APLIMAT 2013 - MATHEMATICS AND ART

# «THE CURVED LINE BELONGS TO GOD»:

## THE USE OF RULED SURFACES IN THE ARCHITECTURE OF ANTONI GAUDÍ.

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**Short abstract in the paper**

We introduce the concept of ruled surface and we describe some examples (namely the *one-sheeted hyperboloid*, the *hyperbolic paraboloid*, the *helicoid* and the *conoid*), illustrating their main technical features. Then, we explain how, where and why (both from the technical and the symbolic point of view) Antoni Gaudí uses such surfaces to get his peculiar architecture, pervaded by curved shapes, which on one hand is the heir of the gothic and traditional Catalan architecture and, on the other hand, anticipates the flavour of the modern topological one. A quick glimpse at the latter concludes the paper.

**Key words**

Geometry, ruled surfaces, topology, architecture.

**Mathematics Subject Classification**

51H00, 54H00, 97M00

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*Abstract:*

We introduce the concept of ruled surface and we describe some examples (namely the *one-sheeted hyperboloid*, the *hyperbolic paraboloid*, the *helicoid* and the *conoid*), illustrating their main technical features. Then, we explain how, where and why (both from the technical and the symbolic point of view) Antoni Gaudí uses such surfaces to get his peculiar architecture, pervaded by curved shapes, which on one hand is the heir of the gothic and traditional Catalan architecture and, on the other hand, anticipates the flavour of the modern topological one. A quick glimpse at the latter concludes the paper.

*Introduction.*

«The straight line belongs to man, the curved line belongs to God», ensures Antoni Plàcid Gaudí i Cornet, the Catalan architect whose undying fame is due to his peculiar buildings in Barcelona and its environs, the Sagrada Família in primis.

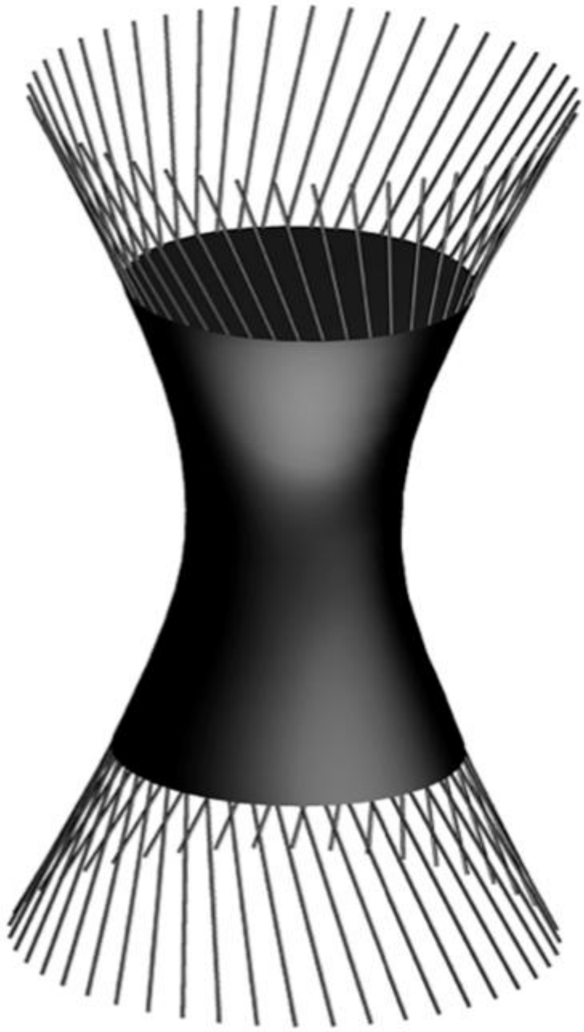
Since Gaudí is an ardent catholic, the whole of his work aims to God's glorification. That's maybe why, in light of the previously recalled quotation, curved shapes are ubiquitous in his buildings. On the other hand, being a man, he is constrained, both for symbolic and for technical reasons, to the use of straight lines while constructing its curved shapes. An apparent contradiction, that Gaudí brilliantly overcomes thanks to a clever use of the so-called *ruled surfaces*. By definition, in fact, a ruled surface is a surface which can be described as the set of points swept by a straight line (also called *ruling*) moving along a prescribed profile. If, for instance, we move a straight line, parallel to itself, along a circle lying in a perpendicular plane, we get a cylinder, which is indeed a ruled surface. Similarly, a cone and a plane are ruled surfaces as well. Nevertheless, those three basic examples are neglected by Gaudí, which is instead attracted by other ruled surfaces, such as the *one-sheeted hyperboloid*, the *hyperbolic paraboloid*, the *helicoid* and the *conoid.*

These surfaces deeply characterize Gaudí's work: we find them in the colums of the Sagrada Família, in the roof of the adjacent Schools (1908-1909), in the church of the Colònia Güell (1898–1916), in the windows of Casa Batlló (1905–1907), in the staircases of Casa Milà (1906-1912) and the list could be very long. Gaudí 's technical motivations are in fact shared by many other architects, contemporaries or not: we find ruled surfaces in the works of Vladimir Shukhov, Eugène Freyssinet, Robert Maillart, EduardoTorroja, Giorgio Baroni, Ícaro de Castro Mello, Le Corbusier, Kenzo Tange, Pier Luigi Nervi, Carlo Mollino, Félix Candela, Santiago Calatrava, etc.

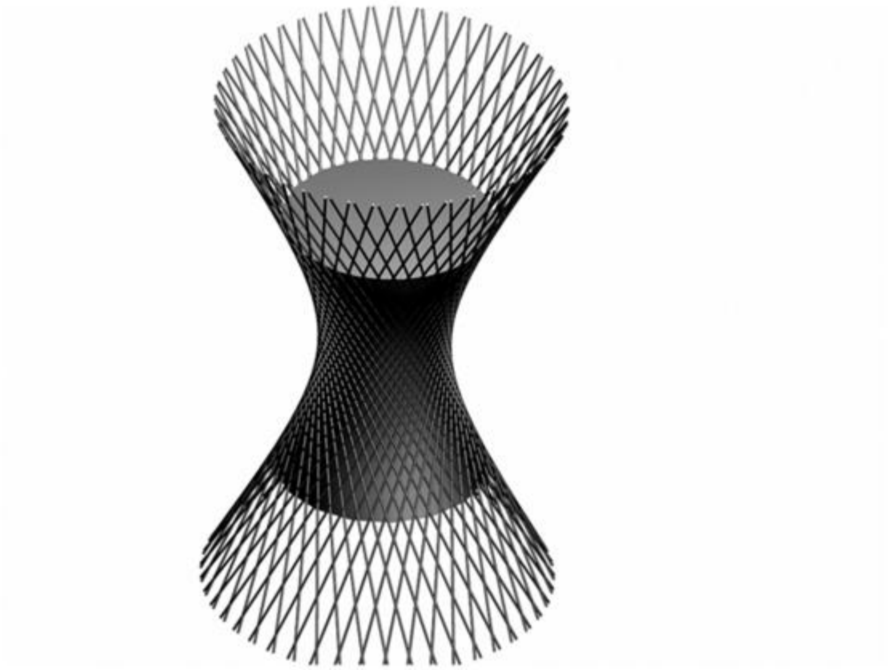
In the following sections we will provide the definition of those ruled surfaces, together with their main technical features. Moreover, we will see how they appear in Gaudí's buildings, as well as in the works of other architects. We refer the interested reader to [1], [2], [5] and [8] for further investigation.

*The one-sheeted hyperboloid.*

Let us consider a cylinder, made of elastic strings, whose ends are locked into two circles, the bases of the cylinder itself. If one twists the upper circle, keeping the bottom fixed, the strings outline a curved surface, which is in fact a one-sheeted hyperboloid. By further twisting, one gets a number (an infinite number, actually) of differently shaped one-sheeted hyperboloids until, by rotating as far as possible, one eventually gets a more familiar ruled surface, the cone.



The definition given is a constructive one, which makes it immediately clear that the one-sheeted hyperboloid is a ruled surface. Indeed, it is a *doubly ruled surface*, which means that it contains two families of rulings. This is clear when one thinks about rotating the upper circle of the starting cylinder in the opposite direction as before: the outer profile one gets is the same, but the family of lines that draws it is different, i.e. it is inclined the other way. The figure below shows both the families of rulings.



An alternative way of defining the same surface, which in fact explains both the name "hyperboloid" and the adjective "one-sheeted", is the following. Let us take an hyperbola and let us rotate it about the perpendicular bisector to the line between the foci: the resulting surface of revolution is the one-sheeted hyperboloid. On the contrary, if one rotates the same hyperbola about the line joining the foci, one obtains a surface of revolution made of two connected components, which is therefore called a *two-sheeted hyperboloid* (and it is not interesting for our purposes, as it is not a ruled surface at all!).

When oriented along the $z$-axis, the one-sheeted circular hyperboloid with skirt radius $a$ has Cartesian equation

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}-\frac{z^2}{c^2}=1\,,$$

while an obvious generalization, the one-sheeted elliptic hyperboloid, has Cartesian equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1\,,$$

where $a$ and $b$ are the semiaxes of the ellipse. In both cases, therefore, it is a *quadratic surface*, that is an algebraic surface described by an equation of degree 2 in $x$, $y$ and $z$.

In the personal symbolism of Gaudí, the hyperboloid represents *the light*: for this reason the architect uses it systematically in the design of the openings through which, in the Sagrada Família, light from the outside illuminates the aisle. One may call them windows if the term, in this specific case, were not too reductive (see Figure 1).

Fig. 1 Hyperboloid vault of the Sagrada Família.

Moreover, being a careful observer of nature, Gaudí notices that the femur, which holds the weight of our body, has approximately the shape of an hyperboloid. Then, yearning for glorifying God through imitation, the Catalan architect uses the one-sheeted hyperboloid to design the columns of its constructions. This explains the peculiar shape of the columns of the Passion façade (Sagrada Família, Figure 2), as well as many others columns in Gaudí's buildings. Their originality is also due to the fact that they are inclined, in awareness of the fact that «the tired wayfarer, when stopping, leans on the inclined stick, as if the stick were vertical, he would not rest at all».

Fig. 2 Inclined hyperbolic columns in the Passion façade of the Sagrada Família.

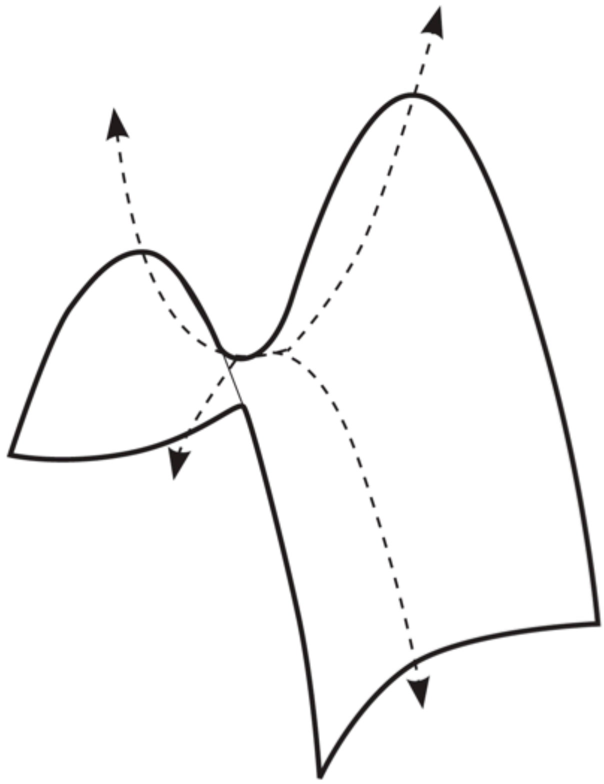
Of course, Gaudí is not alone in using the hyperboloid in architecture: in the same times, in the cold Russian winters, the engineer and scientist Vladimir Shukhov accomplished his first hyperboloid structure, on the occasion of the exhibition in Nizhny Novgorod (Gorky), 1896. It was a tower of steel, consisting of straight bars inclined to connect the two circles at the base and top, to a height of 25 meters. It was a lightweight and rigid structure, perfectly suited to support a water tank capacity of 114,000 liters, to be used for the whole exposition. It was the first in a long series: over the next twenty years Shukhov built hundreds of variations, to be used as water towers, masts for ships or radio antennas, of heights ranging from 20 to 150 meters.

Fig. 3 The Avant-garde radio tower designed by Vladimir Shukhov,1919-21.

Moreover, the nineteenth century architecture has made us familiar with the one-sheeted hyperboloid, as this is the usual form of the cooling towers of nuclear power stations.

*The hyperbolic paraboloid.*

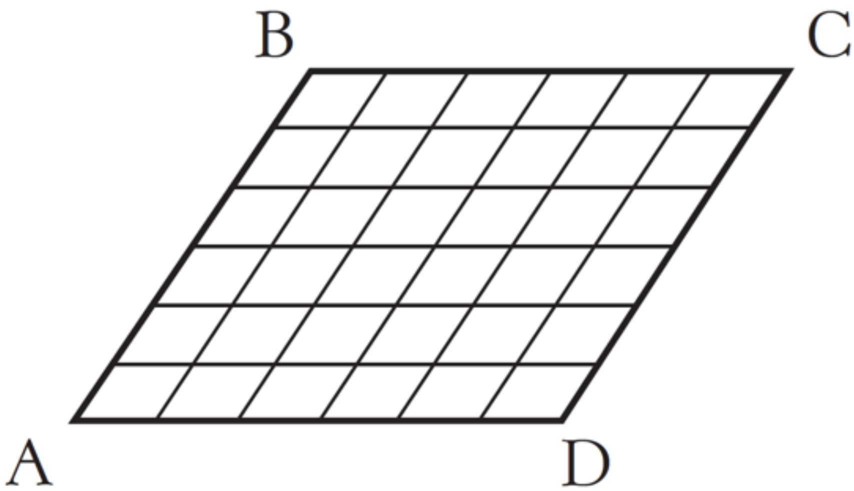
Another ruled surface, also ubiquitous in the works of Gaudí, is the hyperbolic paraboloid, that he called «the father of geometry». This is the saddle-shaped area described by a parabola, which runs parallel to itself with the vertex along another parabola, whose concavity is directed in the opposite direction, as shown in the figure below.



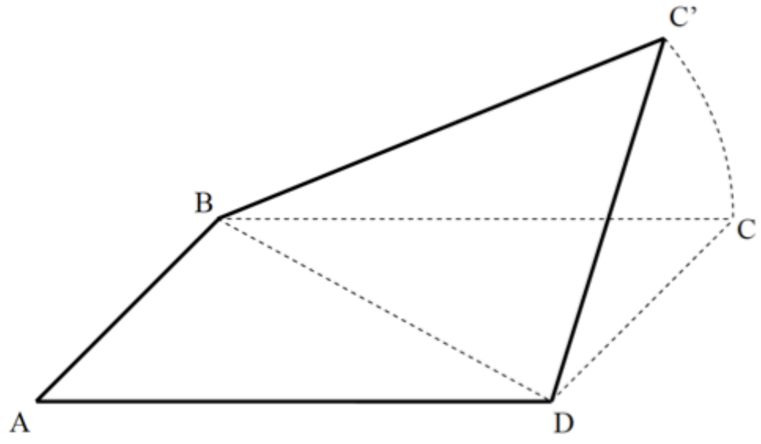
The hyperbolic paraboloid is a quadric as well, given by the Cartesian equation

$$z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$$

Moreover, it is as well a doubly ruled surface, as we will easily prove by the following construction. Let us take a square-shaped structure, with sides made of a rigid material (iron, wood or any other non-elastic one). Let us divide the square into vertical strips through elastic strings, under tension, parallel to the heights, fixed on the lower and upper bases. Then, let us divide it in horizontal strips, using elastic strings parallel to the base, set on the heights, as in the following figure.



Now, take the vertex C and lift it upwards, to the position denoted by C ', leaving the sides AB and AD fixed and thus deforming the square.




The sides of the former square, since they are made of rigid material, will stay straight, and opposite sides, formerly parallel to each other, will become *skew*, that is such that their extensions never meet, because there is no plane containing both of them. On the other hand, the elastic strings will shorten under the effect of the deformation, so that each of them describes the path of minimum length (i.e. the segment) between its ends. The surface of the square, then, will be deformed until it describes another surface, drawn by two families of straight lines, the horizontal and vertical strings. The resulting surface is therefore a doubly ruled surface (since it contains two families of lines), which is exactly the hyperbolic paraboloid we wanted to describe.

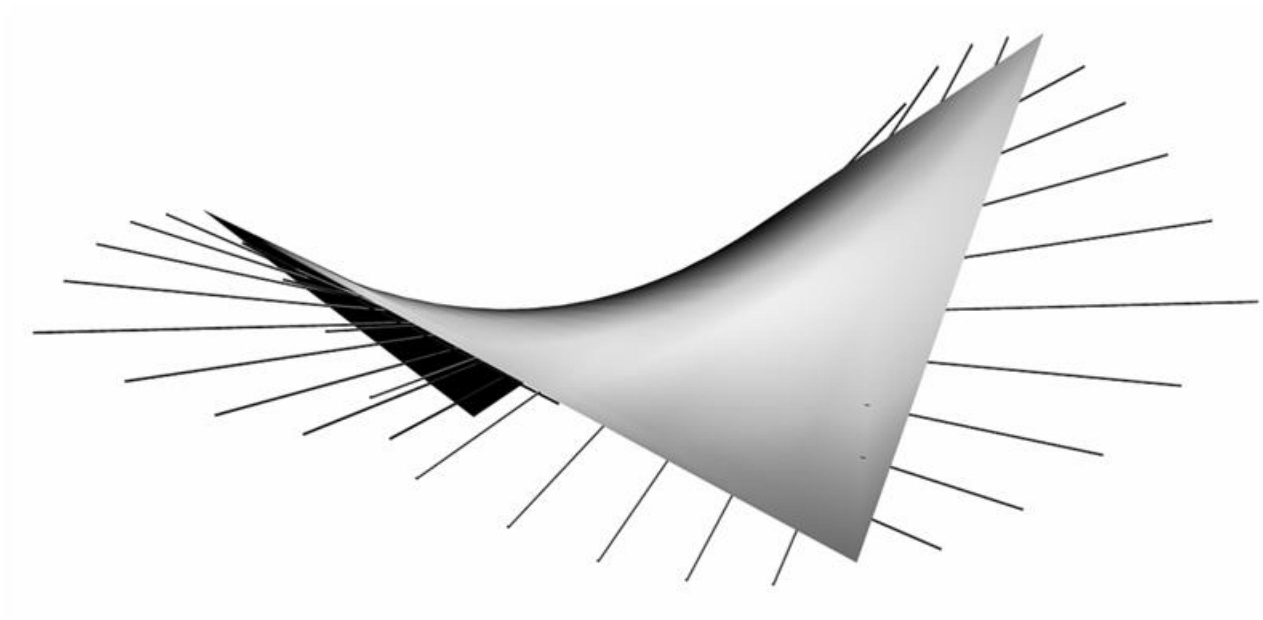


Fig. 4 The hyperbolic paraboloid as a doubly ruled surface

Moreover, the above described construction allows us to view the hyperbolic paraboloid as the surface described by a generating line that moves on two skew straight lines. This can be done in two ways: in the first one, the *generating line* is the vertical string and the *directrices* are the sides

AD and BC', while in the second the generating line is the horizontal string and *directrices* are the sides AB and DC'.

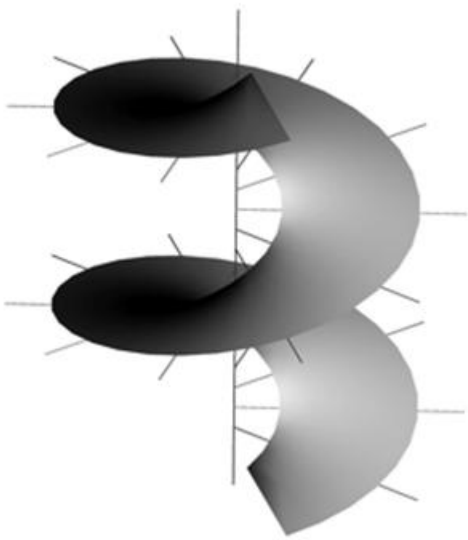
The construction we just described leads Gaudí to interpret the paraboloid as «the perfect symbol of the Trinity», since «a never-ending line represents the Father, the other represents the Son and the third the Holy Spirit, lovingly relating the other two persons».

However, beyond the mystical interpretation, further considerations may explain the good fortune collected by the hyperbolic paraboloid with many other architects, regardless of their religious ardour. The paraboloid is first and foremost an easy surface to construct. To build it, in fact, it is enough to follow the recipe that we just gave, placing two skew lines in space as guides and placing bricks with the help of a rope, supported by those lines.

Gaudi puts the hyperbolic paraboloid everywhere, using it on a small scale to make transitions between different ruled surfaces and on a large scale to make roofs, domes and towers. Later on, many architects of the twentieth century included the hyperbolic paraboloid in their works: Eugene Freyssinet in the hangar for dirigibles at Orly (1916-21), Robert Maillart in a pavilion for the National Exhibition in Zurich (1939), Eduardo Torroja in the tribune of La Zarzuela Hippodrome (1935), Giorgio Baroni in the theater of the Vanzetti foundry in Milan (1937), Ícaro de Castro Mello in the first indoor swimming pool in São Paulo (1953), not to mention Le Corbusier's Philips Pavilion at the Brussels exhibition (1958), Kenzo Tange with Saint Mary's Cathedral in Tokyo (1963), Pier Luigi Nervi with the homonymous Cathedral in San Francisco (1966-71), Carlo Mollino and its coverage of the Teatro Regio in Turin (1965-73), Felix Candela with his hall of cosmic rays in Mexico City (1950-51). And the list could be extended at will…

*The helicoid.*

The helicoid is the surface described by a straight line which wraps around a vertical axis following a helix, as shown in the figure below.



This description makes it clear that it is a ruled surface. As usual, it can also be described in Cartesian coordinates by the following parametric equations:

$$
\begin{aligned}
x &= \rho\cos(\alpha\theta) \\
y &= \rho\sin(\alpha\theta) \\
z &= \theta
\end{aligned},
$$

where $\alpha$ is a constant and $\rho$ and $\theta$ are ranging from $-\infty$ to $+\infty$.

In Gaudí's symbolism, the helicoid represents the upward movement that connects Earth and Heaven. Thus, he uses it as a model to build many of his staircases, including those of La Pedrera and of the towers of the Sagrada Família. However, examples of helicoidal staircases are very common, even in much elder buildings (see also [5]): a very famous one was made by Leonardo da Vinci in the castle of Chambord. Other beautiful examples are provided by those of the Palazzo Contarini del Bovolo in Venice, and those of the Torrazzo of Cremona. The helicoid is also very commonly used for the construction of twisted columns, typical of the Baroque period. Gaudí still manages to take a step further, using the helicoid to shape columns with a very peculiar design. Namely, he takes two helicoids, one winding clockwise and the other one anticlockwise, and he intersects them, getting its peculiar *double twisted column*. We can admire them in the interior of the Sagrada Família, in different sizes which follow a hierarchical order dictated by mechanical and structural constraints.

*The conoid.*

Both the paraboloid and the helicoid are in fact particular cases of a surface, called *conoid*, which is obtained when a generating line slides parallel to a plane (the *directrix plane*), leaning on one side on fixed line (*axis*) and on the other side on a curve (*directrix*). The directrix of the paraboloid, for example, is a straight line, while the directrix of the helicoid is a helix.

Gaudí introduced another conoid, having a sinusoid as its directrix, and he used it to construct the unusual roof of the parish school adjacent to the Sagrada Família (see Figure 4).

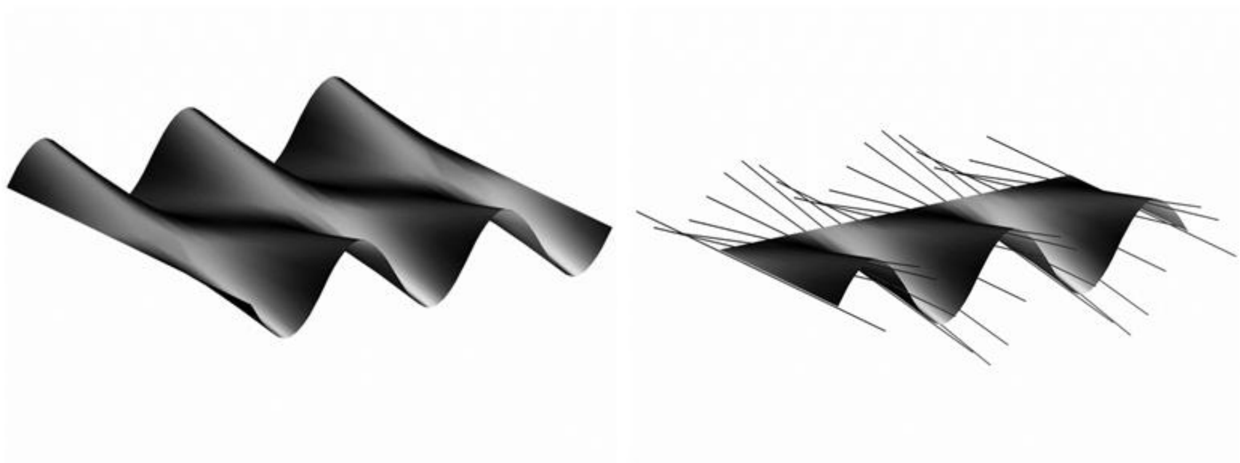


Fig. 5: A sinusoidal conoid.

The result is amazing: Le Corbusier was so charmed by that building, that he defined Gaudí «the best architect in stone of the twentieth century». On the other hand, we'd better skip on Gaudí's comment about his famous admirer. After all, love is not always mutual…

*A step further.*

The use that Gaudí makes of geometry may well be defined a "creative" one: mathematics for him is not only a bare technical tool, but it is also a useful creative support, capable to supply him with inspiration and new ideas. This view is actually shared by many other architects that, overcoming Gaudí's fairly classical geometry, completely disengage from traditional paradigms, to be inspired by a new geometric world, whose principles are those of the non-Euclidean geometries, topology, fractal geometry or high dimensional geometry.

Topology, for instance, takes the lion's share in the projects of the Dutch architects Van Berkel and Bos (see [3] for further discussion). Their Möbius House (Het Gooi, 1998) is inspired by the Möebius strip, the topological surface obtained by a paper strip when giving it a half-twist, and then joining the ends of the strip together to form a loop.

One of its main characteristics is that it has an unique face. Emphasizing this property, the Möebius house has walls which flow and mingle with each other, concrete structures becoming outdoor furniture and windows becoming internal partitions, in a spatial continuity that provides the building with its fascinating dynamism. And the same surface fascinates other architects, for instance the Danish BIG, who designed a Möebius-like futuristic library in Astana, the capital of Kazakhstan (see Figure 6).

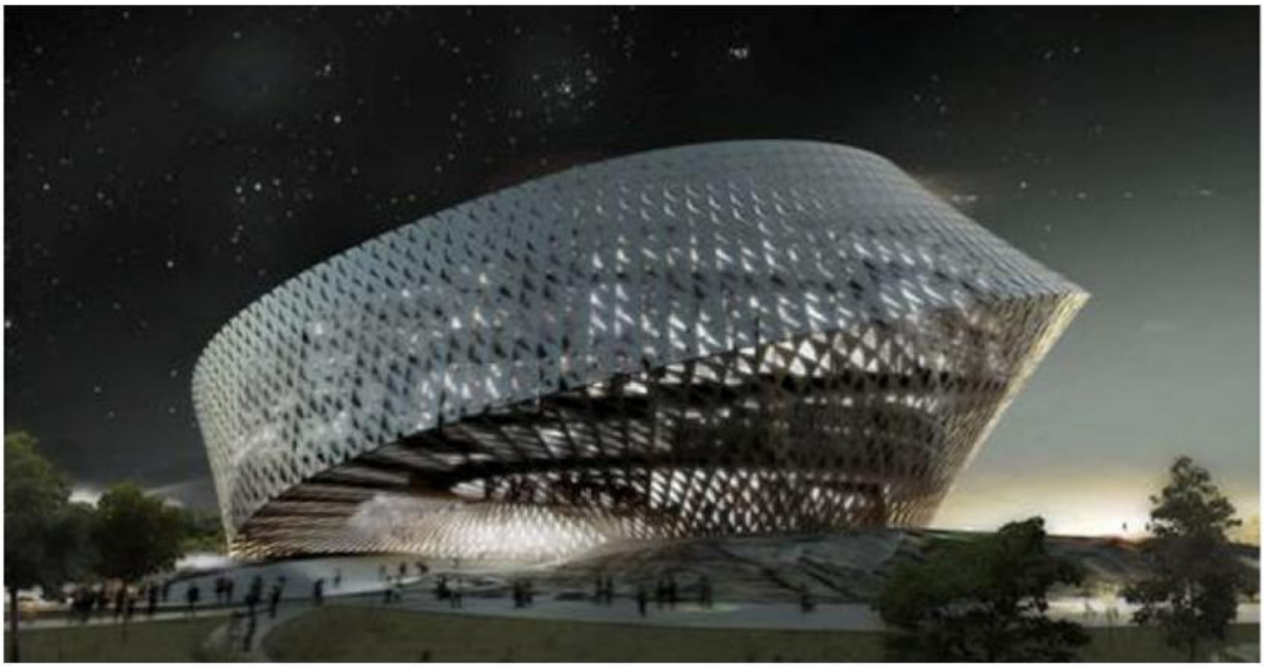


Fig. 6 Astana National Library, BIG.

Van Berkel and Bos go even further, by designing the Arnhem Central station according to a form that mimics the Klein bottle (or better that of its immersion in $R^3$ - since the Klein bottle is in fact a topological surface living in $R^4$).

The Japanese architect Toyo Ito, instead, explores the expressive potential of a variable curvature geometry, that you can admire in the crematorium of Kakamigahara (Tokyo, 2007), better known as "Forest of meditation", build in collaboration with the structural engineer Mutsuro Sasaki. And the same non-linearity concept leads the Canadian Frank O. Gehry in the making of many of his works, one for all the Guggenheim Museum Bilbao (1991 - 1997). And a mention is due to the Spanish Santiago Calatrava, with his impressive City of Arts and Sciences (Valencia, 1996 - 2005).

The German architect Peter Eisenman, instead, plays with fractals and, fascinated by the catastrophe theory of Rene Thom, realizes one of those singularities, the "butterfly", in its Rebstock Park (Frankfurt, 1990). Another elementary catastrophe, the "bend", underlies many buildings, including the Educatorium (Utrecht, 1993) and the VPRO building (Hilversum, 1996) by the Dutch architects of the group MVRDV (see Figure 7).

Fig. 7 A "bend" in the VPRO building by MVRDV.

Among others, Foster+Partners show, morover, how a careful mathematical design could lead to the construction of buildings that, maximizing energy savings, result economically and ecologically sustainable, as well as aesthetically attractive.

Fig. 8 Foster+Partners, 30 St Mary Axe.

And the list could go on at will, including even more original works, whose realization was absolutely unthinkable a few years ago, due to the innovative materials and technologies it requires.

The creative use of mathematics goes beyond architecture, to infiltrate other disciplines, which are apparently even more distant. All done in the spirit of Salvador Dalí, one of the most visionary artists of his time, whose words are a perfect conclusion for our paper:
«You have to use geometry as a guide to the symmetry in the composition of your works. I know that the romantic painters argue that these mathematical frameworks kill the artist's inspiration, giving him too much to think about. Do not hesitate a moment to reply that, on the contrary, it is exactly not to have to think and reflect on certain things, that you use them».

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She has many years of teaching experience, mainly with the architecture students of her University, and she collaborates with several publishing houses in the drafting of texts for high schools and universities. She is the author of a book on *Non-Euclidean Geometries* published by Alphatest in the series Gli Spilli and of the book *Insalate di matematica 3, Sette variazioni su arte, design e architettura*, edited by Sironi.