssim e^{-ct}\, t^{-\frac{1}{2\delta}\left(b+r_j+(k-j)\sigma\right)} \|u_j\|_{L^{q_j}}.$$

We notice that this latter estimate may also be easily obtained by using the property of Fourier transform:

$$\|f\|_{L^q} \lesssim \|\hat{f}\|_{L^{q'}}, \qquad \|\hat{f}\|_{L^{q_j'}} \le \|f\|_{L^{q_j}}, \qquad \text{if } q_j \le 2 \le q,$$

and directly estimating the Fourier transform of the fundamental solution.

This concludes the proof of (64).

4.1. Use of singular estimates to deal with the semilinear problem

The restriction on the space dimension and the related choice of the solution space in our results, is related to the use of high-frequencies linear estimates, which are (possibly) singular at $t = 0$, as done in [39]. This singularity is related to the use of the smoothing effect of the structural damping term at short times (Proposition 4.3).

Lemma 4.3. *Let $2\delta \in (0,\sigma)$, $u_0 \equiv 0$, and $T > 0$. Then, for any $\varepsilon > 0$ and $p > 1$, there exists a sufficiently large $\bar{q} = \bar{q}(p,\varepsilon) \in (2p, \infty)$ such that*

$$\||D|^b \partial_t^k u_{\chi_1}(t,\cdot)\|_{L^q} \lesssim t^{-\frac{1}{2\delta}\left(\frac{n}{2}(\sigma-2\delta)+b+(k-1)\sigma\right)-\varepsilon} \|u_1\|_{L^{\tilde{q}}}, \qquad \forall t \in (0,T], \quad q \in [2,\bar{q}], \tag{68}$$

where

$$\tilde{q} = \min\{q, \bar{q}/p\}.$$

Moreover, if we fix $\eta \in (1,2)$, then

$$\||D|^b \partial_t^k u_{\chi_1}(t,\cdot)\|_{L^q} \lesssim t^{-\frac{1}{2\delta}\left(n\left(\frac{1}{\eta}-\frac{1}{2}\right)(\sigma-2\delta)+b+(k-1)\sigma\right)} \|u_1\|_{L^q}, \qquad \forall t \in (0,T], \quad q \in [\eta, 2]. \tag{69}$$

Proof. To prove (68), we apply (64). Setting $q_1 = \tilde{q}$ for some $\bar{q} \in (2p, \infty)$,

$$\Xi(q_1, q) \le \Xi(\bar{q}/p, \bar{q}) = \frac{1}{2} - \frac{p}{\bar{q}},$$
$$\frac{1}{q_1} - \frac{1}{q} \le \frac{p}{\bar{q}} - \frac{1}{\bar{q}},$$

for any $q \in [2, \bar{q}]$. Therefore, for sufficiently large $\bar{q}$, (68) follows.

To prove (69), it is sufficient to apply (66) for any $q \in [\eta, 2]$.

In particular, the following corollary will be used to manage the high frequencies part of the solution related to the nonlinear term, at short times.

Corollary 4.1. *Let $2\delta \in (0,\sigma)$, $b + k\sigma < \sigma + 2\delta$, and $T > 0$. If we assume that*

$$\frac{n}{2}(\sigma - 2\delta) + b + (k-1)\sigma < 2\delta, \tag{70}$$

then there exists $M \in (2p,\infty)$ such that for any $\bar{q} \in [M,\infty)$, it holds

$$\||D|^b \partial_t^k u_{\chi_1}(t,\cdot)\|_{L^q} \lesssim t^{-\gamma}\|u_1\|_{L^{\tilde{q}}}, \qquad \forall t \in (0,T], \tag{71}$$

for any $q \in (1,\bar{q}]$, where $\tilde{q} = \min\{q, \bar{q}/p\}$, for some power $\gamma < 1$.

4.2. The limit case $\sigma = 2\delta$

In the limit case $2\delta = \sigma$, linear estimates may be easily obtained following the ideas in [39, 42]. The obtained estimates are consistent with the ones derived for $2\delta \in (0,\sigma)$.

Proposition 4.4. *Let $2\delta = \sigma$. Then the solution to* (6) *satisfies:*

- *estimate* (58)*, i.e.*

$$\|\partial_t^k |D|^b u_{\chi_0}(t,\cdot)\|_{L^q} \lesssim \sum_{j=0}^{1} (1+t)^{-\frac{1}{\sigma}\left(n\left(\frac{1}{q_j}-\frac{1}{q}\right)+b\right)+j-k}\|u_j\|_{L^{q_j}}, \tag{72}$$

  for any $q_0, q_1 \geq 1$, $q \in [\max\{q_0,q_1\},\infty]$, $t \geq 0$, $b \geq 0$ and $k \in \mathbb{N}$;

- *decay estimate* (62)*, in particular,*

$$\|\partial_t^k |D|^{m\sigma} u_{\chi_1}(t,\cdot)\|_{L^q} \lesssim e^{-ct}\left(\|u_0\|_{W^{(m+k)\sigma,q}} + \|u_1\|_{W^{(m+k-1)_+\sigma,q}}\right), \qquad \forall t \geq 0,$$

  for $m,k \in \mathbb{N}$, for any $q \in [1,\infty]$;

- *the singular estimate* (64)*, i.e.*

$$\|\partial_t^k |D|^b u_{\chi_1}(t,\cdot)\|_{L^q} \lesssim e^{-ct}\sum_{j=0}^{1} t^{-\frac{1}{\sigma}\left(n\left(\frac{1}{q_j}-\frac{1}{q}\right)+b\right)+j-k}\|u_j\|_{L^{q_j}}, \qquad \forall t > 0, \tag{73}$$

  for any $q_0, q_1 \geq 1$, $q \in [\max\{q_0,q_1\},\infty]$.

Combining together (72) and (73) we derive, in particular,

$$\|\partial_t^k |D|^b u(t,\cdot)\|_{L^q} \lesssim \sum_{j=0}^{1} t^{-\frac{1}{\sigma}\left(n\left(\frac{1}{q_j}-\frac{1}{q}\right)+b\right)+j-k}\|u_j\|_{L^{q_j}}. \tag{74}$$

5. Proof of Theorem 5

To prove Theorem 5, we may follow the ideas in [6].

We may write the (global) solution to the linear Cauchy problem (6) in the form

$$u^{\mathrm{lin}} := K_0(t,x) *_{(x)} u_0(x) + K_1(t,x) *_{(x)} u_1(x),$$

where $K_0(t,x), K_1(t,x)$ are the fundamental solutions to (6). By Duhamel's principle, a function $u \in X$, where X is a suitable space, is a solution to (1) or (2) if, and only if, it satisfies the equality

$$u(t,x) = u^{\mathrm{lin}}(t,x) + \int_0^t K_1(t-s,x) *_{(x)} f(u(s,x))\,ds, \qquad \text{in } X, \tag{75}$$

where, here and in the following, we set $f(u(s,x)) = |\partial_t^j u|^p$, with $j = 0$ in (1) and $j = 1$ in (2). Incidentally, we notice that more general shapes of function f can be considered, since we are dealing with small data solutions, but we restrict to the ones above, for the sake of simplicity.

The proof of our global existence results is based on the following scheme. We fix p in (1) and (2), and we define the space

$$X := C([0,\infty), W^{\sigma,1} \cap W^{\sigma,\infty}) \cap C^1([0,\infty), L^1 \cap L^\infty), \tag{76}$$

with norm given by

$$\|u\|_X := \sup_{t\in[0,\infty)} \Big\{(1+t)^{-1}\|u(t,\cdot)\|_{L^1} + \|(|D|^\sigma u, u_t)(t,\cdot)\|_{L^1} + (1+t)^{\frac{n}{\sigma}-1}\|u(t,\cdot)\|_{L^\infty} + (1+t)^{\frac{n}{\sigma}}\|(|D|^\sigma u, u_t)(t,\cdot)\|_{L^\infty}\Big\}. \tag{77}$$

In particular, by interpolation, any function $u \in X$ satisfies decay estimates with the same powers appearing in (20) and (21) for any $t \in [0,\infty)$ and $q \in [1,\infty]$.

Thanks to Proposition 4.4, $u^{\mathrm{lin}} \in X$ and it satisfies

$$\|u^{\mathrm{lin}}\|_X \leq C\,(\|u_0\|_{\mathcal{A}} + \|u_1\|_{\mathcal{B}})\,. \tag{78}$$

We define the operator F such that, for any $u \in X$,

$$Fu(t,x) := \int_0^t K_1(t-s,x) *_{(x)} f(u(s,x))\,ds\,, \tag{79}$$

then we prove the estimates

$$\|Fu\|_X \leq C\|u\|_X^p\,, \tag{80}$$

$$\|Fu - Fv\|_X \leq C\|u-v\|_X\big(\|u\|_X^{p-1} + \|v\|_X^{p-1}\big)\,. \tag{81}$$

By standard arguments, since u^{lin} satisfies (78) and $p > 1$, from (80) it follows that $F + u^{\mathrm{lin}}$ maps balls of X into balls of X, for small data in $\mathcal{A}\times\mathcal{B}$, and that estimates (80)-(81) lead to the existence of a unique solution to (75), that is, $u = u^{\mathrm{lin}} + Fu$, satisfying (78). We simultaneously gain a local and a global existence result.

Our starting point is the use of the linear estimates in Proposition 4.4. We prove (80), setting $(u_0, u_1) = (0, f(u))$ and replacing t by $t-s$, separately for $(Fu)_{\chi_0}$ and $(Fu)_{\chi_1}$. We omit the proof of (81), since it is analogous to the proof of (80).

The information that u belongs to X plays a fundamental role to estimate $f(u(s,\cdot))$ in suitable norms. We will employ the following well-known result.

Lemma 5.1. *Let $\alpha < 1 < \beta$. Then it holds*

$$\int_0^t (t-s)^{-\alpha}\,(1+s)^{-\beta}\,ds \lesssim (1+t)^{-\alpha}.$$

Lemma 5.1 has been proved in many different versions by many authors. One earlier version of this lemma goes back to [43]. We give a short proof of this result since it is useful to understand the approach employed later to estimate similar integrals.

Proof. First let $t \geq 1$. Splitting the integration interval into $[0,t/2]$ and $[t/2,t]$, we find:

$$\int_0^{t/2} (t-s)^{-\alpha}\,(1+s)^{-\beta}\,ds \approx t^{-\alpha}\int_0^{t/2} (1+s)^{-\beta}\,ds \lesssim t^{-\alpha},$$

$$\int_{t/2}^{t}(t-s)^{-\alpha}\,(1+s)^{-\beta}\,ds \approx (1+t)^{-\beta}\int_{t/2}^{t}(t-s)^{-\alpha}\,ds \lesssim t^{1-\alpha}(1+t)^{-\beta} \leq t^{-\alpha}.$$

On the other hand, for $t \in [0,1]$,

$$\int_{0}^{t}(t-s)^{-\alpha}\,(1+s)^{-\beta}\,ds \leq \int_{0}^{t}(t-s)^{-\alpha}\,ds \lesssim t^{1-\alpha} \leq 1.$$

The role played by Lemma 5.1 is motivated by the following immediate consequence of the definition of (77).

Lemma 5.2. *Let $u \in X$. Then*

$$\| |u(t,\cdot)|^p \|_{L^1} \lesssim (1+t)^{-\beta}\|u\|_X^p, \qquad t \in [0,\infty),$$

for some $\beta > 1$, if $p > p_0$, whereas

$$\| |u_t(t,\cdot)|^p \|_{L^1} \lesssim (1+t)^{-\beta}\|u\|_X^p, \qquad t \in [0,\infty),$$

for some $\beta > 1$, if $p > p_1$.

Proof. Due to

$$\| |\partial_t^j u(t,\cdot)|^p \|_{L^1} \lesssim \|\partial_t^j u(t,\cdot)\|_{L^p}^p,$$

it is sufficient to notice that

$$p\left(\frac{n}{\sigma}\left(1-\frac{1}{p}\right)-1+j\right) > 1 \qquad \Longleftrightarrow \qquad p > p_j,$$

to conclude the proof.

Lemma 5.3. *Let $p > p_0$ in* (1) *or $p > p_1$ in* (2)*. Then:*

$$\| |D|^{m\sigma}\partial_t^k (Fu)(t,\cdot)\|_{L^q} \lesssim (1+t)^{1-\frac{n}{\sigma}\left(1-\frac{1}{q}\right)-k-m}\|u\|_X^p, \tag{82}$$

for $m+k = 0,1$ and $q \in [1,\infty]$.

Proof. It is sufficient to prove (82) for $q = 1$ and $q = \infty$, since we can later interpolate. By virtue of (74) with $q_1 = q = 1$, and Lemma 5.2, we may estimate

$$\| |D|^{m\sigma}\partial_t^k (Fu)(t,\cdot)\|_{L^1} \lesssim \int_0^t (t-s)^{1-k-m}\,\|f(u(s,\cdot))\|_{L^1}\,ds \lesssim \|u\|_X^p \int_0^t (t-s)^{1-k-m}\,(1+s)^{-\beta}\,ds,$$

for some $\beta > 1$. Noticing that $1-k-m \geq 0$ for $m+k = 0,1$, by Lemma 5.1, estimate (82) follows for $q = 1$.

To deal with $q = \infty$ we may proceed as before only if $1-k-m-n/\sigma > -1$. Otherwise, let $t \geq 2$; we use (74) with $q_1 = 1$ in $[0,t/2]$ and (74) with $q_1 = \infty$ in $[t/2,t]$,

$$\| |D|^{m\sigma}\partial_t^k (Fu)(t,\cdot)\|_{L^\infty} \lesssim \int_0^{t/2} (t-s)^{1-k-m-\frac{n}{\sigma}}\,\|f(u(s,\cdot))\|_{L^1}\,ds + \int_{t/2}^t (t-s)^{1-k-m}\,\|f(u(s,\cdot))\|_{L^\infty}\,ds,$$

then we estimate

$$\int_0^{t/2} (t-s)^{1-k-m-\frac{n}{\sigma}}\,\|f(u(s,\cdot))\|_{L^1}\,ds \lesssim \|u\|_X^p\, t^{1-k-m-\frac{n}{\sigma}} \int_0^{t/2} (1+s)^{-\beta}\,ds \lesssim \|u\|_X^p\, t^{1-k-m-\frac{n}{\sigma}},$$

$$\int_{t/2}^t (t-s)^{1-k-m}\,\|f(u(s,\cdot))\|_{L^\infty}\,ds \lesssim \|u\|_X^p\,(1+t)^{(1-j-\frac{n}{\sigma})p} \int_{t/2}^t (t-s)^{1-k-m}\,ds \lesssim \|u\|_X^p\,(1+t)^{(1-j-\frac{n}{\sigma})p+2-k-m}.$$

The proof of (82) follows noticing that

$$\left(1-j-\frac{n}{\sigma}\right)p+2-k-m = -\beta+2-k-m-\frac{n}{\sigma},$$

for some $\beta > 1$, due to $p > p_j$. For $t \in [0,2]$, it is sufficient to use (74) with $q_1 = q = \infty$, to estimate

$$\||D|^{m\sigma}\partial_t^k(Fu)(t,\cdot)\|_{L^\infty} \lesssim \int_0^t (t-s)^{1-k-m}\,\|f(u(s,\cdot))\|_{L^\infty}\,ds \lesssim \|u\|_X^p \int_0^t (t-s)^{1-k-m}\,(1+s)^{-\beta}\,ds \lesssim \|u\|_X^p,$$

where we used Lemma 5.2 once again.

This concludes the proof of Theorem 5.

6. Proof of Theorems 6 and 7

To prove Theorems 6 and 7, we will employ the linear $L^{\tilde q} - L^q$ high frequencies estimates which we prepared in Section 4 to avoid the use of $L^1 - L^1$ and $L^\infty - L^\infty$ high frequencies estimates. We follow the steps of the proof of Theorem 5.

If we consider problem (1), then we fix the space

$$X_0 := C([0,\infty), L^\eta \cap H^\sigma \cap L^{\bar q}) \cap C^1([0,\infty), L^2)$$

for a fixed $\eta \in (1, \min\{2,p\}]$ and for sufficiently large $\bar q$, with norm given by

$$\|u\|_{X_0} := \sup_{t\in[0,\infty)} \Big\{ \max_{q\in[\eta,\bar q]} (1+t)^{\beta_q}\|u(t,\cdot)\|_{L^q} + ((1+t)^{\beta_2+1}\|u_t(t,\cdot)\|_{L^2} + (1+t)^{\gamma_2}\||D|^\sigma u(t,\cdot)\|_{L^2}) \Big\}.$$

If we consider problem (2), then we fix the space

$$X_1 := C([0,\infty), W^{\sigma,\eta} \cap W^{\sigma,\bar q}) \cap C^1([0,\infty), L^\eta \cap L^{\bar q}),$$

for a fixed $\eta \in (1, \min\{2,p\}]$ and for sufficiently large $\bar q$, with norm given by

$$\|u\|_{X_1} := \sup_{t\in[0,\infty)} \max_{q\in[\eta,\bar q]} \Big\{ (1+t)^{\beta_q}\|u(t,\cdot)\|_{L^q} + (1+t)^{\beta_q+1}\|u_t(t,\cdot)\|_{L^q} + (1+t)^{\gamma_q}\||D|^\sigma u(t,\cdot)\|_{L^q} \Big\}.$$

Any function $u \in X_0$ satisfies decay estimates with the same powers appearing in (33), (31) and (32). Moreover, any function $u \in X_1$ satisfies decay estimates with the same powers appearing in (35) and (36).

In the following we denote by X both spaces X_0 and X_1, when there is no need to distinguish among them.

Thanks to Propositions 4.1 and 4.2, $u^{\mathrm{lin}} \in X$, and it satisfies (78). Indeed, setting $b + k\sigma = j\sigma$, in (63), where $j = 0$ if we consider X_0 and $j = 1$ if we consider X_1, then

$$\frac{n}{2}(\sigma - 2\delta) + j\sigma < \sigma + 2\delta,$$

due to the restriction $n \le n_j$, with n_j as in (29) and (34), on the space dimension.

Therefore, to conclude the proof of Theorems 6 and 7, we shall only prove (80) and (81). Again, we only prove the first one, being the proof of the second one analogous. We will separately consider low and high frequencies. We preliminarily notice that Lemma 5.2 is still valid, i.e.

$$\||\partial_t^j u(t,\cdot)|^p\|_{L^1} \lesssim \|\partial_t^j u(t,\cdot)\|_{L^p}^p \lesssim (1+t)^{-p(\beta_p+j)}\,\|u\|_{X_j}^p \lesssim (1+t)^{-\beta}\|u\|_{X_j}^p, \tag{83}$$

for some $\beta > 1$, since $p(\beta_p + j) > 1$ if, and only if, $p > p_j$, $j = 0, 1$ (see Remark 2.2). We are now ready to prove the following.

Lemma 6.1. *Let $p > p_0$ in* (1) *or $p > p_1$ in* (2). *Then:*

$$\|(Fu)_{\chi_0}(t,\cdot)\|_{L^q} \lesssim (1+t)^{-\beta_q}\|u\|_X^p, \quad \forall q \in [\eta, \bar q], \tag{84}$$

for a sufficiently large $\bar q$.

Proof. Let $\bar q$ be sufficiently large that
$$n\left(\frac{p}{\bar q}-\frac{1}{\bar q}\right)<\sigma+2\delta,$$
so that
$$n\left(\frac{1}{\tilde q}-\frac{1}{q}\right)-2\delta\leq n\left(\frac{p}{\bar q}-\frac{1}{\bar q}\right)-2\delta<\sigma, \tag{85}$$
for any $q\in[\eta,\bar q]$, where
$$\tilde q=\min\left\{q,\frac{\bar q}{p}\right\}.$$
We may now prove (84). First, let $\beta_q<1$. In this case, by (83), we get
$$\|(Fu)_{\chi_0}(t,\cdot)\|_{L^q}\lesssim\int_0^t(t-s)^{-\beta_q}\,\|f(u(s,\cdot))\|_{L^1}\,ds\lesssim\|u\|_X^p\int_0^t(t-s)^{-\beta_q}\,(1+s)^{-\beta}\,ds,$$
for some $\beta>1$, and the statement follows by Lemma 5.1. Now let $\beta_q\geq1$, in particular, (22) holds. Due to $\sigma<2(\sigma-\delta)$ (by virtue of $2\delta<\sigma$), thanks to (85), we may now fix $q^\sharp\in[1,\tilde q]$, such that
$$\alpha_q:=\frac{n}{2(\sigma-\delta)}\left(\frac{1}{q^\sharp}-\frac{1}{q}\right)-\frac{2\delta}{2(\sigma-\delta)}\in[0,1).$$
Then we use L^1-L^q estimates for $s\in[0,t/2]$ and $L^{q^\sharp}-L^q$ estimates for $s\in[t/2,t]$.

Due to $\alpha_q\geq0$, we may use linear estimates (58) so that, thanks to (83), we get
$$\begin{aligned}\|(Fu)_{\chi_0}(t,\cdot)\|_{L^q}&\lesssim\int_0^{t/2}(t-s)^{-\beta_q}\,\|f(u(s,\cdot))\|_{L^1}\,ds+\int_{t/2}^t(t-s)^{-\alpha_q}\,\|f(u(s,\cdot))\|_{L^{q^\sharp}}\,ds\\&\lesssim\|u\|_X^p\,(1+t)^{-\beta_q}\int_0^{t/2}(1+s)^{-\beta}\,ds+t^{1-\alpha_q}(1+t)^{-p\beta_{pq^\sharp}-jp}\,\|u\|_{X_j}^p,\end{aligned}$$
for some $\beta>1$. We remark that condition $\alpha_q<1$ played a fundamental role to estimate the integral in $[t/2,t]$.

First, let $j=0$. We now notice that (22) holds with $q=pq^\sharp$, since it holds for $q=p$. Therefore, we may compute:
$$\alpha_q+p\beta_{pq^\sharp}-1=\alpha_q+\frac{n}{2(\sigma-\delta)}\left(\frac{1}{p}-\frac{1}{pq^\sharp}\right)p+p\beta_p-1>\alpha_q+\frac{n}{2(\sigma-\delta)}\left(1-\frac{1}{q^\sharp}\right)=\beta_q,$$
where we used once again $p\beta_p-1>0$ for any $p>p_0$.

If $j=1$, using $p\beta_p+p>1$, we may similarly get
$$\alpha_q+p\beta_{pq^\sharp}+p-1>\beta_q,$$
if we prove that
$$p\beta_{pq^\sharp}+p-1>\frac{n}{2(\sigma-\delta)}\left(\frac{1}{p}-\frac{1}{pq^\sharp}\right)p. \tag{86}$$
Indeed, it is sufficient to distinguish three cases. If (22) holds with $q=p$, then it also holds for $q=pq^\sharp$, and
$$p\beta_{pq^\sharp}+p-1=\frac{n}{2(\sigma-\delta)}\left(\frac{1}{p}-\frac{1}{pq^\sharp}\right)p+p\beta_p+p-1>\frac{n}{2(\sigma-\delta)}\left(\frac{1}{p}-\frac{1}{pq^\sharp}\right)p.$$
If (22) holds with $q=pq^\sharp$, but not with $q=p$, then, due to $\delta<\sigma-\delta$, we get:
$$p\beta_{pq^\sharp}+p-1=\frac{n}{2(\sigma-\delta)}\left(\frac{1}{p}-\frac{1}{pq^\sharp}\right)p+\frac{p}{2(\sigma-\delta)}\left\{n\left(1-\frac{1}{p}\right)-2\delta\right\}+p-1$$

$$> \frac{n}{2(\sigma-\delta)}\left(\frac{1}{p}-\frac{1}{pq^\sharp}\right)p + p\beta_p + p - 1 > \frac{n}{2(\sigma-\delta)}\left(\frac{1}{p}-\frac{1}{pq^\sharp}\right)p.$$

If (22) does not hold with $q = pq^\sharp$, then, due to $2\delta < 2(\sigma-\delta)$, we get:

$$p\beta_{pq^\sharp} + p - 1 = \frac{n}{2\delta}\left(\frac{1}{p}-\frac{1}{pq^\sharp}\right)p + p\beta_p + p - 1 > \frac{n}{2(\sigma-\delta)}\left(\frac{1}{p}-\frac{1}{pq^\sharp}\right)p.$$

Summarizing, we proved (86) in all cases.

This concludes the proof of (84).

Lemma 6.2. *Let $p > p_0$ in* (1) *or $p > p_1$ in* (2)*. Then:*

$$\||D|^\sigma (Fu)_{\chi_0}(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\gamma_2}\|u\|_X^p, \tag{87}$$
$$\|\partial_t (Fu)_{\chi_0}(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\beta_2-1}\|u\|_X^p \tag{88}$$

Moreover, if $u \in X_1$, then:

$$\||D|^\sigma (Fu)_{\chi_0}(t,\cdot)\|_{L^q} \lesssim (1+t)^{-\gamma_q}\|u\|_{X_1}^p, \quad \forall q \in [\eta,\bar q], \tag{89}$$
$$\|\partial_t (Fu)_{\chi_0}(t,\cdot)\|_{L^q} \lesssim (1+t)^{-\beta_q-1}\|u\|_{X_1}^p, \quad \forall q \in [\eta,\bar q], \tag{90}$$

for a sufficiently large $\bar q$.

Proof. Let $\bar q$ be sufficiently large that

$$n\left(\frac{p}{\bar q}-\frac{1}{\bar q}\right) < 2\delta,$$

so that

$$n\left(\frac{1}{\tilde q}-\frac{1}{q}\right) + \sigma - 2\delta \leq n\left(\frac{p}{\bar q}-\frac{1}{\bar q}\right) + \sigma - 2\delta < \sigma, \tag{91}$$

for any $q \in [\eta,\bar q]$, where

$$\tilde q = \min\left\{q, \frac{\bar q}{p}\right\}.$$

We may now prove our estimates. For the sake of brevity, we only prove (89) and (90), for $u \in X_1$, i.e. when the nonlinearity is $|u_t|^p$.

We first consider (89). If $\gamma_q < 1$, by (83), we get

$$\||D|^\sigma (Fu)_{\chi_0}(t,\cdot)\|_{L^q} \lesssim \int_0^t (t-s)^{-\gamma_q}\, \|f(u(s,\cdot))\|_{L^1}\, ds \lesssim \|u\|_X^p \int_0^t (t-s)^{-\gamma_q}\, (1+s)^{-\beta}\, ds,$$

for some $\beta > 1$, and the statement follows by Lemma 5.1.

Now let $\gamma_q \geq 1$. Due to $\sigma < 2(\sigma-\delta)$ (by virtue of $2\delta < \sigma$), thanks to (91), we may now fix $q^\sharp \in [1,\tilde q]$, such that

$$\alpha_q := \frac{n}{2(\sigma-\delta)}\left(\frac{1}{q^\sharp}-\frac{1}{q}\right) + \frac{\sigma-2\delta}{2(\sigma-\delta)} \in [0,1).$$

Then we use $L^1 - L^q$ estimates for $s \in [0,t/2]$ and $L^{q^\sharp} - L^q$ estimates for $s \in [t/2,t]$.

Due to $\alpha_q \geq 0$, we may use linear estimates (58) so that, thanks to (83), we get

$$\begin{aligned}\||D|^\sigma (Fu)_{\chi_0}(t,\cdot)\|_{L^q} &\lesssim \int_0^{t/2} (t-s)^{-\gamma_q}\, \|f(u(s,\cdot))\|_{L^1}\, ds + \int_{t/2}^t (t-s)^{-\alpha_q}\, \|f(u(s,\cdot))\|_{L^{q^\sharp}}\, ds \\ &\lesssim \|u\|_X^p\, (1+t)^{-\gamma_q} \int_0^{t/2} (1+s)^{-\beta}\, ds + t^{1-\alpha_q}(1+t)^{-p\beta_{pq^\sharp}-p}\, \|u\|_{X_1}^p,\end{aligned}$$

for some $\beta > 1$. We remark that condition $\alpha_q < 1$ played a fundamental role to estimate the integral in $[t/2, t]$.

By using (86), we immediately obtain:

$$t^{1-\alpha_q}(1+t)^{-p\beta_{pq^\sharp}-p} \lesssim (1+t)^{-\gamma_q},$$

and this concludes the proof of (89).

We now prove (90). First, assume that (22) does not hold, so that $\beta_q + 1 < 1$. Then, by (83), we get

$$\|\partial_t (Fu)_{\chi_0}(t,\cdot)\|_{L^q} \lesssim \int_0^t (t-s)^{-\beta_q-1}\, \|f(u(s,\cdot))\|_{L^1}\, ds \lesssim \|u\|_X^p \int_0^t (t-s)^{-\beta_q-1}\, (1+s)^{-\beta}\, ds,$$

for some $\beta > 1$, and the statement follows by Lemma 5.1.

Now let us assume that (22) holds. Then, for any $\varepsilon \in (0, 2\delta)$, we may take $q^\sharp \in [1, q]$ such that

$$\alpha_q = \frac{n}{2\delta}\left(\frac{1}{q^\sharp} - \frac{1}{q}\right),$$

verifies $\alpha_q = 1 - \varepsilon/(2\delta) > 0$. Then we use $L^1 - L^q$ estimates for $s \in [0, t/2]$ and $L^{q^\sharp} - L^q$ estimates for $s \in [t/2, t]$. Being $\alpha_q < 1$, we now have to use linear estimates (60) (instead of (58)) so that, thanks to (83), we get

$$\begin{aligned}\|\partial_t (Fu)_{\chi_0}(t,\cdot)\|_{L^q} &\lesssim \int_0^{t/2} (t-s)^{-\beta_q-1}\, \|f(u(s,\cdot))\|_{L^1}\, ds + \int_{t/2}^t (t-s)^{-\alpha_q}\, \|f(u(s,\cdot))\|_{L^{q^\sharp}}\, ds \\ &\lesssim \|u\|_X^p\, (1+t)^{-\beta_q-1} \int_0^{t/2} (1+s)^{-\beta}\, ds + t^{1-\alpha_q}(1+t)^{-p\beta_{pq^\sharp}-p}\, \|u\|_{X_1}^p, \end{aligned} \tag{92}$$

for some $\beta > 1$. We remark that condition $\alpha_q < 1$ played a fundamental role to estimate the integral in $[t/2, t]$. Since the inequality in (86) is strict, for sufficiently small $\varepsilon(\bar{q})$, we may estimate:

$$\alpha_q + p\beta_{pq^\sharp} + p - 1 > \alpha_q + \frac{n}{2(\sigma-\delta)}\left(1 - \frac{1}{q^\sharp}\right) + \varepsilon\left\{\frac{1}{2\delta} - \frac{1}{2(\sigma-\delta)}\right\} = \beta_q + 1,$$

so that we obtain

$$t^{1-\alpha_q}(1+t)^{-p\beta_{pq^\sharp}-p} \lesssim (1+t)^{-\beta_q-1}.$$

This concludes the proof of (90).

Now we may consider the high frequencies estimate. At high frequencies, Corollary 4.1 plays a fundamental role and we will also make use of a modified version of Lemma 5.1.

Lemma 6.3. *Let $c > 0$ and $\alpha \in \mathbb{R}$. Then it holds*

$$\int_0^t e^{-c(t-s)}\, (1+s)^{-\alpha}\, ds \lesssim (1+t)^{-\alpha}.$$

Proof. Splitting the integration interval into $[0, t/2]$ and $[t/2, t]$, we find:

$$\begin{aligned}&\int_0^{t/2} e^{-c(t-s)}\, (1+s)^{-\alpha}\, ds \leq e^{-ct/2} \int_0^{t/2} (1+s)^{-\alpha}\, ds \lesssim e^{-ct/2} \lesssim (1+t)^{-\alpha}, \\ &\int_{t/2}^t e^{-c(t-s)}\, (1+s)^{-\alpha}\, ds \approx (1+t)^{-\alpha} \int_{t/2}^t e^{-c(t-s)}\, ds \lesssim (1+t)^{-\alpha}.\end{aligned}$$

We are now ready to prove the following.

Lemma 6.4. *Let us assume $n \leq n_0$, with n_0 as in* (29)*, and $p > p_0$ in* (1)*, or $n \leq n_1$, with n_1 as in* (34)*, and $p > p_1$ in* (2)*. Then:*

$$\|(Fu)_{\chi_1}(t,\cdot)\|_{L^q} \lesssim (1+t)^{-\beta_q} \|u\|_{X_1}^p, \quad \forall q \in [\eta, \bar{q}], \tag{93}$$

for a sufficiently large $\bar{q}$. Also,

$$\||D|^\sigma (Fu)_{\chi_1}(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\gamma_2} \|u\|_X^p, \tag{94}$$

$$\|\partial_t (Fu)_{\chi_1}(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\beta_2 - 1} \|u\|_X^p \tag{95}$$

Moreover, if $u \in X_1$, then:

$$\||D|^\sigma (Fu)_{\chi_1}(t,\cdot)\|_{L^q} \lesssim (1+t)^{-\gamma_q} \|u\|_{X_1}^p, \quad \forall q \in [\eta, \bar{q}], \tag{96}$$

$$\|\partial_t (Fu)_{\chi_1}(t,\cdot)\|_{L^q} \lesssim (1+t)^{-\beta_q - 1} \|u\|_{X_1}^p, \quad \forall q \in [\eta, \bar{q}], \tag{97}$$

for a sufficiently large $\bar{q}$.

We notice that $n \leq n_0$ if, and only if, (70) holds with $b = k = 0$ in Corollary 4.1, whereas $n \leq n_1$ if, and only if, (70) holds with $(b,k) = (\sigma, 0)$ or, equivalently, with $(b,k) = (0,1)$, in Corollary 4.1.

Proof. We only prove (93), for the sake of brevity. First, let $t \geq 2$. We employ $L^{\tilde{q}} - L^q$ high frequencies estimates, where $\tilde{q} = \min\{q, \bar{q}/p\}$, splitting the integration interval in $[0, t-1]$ and $[t-1, t]$. Then, thanks to Proposition 4.3 and Corollary 4.1, we may estimate

$$\|(Fu)_{\chi_1}(t,\cdot)\|_{L^q} \lesssim \int_0^{t-1} e^{-c(t-s)} \, \|f(u(s,\cdot))\|_{L^{\tilde{q}}} \, ds + \int_{t-1}^t (t-s)^{-\gamma} \, \|f(u(s,\cdot))\|_{L^{\tilde{q}}} \, ds,$$

for some $\gamma < 1$. By Lemma 6.3, we derive

$$\int_0^{t-1} e^{-c(t-s)} \, \|f(u(s,\cdot))\|_{L^{\tilde{q}}} \, ds \lesssim \|u\|_{X_j}^p \int_0^{t-1} e^{-c(t-s)} \, (1+s)^{-p\beta_{p\tilde{q}} - jp} \, ds \lesssim (1+t)^{-p\beta_{p\tilde{q}} - jp} \, \|u\|_{X_j}^p,$$

for $j = 0, 1$, whereas we directly obtain:

$$\int_{t-1}^t (t-s)^{-\gamma} \, \|f(u(s,\cdot))\|_{L^{\tilde{q}}} \, ds \lesssim (1+t)^{-p\beta_{p\tilde{q}} - jp} \, \|u\|_{X_j}^p \int_{t-1}^t (t-s)^{-\gamma} \, ds \approx (1+t)^{-p\beta_{p\tilde{q}} - jp} \, \|u\|_{X_j}^p,$$

thanks to $\gamma < 1$. Due to $p\tilde{q} \geq q$, and $p > 1$, estimate (93) trivially follows. For $t \in [0,2]$, it sufficient to use

$$\||D|^b \partial_t^k (Fu)_{\chi_1}(t,\cdot)\|_{L^q} \lesssim \int_0^t (t-s)^{-\gamma} \, \|f(u(s,\cdot))\|_{L^{\tilde{q}}} \, ds \lesssim \|u\|_X^p, \quad k = 0, 1.$$

This concludes the proof.

Remark 6.1. It is clear that one may prove for the high frequencies part of $(Fu)_{\chi_1}$, better estimates than the ones described in Lemma 6.4. However, this improvement would not influence the profile of the whole term Fu, since its asymptotic behavior as $t \to \infty$ is determined by its low frequencies part, as it happens for the solution to the linear problem.

7. Sketch of the proof of Theorems 8 and 9

We follow the steps in Sections 5 and 6, but now the solution spaces X_0 and X_1 are

$$X_0 := C([0,\infty), L^\eta \cap H^\kappa) \cap C^1([0,\infty), H^{\kappa-\sigma}) \tag{98}$$

$$X_1 := C([0,\infty), W^{\sigma,\eta} \cap H^\kappa) \cap C^1([0,\infty), L^\eta \cap H^{\kappa-\sigma}) \tag{99}$$

where

$$\eta = \min\{2, p\}, \quad \kappa = \max\left\{\frac{n}{2}\left(1-\frac{1}{p}\right), \sigma\right\},$$

with norms given by

$$\|u\|_{X_0} := \sup_{t\in[0,\infty)} \Big\{\max_{q\in[\eta,2]}(1+t)^{\beta_q}\|u(t,\cdot)\|_{L^q} + \max_{b\in[0,\kappa]}(1+t)^{\theta_b}\||D|^b u(t,\cdot)\|_{L^2} + \max_{b\in[0,\kappa-\sigma]}(1+t)^{\theta_b+1}\||D|^b u_t(t,\cdot)\|_{L^2}\Big\},$$

$$\|u\|_{X_1} := \sup_{t\in[0,\infty)} \Big\{\max_{q\in[\eta,2]}(1+t)^{\beta_q}(\|u(t,\cdot)\|_{L^q} + (1+t)\|u_t(t,\cdot)\|_{L^q})$$
$$+ \max_{b\in[0,\kappa]}(1+t)^{\theta_b}\||D|^b u(t,\cdot)\|_{L^2} + \max_{b\in[0,\kappa-\sigma]}(1+t)^{\theta_b+1}\||D|^b u_t(t,\cdot)\|_{L^2}\Big\}.$$

We notice that the first term in $\|u\|_{X_0}$ and the first two terms in $\|u\|_{X_1}$ may be omitted if $p \geq 2$, so that $\eta = 2$, since they are included in the last ones (for $b = 0$).

Again, we denote by X both spaces X_0 and X_1, when there is no need to distinguish among them. Thanks to Propositions 4.1 and 4.2, and noticing that (62) with $q = 2$, can be easily generalized to

$$\|\partial_t^k |D|^b u_{X_1}(t,\cdot)\|_{L^2} \lesssim e^{-ct}\,(\|u_0\|_{H^{k\sigma+b}} + \|u_1\|_{H^{(k\sigma+b-\sigma)_+}}), \qquad \forall t \geq 0, \tag{100}$$

we get that $u^{\mathrm{lin}} \in X$ and it satisfies (78).

Therefore, to prove Theorems 8 and 9, we shall only prove (80) and (81). We only sketch the proof of the first one, being the proof of the second one analogous, and we will separately consider low and high frequencies. We preliminarily notice that Lemma 5.2 is still valid (since we may use (27) if $p > 2$). We are now ready to prove the following.

Lemma 7.1. *Let $p > p_0$ in* (1) *or $p > p_1$ in* (2). *Then:*

$$\||D|^b \partial_t^k (Fu)_{X_0}(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\theta_b-k}\|u\|_X, \tag{101}$$

for any $b \in [0,\kappa]$ if $k = 0$ and for any $b \in [0,\kappa-\sigma]$ if $k = 1$.

Proof. If $\theta_b + k < 1$, then, by (83), we may directly estimate

$$\||D|^b \partial_t^k (Fu)_{X_0}(t,\cdot)\|_{L^2} \lesssim \int_0^t (t-s)^{-\theta_b-k}\,\|f(u(s,\cdot))\|_{L^1}\,ds \lesssim \|u\|_X^p \int_0^t (t-s)^{-\theta_b-k}\,(1+s)^{-\beta}\,ds,$$

for some $\beta > 1$, and the statement follows by Lemma 5.1. If $\theta_b + k \geq 1$, we may distinguish different cases, as it happened in the proof of Lemmas 6.1 and 6.2. For the sake of brevity, we only discuss the easier case $k = 0$ and $b \geq 2\delta$ in (101). In this case, it is sufficient to use $L^1 - L^2$ estimates for $s \in [0, t/2]$ and $L^2 - L^2$ estimates for $s \in [t/2, t]$. In the integral between 0 and $t/2$, we proceed as before. In the integral between $t/2$ and t, using (58), we may estimate

$$\int_{t/2}^t (t-s)^{-\frac{b-2\delta}{2(\sigma-\delta)}}\,\|f(u(s,\cdot))\|_{L^2}\,ds \lesssim t^{1-\frac{b-2\delta}{2(\sigma-\delta)}}\,(1+t)^{-p\beta_{2p}-jp}\,\|u\|_X^p.$$

We notice that we used $(b-2\delta)/(2(\sigma-\delta)) < 1$, which follows as a consequence of $b < \sigma + 2\delta$ and $2\delta < \sigma$. The proof follows, thanks again to $p\beta_p + jp > 1$, by

$$1 - \frac{b-2\delta}{2(\sigma-\delta)} - p\beta_{2p} - jp \leq 1 - \theta_b - p\beta_p - jp < -\theta_b.$$

If $p < 2$, we also use the following.

Lemma 7.2. *If $p < 2$, then* (84) *holds for any $q \in [p, 2]$. Moreover, if we are considering* (2)*, then* (90) *holds for any $q \in [p, 2]$.*

Proof. The proof is completely analogous to the proof of Lemmas 6.1 and 6.2, with $\tilde{q} = q$.

At high frequencies, we may easily prove the following.

Lemma 7.3. *Let us assume* (39) *for some p as in* (37) *in* (1)*, or* (46) *for some p as in* (44) *in* (2)*. Then:*

$$\||D|^b \partial_t^k (Fu)_{\chi_1}(t, \cdot)\|_{L^2} \lesssim (1+t)^{-\theta_b - k} \|u\|_X, \tag{102}$$

for any $b \in [0, \kappa]$ if $k = 0$ and for any $b \in [0, \kappa - \sigma]$ if $k = 1$. Moreover, if $p < 2$, then (93) *holds for any $q \in [p, 2]$ and, if we are considering* (2)*, then* (97) *holds for any $q \in [p, 2]$.*

Proof. The proof is analogous to the proof of Lemma 6.4, but the role played by Corollary 4.1 is now replaced by (67), and, if $p < 2$, by (66) with $b = 0$, for any $q \in [p, 2]$. In particular, due to the right-hand side inequality in (37) or in (44), the singularity $t^{-(b+(k-1)\sigma)/(2\delta)}$ in (67) is integrable, since $b + k\sigma \leq \kappa < \sigma + 2\delta$, and the singularity

$$t^{-n\left|\frac{1}{q}-\frac{1}{2}\right|\left(\frac{\sigma-2\delta}{2\delta}\right)-\frac{1}{2\delta}(b+(k-1)\sigma)},$$

in (66) is integrable, due to assumption (39) or (46).

This concludes the proof of Theorems 8 and 9.

8. The special case $\sigma = 1$

For $\sigma = 1$ and $2\delta \in (0, 1)$, we may relax the restriction on the space dimension in Theorems 6, 7, 8 and 9, when the space dimension is odd. Also, we may take $\eta = 1$ and $\bar{q} = \infty$ in Theorem 6 and 7, thanks to the $L^1 - L^1$ estimates obtained in [39].

With these modifications, Theorems 6 and 8 improve the corresponding result obtained by the authors in Theorem 2 in [8], for the global existence of small data solutions to (1). The modified version of Theorem 6 allows to deal with all supercritical powers, but the maximum space dimension remains the same as in [8]. On the other hand, the modified version of Theorem 8 gives global existence up to a maximum space dimension which is larger, in general, than the one obtained in [8]. We remark that in this case δ is necessarily fractional, being $2\delta \in (0, 1)$; in particular, the nonexistence counterpart of these results is not available.

These improvements are consequence of the following.

Proposition 8.1. *Let $\sigma = 1$ and $2\delta \in (0, 1)$. Then, for $k = 0, 1$ and $b \geq 0$, the solution to* (6) *satisfies the singular estimate*

$$\|\partial_t^k |D|^b u_{\chi_1}(t, \cdot)\|_{L^q} \lesssim t^{\left[\frac{n}{2}\right]\left|\frac{2}{q}-1\right|\left(1-\frac{1}{2\delta}\right)-\frac{b+k}{2\delta}} e^{-ct} \left(\|u_0\|_{L^q} + t^{\frac{1}{2\delta}} \|u_1\|_{L^q}\right), \qquad t \in (0, T], \tag{103}$$

for any $1 \leq q \leq \infty$.

Proof. We first prove (103) for $q = 1$. To prove it, it is sufficient to prove that

$$\left\|\mathfrak{F}^{-1}\left(\partial_t^k |\xi|^b \chi_1(|\xi|) \widehat{K}_j(t, |\xi|)\right)\right\|_{L^1} \lesssim e^{-ct}\, t^{\left[\frac{n}{2}\right]\left(1-\frac{1}{2\delta}\right)-\frac{b+k-j}{2\delta}}, \tag{104}$$

and then use Young inequality. The case $k = 0$ was already proved in [39], see Corollaries 8 and 9. For $k = 1$, thanks to $\lambda_+ \lambda_- = |\xi|^2$ we get

$$\partial_t \widehat{K_0}(t, |\xi|) = \frac{|\xi|^2 \left(e^{\lambda_+ t} - e^{\lambda_- t}\right)}{\lambda_+ - \lambda_-}.$$

We conclude (104), for $j = 0$, by applying Lemma 9 of [39]. Moreover, by computing

$$\partial_t \widehat{K_1}(t, |\xi|) = \frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-} = e^{\lambda_+ t} + e^{\lambda_- t} - \widehat{K_0}(t, |\xi|),$$

we conclude (104), for $j = 1$, by applying Lemma 10 of [39].

Applying Riesz-Thorin interpolation theorem to (103) with $q = 1$, and to (67), we obtain (103) for any $q \in [1, 2]$. For $q \in (2, \infty]$, (103) follows by duality argument (see, for instance, page 95 of [44]).

Estimate (103) coincide with (66) when n is even, but the singularity is weaker when n is odd.

The statements of Lemma 4.3 and Corollary 4.1 are improved accordingly. Also, Proposition 4.2 can be modified to include the cases $q = 1$ and $q = \infty$ in (62).

As a consequence, the statements of Theorems 6 and 7 can be improved in the following way, when $\sigma = 1$:

- the maximum space dimensions n_0 in (29) and n_1 in (34) become:

$$n_0(1, \delta) := 1 + 2 \max \left\{ m \in \mathbb{N} : \ m < \frac{1 + 2\delta}{1 - 2\delta} \right\},$$
$$n_1(1, \delta) := 1 + 2 \max \left\{ m \in \mathbb{N} : \ m < \frac{2\delta}{1 - 2\delta} \right\};$$

- it is possible to take $\eta \in [1, \min\{2, p\}$ and $\bar{q} \in [M, \infty]$.

Similarly, conditions (39) and (46) in Theorems 8 and 9 may be relaxed to:

$$\left[\frac{n}{2}\right] \left(\frac{2}{p} - 1\right)_+ (1 - 2\delta) < \begin{cases} 1 + 2\delta & \text{in Theorem 8,} \\ 2\delta & \text{in Theorem 9.} \end{cases}$$

In particular, condition above holds for any $\delta \in (0, 1/2)$ and $p > p_0$, if $n \leq 5$, consistently with the same bound obtained in [38] for the classical damped wave equation, i.e. for $\delta = 0$.

9. Local existence in the energy space

Several existence results for local solutions to problems (1) and (2) may be given, by using different approaches. For the sake of simplicity, we just investigate the local existence of energy solutions with standard tools.

Proposition 9.1. *We distinguish three cases.*

- *If $2\delta \leq \sigma$, let us fix p in* (1)*, such that*

$$1 < p < 1 + \frac{2(\sigma + 2\delta)}{(n - 2(\sigma + 2\delta))_+}, \tag{105}$$

  or p in (2)*, such that*

$$1 < p < 1 + \frac{4\delta}{(n - 4\delta)_+}. \tag{106}$$

  Then there exists $\varepsilon \in (0, 2\delta]$ and $T > 0$ such that for any data

$$(u_0, u_1) \in H^{\sigma + 2\delta - \varepsilon} \times H^{2\delta - \varepsilon} \tag{107}$$

  there exists a local solution

$$u \in C([0, T], H^{\sigma + 2\delta - \varepsilon}) \cap C^1([0, T], H^{2\delta - \varepsilon})$$

  to (1) *or to* (2)*.*

- *If $\sigma \le 2\delta < 2\sigma$, let us fix p in* (1)*, such that*

$$1 < p < 1 + \frac{4\sigma}{(n-4\sigma)_+}, \tag{108}$$

or p in (2)*, such that* (106) *holds. Then there exists $\varepsilon \in (0, 2\delta]$ and $T > 0$ such that for any data*

$$(u_0, u_1) \in H^{2\sigma-\varepsilon} \times H^{2\delta-\varepsilon} \tag{109}$$

there exists a local solution

$$u \in C([0,T], H^{2\sigma-\varepsilon}) \cap C^1([0,T], H^{2\delta-\varepsilon})$$

to (1) *or to* (2)*.*

- *If $\delta \ge \sigma$, let us fix p in* (1)*, such that*

$$1 < p \le 1 + \frac{4\delta}{(n-4\delta)_+}, \tag{110}$$

or p in (2)*, such that* (106) *holds. Then there exists $\varepsilon \in (0, 2\delta]$ and $T > 0$ such that for any data*

$$(u_0, u_1) \in H^{2\delta} \times H^{2\delta-\varepsilon} \tag{111}$$

there exists a local solution

$$u \in C([0,T], H^{2\delta}) \cap C^1([0,T], H^{2\delta-\varepsilon})$$

to (1) *or to* (2)*.*

It may appear surprising that in the case $\delta \ge \sigma$ a natural solution space is $C([0,T], H^{2\delta}) \cap C^1([0,T], H^{2\delta-\varepsilon})$, but several new effect appears for the regularity of the solution to (6), when $\delta \ge \sigma$. In particular, we address the interested reader to [21], where smoothing effect are investigated in abstract setting, in bounded domains.

Remark 9.1. When $2\delta \le \sigma$, the solution and data spaces in Proposition 9.1 are the same as in Theorems 8 and 9, but we do not require the L^p regularity of the local solutions, when $p < 2$, and we do not take L^1 regularity assumptions on the initial data. Indeed, the L^1 assumption is taken to improve the decay rate of the solution as $t \to \infty$, and so improve the critical exponent of the problem (see Section 2.5), and this aspect is of no interest for local solutions.

It is clear that condition (105) and, respectively, (106) and (108), is related to the continuous embedding of $H^{\sigma+2\delta-\varepsilon}$ and, respectively, $H^{2\delta-\varepsilon}$ and $H^{2\sigma-\varepsilon}$, into L^{2p}, for a sufficiently small $\varepsilon > 0$. Similarly, condition (110) is related to the continuous embedding of $H^{2\delta}$ into L^{2p}. The choice of this regularity and, in particular, the presence of $\varepsilon > 0$, is related to the smoothing effect created by the structural damping.

For this reason, the proof of the local existence result in the energy space is not completely standard, and we sketch it.

Proof. Let X be the solution space with its usual norm $\|\cdot\|_X$; for instance, if $2\delta \le \sigma$ then

$$X := C([0,T], H^{\sigma+2\delta-\varepsilon}) \cap C^1([0,T], H^{2\delta-\varepsilon}),$$
$$\|u\|_X := \max_{t\in[0,T]} \big(\|u(t,\cdot)\|_{H^{\sigma+2\delta-\varepsilon}} + \|u_t(t,\cdot)\|_{H^{2\delta-\varepsilon}}\big).$$

Moreover, let $Y = C([0,T], L^2)$, with the usual norm

$$\|f\|_Y := \max_{t\in[0,T]} \|f(t,\cdot)\|_{L^2}.$$

The local existence of the solution to (1) or (2) follows, by standard arguments (see, for instance, [13]), if one is able to prove the following:

(a) given the initial data as in the statement, the solution to the linear problem is in X;

(b) for any $u, v \in X$, it holds
$$\||u|^p - |v|^p\|_Y \leq C(\|u\|_X, \|v\|_X)\, \|u - v\|_X,$$
or, respectively,
$$\||u_t|^p - |v_t|^p\|_Y \leq C(\|u\|_X, \|v\|_X)\, \|u - v\|_X;$$

(c) for any $f \in Y$, it holds:
$$\left\| \int_0^t K_1(t-s,\cdot) *_{(x)} f(s,\cdot)\, ds \right\|_X \leq C(T)\, \|f\|_Y$$
where K_1 is the fundamental solution to (1) or, respectively, to (2), as in Section 4.

An estimate from below on the maximal existence time of the solution may depend, in general, on the size of the initial data, and on the power p, as well as on $\sigma, \delta, \varepsilon$.

Property (b) is an immediate consequence of Hölder inequality and of the Sobolev embedding, for a sufficiently small ε, with respect to p. To prove property (a) and (c), we use standard energy estimates; in the second case, the smoothing effect comes into play and singular estimates are also considered.

After performing the Fourier transform, taking into account of the behavior of the roots of $\lambda^2 + |\xi|^{2\delta}\lambda + |\xi|^{2\sigma} = 0$ as $|\xi| \to \infty$, it is easy to obtain the desired estimates. To prove (a), it is sufficient to notice that, for any $t \geq 0$ and for sufficiently large $|\xi|$, it holds:
$$\begin{aligned}
\hat{u}(t,\xi) &\lesssim \hat{u}_0(\xi) + |\xi|^{-\max\{\sigma,2\delta\}}\hat{u}_1(\xi),\\
\hat{u}_t(t,\xi) &\lesssim |\xi|^{2\sigma-\max\{\sigma,2\delta\}}\hat{u}_0(\xi) + \hat{u}_1(\xi),
\end{aligned}$$
so that (a) follows by a direct application of Plancherel theorem.

It remains to prove (c). First, let $2\delta \leq \sigma$. Then, for any $t > 0$ and for sufficiently large $|\xi|$, we have (see Section 4):
$$\begin{aligned}
\hat{K}_1(t,\xi) &\lesssim e^{-ct|\xi|^{2\delta}}|\xi|^{-\sigma} \lesssim t^{-1+\frac{\varepsilon}{2\delta}}\, |\xi|^{\varepsilon-2\delta-\sigma},\\
\partial_t\hat{K}_1(t,\xi) &\lesssim e^{-ct|\xi|^{2\delta}} \lesssim t^{-1+\frac{\varepsilon}{2\delta}}\, |\xi|^{\varepsilon-2\delta},
\end{aligned}$$
for any $\varepsilon \in (0, 2\delta]$. This immediately leads to:
$$\begin{aligned}
\left\| \int_0^t K_1(t-s,\cdot) *_{(x)} f(s,\cdot)\, ds \right\|_X &\leq \max_{t\in[0,T]} \int_0^t \Big(\big\|K_1(t-s,\cdot) *_{(x)} f(s,\cdot)\big\|_{H^{\sigma+2\delta-\varepsilon}} + \big\|\partial_t K_1(t-s,\cdot) *_{(x)} f(s,\cdot)\big\|_{H^{2\delta-\varepsilon}} \Big)\, ds\\
&\lesssim \max_{t\in[0,T]} \int_0^t (t-s)^{-1+\frac{\varepsilon}{2\delta}}\, \|f(s,\cdot)\|_{L^2}\, ds \leq C(T,\varepsilon)\|f\|_Y.
\end{aligned}$$
Now, let $2\delta > \sigma$. In this case, the roots of $\lambda^2 + |\xi|^{2\delta}\lambda + |\xi|^{2\sigma} = 0$ as $|\xi| \to \infty$ verifies
$$\lambda_+ \sim -|\xi|^{2(\sigma-\delta)}, \qquad \lambda_- \sim -|\xi|^{2\delta},$$
so that
$$\begin{aligned}
\hat{K}_1(t,\xi) &\lesssim |\xi|^{-2\delta}(e^{-ct|\xi|^{2(\sigma-\delta)}} + e^{-ct|\xi|^{2\delta}}),\\
\partial_t\hat{K}_1(t,\xi) &\lesssim e^{-ct|\xi|^{2\delta}} + |\xi|^{-2(2\delta-\sigma)}\, e^{-ct|\xi|^{2(\sigma-\delta)}}.
\end{aligned}$$
In particular, when $\sigma \leq \delta$, the root λ_+ remains bounded or tends to 0, so that the smoothing effect does not appear in the related part of the solution. We first consider the case $\sigma \leq 2\delta < 2\sigma$. Then, recalling that $\delta \geq \sigma - \delta$, we obtain
$$\hat{K}_1(t,\xi) \lesssim (t^{-1+\frac{\varepsilon}{2(\sigma-\delta)}} + t^{-1+\frac{\varepsilon+2(2\delta-\sigma)}{2\delta}})\, |\xi|^{\varepsilon-2\sigma},$$

$$\partial_t \hat{K}_1(t,\xi) \lesssim (t^{-1+\frac{\varepsilon}{2\delta}} + t^{-1+\frac{\varepsilon}{2(\sigma-\delta)}})\,|\xi|^{\varepsilon-2\delta},$$

for any $\varepsilon \in (0, 2(\sigma-\delta)]$, and this leads to:

$$\begin{aligned}\left\| \int_0^t K_1(t-s,\cdot) *_{(x)} f(s,\cdot)\,ds \right\|_X &\leq \max_{t\in[0,T]} \int_0^t \left(\left\| K_1(t-s,\cdot) *_{(x)} f(s,\cdot) \right\|_{H^{2\sigma-\varepsilon}} + \left\| \partial_t K_1(t-s,\cdot) *_{(x)} f(s,\cdot) \right\|_{H^{2\delta-\varepsilon}} \right) ds \\ &\lesssim \max_{t\in[0,T]} \int_0^t (t-s)^{-1+\frac{\varepsilon}{2(\sigma-\delta)}} \|f(s,\cdot)\|_{L^2}\,ds \leq C(T,\varepsilon)\|f\|_Y.\end{aligned}$$

Now, let $\sigma \leq \delta$. In this case, we drop the exponential $e^{-ct|\xi|^{2(\sigma-\delta)}}$, since it gives no smoothing. We obtain:

$$\begin{aligned}\hat{K}_1(t,\xi) &\lesssim |\xi|^{-2\delta}(1 + e^{-ct|\xi|^{2\delta}}) \lesssim |\xi|^{-2\delta}, \\ \partial_t \hat{K}_1(t,\xi) &\lesssim e^{-ct|\xi|^{2\delta}} + |\xi|^{-2(2\delta-\sigma)} \lesssim t^{-1+\frac{\varepsilon}{2\delta}}\,|\xi|^{\varepsilon-2\delta} + |\xi|^{-2(2\delta-\sigma)},\end{aligned}$$

for any $\varepsilon \in (0, 2\delta]$, and, recalling that $2\delta - \sigma \geq \delta$, this leads to:

$$\begin{aligned}\left\| \int_0^t K_1(t-s,\cdot) *_{(x)} f(s,\cdot)\,ds \right\|_X &\leq \max_{t\in[0,T]} \int_0^t \left(\left\| K_1(t-s,\cdot) *_{(x)} f(s,\cdot) \right\|_{H^{2\delta}} + \left\| \partial_t K_1(t-s,\cdot) *_{(x)} f(s,\cdot) \right\|_{H^{2\delta-\varepsilon}} \right) ds \\ &\lesssim \max_{t\in[0,T]} \int_0^t (t-s)^{-1+\frac{\varepsilon}{2\delta}} \|f(s,\cdot)\|_{L^2}\,ds \leq C(T,\varepsilon)\|f\|_Y.\end{aligned}$$

The proof of (c) is completed, and this concludes the proof.

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References

[1] R.C. Charão, C.R. da Luz, *Asymptotic properties for a semilinear plate equation in unbounded domains*, J. Hyperbolic Differ. Equ. **6** (2009) 269–294.

[2] R.C. Charão, C.R. da Luz, R. Ikehata, *Sharp Decay Rates for Wave Equations with a Fractional Damping via New Method in the Fourier Space*, Journal of Math. Anal. and Appl., **408** (2013), 1, 247–255, http://dx.doi.org/10.1016/j.jmaa.2013.06.016.

[3] R.C. Charão, C.R. da Luz, R. Ikehata, *New decay rates for a problem of plate dynamics with fractional damping*, J. Hyperbolic Differ. Equ. **10**, 3 (2013), 563–575, http://dx.doi.org/10.1142/S0219891613500203.

[4] R.C. Charão, C.R. da Luz, R. Ikehata, *Asymptotic behavior for abstract evolution differential equations of second order,* J. Differential Equations **259** (2015), 5017–5039.

[5] R.C. Charão, J.L. Horbach, R. Ikehata, *Optimal decay rates and asymptotic profile for the plate equation with structural damping*, J. Math. Anal. Appl. **440** (2016), 529–560.

[6] M. D'Abbicco, *A benefit from the L^1 smallness of initial data for the semilinear wave equation with structural damping*, in Current Trends in Analysis and its Applications, 2015, 209–216. Proceedings of the 9th ISAAC Congress, Krakow. Eds V. Mityushev and M. Ruzhansky, http://www.springer.com/br/book/9783319125763.

[7] M. D'Abbicco, M.R. Ebert, *Diffusion phenomena for the wave equation with structural damping in the $L^p - L^q$ framework*, J. Differential Equations, **256** (2014), 2307–2336, http://dx.doi.org/10.1016/j.jde.2014.01.002.

[8] M. D'Abbicco, M.R. Ebert, *An application of $L^p - L^q$ decay estimates to the semilinear wave equation with parabolic-like structural damping*, Nonlinear Analysis **99** (2014), 16–34, http://dx.doi.org/10.1016/j.na.2013.12.021.

[9] M. D'Abbicco, M.R. Ebert, *A classification of structural dissipations for evolution operators*, Math. Meth. Appl. Sci. **39** (2016), 2558-2582, http://dx.doi.org/10.1002/mma.3713.

[10] M. D'Abbicco, M.R. Ebert, T. Picon, *Long time decay estimates in real Hardy spaces for evolution equations with structural dissipation*, J. Pseudo-differential Operators and Appl., **7**, 2 (2016), 261–293, http://dx.doi.org/10.1007/s11868-015-0141-9.

[11] M. D'Abbicco, E. Jannelli, *A damping term for higher-order hyperbolic equations*, Ann. Mat. Pura ed Appl. **195**, 2, 2016, 557–570, http://dx.doi.org/10.1007/s10231-015-0477-z.

[12] M. D'Abbicco, S. Lucente, *NLWE with a special scale-invariant damping in odd space dimension*, Discr. Cont. Dynamical Systems, AIMS Proceedings, 2015, 312–319, http://dx.doi.org/10.3934/proc.2015.0312.

[13] M. D'Abbicco, S. Lucente, *The beam equation with nonlinear memory*, Z. Angew. Math. Phys. (2016) 67:60, http://dx.doi.org/10.1007/s00033-016-0655-x.

[14] M. D'Abbicco, S. Lucente, M. Reissig, *A shift in the critical exponent for semilinear wave equations with a not effective damping*, Journal of Differential Equations **259**, 2015, 5040–5073, http://dx.doi.org/10.1016/j.jde.2015.06.018.

[15] M. D'Abbicco, M. Reissig, *Semilinear structural damped waves*, Math. Methods in Appl. Sc., **37** (2014), 1570–1592, http://dx.doi.org/10.1002/mma.2913.

[16] L. D'Ambrosio, S. Lucente, *Nonlinear Liouville theorems for Grushin and Tricomi operators*, J. Differential Equations **123** (2003), 511–541.

[17] Yu.V. Egorov, V.A. Galaktionov, V.A. Kondratiev, and S.I. Pohozaev, *On the necessary conditions of existence to a quasilinear inequality in the half-space*, Comptes Rendus Acad. Sci. Paris, Série I **330** (2000), 93–98.

[18] V.A. Galaktionov, E.L. Mitidieri, S.I. Pohozaev, *Blow-up for Higher-Order Parabolic, Hyperbolic, Dispersion and Schrodinger Equations*, Monogr. Res. Notes Math., Chapman and Hall/CRC, 2014, ISBN 9781482251722.

[19] V. Georgiev, *Weighted estimate for the wave equation*, Nonlinear Waves, Proceedings of the Fourth MSJ International Research Institute, vol. **1**, Hokkaido Univ., 1996, pp. 71–80.

[20] V. Georgiev, H. Lindblad, C. D. Sogge, *Weighted Strichartz estimates and global existence for semilinear wave equations*, Amer. J. Math., **119** (1997), 1291–1319.

[21] M. Ghisi, M. Gobbino, A. Haraux, *Local and global smoothing effects for some linear hyperbolic equations with a strong dissipation*, Trans. Amer. Math. Soc. **368**, 3 (2016), 2039–2079.

[22] Han Yang, A. Milani, *On the diffusion phenomenon of quasilinear hyperbolic waves*, Bull. Sci. math. **124**, 5 (2000) 415–433.

[23] Hsiao L., Liu Tai-ping, *Convergence to nonlinear diffusion waves for solutions of a system of hyperbolic conservations with damping*, Comm. Math. Phys. **143** (1992), 599–605.

[24] R. Ikehata, *Asymptotic Profiles for Wave Equations with Strong Damping*, J. Differential Equations **257** (2014), 2159–2177, http://dx.doi.org/10.1016/j.jde.2014.05.031

[25] R. Ikehata, Y. Mayaoka, T. Nakatake, *Decay estimates of solutions for dissipative wave equations in $\mathbb{R}^N$ with lower power nonlinearities*, J. Math. Soc. Japan, **56** (2004), 2, 365–373.

[26] R. Ikehata, M. Ohta, *Critical exponents for semilinear dissipative wave equations in $\mathbb{R}^N$*, J. Math. Anal. Appl. **269** (2002), 87–97.

[27] F. John, *Blow-up of solutions of nonlinear wave equations in three space dimensions*, Manuscripta Math., **28** (1979), 235–268.

[28] G. Karch, *Selfsimilar profiles in large time asymptotics of solutions to damped wave equations*, Studia Mathematica **143** (2000), 2, 175–197.

[29] T. Kato, *Blow-up of solutions of some nonlinear hyperbolic equations*, Comm. Pure Appl. Math., **32** (1980), 501–505.

[30] P. Marcati, K. Nishihara, *The L^p-L^q estimates of solutions to one-dimensional damped wave equations and their application to the compressible flow through porous media*, J. Differential Eq. **191** (2003), 445–469.

[31] E. Mitidieri, S. I. Pohozaev, *The absence of Global Positive Solutions to Quasilinear Elliptic Inequalities*, Doklady Mathematics **57** (1998), 250–253.

[32] E. Mitidieri, S. I. Pohozaev, *Nonexistence of Positive Solutions for a Systems of Quasilinear Elliptic Equations and Inequalities in $\mathbb{R}^n$*, Doklady Mathematics **59** (1999), 1351–1355.

[33] E. Mitidieri, S.I. Pohozaev, *Non-existence of weak solutions for some degenerate elliptic and parabolic problems on $\mathbb{R}^n$*, J. Evolution Equations **1** (2001) 189–220.

[34] E. Mitidieri, S.I. Pohozaev, *Nonexistence of Weak Solutions for Some Degenerate and Singular Hyperbolic Problems on $\mathbb{R}^n$*, Proc. Steklov Institute of Mathematics **232** (2001), 240–259.

[35] E. Mitidieri, S. I. Pokhozhaev *Lifespan Estimates for Solutions of Some Evolution Inequalities*, Differential Equations **45**, 10 (2009), 1473–1484.

[36] A. Miyachi, *On some Fourier multipliers for $H^p(\mathbb{R}^n)$*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), 157–179.

[37] A. Miyachi, *On some estimates for the wave equation in L^p and H^p*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), 331–354.

[38] T. Narazaki, *$L^p - L^q$ estimates for damped wave equations and their applications to semilinear problem*, J. Math. Soc. Japan **56** (2004), 586–626.

[39] T. Narazaki, M. Reissig, *L^1 estimates for oscillating integrals related to structural damped wave models*, in Studies in Phase Space Analysis with Applications to PDEs, Cicognani M, Colombini F, Del Santo D (eds), Progress in Nonlinear Differential Equations and Their Applications. Birkhäuser, 2013; 215–258.

[40] K. Nishihara, *$L^p - L^q$ estimates for solutions to the damped wave equations in 3-dimensional space and their applications*, Math. Z. **244** (2003), 631–649.

[41] J.C. Peral, *L^p Estimates for the Wave Equation*, J. of Functional Anal. **36** (1980), 114–145.

[42] Pham Trieu Duong, M. Kainane and M. Reissig, *Global existence for semi-linear structurally damped σ-evolution models*, J. Math. Anal. Appl. **431** (2015), 569–596, doi:10.1016/j.jmaa.2015.06.001.

[43] I.E. Segal, *Quantization and dispersion for nonlinear relativistic equations*, Mathematical Theory of Elementary Particles, M. I. T. Press, Cambridge, Mass., 1966, 79-108.

[44] E.M. Stein, *Singular integrals and differentiability properties of functions,* Princeton, 1970.

[45] Y. Sugitani and S. Kawashima, *Decay estimates of solutions to a semi-linear dissipative plate equation*, J. Hyperbolic Differ. Equ. **7** (2010) 471–501.

[46] G. Todorova, B. Yordanov, *Critical Exponent for a Nonlinear Wave Equation with Damping*, J. Differential Equations **174** (2001), 464–489.

[47] Qi S. Zhang, *A blow-up result for a nonlinear wave equation with damping: the critical case*, C. R. Acad. Sci. Paris Sér. I Math. **333** (2001), 109–114.