Constrained BSDEs representation of the value function in optimal control of pure jump Markov processes

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Abstract

We consider a classical finite horizon optimal control problem for continuous-time pure jump Markov processes described by means of a rate transition measure depending on a control parameter and controlled by a feedback law. For this class of problems the value function can often be described as the unique solution to the corresponding Hamilton-Jacobi-Bellman equation. We prove a probabilistic representation for the value function, known as nonlinear Feynman-Kac formula. It relates the value function with a backward stochastic differential equation (BSDE) driven by a random measure and with a sign constraint on its martingale part. We also prove existence and uniqueness results for this class of constrained BSDEs. The connection of the control problem with the constrained BSDE uses a control randomization method recently developed by several authors. This approach also allows to prove that the value function of the original non-dominated control problem coincides with the value function of an auxiliary dominated control problem, expressed in terms of equivalent changes of probability measures.

Keywords: Backward stochastic differential equations, optimal control problems, pure jump Markov processes, marked point processes, randomization.

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1 Introduction

The main aim of this paper is to prove that the value function in a classical optimal control problem for pure jump Markov processes can be represented by means of an appropriate Backward Stochastic Differential Equation (BSDE) that we introduce and for which we prove an existence and uniqueness result. Optimal control of pure jump processes has a long tradition, dating back at least to [31] (see also the references therein and the monograph [7]) and the reader may find in [21] an updated exposition. A renewed interest arose in the context of Mathematical Finance, as many financial market models are described by processes in this class: see for instance [24] where an extensive list of references can be found, or [20] for a related optimization problem.

We start by describing our setting in an informal way. A pure jump Markov process X in a general measurable state space $(E, \mathcal{E})$ can be described by means of a rate transition measure, or intensity measure, $\nu(t, x, B)$ defined for $t \geq 0$, $x \in E$, $B \in \mathcal{E}$. The process starts at time $t \geq 0$ from some initial point $x \in E$ and stays there up to a random time T_1 such that

$$\mathbb{P}(T_1 > s) = \exp\left(-\int_t^s \nu(r, x, E)\, dr\right), \qquad s \geq t.$$

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At time T_1, the process jumps to a new point X_{T_1} chosen with probability $\nu(T_1, x, \cdot)/\nu(T_1, x, E)$ (conditionally to T_1) and then it stays again at X_{T_1} up to another random time T_2 such that

$$\mathbb{P}(T_2 > s \mid T_1, X_{T_1}) = \exp\left(-\int_{T_1}^{s} \nu(r, X_{T_1}, E)\,dr\right), \qquad s \geq T_1,$$

and so on.

A controlled pure jump Markov process is obtained starting from a rate measure $\lambda(x, a, B)$ defined for $x \in E$, $a \in A$, $B \in \mathcal{E}$, i.e., depending on a control parameter a taking values in a measurable space of control actions $(A, \mathcal{A})$. The control strategies we consider consist in the choice of a feedback control law, which is a measurable function $\alpha : [0, \infty) \times E \to A$. $\alpha(t, x) \in A$ is the control action selected at time t if the system is in state x. The controlled Markov process X is simply the one corresponding to the rate transition measure $\lambda(x, \alpha(t, x), B)$, and we denote by $\mathbb{P}_\alpha^{t,x}$ the corresponding law, where t, x are the initial time and starting point. This is the natural way to control a pure jump Markov process, see for instance [21] and Remark 2.3.

We note that an alternative construction of (controlled or uncontrolled) Markov processes consists in defining them as solutions to stochastic equations driven by some noise (for instance, by a Poisson process) and with appropriate coefficients depending on a control process. In the context of pure jump processes, our approach based on the introduction of the controlled rate measure $\lambda(x, a, B)$ often leads to more general results and it is more natural in several contexts.

In the classical finite horizon control problem one seeks to maximize over all control laws α a functional of the form

$$J(t, x, \alpha) = \mathbb{E}_\alpha^{t,x}\left[\int_t^T f(s, X_s, \alpha(s, X_s))\,ds + g(X_T)\right], \tag{1.1}$$

where a deterministic finite horizon $T > 0$ is given and f, g are given real functions, defined on $[0, T] \times E \times A$ and E, representing the running cost and the terminal cost, respectively. The value function of the control problem is defined in the usual way:

$$V(t, x) = \sup_\alpha J(t, x, \alpha), \qquad t \in [0, T],\ x \in E. \tag{1.2}$$

We will only consider the case when the controlled rate measure λ and the costs f, g are bounded. Then, under some technical assumptions, V is known to be the unique solution on $[0, T] \times E$ to the Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{cases} -\dfrac{\partial v}{\partial t}(t, x) &= \sup\limits_{a \in A}\left(\displaystyle\int_E (v(t, y) - v(t, x))\,\lambda(x, a, dy) + f(t, x, a)\right), \\ v(T, x) &= g(x), \end{cases} \tag{1.3}$$

and if the supremum is attained at some $\alpha(t, x) \in A$ depending measurably on (t, x) then α is an optimal feedback law. Note that the right-hand side of (1.3) is an integral operator: this allows for easy notions of solutions to the HJB equation, that do not in particular need the use of the theory of viscosity solutions.

Our purpose is to relate the value function $V(t, x)$ to an appropriate BSDE. We wish to extend to our framework the theory developed in the context of classical optimal control for diffusion processes, constructed as solutions to stochastic differential equations of Itô type driven by Brownian motion, where representation formulae for the solution to the HJB equation exist and are often called nonlinear Feynman-Kac formulae. The majority of those results requires that only the drift coefficient of the stochastic equation depends on the control parameter, so that in this case the HJB equation is a second-order semi-linear partial differential equation and the

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nonlinear Feyman-Kac formula is well known, see e.g. [17]. Generally, in this case the laws of the corresponding controlled processes are all absolutely continuous with respect to the law of a given, uncontrolled process, so that they form a dominated model.

A natural extension to our framework could be obtained imposing conditions implying that the set of probability laws $\{\mathbb{P}_\alpha^{t,x}\}_\alpha$, when α varies over all feedback laws, is a dominated model. This is the point of view taken in [11], where an appropriate BSDE is introduced and solved and a Feyman-Kac formula for the value function is proved in a restricted framework. Extensions are given in [1] to controlled semi-Markov processes and in [10] to more general non-Markovian cases.

In the present paper we want to consider the general case when $\{\mathbb{P}_\alpha^{t,x}\}_\alpha$ is not a dominated model. Even for finite state space E, by a proper choice of the measure $\lambda(x, a, B)$ it is easy to formulate quite natural control problems for which this is the case.

In the context of controlled diffusions, probabilistic formulae for the value function for non-dominated models have been discovered only in recent years. We note that in this case the HJB equation is a fully nonlinear partial differential equation. To our knowledge, there are only a few available techniques. One possibility is to use the theory of second-order BSDEs, see for instance [9], [33]. Another possibility relies on the use of the theory of G-expectations, see e.g. [30]. Both theories have been largely developed by several authors. In this paper we rather follow another approach which was introduced in [5], and then followed by [14], [15], [16], [25] in various contexts of stochastic optimization problems. Here we mainly follow the systematic exposition contained in [26]. Related extensions and applications can be found in [19], [12], [13]. It consists in a *control randomization method* (not to be confused with the use of relaxed controls) which can be described informally as follows, in our framework of controlled pure jump Markov processes.

We note that for any choice of a feedback law α the pair of stochastic processes $(X_s, \alpha(s, X_s))$ represents the state trajectory and the associated control process. In a first step, for any initial time $t \geq 0$ and starting points $x \in E$, $a \in A$, we replace it by an (uncontrolled) Markovian pair of pure jump stochastic processes (X_s, I_s), possibly constructed on a different probability space $(\Omega, \mathcal{F}, \mathbb{P}^{t,x,a})$, in such a way that the process I is a Poisson process with values in the space of control actions A with an intensity measure $\lambda_0(db)$ which is arbitrary but finite and with full support, and $X_t = x$, $I_t = a$ (see Remark 3.3 for further details). Next we formulate an auxiliary optimal control problem where we control the intensity of the process I: for any predictable, bounded and positive random field $\nu_t(b)$ on $(0, \infty) \times A$, by means of a theorem of Girsanov type we construct a probability measure $\mathbb{P}_\nu$ under which the compensator of I is the random measure $\nu_t(b)\, \lambda_0(db)\, dt$ (under $\mathbb{P}_\nu$ the law of X also changes: see Remark 3.3 for further details) and then we maximize the functional

$$\mathbb{E}_\nu \left[g(X_T) + \int_t^T f(s,\, X_s,\, I_s)\, ds \right],$$

over all possible choices of the process ν. Following the terminology of [26], this will be called the *dual* control problem. Its value function, denoted $V^*(t, x, a)$, also depends *a priori* on the starting point $a \in A$ of the process I (in fact we should write $\mathbb{P}_\nu^{t,x,a}$ instead of $\mathbb{P}_\nu$, but in this discussion we drop this dependence for simplicity) and the family $\{\mathbb{P}_\nu\}_\nu$ is a dominated model. As in [26] we are able to show that the value functions for the original problem and the dual one are the same: $V(t, x) = V^*(t, x, a)$, so that the latter does not in fact depend on a. In particular we have replaced the original control problem by a dual one that corresponds to a dominated model and has the same value function. Moreover, we can introduce a well-posed BSDE that represents $V^*(t, x, a)$ (and hence $V(t, x)$). It is an equation on the time interval $[t, T]$ of the form

$$Y_s \;\; = \;\; g(X_T) + \int_s^T f(r, X_r, I_r)\, dr + K_T - K_s$$

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$$-\int_s^T \int_{E \times A} Z_r(y,\, b)\, q(dr\, dy\, db) - \int_s^T \int_A Z_r(X_r,\, b)\, \lambda_0(db)\, dr, \tag{1.4}$$

with unknown triple (Y, Z, K) (depending also on (t, x, a)), where q is the compensated random measure associated to (X, I), Z is a predictable random field and K a predictable increasing càdlàg process, where we additionally add the sign constraint

$$Z_s(X_{s-}, b) \leqslant 0, \qquad ds \otimes d\mathbb{P}^{t,x,a} \otimes \lambda_0(db)\text{-a.e.} \tag{1.5}$$

It turns out that this equation has a unique minimal solution, in an appropriate sense, and that the value of the process Y at the initial time represents both the original and the dual value function:

$$Y_t = V(t, x) = V^*(t, x, a). \tag{1.6}$$

This is the desired BSDE representation of the value function for the original control problem and a Feyman-Kac formula for the general HJB equation (1.3).

We note that, in analogy with the classical diffusive case, a BSDE representation of the value function should be possible even in the more general context of controlled pure jump non-Markovian processes. Clearly, in this case the formulation of the control problem should be appropriately redefined, and in particular it should include more general control strategies than feedback laws, see for instance [4]. This will be the object of future work.

The paper is organized as follows. Section 2 is essentially devoted to lay down a setting where the classical optimal control problem (1.2) is solved by means of the corresponding HJB equation (1.3). We first recall the general construction of a Markov process given its rate transition measure. Having in mind to apply techniques based on BSDEs driven by random measures we need to work in a canonical setting and use a specific filtration, see Remark 2.2. Therefore the construction we present is based on the well-posedness of the martingale problem for multivariate (marked) point processes studied in [22] and it is exposed in detail. This general construction is then used to formulate in a precise way the optimal control problem for the jump Markov process and it is used again in the subsequent section when we define the pair (X, I) mentioned above. Still in section 2, we present classical results on existence and uniqueness of the solution to the HJB equation (1.3) and its identification with the value function V. These results are similar to those in [31], a place where we could find a clear and complete exposition of all the basic theory and to which we refer for further references and related results. We note that the compactness of the space of control actions A, together with suitable upper-semicontinuity conditions of the coefficients of the control problem, is one of the standard assumptions needed to ensure the existence of an optimal control, which is usually constructed by means of an appropriate measurable selection theorem. Since our main aim was only to find a representation formula for the value function we wished to avoid the compactness condition. This was made possible by the use of a different measurable selection result, that however requires lower-semicontinuity conditions. Although this is not usual in the context of maximization problems, this turned out to be the right condition that allows to dispense with compactness assumptions and to prove well-posedness of the HJB equation and a verification theorem. A small variation of the proofs recovers the classical results in [31], and even with slightly weaker assumptions: see Remark 2.11 for a more detailed comparison.

In section 3 we start to develop the control randomization method: we introduce the auxiliary process (X, I) and formulate the dual control problem under appropriate conditions. Finding the correct formulation required some efforts; in particular we could not mimic the approach of previous works on control randomization mentioned above, since we are not dealing with processes defined as solutions to stochastic equations.

In section 4 we introduce the constrained BSDE (1.4)-(1.5) and we prove, under suitable conditions, that it has a unique minimal solution (Y, Z, K) in a certain class of processes. Moreover,

the value of Y at the initial time coincides with the value function of the dual optimal control problem. This is the content of the first of our main results, Theorem 4.2. The proof relies on a penalization approach and a monotonic passage to the limit, and combines BSDE techniques with control-theoretic arguments: for instance, a "penalized" dual control problem is also introduced in order to obtain certain uniform upper bounds. In [26], in the context of diffusion processes, a more general result is proved, in the sense that the generator f may also depend on (Y, Z); similar generalizations are possible in our context as well, but they seem less motivated and in any case they are not needed for the applications to optimal control.

Finally, in section 5 we prove the second of our main results, Theorem 5.1. It states that the initial value of the process Y in (1.4)-(1.5) coincides with the value function $V(t, x)$. As a consequence, the value function is the same for the original optimal control problem and for the dual one and we have the nonlinear Feynman-Kac formula (1.6).

The assumptions in Theorem 5.1 are fairly general: the state space E and the control action space A are Borel spaces, the controlled kernel λ is bounded and has the Feller property, and the cost functions f, g are continuous and bounded. No compactness assumption is required. When E is finite or countable we have the special case of (continuous-time) controlled Markov chains. A large class of optimization problems for controlled Markovian queues falls under the scope of our result.

In recent years there has been much interest in numerical approximation of the value function in optimal control of Markov processes, see for instance the book [21] in the discrete state case. The Feynman-Kac formula (1.6) can be used to design algorithms based on numerical approximation of the solution to the constrained BSDE (1.4)-(1.5). Numerical schemes for this kind of equations have been proposed and analyzed in the context of diffusion processes, see [27], [28]. We hope that our results may be used as a foundation for similar methods in the context of pure jump processes as well.

2   Pure jump controlled Markov processes

2.1   The construction of a jump Markov process given the rate transition measure

Let E be a Borel space, i.e., a topological space homeomorphic to a Borel subset of a compact metric space (some authors call it a Lusin space); in particular, E could be a Polish space. Let $\mathcal{E}$ denote the corresponding Borel σ-algebra.

We will often need to construct a Markov process in E with a given (time dependent) rate transition measure, or intensity measure, denoted by ν. With this terminology we mean that $B \mapsto \nu(t, x, B)$ is a nonnegative measure on $(E, \mathcal{E})$ for every $(t, x) \in [0, \infty) \times E$ and $(t, x) \mapsto \nu(t, x, B)$ is a Borel measurable function on $[0, \infty) \times E$ for every $B \in \mathcal{E}$. We assume that

$$\sup_{t \geq 0, \, x \in E} \nu(t, x, E) < \infty. \tag{2.1}$$

We recall the main steps in the construction of the corresponding Markov process. We note that (2.1) allows to construct a non-explosive process. Since ν depends on time the process will not be time-homogeneous in general. Although the existence of such a process is a well known fact, we need special care in the choice of the corresponding filtration, since this will be crucial when we solve associated BSDEs and implicitly apply a version of the martingale representation theorem in the sections that follow: see also Remark 2.2 below. So in the following we will use an explicit construction that we are going to describe. Many of the techniques we are going to

use are borrowed from the theory of multivariate (marked) point processes. We will often follow [22], but we also refer the reader to the treatise [6] for a more systematic exposition.

We start by constructing a suitable sample space to describe the jumping mechanism of the Markov process. Let Ω' denote the set of sequences $\omega' = (t_n, e_n)_{n \geq 1}$ in $((0, \infty) \times E) \cup \{(\infty, \Delta)\}$, where $\Delta \notin E$ is adjoined to E as an isolated point, satisfying in addition

$$t_n \leq t_{n+1}; \qquad t_n < \infty \implies t_n < t_{n+1}. \tag{2.2}$$

To describe the initial condition we will use the measurable space $(E, \mathcal{E})$. Finally, the sample space for the Markov process will be $\Omega = E \times \Omega'$. We define canonical functions $T_n : \Omega \to (0, \infty]$, $E_n : \Omega \to E \cup \{\Delta\}$ as follows: writing $\omega = (e, \omega')$ in the form $\omega = (e, t_1, e_1, t_2, e_2, \ldots)$ we set for $t \geq 0$ and for $n \geq 1$

$$T_n(\omega) = t_n, \qquad E_n(\omega) = e_n, \qquad T_\infty(\omega) = \lim_{n \to \infty} t_n, \qquad T_0(\omega) = 0, \qquad E_0(\omega) = e.$$

We also define $X : \Omega \times [0, \infty) \to E \cup \{\Delta\}$ setting $X_t = 1_{[0, T_1]}(t) E_0 + \sum_{n \geq 1} 1_{(T_n, T_{n+1}]}(t) E_n$ for $t < T_\infty$, $X_t = \Delta$ for $t \geq T_\infty$.

In Ω we introduce for all $t \geq 0$ the σ-algebras $\mathcal{G}_t = \sigma(N(s, A) : s \in (0, t], A \in \mathcal{E})$, i.e. generated by the counting processes defined as $N(s, A) = \sum_{n \geq 1} 1_{T_n \leq s} 1_{E_n \in A}$. To take into account the initial condition we also introduce the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$, where $\mathcal{F}_0 = \mathcal{E} \otimes \{\emptyset, \Omega'\}$, and for all $t \geq 0$ $\mathcal{F}_t$ is the σ-algebra generated by $\mathcal{F}_0$ and $\mathcal{G}_t$. $\mathbb{F}$ is right-continuous and will be called the natural filtration. In the following all concepts of measurability for stochastic processes (adaptedness, predictability etc.) refer to $\mathbb{F}$. We denote by $\mathcal{F}_\infty$ the σ-algebra generated by all the σ-algebras $\mathcal{F}_t$. The symbol $\mathcal{P}$ denotes the σ-algebra of $\mathbb{F}$-predictable subsets of $[0, \infty) \times \Omega$. The initial distribution of the process X will be described by a probability measure μ on $(E, \mathcal{E})$. Since $\mathcal{F}_0 = \{A \times \Omega' : A \in \mathcal{E}\}$ is isomorphic to $\mathcal{E}$, μ will be identified with a probability measure on $\mathcal{F}_0$, denoted by the same symbol (by abuse of notation) and such that $\mu(A \times \Omega') = \mu(A)$.

On the filtered sample space $(\Omega, \mathbb{F})$ we have so far introduced the canonical marked point process $(T_n, E_n)_{n \geq 1}$. The corresponding random measure p is, for any $\omega \in \Omega$, a σ-finite measure on $((0, \infty) \times E, \mathcal{B}(0, \infty) \otimes \mathcal{E})$ defined as

$$p(\omega, ds\, dy) = \sum_{n \geq 1} 1_{T_n(\omega) < \infty}\, \delta_{(T_n(\omega), E_n(\omega))}(ds\, dy),$$

where δ_k denotes the Dirac measure at point $k \in (0, \infty) \times E$.

Now let ν denote a time-dependent rate transition measure as before, satisfying (2.1). We need to introduce the corresponding generator and transition semigroup as follows. We denote by $B_b(E)$ the space of $\mathcal{E}$-measurable bounded real functions on E and for $\phi \in B_b(E)$ we set

$$\mathcal{L}_t \phi(x) = \int_E (\phi(y) - \phi(x))\, \nu(t, x, dy), \qquad t \geq 0,\, x \in E.$$

For any $T \in (0, \infty)$ and $g \in B_b(E)$ we consider the Kolmogorov equation on $[0, T] \times E$:

$$\begin{cases} \dfrac{\partial v}{\partial s}(s, x) + \mathcal{L}_s v(s, x) = 0, \\ v(T, x) = g(x). \end{cases} \tag{2.3}$$

It is easily proved that there exists a unique measurable bounded function $v : [0, T] \times E$ such that $v(T, \cdot) = g$ on E and, for all $x \in E$, $s \mapsto v(s, x)$ is an absolutely continuous map on $[0, T]$ and the

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first equation in (2.3) holds for almost all $s \in [0,T]$ with respect to the Lebesgue measure. To verify this we first write (2.3) in the equivalent integral form

$$v(s,x) = g(x) + \int_s^T \mathcal{L}_r v(r,x)\, dr, \qquad s \in [0,T],\ x \in E.$$

Then, noting the inequality $|\mathcal{L}_t \phi(x)| \leq 2 \sup_{y \in E} |\phi(y)| \sup_{t \in [0,T], y \in E} \nu(t,y,E)$, a solution to the latter equation can be obtained by a standard fixed point argument in the space of bounded measurable real functions on $[0,T] \times E$ endowed with the supremum norm.

This allows to define the transition operator $P_{sT} : B_b(E) \to B_b(E)$, for $0 \leq s \leq T$, letting $P_{sT}[g](x) = v(s,x)$, where v is the solution to (2.3) with terminal condition $g \in \mathcal{B}_b(E)$.

Proposition 2.1. *Let (2.1) hold and let us fix $t \in [0,\infty)$ and a probability measure μ on $(E,\mathcal{E})$.*

1. *There exists a unique probability measure on $(\Omega, \mathcal{F}_\infty)$, denoted by $\mathbb{P}^{t,\mu}$, such that its restriction to $\mathcal{F}_0$ is μ and the $\mathbb{F}$-compensator (or dual predictable projection) of the measure p under $\mathbb{P}^{t,\mu}$ is the random measure $\tilde{p}(ds\, dy) := 1_{[t,T_\infty)}(s)\nu(s, X_{s-}, dy)\, ds$. Moreover, $\mathbb{P}^{t,\mu}(T_\infty = \infty) = 1$.*

2. *In the probability space $\{\Omega, \mathcal{F}_\infty, \mathbb{P}^{t,\mu}\}$ the process X has distribution μ at time t and it is Markov on the time interval $[t,\infty)$ with respect to $\mathbb{F}$ with transition operator P_{sT}: explicitly, for every $t \leq s \leq T$ and for every $g \in \mathcal{B}_b(E)$,*

$$\mathbb{E}^{t,\mu}\left[g(X_T) \mid \mathcal{F}_s\right] = P_{sT}[g](X_s), \qquad \mathbb{P}^{t,\mu} - a.s.$$

Proof. Point 1 follows from a direct application of [22], Theorem 3.6. The non-explosion condition $\mathbb{P}^{t,\mu}(T_\infty = \infty) = 1$ follows from the fact that λ is bounded.

To prove point 2 we denote $v(s,x) = P_{sT}[g](x)$ the solution to the Kolmogorov equation (2.3) and note that

$$v(T, X_T) - v(s, X_s) = \int_s^T \frac{\partial v}{\partial r}(r, X_r)\, dr + \int_{(s,T]} \int_E (v(r,y) - v(r, X_{r-}))\, p(dr\, dy).$$

This identity is easily proved taking into account that X is constant among jump times and using the definition of the random measure p. Recalling the form of the $\mathbb{F}$-compensator $\tilde{p}$ of p under $\mathbb{P}^{t,\mu}$ we have, $\mathbb{P}^{t,\mu}$-a.s.,

$$\mathbb{E}^{t,\mu}\left[\int_{(s,T]} \int_E (v(r,y) - v(r, X_{r-}))\, p(dr\, dy) \mid \mathcal{F}_s\right]$$
$$= \mathbb{E}^{t,\mu}\left[\int_{(s,T]} \int_E (v(r,y) - v(r, X_{r-}))\, \tilde{p}(dr\, dy) \mid \mathcal{F}_s\right]$$
$$= \mathbb{E}^{t,\mu}\left[\int_{(s,T]} \int_E (v(r,y) - v(r, X_r))\, \nu(r, X_r, dy)\, dr \mid \mathcal{F}_s\right]$$
$$= \mathbb{E}^{t,\mu}\left[\int_{(s,T]} \mathcal{L}_r v(r, X_r)\, dr \mid \mathcal{F}_s\right]$$

and we finally obtain

$$\mathbb{E}^{t,\mu}\left[g(X_T) \mid \mathcal{F}_s\right] - P_{sT}[g](X_s) = \mathbb{E}^{t,\mu}[v(T, X_T) \mid \mathcal{F}_s] - v(s, X_s)$$
$$= \mathbb{E}^{t,\mu}\left[\int_s^T \left(\frac{\partial v}{\partial r}(r, X_r) + \mathcal{L}_r v(r, X_r)\right) dr \mid \mathcal{F}_s\right] = 0.$$

$\square$

In the following we will mainly consider initial distributions μ concentrated at some point $x \in E$, i.e. $\mu = \delta_x$. In this case we use the notation $\mathbb{P}^{t,x}$ rather than $\mathbb{P}^{t,\delta_x}$. Note that, $\mathbb{P}^{t,x}$-a.s., we have $T_1 > t$ and therefore $X_s = x$ for all $s \in [0, t]$.

Remark 2.2. Since the process X is $\mathbb{F}$-adapted, its natural filtration $\mathbb{F}^X = (\mathcal{F}_t^X)_{t \geq 0}$ defined by $\mathcal{F}_t^X = \sigma(X_s : s \in [0, t])$ is smaller than $\mathbb{F}$. The inclusion may be strict, and may remain such if we consider the corresponding completed filtrations. The reason is that the random variables E_n and E_{n+1} introduced above may coincide on a set of positive probability, for some n, and therefore knowledge of a trajectory of X does not allow to reconstruct the trajectory (T_n, E_n).

In order to have $\mathcal{F}_s = \mathcal{F}_s^X$ up to $\mathbb{P}^{t,\mu}$-null sets one could require that $\nu(t, x, \{x\}) = 0$, i.e. that T_n are in fact jump times of X, but this would impose unnecessary restrictions in some constructs that follow.

Clearly, the Markov property with respect to $\mathbb{F}$ implies the Markov property with respect to $\mathbb{F}^X$ as well.

2.2 Optimal control of pure jump Markov processes

In this section we formulate and solve an optimal control problem for a Markov process with a state space E, which is still assumed to be a Borel space with its Borel σ-algebra $\mathcal{E}$. The other data of the problem will be another Borel space A, endowed with its Borel σ-algebra $\mathcal{A}$ and called the space of control actions; a finite time horizon, i.e. a (deterministic) element $T \in (0, \infty)$; two real valued functions f and g, defined on $[0, T] \times E \times A$ and E and called running and terminal cost functions respectively; and finally a measure transition kernel λ from $(E \times A, \mathcal{E} \otimes \mathcal{A})$ to $(E, \mathcal{E})$: namely $B \mapsto \lambda(x, a, B)$ is a nonnegative measure on $(E, \mathcal{E})$ for every $(x, a) \in E \times A$ and $(x, a) \mapsto \lambda(x, a, B)$ is a Borel measurable function for every $B \in \mathcal{E}$. We assume that λ satisfies the following condition:

$$\sup_{x \in E, a \in A} \lambda(x, a, E) < \infty. \tag{2.4}$$

The requirement that $\lambda(x, a, \{x\}) = 0$ for all $x \in E$ and $a \in A$ is natural in many applications, but it is not needed. The kernel λ depending on the control parameter $a \in A$ plays the role of a controlled intensity measure for a controlled Markov process. Roughly speaking, we may control the dynamics of the process by changing its jump intensity dynamically. For a more precise definition, we first construct Ω, $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$, $\mathcal{F}_\infty$ as in the previous paragraph. Then we introduce the class of admissible control laws $\mathcal{A}_{ad}$ as the set of all Borel-measurable maps $\alpha : [0, T] \times E \to A$. To any such α we associate the rate transition measure $\nu^\alpha(t, x, dy) := \lambda(x, \alpha(t, x), dy)$.

For every starting time $t \in [0, T]$ and starting point $x \in E$, and for each $\alpha \in \mathcal{A}_{ad}$, we construct as in the previous paragraph the probability measure on $(\Omega, \mathcal{F}_\infty)$, that will be denoted $\mathbb{P}_\alpha^{t,x}$, corresponding to t, to the initial distribution concentrated at x and to the the rate transition measure ν^α. According to Proposition 2.1, under $\mathbb{P}_\alpha^{t,x}$ the process X is Markov with respect to $\mathbb{F}$ and satisfies $X_s = x$ for every $s \in [0, T]$; moreover, the restriction of the measure p to $(t, \infty) \times E$ admits the compensator $\lambda(X_{s-}, \alpha(s, X_{s-}), dy) \, ds$. Denoting by $\mathbb{E}_\alpha^{t,x}$ the expectation under $\mathbb{P}_\alpha^{t,x}$ we finally define, for $t \in [0, T]$, $x \in E$ and $\alpha \in \mathcal{A}_{ad}$, the gain functional

$$J(t, x, \alpha) = \mathbb{E}_\alpha^{t,x} \left[\int_t^T f(s, X_s, \alpha(s, X_s)) \, ds + g(X_T) \right], \tag{2.5}$$

and the value function of the control problem

$$V(t, x) = \sup_{\alpha \in \mathcal{A}_{ad}} J(t, x, \alpha). \tag{2.6}$$

Since we will assume below that f and g are at least Borel-measurable and bounded, both J and V are well defined and bounded.

Remark 2.3. In this formulation the only control strategies that we consider are control laws of feedback type, i.e., the control action $\alpha(t, x)$ at time t only depends on t and on the state x for the controlled system at the same time. This is a natural and frequently adopted formulation. Different formulations are possible, but usually the corresponding value function is the same and, if an optimal control exists, it is of feedback type.

Remark 2.4. All the results that follows admit natural extensions to slightly more general cases. For instance, λ might depend on time, or the set of admissible control actions may depend on the present state (so admissible control laws should satisfy $\alpha(t, x) \in A(x)$, where $A(x)$ is a given subset of A) provided appropriate measurability conditions are satisfied. We limit ourselves to the previous setting in order to simplify the notation.

Let us consider the Hamilton-Jacobi-Bellman equation (for short, HJB equation) related to the optimal control problem: this is the following nonlinear integro-differential equation on $[0, T] \times E$:

$$-\frac{\partial v}{\partial t}(t, x) \;=\; \sup_{a \in A} \left(\mathcal{L}_E^a v(t, x) + f(t, x, a) \right), \tag{2.7}$$

$$v(T, x) \;=\; g(x), \tag{2.8}$$

where the operator $\mathcal{L}_E^a$ is defined by

$$\mathcal{L}_E^a \phi(x) = \int_E (\phi(y) - \phi(x)) \, \lambda(x, a, dy) \tag{2.9}$$

for all $(t, x, a) \in [0, T] \times E \times A$ and every $\phi \in B_b(E)$.

Definition 2.1. We say that a Borel-measurable bounded function $v : [0, T] \times E \to \mathbb{R}$ is a solution to the HJB equation if the right-hand side of (2.7) is Borel-measurable and, for every $x \in E$, (2.8) holds, the map $t \mapsto v(t, x)$ is absolutely continuous in $[0, T]$ and (2.7) holds almost everywhere on $[0, T]$ (the null set of points where it possibly fails may depend on x).

In the analysis of the HJB equation and the control problem we will use the following function spaces, defined for any metric space S:

1. $C_b(S) = \{\phi : S \to \mathbb{R} \text{ continuous and bounded}\}$,

2. $LSC_b(S) = \{\phi : S \to \mathbb{R} \text{ lower semi-continuous and bounded}\}$.

3. $USC_b(S) = \{\phi : S \to \mathbb{R} \text{ upper semi-continuous and bounded}\}$.

$C_b(S)$, equipped with the supremum norm $\|\phi\|_\infty$, is a Banach space. $LSC_b(S)$ and $USC_b(S)$ are closed subsets of $C_b(S)$, hence complete metric spaces with the induced distance.

In the sequel we need the following classical selection theorem. For a proof we refer for instance to [3], Propositions 7.33 and 7.34, where a more general statement can also be found.

Proposition 2.5. *Let U be a metric space, V a metric separable space. For $F : U \times V \to \mathbb{R}$ set*

$$F^*(u) = \sup_{v \in V} F(u, v), \qquad u \in U.$$

1. *If $F \in USC_b(U \times V)$ and V is compact then $F^* \in USC_b(U)$ and there exists a Borel-measurable $\phi : U \to V$ such that*

$$F(u, \phi(u)) = F^*(u), \qquad u \in U.$$

2. *If $F \in LSC_b(U \times V)$ then $F^* \in LSC_b(U)$ and for every $\epsilon > 0$ there exists a Borel-measurable $\phi_\epsilon : U \to V$ such that*

$$F(u, \phi_\epsilon(u)) \geq F^*(u) - \epsilon, \qquad u \in U.$$

Next we present a well-posedness result and a verification theorem for the HJB equation in the space $LSC_b([0, T] \times E)$, Theorems 2.6 and 2.9 below. The use of lower semi-continuous bounded functions was already commented in the introduction and will be useful for the results in section 5. A small variation of our arguments also yields corresponding results in the class of upper semi-continuous functions, which are more natural when dealing with a maximization problem, see Theorems 2.7 and 2.10 that slightly generalize classical results. We first formulate the assumptions we need.

$$\lambda \text{ is a Feller transition kernel.} \tag{2.10}$$

We recall that this means that for every $\phi \in C_b(E)$ the function $(x, a) \to \int_E \phi(y) \, \lambda(x, a, dy)$ is continuous (hence it belongs to $C_b(E \times A)$ by (2.4)).

Next we will assume either that

$$f \in LSC_b([0, T] \times E \times A), \quad g \in LSC_b(E), \tag{2.11}$$

or

$$f \in USC_b([0, T] \times E \times A), \quad g \in USC_b(E) \text{ and } A \text{ is a compact metric space.} \tag{2.12}$$

Theorem 2.6. *Under the assumptions (2.4), (2.10), (2.11) there exists a unique solution $v \in LSC_b([0, T] \times E)$ to the HJB equation (in the sense of Definition 2.1).*

Proof. We first make a change of unknown function setting $\tilde{v}(t, x) = e^{-\Lambda t} v(t, x)$, where $\Lambda := \sup_{x \in E, a \in A} \lambda(x, a, E)$ is finite by (2.4). It is immediate to check that v is a solution to (2.7)-(2.8) if and only if $\tilde{v}$ is a solution to

$$
\begin{aligned}
-\frac{\partial \tilde{v}}{\partial t}(t, x) &= \sup_{a \in A} \left(\mathcal{L}_E^a \tilde{v}(t, x) + e^{-\Lambda t} f(t, x, a) + \Lambda \tilde{v}(t, x) \right) \\
&= \sup_{a \in A} \left(\int_E \tilde{v}(t, y) \, \lambda(x, a, dy) + (\Lambda - \lambda(x, a, E)) \tilde{v}(t, x) + e^{-\Lambda t} f(t, x, a) \right), \tag{2.13}
\end{aligned}
$$

$$\tilde{v}(T, x) = e^{-\Lambda T} g(x). \tag{2.14}$$

The notion of solution we adopt for (2.13)-(2.14) is completely analogous to Definition 2.1 and need not to be repeated. We set $\Gamma_{\tilde{v}}(t, x) := \int_t^T \sup_{a \in A} \gamma_{\tilde{v}}(s, x, a) \, ds$ where

$$\gamma_{\tilde{v}}(t, x, a) := \int_E \tilde{v}(t, y) \, \lambda(x, a, dy) + (\Lambda - \lambda(x, a, E)) \tilde{v}(t, x) + e^{-\Lambda t} f(t, x, a) \tag{2.15}$$

and note that solving (2.13)-(2.14) is equivalent to finding $\tilde{v} \in LSC_b([0, T] \times E)$ satisfying

$$\tilde{v}(t, x) = g(x) + \Gamma_{\tilde{v}}(t, x), \qquad t \in [0, T], \ x \in E.$$

We will prove that $\tilde{v} \mapsto g + \Gamma_{\tilde{v}}$ is a well defined map of $LSC_b([0, T] \times E)$ into itself and it has a unique fixed point, which is therefore the required solution.

Fix $\tilde{v} \in LSC_b([0, T] \times E)$. It follows easily from (2.4) that $\gamma_{\tilde{v}}$ is bounded and, if $\sup_{a \in A} \gamma_{\tilde{v}}(\cdot, \cdot, a)$ is Borel-measurable, $\Gamma_{\tilde{v}}$ is bounded as well. Next we prove that $\gamma_{\tilde{v}}$ and $\Gamma_{\tilde{v}}$ are lower semi-continuous. Note that $(x, a) \mapsto \Lambda - \lambda(x, a, E)$ continuous and *nonnegative* (this is the reason why we introduced the equation for $\tilde{v}$), so

$$(t, x, a) \mapsto (\Lambda - \lambda(x, a, E)) \tilde{v}(t, x) + e^{-\Lambda t} f(t, x, a)$$

is in $LSC_b([0,T] \times E \times A)$. Since λ is Feller, it is known that the map

$$(t,x,a) \mapsto \int_E \tilde{v}(t,y)\,\lambda(x,a,dy) \tag{2.16}$$

is continuous when $\tilde{v} \in C_b([0,T] \times E)$ (see [3], Proposition 7.30). For general $\tilde{v} \in LSC_b([0,T] \times E)$, there exists a uniformly bounded and increasing sequence $\tilde{v}_n \in C_b([0,T] \times E)$ such that $\tilde{v}_n \to \tilde{v}$ pointwise (see [3], Lemma 7.14). From the Fatou Lemma we deduce that the map (2.16) is in $LSC_b([0,T] \times E \times A)$ and we conclude that $\gamma_{\tilde{v}} \in LSC_b([0,T] \times E \times A)$ as well. Therefore $\sup_{a \in A} \gamma_{\tilde{v}}(\cdot,\cdot,a)$, which equals the right-hand side of (2.13), is lower semi-continuous and hence Borel-measurable. To prove lower semi-continuity of $\Gamma_{\tilde{v}}$ suppose $(t_n, x_n) \to (t,x)$; then

$$
\begin{aligned}
\Gamma_{\tilde{v}}(t_n, x_n) - \Gamma_{\tilde{v}}(t,x) &= \int_{t_n}^{t} \sup_{a \in A} \gamma_{\tilde{v}}(s, x_n, a)\,ds + \int_t^T \left(\sup_{a \in A} \gamma_{\tilde{v}}(s, x_n, a) - \sup_{a \in A} \gamma_{\tilde{v}}(s, x, a) \right) ds \\
&\geq -|t - t_n|\,\|\gamma_{\tilde{v}}\|_\infty + \int_t^T \left(\sup_{a \in A} \gamma_{\tilde{v}}(s, x_n, a) - \sup_{a \in A} \gamma_{\tilde{v}}(s, x, a) \right) ds.
\end{aligned}
$$

By the Fatou Lemma

$$\liminf_{n \to \infty} \Gamma_{\tilde{v}}(t_n, x_n) - \Gamma_{\tilde{v}}(t,x) \geq \int_t^T \liminf_{n \to \infty} \left(\sup_{a \in A} \gamma_{\tilde{v}}(s, x_n, a) - \sup_{a \in A} \gamma_{\tilde{v}}(s, x, a) \right) ds \geq 0,$$

where in the last inequality we have used the lower semi-continuity of $\sup_{a \in A} \gamma_{\tilde{v}}(\cdot, \cdot, a)$.

Since we assume that $g \in LSC_b(E)$ we have thus checked that $\tilde{v} \mapsto g + \Gamma_{\tilde{v}}$ maps $LSC_b([0,T] \times E)$ into itself. To prove that it has a unique fixed point we note the easy estimate based on (2.4), valid for every $\tilde{v}', \tilde{v}'' \in LSC_b([0,T] \times E)$:

$$
\begin{aligned}
\left| \sup_{a \in A} \gamma_{\tilde{v}'}(t, x, a) - \sup_{a \in A} \gamma_{\tilde{v}''}(t, x, a) \right| &\leq \sup_{a \in A} |\gamma_{\tilde{v}'}(t, x, a) - \gamma_{\tilde{v}''}(t, x, a)| \\
&\leq \sup_{a \in A} \left(\int_E |\tilde{v}'(t,y) - \tilde{v}''(t,y)|\,\lambda(x, a, dy) + |\tilde{v}'(t,x) - \tilde{v}''(t,x)|\,\lambda(x, a, E) \right) \\
&\leq 2\Lambda\,\|\tilde{v}' - \tilde{v}''\|_\infty.
\end{aligned}
$$

By a standard technique one proves that a suitable iteration of the map $\tilde{v} \mapsto g + \Gamma_{\tilde{v}}$ is a contraction with respect to the distance induced by the supremum norm, and hence that map has a unique fixed point. $\square$

Theorem 2.7. *Under the assumptions* (2.4), (2.10), (2.12) *there exists a unique solution* $v \in USC_b([0,T] \times E)$ *to the HJB equation.*

Proof. The proof is almost the same as in the previous Theorem, replacing LSC_b with USC_b with obvious changes. We introduce $\tilde{v}$, $\gamma_{\tilde{v}}$ and $\Gamma_{\tilde{v}}$ as before and we prove in particular that $\gamma_{\tilde{v}} \in USC_b([0,T] \times E \times A)$. The only difference is that we can not immediately conclude that $\sup_{a \in A} \gamma_{\tilde{v}}(\cdot, \cdot, a)$ is upper semi-continuous as well. However, at this point we can apply point 1 of Proposition 2.5 choosing $U = [0,T] \times E$, $V = A$ and $F = \gamma_{\tilde{v}}$ and we deduce that in fact $\sup_{a \in A} \gamma_{\tilde{v}}(\cdot, \cdot, a) \in USC_b([0,T] \times E)$. The rest of the proof is the same. $\square$

Corollary 2.8. *Under the assumptions* (2.4), (2.10), *if* $f \in C_b([0,T] \times E \times A)$, $g \in C_b(E)$ *and A is a compact metric space then the solution v to the HJB equation belongs to* $C_b([0,T] \times E)$.

The Corollary follows immediately from the two previous results. We proceed to a verification theorem for the HJB equation.

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Theorem 2.9. *Under the assumptions* (2.4), (2.10), (2.11) *the unique solution* $v \in LSC_b([0, T] \times E)$ *to the HJB equation coincides with the value function* V.

Proof. Let us fix $(t, x) \in [0, T] \times E$. As in the proof of Proposition 2.1 we have the identity

$$g(X_T) - v(t, X_t) = \int_t^T \frac{\partial v}{\partial r}(r, X_r) \, dr + \int_{(t,T]} \int_E (v(r, y) - v(r, X_{r-})) \, p(dr \, dy),$$

which follows from the absolute continuity of $t \mapsto v(t, x)$, taking into account that X is constant among jump times and using the definition of the random measure p. Given an arbitrary admissible control $\alpha \in \mathcal{A}_{ad}$ we take the expectation with respect to the corresponding probability $\mathbb{P}_\alpha^{t,x}$. Recalling that the compensator under $\mathbb{P}^{t,x}$ is $1_{[t,\infty)}(s)\lambda(X_{s-}, \alpha(s, X_{s-}), dy) \, ds$ we obtain

$$
\begin{aligned}
\mathbb{E}_\alpha^{t,x}[g(X_T)] - v(t, X_t) \;=\; & \int_t^T \frac{\partial v}{\partial r}(r, X_r) \, dr \\
& + \int_{(t,T]} \int_E (v(r, y) - v(r, X_{r-})) \, \lambda(X_{r-}, \alpha(r, X_{r-}), dy) \, dr \\
=\; & \int_t^T \left(\frac{\partial v}{\partial r}(r, X_r) + \mathcal{L}_E^{\alpha(r, X_r)} v(r, X_r) \right) dr.
\end{aligned}
$$

Adding $\mathbb{E}_\alpha^{t,x} \int_t^T f(r, X_r, \alpha(r, X_r)) \, dr$ to both sides and rearranging terms we obtain

$$v(t, x) = J(t, x, \alpha) - \mathbb{E}_\alpha^{t,x} \int_t^T \left\{ \frac{\partial v}{\partial r}(r, X_r) + \mathcal{L}_E^{\alpha(r, X_r)} v(r, X_r) + f(r, X_r, \alpha(r, X_r)) \right\} dr. \quad (2.17)$$

Recalling the HJB equation and taking into account that X has piecewise constant trajectories we conclude that the term in curly brackets $\{\dots\}$ is nonpositive and therefore we have $v(t, x) \geq J(t, x, \alpha)$ for every admissible control.

Now we recall that in the proof of Theorem 2.6 we showed that the function $\gamma_{\tilde{v}}$ defined in (2.15) belongs to $LSC_b([0, T] \times E \times A)$. Therefore the function

$$F(t, x, a) := e^{\Lambda t} \gamma_{\tilde{v}}(t, x, a) = \mathcal{L}_E^a v(t, x) + f(t, x, a) + \Lambda v(t, x)$$

is also lower semi-continuous and bounded. Applying point 2 of Proposition 2.5 with $U = [0, T] \times E$ and $V = A$ we see that for every $\epsilon > 0$ there exists a Borel-measurable $\alpha_\epsilon : [0, T] \times E \to A$ such that $F(t, x, \alpha_\epsilon(t, x)) \geq \inf_{a \in A} F(t, x, a) - \epsilon$ for all $t \in [0, T]$, $x \in E$. Taking into account the HJB equation we conclude that for every $x \in E$ we have

$$\mathcal{L}_E^{\alpha_\epsilon(t,x)} v(t, x) + f(t, x, \alpha_\epsilon(t, x)) \geq -\frac{\partial v}{\partial t}(t, x) - \epsilon$$

for almost all $t \in [0, T]$. Noting that α_ϵ is an admissible control and choosing $\alpha = \alpha_\epsilon$ in (2.17) we obtain $v(t, x) \leq J(t, x, \alpha_\epsilon) + \epsilon (T - t)$. Since we know that $v(t, x) \geq J(t, x, \alpha)$ for every $\alpha \in \mathcal{A}_{ad}$ we conclude that v coincides with the value function V. $\square$

Theorem 2.10. *Under assumptions* (2.4), (2.10), (2.12) *the unique solution* $v \in USC_b([0, T] \times E)$ *to the HJB equation coincides with the value function* V. *Moreover there exists an optimal control* α, *which is given by any function satisfying*

$$\mathcal{L}_E^{\alpha(t,x)} v(t, x) + f(t, x, \alpha(t, x)) = \sup_{a \in A} \left(\mathcal{L}_E^a v(t, x) + f(t, x, a) \right). \quad (2.18)$$

Proof. We proceed as in the previous proof, but we can now apply point 2 of Proposition 2.5 to the function F and deduce that there exists a Borel-measurable $\alpha : [0, T] \times E \to A$ such that (2.18) holds. Any such control α is optimal: in fact we obtain for every $x \in E$,

$$\mathcal{L}_E^{\alpha(t,x)} v(t, x) + f(t, x, \alpha(t, x)) = -\frac{\partial v}{\partial t}(t, x)$$

for almost all $t \in [0, T]$ and so $v(t, x) = J(t, x, \alpha)$. □

Remark 2.11. As already mentioned, Theorems 2.7 and 2.10 are similar to classical results: compare for instance [31], Theorems 10, 12, 13, 14. In that paper the author solves the HJB equations by means of a general result on nonlinear semigroups of operators, and for this he requires some more functional-analytic structure, for instance he embeds the set of decision rules into a properly chosen topological vector space. He also has more stringent conditions of the kernel λ, for instance $\lambda(x, a, B)$ should be strictly positive and continuous in (x, a) for each fixed $B \in \mathcal{E}$.

3 Control randomization and dual optimal control problem

In this section we start to implement the control randomization method. In the first step, for any initial time $t \geq 0$ and starting point $x \in E$, we construct an (uncontrolled) Markovian pair of pure jump stochastic processes (X, I) with values in $E \times A$, by specifying its rate transition measure Λ as in (3.3) below. Next we formulate an auxiliary optimal control problem where, roughly speaking, we optimize a cost functional by modifying the intensity of the process (X, I) over a suitable family. This "dual" control problem will be studied in section 4 by an approach based on BSDEs. In section 5 we will prove that the dual value function coincides with the one introduced in the previous section.

3.1 A randomized control system

Let E, A be Borel spaces with corresponding Borel σ-algebras $\mathcal{E}$, $\mathcal{A}$ and let λ be a measure transition kernel from $(E \times A, \mathcal{E} \otimes \mathcal{A})$ to $(E, \mathcal{E})$ as before. As another basic datum we suppose we are given a finite measure λ_0 on $(A, \mathcal{A})$ with full topological support, i.e., it is strictly positive on any non-empty open subset of A. Note that since A is metric separable such a measure can always be constructed, for instance supported on a dense discrete subset of A. We still assume (2.4), so we formulate the following assumption:

(Hλ) λ_0 is a finite measure on $(A, \mathcal{A})$ with full topological support and λ satisfies

$$\sup_{x \in E, a \in A} \lambda(x, a, E) < \infty. \tag{3.1}$$

On the contrary, the Feller property (2.10) earlier imposed on λ is not needed in this Section nor in Section 4; it will be required again for the results of Section 5.

We wish to construct a Markov process as in section 2.1, but with state space $E \times A$. Accordingly, let Ω' denote the set of sequences $\omega' = (t_n, e_n, a_n)_{n \geq 1}$ contained in $((0, \infty) \times E \times A) \cup \{(\infty, \Delta, \Delta')\}$, where $\Delta \notin E$ (respectively, $\Delta' \notin A$) is adjoined to E (respectively, to A) as an isolated point, satisfying (2.2) In the sample space $\Omega = E \times A \times \Omega'$ we define $T_n : \Omega \to (0, \infty]$, $E_n : \Omega \to E \cup \{\Delta\}$, $A_n : \Omega \to A \cup \{\Delta'\}$, as follows: writing $\omega = (e, a, \omega')$ in the form $\omega = (e, a, t_1, e_1, t_2, e_2, \dots)$ we set for $t \geq 0$ and for $n \geq 1$

$$T_n(\omega) = t_n, \qquad T_\infty(\omega) = \lim_{n \to \infty} t_n, \qquad T_0(\omega) = 0,$$
$$E_n(\omega) = e_n, \qquad A_n(\omega) = a_n, \qquad E_0(\omega) = e, \qquad A_0(\omega) = a.$$

We also define processes $X : \Omega \times [0, \infty) \to E \cup \{\Delta\}$, $I : \Omega \times [0, \infty) \to A \cup \{\Delta'\}$ setting

$$X_t = 1_{[0,T_1]}(t) E_0 + \sum_{n \geq 1} 1_{(T_n, T_{n+1}]}(t) E_n, \qquad I_t = 1_{[0,T_1]}(t) A_0 + \sum_{n \geq 1} 1_{(T_n, T_{n+1}]}(t) A_n,$$

for $t < T_\infty$, $X_t = \Delta$ and $I_t = \Delta'$ for $t \geq T_\infty$.

In Ω we introduce for all $t \geq 0$ the σ-algebras $\mathcal{G}_t = \sigma(N(s, B) : s \in (0, t], B \in \mathcal{E} \otimes \mathcal{A})$ generated by the counting processes $N(s, B) = \sum_{n \geq 1} 1_{T_n \leq s} 1_{(E_n, A_n) \in B}$ and the σ-algebra $\mathcal{F}_t$ generated by $\mathcal{F}_0$ and $\mathcal{G}_t$, where $\mathcal{F}_0 := \mathcal{E} \otimes \mathcal{A} \otimes \{\emptyset, \Omega'\}$. We still denote $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ and $\mathcal{P}$ the corresponding filtration and predictable σ-algebra. By abuse of notation we also denote by the same symbol the trace of $\mathcal{P}$ on subsets of the form $[0, T] \times \Omega$ or $[t, T] \times \Omega$, for deterministic times $0 \leq t \leq T < \infty$.

The random measure p is now defined on $(0, \infty) \times E \times A$ as

$$p(ds\,dy\,db) = \sum_{n \in \mathbb{N}} 1_{\{T_n < \infty\}} \delta_{\{T_n, E_n, A_n\}}(ds\,dy\,db). \tag{3.2}$$

By means of λ and λ_0 satisfying assumption $(\mathbf{H}\lambda)$ we define a (time-independent) rate transition measure on $E \times A$ given by

$$\Lambda(x, a; dy\,db) = \lambda(x, a, dy)\, \delta_a(db) + \lambda_0(db)\, \delta_x(dy). \tag{3.3}$$

and the corresponding generator $\mathcal{L}$:

$$\begin{aligned}
\mathcal{L}\varphi(x, a) &:= \int_{E \times A} (\varphi(y, b) - \varphi(x, a))\, \Lambda(x, a; dy\,db) \\
&= \int_E (\varphi(y, a) - \varphi(x, a))\, \lambda(x, a, dy) + \int_A (\varphi(x, b) - \varphi(x, a))\, \lambda_0(db),
\end{aligned}$$

for all $(x, a) \in E \times A$ and every function $\varphi \in B_b(E \times A)$.

Given any starting time $t \geq 0$ and starting point $(x, a) \in E \times A$, an application of Proposition 2.1 provides a probability measure on $(\Omega, \mathcal{F}_\infty)$, denoted by $\mathbb{P}^{t,x,a}$, such that (X, I) is a Markov process on the time interval $[t, \infty)$ with respect to $\mathbb{F}$ with transition probabilities associated to $\mathcal{L}$. Moreover, $\mathbb{P}^{t,x,a}$-a.s., $X_s = x$ and $I_s = a$ for all $s \in [0, t]$. Finally, the restriction of the measure p to $(t, \infty) \times E \times A$ admits as $\mathbb{F}$-compensator under $\mathbb{P}^{t,x,a}$ the random measure

$$\tilde{p}(ds\,dy\,db) := \lambda_0(db)\, \delta_{\{X_{s-}\}}(dy)\, ds + \lambda(X_{s-}, I_{s-}, dy)\, \delta_{\{I_{s-}\}}(db)\, ds.$$

We denote $q := p - \tilde{p}$ the compensated martingale measure associated to p.

Remark 3.1. Note that $\Lambda(x, a; \{x, a\}) = \lambda_0(\{a\}) + \lambda(x, a, \{x\})$. So even if we assumed that $\lambda(x, a, \{x\}) = 0$, in general the rate measure Λ would not satisfy the corresponding condition $\Lambda(x, a; \{x, a\}) = 0$. We remark that imposing the additional requirement that $\lambda_0(\{a\}) = 0$ is too restrictive since, due to the assumption that λ_0 has full support, it would rule out the important case when the space of control actions A is finite or countable.

3.2 The dual optimal control problem

We introduce a dual control problem associated to the process (X, I) and formulated in a weak form. For fixed (t, x, a), it consists in defining a family of probability measures $\{\mathbb{P}_\nu^{t,x,a}, \nu \in \mathcal{V}\}$ in the space $(\Omega, \mathcal{F}_\infty)$, all absolutely continuous with respect to $\mathbb{P}^{t,x,a}$, whose effect is to change the stochastic intensity of the process (X, I) (more precisely, under each $\mathbb{P}_\nu^{t,x,a}$ the compensator of the associated point process takes a desired form), with the aim of maximizing a cost depending

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on f, g. We note that $\{\mathbb{P}^{t,x,a}_\nu, \ \nu \in \mathcal{V}\}$ is a dominated family of probability measures. We proceed with precise definitions.

We still assume that $(\mathbf{H}\lambda)$ holds. Let us define

$$\mathcal{V} \quad = \quad \{\nu : \Omega \times [0, \infty) \times A \to (0, \infty), \ \mathcal{P} \otimes \mathcal{A}\text{-measurable and bounded}\}.$$

For every $\nu \in \mathcal{V}$, we consider the predictable random measure

$$\tilde{p}^\nu(ds\, dy\, db) := \nu_s(b)\, \lambda_0(db)\, \delta_{\{X_{s-}\}}(dy)\, ds + \lambda(X_{s-},\, I_{s-},\, dy)\, \delta_{\{I_{s-}\}}(db)\, ds. \tag{3.4}$$

Now we fix $t \in [0, T]$, $x \in E$, $a \in A$ and, with the help of a theorem of Girsanov type, we will show how to construct a probability measure on $(\Omega, \mathcal{F}_\infty)$, equivalent to $\mathbb{P}^{t,x,a}$, under which $\tilde{p}^\nu$ is the compensator of the measure p on $(0, T] \times E \times A$. By the Radon-Nikodym theorem one can find two nonnegative functions d_1, d_2 defined on $\Omega \times [0, \infty) \times E \times A$, measurable with respect to $\mathcal{P} \otimes \mathcal{E} \otimes \mathcal{A}$ such that

$$
\begin{aligned}
\lambda_0(db)\, \delta_{\{X_{t-}\}}(dy)\, dt &= d_1(t, y, b)\, \tilde{p}(dt\, dy\, db) \\
\lambda(X_{t-},\, I_{t-},\, dy)\, \delta_{\{I_{t-}\}}(db)\, dt &= d_2(t, y, b)\, \tilde{p}(dt\, dy\, db), \\
d_1(t, y, b) + d_2(t, y, b) &= 1, \qquad \tilde{p}(dt\, dy\, db)\text{-a.e.}
\end{aligned}
$$

and we have $d\tilde{p}^\nu = (\nu\, d_1 + d_2)\, d\tilde{p}$. For any $\nu \in \mathcal{V}$, consider then the Doléans-Dade exponential local martingale L^ν defined setting $L^\nu_s = 1$ for $s \in [0, t]$ and

$$
\begin{aligned}
L^\nu_s &= \exp\left(\int_t^s \int_{E \times A} \log(\nu_r(b)\, d_1(r, y, b) + d_2(r, y, b))\, p(dr\, dy\, db) - \int_t^s \int_A (\nu_r(b) - 1)\lambda_0(db)\, dr\right) \\
&= e^{\int_t^s \int_A (1 - \nu_r(b))\lambda_0(db)\, dr} \prod_{n \geq 1 : T_n \leq s} (\nu_{T_n}(A_n)\, d_1(T_n, E_n, A_n) + d_2(T_n, E_n, A_n))
\end{aligned}
$$

for $s \in [t, T]$, where $q = p - \tilde{p}$. When L^ν is a true martingale, i.e., $\mathbb{E}^{t,x,a}[L^\nu_T] = 1$, we can define a probability measure $\mathbb{P}^{t,x,a}_\nu$ equivalent to $\mathbb{P}^{t,x,a}$ on $(\Omega, \mathcal{F}_\infty)$ setting $\mathbb{P}^{t,x,a}_\nu(d\omega) = L^\nu_T(\omega)\mathbb{P}^{t,x,a}(d\omega)$. By the Girsanov theorem for point processes ([22], Theorem 4.5) the restriction of the random measure p to $(0, T] \times E \times A$ admits $\tilde{p}^\nu = (\nu\, d_1 + d_2)\, \tilde{p}$ as compensator under $\mathbb{P}^{t,x,a}_\nu$. We denote by $\mathbb{E}^{t,x,a}_\nu$ the expectation operator under $\mathbb{P}^{t,x,a}_\nu$ and by $q^\nu := p - \tilde{p}^\nu$ the compensated martingale measure of p under $\mathbb{P}^{t,x,a}_\nu$. The validity of the condition $\mathbb{E}^{t,x,a}[L^\nu_T] = 1$ under our assumptions, as well as other useful properties, are proved in the following proposition.

Lemma 3.2. *Let assumption $(\mathbf{H}\lambda)$ hold. Then, for every $t \in [0, T]$, $x \in E$ and $\nu \in \mathcal{V}$, under the probability $\mathbb{P}^{t,x,a}$ the process L^ν is a martingale on $[0, T]$ and L^ν_T is square integrable.*

In addition, for every $\mathcal{P} \otimes \mathcal{E} \otimes \mathcal{A}$-measurable function $H : \Omega \times [t, T] \times E \times A \to \mathbb{R}$ such that $\mathbb{E}^{t,x,a}\left[\int_t^T \int_{E \times A} |H_s(y, b)|^2\, \tilde{p}(ds\, dy\, db)\right] < \infty$, the process $\int_t^\cdot \int_{E \times A} H_s(y, b)\, q^\nu(ds\, dy\, db)$ is a $\mathbb{P}^{t,x,a}_\nu$-martingale on $[t, T]$.

Proof. The first part of the proof is inspired by Lemma 4.1 in [26]. In particular, since ν is bounded and $\lambda_0(A) < \infty$, we see that

$$S^\nu_T = \exp\left(\int_t^T \int_A |\nu_s(b) - 1|^2 \lambda_0(db)\, ds\right)$$

is bounded. Therefore, from Theorem 8, see also Theorem 9, in [32], follows the martingale property of L^ν together with its uniform integrability. Concerning the square integrability of L^ν_T,

set $\ell(x, \lambda) := 2\ln(x\lambda + 1 - \lambda) - \ln(x^2\lambda + 1 - \lambda)$, for any $x \geq 0$ and $\lambda \in [0, 1]$. From the definition of L^ν we have (recalling that $d_2(s, y, b) = 1 - d_1(s, y, b)$)

$$|L_T^\nu|^2 = L_T^{\nu 2} S_T^\nu \exp\left(\int_t^T \int_{E \times A} \ell(\nu_s(b), d_1(s, y, b))\, p(ds\, dy\, db)\right) \leq L_T^{\nu 2} S_T^\nu,$$

where the last inequality follows from the fact that ℓ is nonpositive. This entails that L_T^ν is square integrable.

Let us finally fix a predictable function H such that $\mathbb{E}^{t,x,a}\left[\int_t^T \int_{E \times A} |H_s(y, b)|^2\, \tilde{p}(ds\, dy\, db)\right] < \infty$. The process $\int_t^\cdot \int_{E \times A} H_s(y, b)\, q^\nu(ds\, dy\, db)$ is a $\mathbb{P}_\nu^{t,x,a}$-local martingale, and the uniform integrability follows from the Burkholder-Davis-Gundy and Cauchy Schwarz inequalities, together with the square integrability of L_T^ν. □

To complete the formulation of the dual optimal control problem we specify the conditions that we will assume for the cost functions f, g (given a general Borel space S, we denote by $B_b(S)$ the space of Borel-measurable bounded real functions on S):

(Hfg) $\quad f \in B_b([0, T] \times E \times A)$ and $g \in B_b(E)$.

For every $t \in [0, T]$, $x \in E$, $a \in A$ and $\nu \in \mathcal{V}$ we finally introduce the dual gain functional

$$J(t, x, a, \nu) = \mathbb{E}_\nu^{t,x,a}\left[g(X_T) + \int_t^T f(s,\, X_s,\, I_s) ds\right],$$

and the dual value function

$$V^*(t, x, a) = \sup_{\nu \in \mathcal{V}} J(t, x, a, \nu). \tag{3.5}$$

Remark 3.3. An interpretation of the dual optimal control problem can be given as follows. Suppose that $\lambda(x, a, \{x\})) = 0$ for all $x \in E$ and $a \in A$, let $\{R_n\} \subset \{T_n\}$ denote the sequence of jump times of X and let $\{S_n\}$ denote (a renumbering of) the remaining elements of $\{T_n\}$. It can be proved that, under $\mathbb{P}_\nu^{t,x,a}$, the compensators of the corresponding random measures $\mu^I(ds\, db) = \sum_n \delta_{(S_n, I_{S_n})}(ds\, db)$ on $(0, \infty) \times A$ and $\mu^X(ds\, dy) = \sum_n \delta_{(R_n, X_{R_n})}(ds\, dy)$ on $(0, \infty) \times E$ are

$$\tilde{\mu}^I(ds\, db) = \nu_s(b)\, \lambda_0(db)\, 1_{(t,\infty)}(s)\, ds, \qquad \tilde{\mu}^X(ds\, dy) = \lambda(X_{s-}, I_{s-}, dy)\, 1_{(t,\infty)}(s)\, ds.$$

Thus, the effect of choosing ν is to change the intensity of the I-component. We leave the proofs of these facts to the reader since they will not be used in the sequel.

4 Constrained BSDE and representation of the dual value function

In this section we introduce a BSDE, with a sign constrain on its martingale part, and prove existence and uniqueness of a minimal solution, in an appropriate sense. The BSDE is then used to give a representation formula for the dual value function introduced above.

Throughout this section we assume that the assumptions **(Hλ)** and **(Hfg)** are satisfied and we use the randomized control setting introduced above: $\Omega, \mathbb{F}, X, \mathbb{P}^{t,x,a}$ as well as the random measures $p, \tilde{p}, q$ are the same as in subsection 3.1. For any $(t, x, a) \in [0, T] \times E \times A$, we introduce the following notation.

- $\mathbf{L}^2(\lambda_0)$, the set of $\mathcal{A}$-measurable maps $\psi : A \to \mathbb{R}$ such that

$$|\psi|^2_{\mathbf{L}^2(\lambda_0)} \quad := \quad \int_A |\psi(b)|^2\, \lambda_0(db) < \infty.$$

- $\mathbf{L^2_{t,x,a}}(\mathcal{F}_\tau)$, the set of $\mathcal{F}_\tau$-measurable random variable X such that $\mathbb{E}^{t,x,a}\left[|X|^2\right] < \infty$; here τ is an $\mathbb{F}$-stopping time with values in $[t, T]$.

- $\mathbf{S^2_{t,x,a}}$ the set of real valued càdlàg adapted processes $Y = (Y_s)_{t \leqslant s \leqslant T}$ such that

$$||Y||^2_{\mathbf{S^2_{t,x,a}}} := \mathbb{E}^{t,x,a}\left[\sup_{t \leqslant s \leqslant T} |Y_s|^2\right] < \infty.$$

- $\mathbf{L^2_{t,x,a}}(q)$, the set of $\mathcal{P} \otimes \mathcal{E} \otimes \mathcal{A}$-measurable maps $Z : \Omega \times [t, T] \times E \times A \to \mathbb{R}$ such that

$$||Z||^2_{\mathbf{L^2_{t,x,a}}(q)} := \mathbb{E}^{t,x,a}\left[\int_t^T \int_{E \times A} |Z_s(y, b)|^2 \, \tilde{p}(ds \, dy \, db)\right]$$
$$= \mathbb{E}^{t,x,a}\left[\int_t^T \int_E |Z_s(I_s, y)|^2 \, \lambda(X_s, I_s, dy) \, ds + \int_t^T \int_A |Z_s(X_s, b)|^2 \, \lambda_0(db) \, ds\right] < \infty.$$

- $\mathbf{K^2_{t,x,a}}$ the set of nondecreasing predictable processes $K = (K_s)_{t \leqslant s \leqslant T} \in \mathbf{S^2_{t,x,a}}$ with $K_t = 0$, with the induced norm

$$||K||^2_{\mathbf{K^2_{t,x,a}}} = \mathbb{E}^{t,x,a}\left[|K_T|^2\right].$$

We are interested in studying the following family of BSDEs parametrized by (t, x, a): $\mathbb{P}^{t,x,a}$-a.s.,

$$Y_s^{t,x,a} = g(X_T) + \int_s^T f(r, X_r, I_r) \, dr + K_T^{t,x,a} - K_s^{t,x,a} \tag{4.1}$$
$$- \int_s^T \int_{E \times A} Z_r^{t,x,a}(y, b) \, q(dr \, dy \, db) - \int_s^T \int_A Z_r^{t,x,a}(X_r, b) \, \lambda_0(db) \, dr, \quad s \in [t, T],$$

with the sign constraint

$$Z_s^{t,x,a}(X_{s-}, b) \leqslant 0, \qquad ds \otimes d\mathbb{P}^{t,x,a} \otimes \lambda_0(db)\text{-a.e. on } [t, T] \times \Omega \times A. \tag{4.2}$$

This constraint can be seen as a sign condition imposed on the jumps of the corresponding stochastic integral.

Definition 4.1. *A solution to the equation* (4.1)-(4.2) *is a triple* $(Y, Z, K) \in \mathbf{S^2_{t,x,a}} \times \mathbf{L^2_{t,x,a}}(q) \times \mathbf{K^2_{t,x,a}}$ *that satisfies* (4.1)-(4.2).

A solution (Y, Z, K) *is called minimal if for any other solution* $(\tilde{Y}, \tilde{Z}, \tilde{K})$ *we have,* $\mathbb{P}^{t,x,a}$-a.s.,

$$Y_s \leqslant \tilde{Y}_s, \qquad s \in [t, T].$$

Proposition 4.1. *Under assumptions* (**H**λ) *and* (**Hfg**)*, for any* $(t, x, a) \in [0, T] \times E \times A$*, if there exists a minimal solution on* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^{t,x,a})$ *to the BSDE* (4.1)-(4.2)*, then it is unique.*

Proof. Let (Y, Z, K) and (Y', Z', K') be two minimal solutions of (4.1)-(4.2). The component Y is unique by definition, and the difference between the two backward equations gives: $\mathbb{P}^{t,x,a}$-a.s.

$$\int_t^s \int_{E \times A} (Z_r(y, b) - Z_r'(y, b)) \, p(dr \, dy \, db)$$
$$= K_s - K_s' + \int_t^s \int_E (Z_r(y, I_{r-}) - Z_r'(y, I_{r-})) \, \lambda(X_{r-}, I_{r-} dy) \, dr, \quad \forall t \leqslant s \leqslant T.$$

The right hand is a predictable process, in particular it has no totally inaccessible jumps (see, e.g., Proposition 2.24, Chapter I, in [23]), while the left side is a pure jump process with totally inaccessible jumps, unless $Z = Z'$. This implies the uniqueness of the component Z, and as a consequence the component K is unique as well. $\qquad\square$

We now state the main result of the section.

Theorem 4.2. *Under the assumptions* **(Hλ)** *and* **(Hfg)**, *for all* $(t, x, a) \in [0, T] \times E \times A$ *there exists a unique minimal solution* $Y^{t,x,a}$ *to* (4.1)-(4.2). *Moreover, for all* $s \in [t, T]$, $Y_s^{t,x,a}$ *has the explicit representation:*

$$Y_s^{t,x,a} = \operatorname*{esssup}_{\nu \in \mathcal{V}} \mathbb{E}_\nu^{t,x,a} \left[g(X_T) + \int_s^T f(r, X_r, I_r) \, dr \,\Big|\, \mathcal{F}_s \right], \quad s \in [t, T]. \tag{4.3}$$

In particular, setting $s = t$, *we have the following representation formula for the value function of the dual control problem:*

$$v^*(t, x, a) = Y_t^{t,x,a}, \qquad (t, x, a) \in [0, T] \times E \times A. \tag{4.4}$$

The rest of this section is devoted to prove Theorem 4.2. To this end we will use a penalization approach presented in the following subsections. Here we only note that for the solvability of the BSDE the use of the filtration $\mathbb{F}$ introduced above is essential, since it involves application of martingale representation theorems for multivariate point processes (see e.g. Theorem 5.4 in [22]).

4.1 Penalized BSDE and associated dual control problem

Let us consider the family of penalized BSDEs associated to (4.1)-(4.2), parametrized by the integer $n \geqslant 1$: $\mathbb{P}^{t,x,a}$-a.s.,

$$\begin{aligned}
Y_s^{n,t,x,a} &= g(X_T) + \int_s^T f(r, X_r, I_r) \, dr + K_T^{n,t,x,a} - K_s^{n,t,x,a} \tag{4.5} \\
&\quad - \int_s^T \int_{E \times A} Z_r^{n,t,x,a}(y, b) \, q(dr \, dy \, db) - \int_s^T \int_A Z_r^{n,t,x,a}(X_r, b) \, \lambda_0(db) \, dr, \quad s \in [t, T],
\end{aligned}$$

where K^n is the nondecreasing process in $\mathbf{K}_{\mathbf{t,x,a}}^{\mathbf{2}}$ defined by

$$K_s^n = n \int_t^s \int_A [Z_r^n(X_r, b)]^+ \lambda_0(db) \, dr.$$

Here we denote by $[u]^+$ the positive part of u. The penalized BSDE (4.5) can be rewritten in the equivalent form: $\mathbb{P}^{t,x,a}$-a.s.,

$$\begin{aligned}
Y_s^{n,t,x,a} &= g(X_T) + \int_s^T f^n(r, X_r, I_r, Z_r^{n,t,x,a}(X_r, \cdot)) \, ds \\
&\quad - \int_s^T \int_{E \times A} Z_r^{n,t,x,a}(y, b) \, q(dr \, dy \, db), \quad s \in [t, T].
\end{aligned}$$

where the generator f^n is defined by

$$f^n(t, x, a, \psi) := f(t, x, a) + \int_A \left\{ n \, [\psi(b)]^+ - \psi(b) \right\} \lambda_0(db), \tag{4.6}$$

for all (t, x, a) in $[0, T] \times E \times A$, and $\psi \in \mathbf{L}^{\mathbf{2}}(\lambda_0)$. We note that under **(Hλ)** and **(Hfg)** f^n is Lipschitz continuous in ψ with respect to the norm of $\mathbf{L}^{\mathbf{2}}(\lambda_0)$, uniformly in (t, x, a), i.e., for every $n \in \mathbb{N}$ there exists a constant L_n depending only on n such that for every $(t, x, a) \in [0, T] \times E \times A$ and $\psi, \psi' \in \mathbf{L}^{\mathbf{2}}(\lambda_0)$,

$$|f^n(t, x, a, \psi') - f^n(t, x, a, \psi)| \leqslant L_n \, |\psi - \psi'|_{\mathbf{L}^{\mathbf{2}}(\lambda_0)}.$$

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The use of the natural filtration $\mathbb{F}$ allows to use well known integral representation results for $\mathbb{F}$-martingales (see, e.g., Theorem 5.4 in [22]) and we have the following proposition, whose proof is standard and is therefore omitted (similar proofs can be found in [34] Theorem 3.2, [2] Proposition 3.2, [11] Theorem 3.4).

Proposition 4.3. *Let assumptions* **(Hλ)** *and* **(Hfg)** *hold. For every initial condition* $(t, x, a) \in [0, T] \times E \times A$, *and for every* $n \in \mathbb{N}$, *there exists a unique solution* $(Y_s^{n,t,x,a}, Z_s^{n,t,x,a})_{s \in [t,T]} \in \mathbf{S}_{\mathbf{t,x,a}}^2 \times \mathbf{L}_{\mathbf{t,x,a}}^2(q)$ *satisfying the penalized BSDE* (4.5).

Next we show that the solution to the penalized BSDE (4.5) provides an explicit representation of the value function of a corresponding dual control problem depending on n. This is the content of Lemma 4.4 which will allow to deduce some estimates uniform with respect to n.

For every $n \geq 1$, let $\mathcal{V}^n$ denote the subset of elements $\nu \in \mathcal{V}$ that take values in $(0, n]$.

Lemma 4.4. *Let assumptions* **(Hλ)** *and* **(Hfg)** *hold. For all* $n \geq 1$ *and* $s \in [t, T]$,

$$Y_s^{n,t,x,a} = \operatorname*{esssup}_{\nu \in \mathcal{V}^n} \mathbb{E}_\nu^{t,x,a} \left[g(X_T) + \int_s^T f(r, X_r, I_r) \, dr \,\Big|\, \mathcal{F}_s \right], \qquad \mathbb{P}^{t,x,a} - a.s. \qquad (4.7)$$

Proof. We fix $n \geq 1$ and for any $\nu \in \mathcal{V}^n$ we introduce the compensated martingale measure $q^\nu(ds \, dy \, db) = q(ds \, dy \, db) - (\nu_s(b) - 1) \, d_1(s, y, b) \, \tilde{p}(ds \, dy \, db)$ under $\mathbb{P}_\nu^{t,x,a}$. We see that the solution (Y^n, Z^n) to the BSDE (4.5) satisfies: $\mathbb{P}^{t,x,a}$-a.s.,

$$\begin{aligned}
Y_s^n &= g(X_T) + \int_s^T f(r, X_r, I_r) \, dr + \int_s^T \int_A \{n[Z_r^n(X_r, b)]^+ - \nu_r(b) \, Z_r^n(X_r, b)\} \, \lambda_0(db) \, dr \\
&\quad - \int_s^T \int_{E \times A} Z_r^n(y, b) \, q^\nu(dr \, dy \, db), \quad s \in [t, T].
\end{aligned} \qquad (4.8)$$

By taking conditional expectation in (4.8) under $\mathbb{P}_\nu^{t,x,a}$ and applying Lemma 3.2 we get, for any $s \in [t, T]$,

$$\begin{aligned}
Y_s^{n,t,x,a} &= \mathbb{E}_\nu^{t,x,a} \left[g(X_T) + \int_s^T f(r, X_r, I_r) \, dr \,\Big|\, \mathcal{F}_s \right] \\
&\quad + \mathbb{E}_\nu^{t,x,a} \left[\int_s^T \int_A \{n[Z_r^{n,t,x,a}(X_r, b)]^+ - \nu_r(b) \, Z_r^{n,t,x,a}(X_r, b)\} \, \lambda_0(db) \, dr \,\Big|\, \mathcal{F}_s \right],
\end{aligned} \qquad (4.9)$$

$\mathbb{P}_\nu^{t,x,a}$-a.s. From the elementary numerical inequality: $n[u]^+ - \nu u \geq 0$ for all $u \in \mathbb{R}$, $\nu \in (0, n]$, we deduce by (4.9) that

$$Y_s^{n,t,x,a} \geq \operatorname*{esssup}_{\nu \in \mathcal{V}^n} \mathbb{E}_\nu^{t,x,a} \left[g(X_T) + \int_s^T f(r, X_r, I_r) \, dr \,\Big|\, \mathcal{F}_s \right]. \qquad (4.10)$$

On the other hand, for $\epsilon \in (0, 1)$, let us consider the process $\nu^\epsilon \in \mathcal{V}^n$ defined by

$$\nu_s^\epsilon(b) = n \, \mathbb{1}_{\{Z_s^{n,t,x,a}(X_{s-}, b) \geq 0\}} + \epsilon \, \mathbb{1}_{\{-1 < Z_s^{n,t,x,a}(X_{s-}, b) < 0\}} - \epsilon \, Z_s^{n,t,x,a}(X_{s-}, b)^{-1} \, \mathbb{1}_{\{Z_s^{n,t,x,a}(X_{s-}, b) \leq -1\}}.$$

By construction, we have

$$n[Z_s^{n,t,x,a}(X_{s-}, b)]^+ - \nu_s^\epsilon(b) \, Z_s^{n,t,x,a}(X_{s-}, b) \leq \epsilon, \qquad s \in [t, T], \, b \in A,$$

and thus for the choice of $\nu = \nu^\epsilon$ in (4.9):

$$Y_s^{n,t,x,a} \leq \mathbb{E}_{\nu^\epsilon}^{t,x,a} \left[g(X_T) + \int_s^T f(r, X_r, I_r) \, dr \,\Big|\, \mathcal{F}_s \right] + \epsilon T \, \lambda_0(A)$$

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$$\leqslant \operatorname*{esssup}_{\nu \in \mathcal{V}^n} \mathbb{E}_\nu^{t,x,a} \left[g(X_T) + \int_s^T f(r, X_r, I_r) \, dr \,\Big|\, \mathcal{F}_s \right] + \epsilon T \, \lambda_0(A).$$

Together with (4.10), this is enough to prove the required representation of Y^n. Note that we could not take $\nu_s(b) = n \mathbb{1}_{\{Z_s^n(X_{s-},b) \geqslant 0\}}$, since this process does not belong to $\mathcal{V}^n$ because of the requirement of strict positivity. $\qquad \square$

4.2 Limit behavior of the penalized BSDEs and conclusion of the proof of Theorem 4.2

As a consequence of the representation (4.7) we immediately obtain the following estimates:

Lemma 4.5. *Let assumptions* **(Hλ)** *and* **(Hfg)** *hold. There exists a constant C, depending only on T, f, g, such that for any $(t, x, a) \in [0, T] \times E \times A$ and $n \geq 1$, $\mathbb{P}^{t,x,a}$-a.s.,*

$$Y_s^{n,t,x,a} \leqslant Y_s^{n+1,t,x,a}, \qquad |Y_s^{n,t,x,a}| \leqslant C, \qquad s \in [t, T].$$

Proof. For fixed $s \in [t, T]$, the almost sure monotonicity of $Y^{n,t,x,a}$ follows from the representation formula (4.7), since by definition $\mathcal{V}^n \subset \mathcal{V}^{n+1}$; moreover, the same formula shows that we can take $C = \|g\|_\infty + T \, \|f\|_\infty$. Finally, these inequalities hold for every $s \in [t, T]$ outside a null set, since the processes $Y^{n,t,x,a}$ are càdlàg. $\qquad \square$

Moreover, the following a priori uniform estimate on the sequence $(Y^{n,t,x,a}, Z^{n,t,x,a}, K^{n,t,x,a})$ holds:

Lemma 4.6. *Let assumptions* **(Hλ)** *and* **(Hfg)** *hold. For all $(t, x, a) \in [0, T] \times E \times A$ and $n \in \mathbb{N}$, there exists a positive constant C' depending only on T, f, g such that*

$$\|Y^{n,t,x,a}\|_{\mathbf{S}_{\mathbf{t,x,a}}^2}^2 + \|Z^{n,t,x,a}\|_{\mathbf{L}_{\mathbf{t,x,a}}^2(q)}^2 + \|K^{n,t,x,a}\|_{\mathbf{K}_{\mathbf{t,x,a}}^2}^2 \leqslant C'. \tag{4.11}$$

Proof. In the following we omit for simplicity of notation the dependence on (t, x, a) for the triple $(Y^{n,t,x,a}, Z^{n,t,x,a}, K^{n,t,x,a})$. The estimate on Y^n follows immediately from the previous lemma:

$$\|Y^n\|_{\mathbf{S}_{\mathbf{t,x,a}}^2}^2 = \mathbb{E}^{t,x,a} \left[\sup_{s \in [t,T]} |Y_s^n|^2 \right] \leqslant C^2. \tag{4.12}$$

Next we notice that, since K^n is continuous, the jumps of Y^n are given by the formula

$$\Delta Y_s^n = \int_{E \times A} Z_s^n(y, b) \, p(\{s\}, dy \, db).$$

The Itô formula applied to $|Y_t^n|^2$ gives:

$$
\begin{aligned}
d|Y_r^n|^2 &= 2 \, Y_{r-}^n \, dY_r^n + |\Delta Y_r^n|^2 \\
&= -2 \, Y_{r-}^n \, f(X_{r-}, I_{r-}) \, dr - 2 \, Y_{r-}^n \, dK_r^n \\
&\quad + 2 \, Y_{r-}^n \int_{E \times A} Z_r^n(y, b) \, q(dr \, dy \, db) + 2 \, Y_{r-}^n \int_A Z_r^n(X_{r-}, b) \, \lambda_0(db) \, dr \\
&\quad + \int_{E \times A} |Z_r^n(y, b)|^2 \, p(\{r\} \, dy \, db).
\end{aligned}
\tag{4.13}
$$

Integrating (4.13) on $[s, T]$, for every $s \in [t, T]$, and recalling the elementary inequality $2ab \leqslant \frac{1}{\delta}a^2 + \delta b^2$ for any constant $\delta > 0$, and that

$$\mathbb{E}^{t,x,a}\left[\int_s^T \int_A |Z_r^n(X_{r-}, b)|^2 \lambda_0(db)\, dr\right] \leqslant \mathbb{E}^{t,x,a}\left[\int_s^T \int_{E \times A} |Z_r^n(y, b)|^2 \tilde{p}(dr\, dy\, db)\right], \qquad (4.14)$$

we have:

$$\mathbb{E}^{t,x,a}\left[|Y_s|^2\right] + \mathbb{E}^{t,x,a}\left[\int_s^T \int_{E \times A} |Z_r^n(y, b)|^2 \tilde{p}(dr\, dy\, db)\right]$$

$$\leqslant \mathbb{E}^{t,x,a}\left[|g(X_T)|^2\right]$$

$$+ \frac{1}{\beta}\mathbb{E}^{t,x,a}\left[\int_s^T |f(r, X_r, I_r)|^2\, dr\right] + \beta \mathbb{E}^{t,x,a}\left[\int_s^T |Y_r^n|^2\, dr\right]$$

$$+ \frac{T\lambda_0(A)}{\gamma}\mathbb{E}^{t,x,a}\left[\int_s^T \int_{E \times A} |Z_r^n(y, b)|^2 \tilde{p}(dr\, dy\, db)\right] + \gamma \mathbb{E}^{t,x,a}\left[\int_s^T |Y_r^n|^2\, dr\right]$$

$$+ \frac{1}{\alpha}\mathbb{E}^{t,x,a}\left[\sup_{s \in [t,T]} |Y_s^n|^2\right] + \alpha \mathbb{E}^{t,x,a}\left[|K_T^n - K_s^n|^2\right], \quad s \in [t, T], \qquad (4.15)$$

for some $\alpha, \beta, \gamma > 0$, Now, from the equation (4.5) we obtain:

$$K_T^n - K_s^n = Y_s^n - g(X_T) - \int_s^T f(r, X_r, I_r)dr$$

$$+ \int_s^T \int_A Z_r^n(X_r, b)\, \lambda_0(db)\, dr$$

$$+ \int_s^T \int_{E \times A} Z_r^n(y, b)\, q(dr\, dy\, db), \quad s \in [t, T].$$

Next we note the equality

$$\mathbb{E}^{t,x,a}\left[\left|\int_s^T \int_{E \times A} Z_r^n(y, b)\, q(dr\, dy\, db)\right|^2\right] = \mathbb{E}^{t,x,a}\left[\int_s^T \int_{E \times A} |Z_r^n(y, b)|^2\, p(dr\, dy\, db)\right]$$

$$= \mathbb{E}^{t,x,a}\left[\int_s^T \int_{E \times A} |Z_r^n(y, b)|^2 \tilde{p}(dr\, dy\, db)\right]$$

that can be proved applying the Itô formula as before to the square of the martingale $u \mapsto \int_s^u \int_{E \times A} Z_r^n(y, b)\, q(dr\, dy\, db)$, $u \in [s, T]$ (or by considering its quadratic variation). Recalling again (4.14) we see that there exists some positive constant B such that

$$\mathbb{E}^{t,x,a}\left[|K_T^n - K_s^n|^2\right] \leqslant B\left(\mathbb{E}^{t,x,a}\left[|Y_s^n|^2\right] + \mathbb{E}^{t,x,a}\left[|g(X_T)|^2\right] + \mathbb{E}^{t,x,a}\left[\int_s^T |f(r, X_r, I_r)|^2\, dr\right]\right.$$

$$\left. + \mathbb{E}^{t,x,a}\left[\int_s^T \int_{E \times A} |Z_r^n(y, b)|^2 \tilde{p}(dr\, dy\, db)\right]\right), \quad s \in [t, T]. \qquad (4.16)$$

Plugging (4.16) into (4.15), and recalling the uniform estimation (4.12) on Y^n, we get

$$(1 - \alpha B)\, \mathbb{E}^{t,x,a}\left[|Y_s|^2\right] + \left(1 - \left[\alpha B + \frac{T\lambda_0(A)}{\gamma}\right]\right)\mathbb{E}^{t,x,a}\left[\int_s^T \int_{E \times A} |Z_r^n(y, b)|^2 \tilde{p}(dr\, dy\, db)\right]$$

$$\leqslant (1 + \alpha B)\, \mathbb{E}^{t,x,a}\left[|g(X_T)|^2\right] + \left(\alpha B + \frac{1}{\beta}\right)\mathbb{E}^{t,x,a}\left[\int_s^T |f(r, X_r, I_r)|^2\, dr\right]$$

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$$+\frac{C^2}{\alpha}+(\gamma+\beta)\,\mathbb{E}^{t,x,a}\left[\int_s^T |Y_r^n|^2\,dr\right],\quad s\in[t,T].$$

Hence, by choosing $\alpha\in\left(0,\frac{1}{B}\right)$, $\gamma>\frac{T\lambda_0(A)}{1-\alpha B}$, $\beta>0$, and applying Gromwall's lemma to $s\to\mathbb{E}^{t,x,a}\left[|Y_s^n|^2\right]$, we obtain:

$$\sup_{s\in[t,T]}\mathbb{E}^{t,x,a}\left[|Y_s|^2\right]+\mathbb{E}^{t,x,a}\left[\int_t^T\int_{E\times A}|Z_s^n(y,b)|^2\,\tilde{p}(ds\,dy\,db)\right]$$

$$\leqslant C'\left(\mathbb{E}^{t,x,a}\left[|g(X_T)|^2\right]+\mathbb{E}^{t,x,a}\left[\int_t^T |f(s,X_s,I_s)|^2\,ds\right]+C^2\right),\qquad(4.17)$$

for some $C'>0$ depending only on T, which gives the required uniform estimate for (Z^n) and also (K^n) by (4.16). $\qquad\square$

We can finally present the conclusion of the proof of Theorem 4.2:

Proof. Let $(t,x,a)\in[0,T]\times E\times A$. We first show that (Y^n,Z^n,K^n) (we omit the dependence on (t,x,a) for simplicity of notation) solution to (4.5) converges in a suitable way to some process (Y,Z,K) solution to the constrained BSDE (4.1)-(4.2). By Lemma 4.5, $(Y^n)_n$ converges increasingly to some adapted process Y, which moreover satisfies $\mathbb{E}^{t,x,a}\left[\sup_{s\in[t,T]}|Y_s|^2\right]<\infty$ by the uniform estimate for $(Y^n)_n$ in Lemma 4.6 and Fatou's lemma. Furthermore, by the dominated convergence theorem, we also have $\mathbb{E}\int_0^T |Y_t^n-Y_t|^2\,dt\to 0$. Next, we prove that there exists $(Z,K)\in\mathbf{L}_{t,x,a}^2(q)\times\mathbf{K}_{t,x,a}^2$ with K predictable, such that

(i) Z is the weak limit of $(Z^n)_n$ in $\mathbf{L}_{t,x,a}^2(q)$;

(ii) K_τ is the weak limit of $(K_\tau^n)_n$ in $\mathbf{L}_{t,x,a}^2(\mathcal{F}_\tau)$, for any stopping time τ valued in $[t\;T]$;

(iii) $\mathbb{P}^{t,x,a}$-a.s.,

$$Y_s = g(X_T)+\int_s^T f(r,X_r,I_r)\,dr+K_T-K_s$$
$$-\int_s^T\int_{E\times A}Z_r(y,b)\,q(dr\,dy\,db)-\int_s^T\int_A Z_r(X_r,b)\,\lambda_0(db)\,dr,\ s\in[t,T],$$

with

$$Z_s(X_{s-},b)\leqslant 0,\qquad ds\otimes d\mathbb{P}^{t,x,a}\otimes\lambda_0(db)-\text{a.e.}$$

Let define the following mappings from $\mathbf{L}_{t,x,a}^2(q)$ to $\mathbf{L}_{t,x,a}^2(\mathcal{F}_\tau)$:

$$I_\tau^1:\quad Z\mapsto\int_t^\tau\int_{E\times A}Z_s(y,b)\,q(ds\,dy\,db),$$

$$I_\tau^2:\quad Z\mapsto\int_t^\tau\int_A Z_s(X_s,b)\,\lambda_0(db)\,ds,$$

for each $\mathbb{F}$-stopping time τ with values in $[t,T]$. We wish to prove that $I_\tau^1 Z^n$ and $I_\tau^2 Z^n$ converge weakly in $\mathbf{L}_{t,x,a}^2(\mathcal{F}_\tau)$ to $I_\tau^1 Z$ and $I_\tau^2 Z$ respectively. Indeed, by the uniform estimates for $(Z^n)_n$ in Lemma 4.6, there exists a subsequence, denoted $(Z^{n_k})_k$, which converges weakly in $\mathbf{L}_{t,x,a}^2(q)$.

Since I_1 and I_2 are linear continuous operators they are also weakly continuous so that we have $I_\tau^1 Z^{n_k} \to I_\tau^1 Z$ and $I_\tau^2 Z^{n_k} \to I_\tau^2 Z$ weakly in $\mathbf{L}^2_{\mathbf{t,x,a}}(\mathcal{F}_\tau)$ as $k \to \infty$. Since we have from (4.5)

$$
\begin{aligned}
K_\tau^{n_k} &= -Y_\tau^{n_k} + Y_t^{n_k} - \int_t^\tau f(r, X_r, I_r)\, dr \\
&\quad + \int_t^\tau \int_A Z_r^{n_k}(X_r, b)\, \lambda_0(db)\, dr + \int_t^\tau \int_{E \times A} Z_r^{n_k}(y, b)\, q(dr\, dy\, db),
\end{aligned}
$$

we also obtain the weak convergence in $\mathbf{L}^2_{\mathbf{t,x,a}}(\mathcal{F}_\tau)$ as $k \to \infty$

$$
\begin{aligned}
K_\tau^{n_k} \rightharpoonup K_\tau &:= -Y_\tau + Y_t - \int_t^\tau f(r, X_r, I_r)\, dr \\
&\quad + \int_t^\tau \int_A Z_r(X_r, b)\, \lambda_0(db)\, dr + \int_t^\tau \int_{E \times A} Z_r(y, b)\, q(dr\, dy\, db). \quad (4.18)
\end{aligned}
$$

Arguing as in [29] proof of Theorem 2.1, or [25] Lemma 3.5, [18] Theorem 3.1 we see that K inherits from K^{n_k} the properties of having nondecreasing paths and of being square integrable and predictable. Finally, from Lemma 2.2 in [29] it follows that K and Y are càdlàg, so that $K^{t,x,a} \in \mathbf{K}^2_{\mathbf{t,x,a}}$ and $Y^{t,x,a} \in \mathbf{S}^2_{\mathbf{t,x,a}}$.

Notice that the processes Z and K in (4.18) are uniquely determined. Indeed, if (Z, K) and (Z', K') satisfy (4.18), then the predictable processes Z and Z' coincide at the jump times and can be identified almost surely with respect to $\tilde{p}(\omega, ds\, dy\, db)\, \mathbb{P}^{t,x,a}(d\omega)$ (a similar argument can be found in the proof of Proposition 4.1 to which we refer for more details). Finally, recalling that the jumps of p are totally inaccessible, we also obtain the uniqueness of the component K. The uniqueness of Z and K entails that all the sequences $(Z^n)_n$ and $(K^n)_n$ respectively converge (in the sense of points (i) and (ii) above) to Z and K.

It remains to show that the jump constraint (4.2) is satisfied. To this end, we consider the functional on $\mathbf{L}^2_{\mathbf{t,x,a}}(q)$ given by

$$
G: \quad Z \quad \mapsto \quad \mathbb{E}^{t,x,a}\left[\int_t^T \int_A [Z_s(X_{s-}, b)]^+\, \lambda_0(db)\, ds\right].
$$

From uniform estimate (4.11), we see that $G(Z^n) \to 0$ as $n \to \infty$. Since G is convex and strongly continuous in the strong topology of $\mathbf{L}^2_{\mathbf{t,x,a}}(q)$, then G is lower semicontinuous in the weak topology of $\mathbf{L}^2_{\mathbf{t,x,a}}(q)$, see, e.g., Corollary 3.9 in [8]. Therefore, we find

$$
G(Z) \leqslant \liminf_{n \to \infty} G(Z^n) = 0,
$$

from which follows the validity of the jump constraint (4.2) on $[t, T]$. We have then showed that (Y, Z, K) is a solution to the constrained BSDE (4.1)-(4.2). It remains to prove that this is the minimal solution. To this end, fix $n \in \mathbb{N}$ and consider a triple $(\bar{Y}, \bar{Z}, \bar{K}) \in \mathbf{S}^2_{\mathbf{t,x,a}} \times \mathbf{L}^2_{\mathbf{t,x,a}}(q) \times \mathbf{K}^2_{\mathbf{t,x,a}}$ satisfying (4.1)-(4.2). For any $\nu \in \mathcal{V}^n$, by introducing the compensated martingale measure q^ν, we see that the solution $(\bar{Y}, \bar{Z}, \bar{K})$ satisfies: $\mathbb{P}^{t,x,a}$-a.s.,

$$
\bar{Y}_s = g(X_T) + \int_s^T f(r, X_r, I_r)\, dr + \bar{K}_T - \bar{K}_s \quad (4.19)
$$

$$
- \int_s^T \int_{E \times A} \bar{Z}_r(y, b)\, q^\nu(dr\, dy\, db) - \int_s^T \int_A \nu_r(b)\, \bar{Z}_r(X_r, b)\, \lambda_0(db)\, dr \quad s \in [t, T].
$$

By taking the expectation under $\mathbb{P}^{t,x,a}_\nu$ in (4.19), recalling Lemma 3.2, and that $\bar{K}$ is nondecreasing, we have

$$
\bar{Y}_s \geqslant \mathbb{E}^{t,x,a}_\nu\left[g(X_T) + \int_s^T f(r, X_r, I_r)\, dr\right] - \mathbb{E}^{t,x,a}_\nu\left[\int_s^T \int_A \nu_r(b)\, \bar{Z}_r(X_r, b)\, \lambda_0(db)\, dr\right]
$$

$$\geqslant \; \mathbb{E}_\nu^{t,x,a}\left[g(X_T) + \int_s^T f(r, X_r, I_r)\, dr \right] \qquad s \in [t, T], \tag{4.20}$$

since ν is valued in $(0, n]$ and Z satisfies constraint (4.2). As ν is arbitrary in $\mathcal{V}^n$, we get from the representation formula (4.7) that $\bar{Y}_s \geqslant Y_s^n$, $\forall s \in [t, T]$, $\forall n \in \mathbb{N}$. In particular, $Y_s = \lim_{n \to \infty} Y_s^n \leqslant \bar{Y}_s$, i.e., the minimality property holds. The uniqueness of the minimal solution straightly follows from Proposition 4.1.

To conclude the proof, we argue on the limiting behavior of the dual representation for Y^n when n goes to infinity. Since $\mathcal{V}^n \subset \mathcal{V}$, it is clear from the representation (4.7) that, for all n and $s \in [t, T]$, $Y_s^n \leqslant \operatorname{esssup}_{\nu \in \mathcal{V}} \mathbb{E}_\nu^{t,x,a}\left[g(X_T) + \int_s^T f(r, X_r, I_r)\, dr \,\middle|\, \mathcal{F}_s \right]$. Moreover, being Y the pointwise limit of Y^n, we deduce that

$$Y_s = \lim_{n \to \infty} Y_s^n \leqslant \operatorname{esssup}_{\nu \in \mathcal{V}} \mathbb{E}_\nu^{t,x,a}\left[g(X_T) + \int_s^T f(r, X_r, I_r)\, dr \,\middle|\, \mathcal{F}_s \right]. \tag{4.21}$$

On the other hand, for any $\nu \in \mathcal{V}$, introducing the compensated martingale measure q^ν under $\mathbb{P}^\nu$ as usual, we see that (Y, Z, K) satisfies

$$\begin{aligned} Y_s \;=\;& g(X_T) + \int_s^T f(r, X_r, I_r)\, dr + K_T - K_s \\ &- \int_s^T \int_{E \times A} Z_r(y, b)\, q^\nu(dr\, dy\, db) - \int_s^T \int_A Z_r(X_r, b)\, \nu_r(b) \lambda_0(db)\, dr, \qquad s \in [t, T]. \end{aligned} \tag{4.22}$$

Arguing in the same way as in (4.20), we obtain

$$Y_s \;\geqslant\; \mathbb{E}_\nu^{t,x,a}\left[g(X_T) + \int_s^T f(r, X_r, I_r)\, dr \,\middle|\, \mathcal{F}_s \right],$$

so that $Y_s \geqslant \operatorname{esssup}_{\nu \in \mathcal{V}} \mathbb{E}_\nu^{t,x,a}\left[g(X_T) + \int_s^T f(r, X_r, I_r)\, dr \,\middle|\, \mathcal{F}_s \right]$ by the arbitrariness of $\nu \in \mathcal{V}$. Together with (4.21) this gives the required equality. $\qquad \square$

5 A BSDE representation for the value function

In this section we conclude the last step in the method of control randomization and we show that the minimal solution to the constrained BSDE (4.1)-(4.2) actually provides a nonlinear Feynman-Kac representation of the solution to the Hamilton-Jacobi-Bellman (HJB) equation (2.7)-(2.8), that we re-write here:

$$-\frac{\partial v}{\partial t}(t, x) = \sup_{a \in A}\left(\mathcal{L}_E^a v(t, x) + f(t, x, a) \right), \qquad v(T, x) = g(x).$$

As a consequence of the dual representation in Theorem 4.2 it follows that the value function of the original optimal control problem can be identified with the dual one, which in particular turns out to be independent on the variable a.

For our result we need the following conditions:

$$\sup_{x \in E, a \in A} \lambda(x, a, E) < \infty, \tag{5.1}$$

$$\lambda \text{ is a Feller transition kernel}, \tag{5.2}$$

$$f \in C_b([0,T] \times E \times A), \quad g \in C_b(E). \tag{5.3}$$

We note that these assumptions are stronger that those required in Theorem 2.6 and therefore they imply that there exists a unique solution $v \in LSC_b([0,T] \times E)$ to the HJB equation in the sense of Definition 2.1. If, in addition, A is a compact metric space then $v \in C_b([0,T] \times E)$ by Corollary 2.8.

Let us consider again the Markov process (X, I) in $E \times A$ constructed in section 3.1, with corresponding family of probability measures $\mathbb{P}^{t,x,a}$ and generator $\mathcal{L}$ introduced in (3.4). Since (5.1)-(5.3) are also stronger than $(\mathbf{H}\lambda)$ and $(\mathbf{Hfg})$, by Theorem 4.2 there exists a unique solution to the BSDE (4.1)-(4.2).

Our main result is as follows:

Theorem 5.1. *Assume* (5.1), (5.2), (5.3). *Let v be the unique solution to the Hamilton-Jacobi-Bellman equation provided by Theorem 2.6. Then for every $(t,x,a) \in [0,T] \times E \times A$,*

$$v(t,x) = Y_t^{t,x,a},$$

where $Y^{t,x,a}$ is the first component of the minimal solution to the constrained BSDE with nonpositive jumps (4.1)-(4.2).

More generally, we have $\mathbb{P}^{t,x,a}$-a.s.,

$$v(s, X_s) = Y_s^{t,x,a}, \qquad s \in [t,T].$$

Finally, for the value function V of the optimal control problem defined in (2.6) *and the dual value function V^* defined in* (3.5) *we have the equalities*

$$V(t,x) = v(t,x) = Y_t^{t,x,a} = V^*(t,x,a).$$

In particular, the latter functions do not depend on a.

Remark 5.2. By similar arguments our result admits possible extensions to cases where condition (5.1) and the boundedness requirements on the gain functions f and g are relaxed. We stick to the previous setting in order to avoid too many technicalities.

The rest of this section is devoted to prove Theorem 5.1.

5.1 A penalized HJB equation

Let us recall the penalized BSDE associated to (4.1)-(4.2): $\mathbb{P}^{t,x,a}$-a.s.,

$$Y_s^{n,t,x,a} = g(X_T) + \int_s^T f(r, X_r, I_r) \, ds - \int_s^T \int_{E \times A} Z_r^{n,t,x,a}(y,b) \, q(dr \, dy \, db) \tag{5.4}$$

$$+ \int_s^T \int_A \left\{ n \, [Z_r^{n,t,x,a}(X_r, b)]^+ - Z_r^{n,t,x,a}(X_r, b) \right\} \lambda_0(db) \, dr, \qquad s \in [t,T].$$

Let us now consider the parabolic semi-linear penalized integro-differential equation, of HJB type: for any $n \geq 1$,

$$\frac{\partial v^n}{\partial t}(t,x,a) + \mathcal{L}v^n(t,x,a) + f(t,x,a) \tag{5.5}$$

$$+ \int_A \left\{ n \, [v^n(t,x,b) - v^n(t,x,a)]^+ - (v^n(t,x,b) - v^n(t,x,a)) \right\} \lambda_0(db) = 0 \quad \text{on } [0,T) \times E \times A,$$

$$v^n(T,x,a) = g(x) \quad \text{on } E \times A, \tag{5.6}$$

The following lemma states that the solution of (5.5)-(5.6) can be represented probabilistically by means of the solution to the penalized BSDE (5.4):

Lemma 5.3. *Assume* (5.1), (5.2), (5.3). *Then there exists a unique function $v^n \in C_b([0,T] \times E \times A)$ such that $t \mapsto v^n(t,x,a)$ is continuously differentiable on $[0,T]$ and (5.5)-(5.6) hold for every $(t,x,a) \in [0,T] \times E \times A$.*

Moreover, for every $(t,x,a) \in [0,T] \times E \times A$ and for every $n \in \mathbb{N}$,

$$Y_s^{n,t,x,a} = v^n(s, X_s, I_s) \tag{5.7}$$

$$Z_s^{n,t,x,a}(y,b) = v^n(s,y,b) - v^n(s, X_{s-}, I_{s-}), \tag{5.8}$$

(to be understood as an equality between elements of the space $\mathbf{S}^2_{t,x,a} \times \mathbf{L}^2_{t,x,a}(q)$) so that in particular $v^n(t,x,a) = Y_t^{n,t,x,a}$.

Proof. We first note that $v^n \in C_b([0,T] \times E \times A)$ is the required solution if and only if

$$v^n(t,x,a) = g(x) + \int_t^T \mathcal{L}v^n(s,x,a)\,ds + \int_t^T f^n(s,x,a, v^n(s,x,\cdot) - v^n(s,x,a)) \tag{5.9}$$

for $t \in [0,,T)$, $x \in E$, $a \in A$, where $f^n(t,x,a,\psi)$ is the map defined in (4.6). We use a fixed point argument, introducing a map Γ from $C_b([0,T] \times E \times A)$ to itself setting $v = \Gamma(w)$ where

$$v(t,x,a) = g(x) + \int_t^T \mathcal{L}w(s,x,a)\,ds + \int_t^T f^n(s,x,a, w(s,x,\cdot) - w(s,x,a))\,ds.$$

Using the boundedness assumptions on λ and λ_0 it can be shown by standard arguments that some iteration of the above map is a contraction in the space of bounded measurable real functions on $[0,T] \times E \times A$ endowed with the supremum norm and therefore the map Γ has a unique fixed point, which is the required solution v^n.

We finally prove the identifications (5.7)-(5.8). Since $v^n \in C_b([0,T] \times E \times A)$ we can apply the Itô formula to the process $v(s, X_s, I_s)$, $s \in [t,T]$, obtaining, $\mathbb{P}^{t,x,a}$-a.s.,

$$v^n(s, X_s, I_s) = v^n(t,x,a) + \int_t^s \left(\frac{\partial v^n}{\partial r}(r, X_r, I_r) + \mathcal{L}_r^I v^n(r, X_r, I_r) \right) dr$$
$$+ \int_t^s \int_{E \times A} (v^n(r,y,b) - v^n(r, X_{r-}, I_{r-}))\, q(dr\,dy\,db), \qquad s \in [t,T].$$

Taking into account that v^n satisfies (5.5)-(5.6) and that (X, I) has piecewise constant trajectories, we obtain $\mathbb{P}^{t,x,a}$-a.s.,

$$\frac{\partial v^n}{\partial r}(r, X_r, I_r) + \mathcal{L}v^n(r, X_r, I_r) + f^n(r, X_r, I_r, v^n(r, X_r, \cdot) - v^n(r, X_r, I_r)) = 0,$$

for almost all $r \in [t,T]$. It follows that, $\mathbb{P}^{t,x,a}$-a.s.,

$$v^n(s, X_s, I_s) = v^n(t,x,a) - \int_t^s f^n(r, X_r, I_r, v^n(r, X_r, \cdot) - v^n(r, X_r, I_r))\, dr$$
$$+ \int_t^s \int_{E \times A} (v^n(r,y,b) - v(r, X_{r-}, I_{r-}))\, q(dr\,dy\,db), \qquad s \in [t,T].$$

Since $v^n(T,x,a) = g(x)$ for all $(x,a) \in E \times A$, simple passages show that

$$v^n(s, X_s, I_s) = g(X_T) + \int_t^s f^n(r, X_r, I_r, v^n(r, X_r, \cdot) - v^n(r, X_r, I_r))\, dr$$
$$- \int_t^s \int_{E \times A} (v^n(r,y,b) - v(r, X_{r-}, I_{r-}))\, q(dr\,dy\,db), \qquad s \in [t,T].$$

Therefore the pairs $(Y_s^{n,t,x,a}, Z_s^{n,t,x,a}(y,b))$ and $(v^n(s,X_s,I_s), v^n(s,y,b) - v^n(s,X_{s-},I_{s-}))$ are both solutions to the same BSDE under $\mathbb{P}^{t,x,a}$, and thus they coincide as members of the space $\mathbf{S}^2_{\mathbf{t,x,a}} \times \mathbf{L}^2_{\mathbf{t,x,a}}(q)$. The required equalities (5.7)-(5.8) follow. In particular we have that $v^n(t,x,a) = Y_t^{n,t,x,a}$. $\qquad\square$

5.2 Convergence of the penalized solutions and conclusion of the proof

We study the behavior of the functions v^n as $n \to \infty$. To this end we first show that they are bounded above by the solution to the HJB equation. Similar bounds may be obtained even without the requirement that the cost functions f, g should be bounded, but we limit ourselves to the previous setting.

Lemma 5.4. *Assume (5.1), (5.2), (5.3). Let v denote the solution to the HJB equation as provided by Theorem 2.6 and v^n the solution to (5.5)-(5.6) as provided in Lemma 5.3. Then, for all $(t,x,a) \in [0,T] \times E \times A$ and $n \geq 1$,*

$$v(t,x) \geq v^n(t,x,a).$$

Proof. Let $v : [0,T] \times E \to \mathbb{R}$ be a solution to the HJB equation. As in the proof of Proposition 2.1 we have the identity

$$g(X_T) - v(t,X_t) = \int_t^T \frac{\partial v}{\partial r}(r,X_r)\,dr + \int_{(t,T]}\int_{E \times A}(v(r,y) - v(r,X_{r-}))\,p(dr\,dy\,db),$$

which follows from the absolute continuity of $t \mapsto v(t,x)$, taking into account that X is constant among jump times and using the definition of the random measure p defined in (3.2) and the fact that v depends on t,x only. Since v is a solution to the HJB equation we have, for all $x \in E$ $a \in A$,

$$-\frac{\partial v}{\partial t}(t,x) \geq \mathcal{L}_E^a v(t,x) + f(t,x,a) = \int_E (v(t,y) - v(t,x))\,\lambda(x,a,dy) + f(t,x,a),$$

almost surely on $[0,T]$. Taking into account that (X,I) has piecewise constant trajectories we obtain

$$
\begin{aligned}
g(X_T) - v(t,X_t) \;\leq\; & \int_{(t,T]}\int_{E \times A}(v(r,y) - v(r,X_{r-}))\,p(dr\,dy\,db) \qquad\qquad (5.10) \\
& - \int_t^T \int_E (v(r,y) - v(r,X_r))\,\lambda(X_r,I_r,dy)\,dr - \int_t^T f(r,X_r,I_r)\,dr.
\end{aligned}
$$

Then, for any $n \geq 1$ and $\nu \in \mathcal{V}^n$ let us consider the probability $\mathbb{P}_\nu^{t,x,a}$ introduced above and recall that under $\mathbb{P}_\nu^{t,x,a}$ the compensator of the random measure $p(dr\,dy\,db)$ is $\tilde{p}^\nu(dr\,dy\,db) = \nu_r(b)\,\lambda_0(db)\,\delta_{\{X_{r-}\}}(dy)\,dr + \lambda(X_{r-}, I_{r-}, dy)\,\delta_{\{I_{r-}\}}(db)\,dr$. Noting that $v(r,y) - v(r,X_{r-})$ is predictable, taking the expectation in (5.10) we obtain

$$\mathbb{E}_\nu^{t,x,a}[g(X_T)] - v(t,x) \leq -\mathbb{E}_\nu^{t,x,a}\int_t^T f(r,X_r,I_r)\,dr.$$

Since $\nu \in \mathcal{V}^n$ was arbitrary, and recalling (4.7), we conclude that

$$v(t,x) \geq \sup_{\nu \in \mathcal{V}^n} \mathbb{E}_\nu^{t,x,a}\left[g(X_T) + \int_t^T f(r,X_r,I_r)\,dr\right] = v^n(t,x,a).$$

$\qquad\square$

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From Lemma 5.3 we know that $v^n(t, x, a) = Y^{n,t,x,a}_t$, and from Lemma 4.5 we know that $v^n(t, x, a)$ is monotonically increasing and uniformly bounded. Therefore we can define

$$\bar{v}(t, x, a) := \lim_{n \to \infty} v^n(t, x, a), \qquad t \in [0, T], \ x \in E, \ a \in A.$$

$\bar{v}$ is bounded, and from Lemma 5.4 we deduce that $\bar{v} \leq v$. As an increasing limit of continuous functions, $\bar{v}$ is lower semi-continuous. Further properties of $\bar{v}$ are proved in the following lemma. In particular, (5.11) (or (5.12)) means that $\bar{v}$ is a supersolution to the HJB equation.

Lemma 5.5. *Assume* (5.1), (5.2), (5.3) *and let $\bar{v}$ be the increasing limit of v^n. Then $\bar{v}$ does not depend on a, i.e. $\bar{v}(t, x, a) = \bar{v}(t, x, b)$ for every $t \in [0, T]$, $x \in E$ and $a, b \in A$. Moreover, setting $\bar{v}(t, x) = \bar{v}(t, x, a)$ we have*

$$\bar{v}(t, x) - \bar{v}(t', x) \ \geq \ \int_t^{t'} (\mathcal{L}^a_E \bar{v}(s, x) + f(s, x, a))\, ds, \quad 0 \leq t \leq t' \leq T, \ x \in E, \ a \in A. \ (5.11)$$

More generally, for arbitrary Borel-measurable $\alpha : [0, T] \to A$ we have

$$\bar{v}(t, x) - \bar{v}(t', x) \geq \int_t^{t'} (\mathcal{L}^{\alpha(s)}_E \bar{v}(s, x) + f(s, x, \alpha(s)))\, ds, \qquad 0 \leq t \leq t' \leq T, \ x \in E, \ a \in A. \ (5.12)$$

Proof. v^n satisfies the integral equation (5.9), namely

$$\begin{aligned}
v^n(t, x, a) \ = \ & g(x) + \int_t^T \int_E (v^n(s, y, a) - v^n(s, x, a))\, \lambda(x, a, dy)\, ds \\
& + \int_t^T f(s, x, a)\, ds + n \int_t^T \int_A [v^n(s, x, b) - v^n(s, x, a)]^+\, \lambda_0(db)\, ds.
\end{aligned}$$

Since v^n is a bounded sequence in $C_b([0, T] \times E \times A)$ converging pointwise to $\bar{v}$, setting $t = 0$, dividing by n and letting $n \to \infty$ we obtain

$$\int_0^T \int_A [\bar{v}(s, x, b) - \bar{v}(s, x, a)]^+\, \lambda_0(db)\, ds = 0. \tag{5.13}$$

Next we claim that $\bar{v}$ is right-continuous in t on $[0, T)$, for fixed $x \in E$, $a \in A$. To prove this we first note that, neglecting the term with the positive part $[\ldots]^+$ we have

$$\begin{aligned}
v^n(t', x, a) - v^n(t, x, a) \ \leq \ & -\int_t^{t'} \int_E (v^n(s, y, a) - v^n(s, x, a))\, \lambda(x, a, dy)\, ds - \int_t^{t'} f(s, x, a)\, ds \\
\leq \ & C_0(t' - t),
\end{aligned} \tag{5.14}$$

for some constant $C_0 > 0$ and for all $0 \leq t \leq t' \leq T$ and $n \geq 1$, where we have used again the fact that v^n is uniformly bounded. Now fix $t \in [0, T)$. Since, as already noticed, $\bar{v}$ is lower semi-continuous we have $\bar{v}(t, x, a) \leq \liminf_{s \downarrow t} \bar{v}(s, x, a)$. The required right continuity follows if we can show that $\bar{v}(t, x, a) \geq \limsup_{s \downarrow t} \bar{v}(s, x, a)$. Suppose not. Then there exists $s_k \downarrow t$ such that $\bar{v}(s_k, x, a)$ tends to some limit $l > \bar{v}(t)$. It follows that $\bar{v}(s_k, x, a) - \bar{v}(t, x, a) > C_0(s_k - t)$ for some k sufficiently large, and therefore also $v^n(s_k, x, a) - v^n(t, x, a) > C_0(s_k - t)$ for some n sufficiently large, contradicting (5.14). This contradiction shows that $\bar{v}$ is right-continuous in t on $[0, T)$.

Then it follows from (5.13) that $\int_A [\bar{v}(t, x, b) - \bar{v}(t, x, a)]^+\, \lambda_0(db) = 0$ for every $x \in E$, $a \in A$, $t \in [0, T]$. Therefore there exists $B \subset A$ (dependent on t, x, a) such that B is a Borel set with $\lambda_0(B) = 0$, and

$$\bar{v}(t, x, a) \ \geq \ \bar{v}(t, x, b'), \qquad b' \notin B. \tag{5.15}$$

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Since λ_0 has full support, B cannot contain any open ball. So given an arbitrary $b \in A$ we can find a sequence $b_n \to b$, $b_n \notin B$. Writing (5.15) with b_n instead of b' and using the lower semi-continuity of $\bar{v}$ we deduce that $\bar{v}(t,x,a) \geq \liminf_n \bar{v}(t,x,b_n) \geq \bar{v}(t,x,b)$. Since a and b were arbitrary we finally conclude that $\bar{v}(t,x,a) = \bar{v}(t,x,b)$ for every $t \in [0,T]$, $x \in E$ and $a,b \in A$, so that $\bar{v}(t,x,a)$ does not depend on a and we can define $\bar{v}(t,x) = \bar{v}(t,x,a)$.

Passing to the limit as $n \to \infty$ in the first inequality of (5.14) we immediately obtain (5.11), so it remains to prove (5.12). Let $\mathcal{A}(\bar{v})$ denote the set of all Borel-measurable $\alpha : [0,T] \to A$ such that (5.12) holds, namely for every $0 \leq t \leq t' \leq T$, $x \in E$, $a \in A$,

$$\bar{v}(t,x) - \bar{v}(t',x) \;\; \geq \;\; \int_t^{t'} \int_E \bar{v}(s,y)\, \lambda(x,\alpha(s),dy)\, ds \tag{5.16}$$

$$-\int_t^{t'} \bar{v}(s,x)\, \lambda(x,\alpha(s),E)\, ds + \int_t^{t'} f(s,x,\alpha(s))\, ds. \tag{5.17}$$

Suppose that $\alpha_n \in \mathcal{A}(\bar{v})$, $\alpha : [0,T] \to A$ is Borel-measurable and $\alpha_n(t) \to \alpha_n(t)$ for almost all $t \in [0,T]$. Note that

$$\int_E \bar{v}(t,y)\, \lambda(x,a,dy) = \lim_{n\to\infty} \int_E \bar{v}^n(t,y,a)\, \lambda(x,a,dy) \tag{5.18}$$

and the latter is an increasing limit. Since $v^n \in C_b([0,T] \times E \times A)$ and λ is Feller, for any $n \geq 1$ the functions in the right-hand side of (5.18) are continuous in (t,x,a) (see e.g. [3], Proposition 7.30) and therefore the left-hand side is a lower semicontinuous function of (t,x,a). It follows from this and the Fatou Lemma that

$$\int_t^{t'} \int_E \bar{v}(s,y)\, \lambda(x,\alpha(s),dy)\, ds \;\; \leq \;\; \int_t^{t'} \liminf_{n\to\infty} \left[\int_E \bar{v}(s,y)\, \lambda(x,\alpha_n(s),dy) \right] ds$$

$$\leq \;\; \liminf_{n\to\infty} \int_t^{t'} \int_E \bar{v}(s,y)\, \lambda(x,\alpha_n(s),dy)\, ds.$$

Using this inequality and the continuity and boundedness of the maps $a \mapsto \lambda(x,a,E)$, $a \mapsto f(t,x,a)$ we see that assuming the validity of inequality (5.16) for α_n implies that it also holds for α, hence $\alpha \in \mathcal{A}(\bar{v})$.

Next we note that $\mathcal{A}(\bar{v})$ contains all piecewise constant functions of the form $\alpha(t) = \sum_{i=1}^k a_i 1_{[t_i,t_{i+1})}(t)$ with $k \geq 1$, $0 = t_1 < t_2 < \ldots < t_{k+1} = T$, $a_i \in A$: indeed, it is enough to write down (5.11) with $[t,t'] = [t_i,t_{i+1}]$ and sum up over i to get (5.12) for $\alpha(\cdot)$ and therefore conclude that $\alpha(\cdot) \in \mathcal{A}(\bar{v})$. Since we have already proved that the class $\mathcal{A}(\bar{v})$ is stable under almost sure pointwise limits it follows that $\mathcal{A}(\bar{v})$ contains all Borel-measurable functions $\alpha : [0,T] \to A$ as required. $\qquad\square$

We are now ready to conclude the proof of our main result.

Proof of Theorem 5.1. We will prove the inequality

$$\bar{v}(t,x) \;\; \geq \;\; V(t,x), \qquad t \in [0,T], x \in E, \tag{5.19}$$

where $\bar{v} = \lim_{n\to\infty} v^n$ was introduced before Lemma 5.5. Since we know that $\bar{v} \leq v$ and, by Theorem 2.9, $v = V$ it follows from (5.19) that $\bar{v} = v = V$. Passing to the limit as $n \to \infty$ in (5.7) and recalling (4.4) all the other equalities follow immediately.

To prove (5.19) we fix $t \in [0,T]$, $x \in E$ and a Borel-measurable map $\alpha : [0,T] \times E \to A$, i.e. an element of $\mathcal{A}_{ad}$, the set of admissible control laws for the primal control problem, and denote by $\mathbb{P}_\alpha^{t,x}$ the associated probability measure on $(\Omega, \mathcal{F}_\infty)$, for the controlled system started at time t

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from point x, as in section 2.2. We will prove that $\bar{v}(t, x) \geq J(t, x, \alpha)$, the gain functional defined in (2.5). Recall that in Ω we had defined a canonical marked point process $(T_n, E_n)_{n \geq 1}$ and the associated random measure p. Fix $\omega \in \Omega$ and consider the points $T_n(\omega)$ lying in $(t, T]$, which we rename S_i; thus, $t < S_1 < \ldots S_k \leq T$, for some k (also depending on ω). Recalling that $\bar{v}(T, x) = g(x)$ we have

$$
\begin{aligned}
g(X_T) - \bar{v}(t, x) &= g(X_T) - \bar{v}(S_k, X_{S_k}) + \sum_{i=1}^{k} [\bar{v}(S_i, X_{S_i}) - \bar{v}(S_i, X_{S_i-})] \\
&\quad + \sum_{i=2}^{k} [\bar{v}(S_i, X_{S_i-}) - \bar{v}(S_{i-1}, X_{S_{i-1}})] + \bar{v}(S_1, X_{S_1-}) - \bar{v}(t, x).
\end{aligned}
$$

$\mathbb{P}_\alpha^{t,x}$-a.s we have $X_{S_i-} = X_{S_{i-1}}$ $(2 \leq i \leq k)$ and $X_{S_1-} = x$, so we obtain

$$
\begin{aligned}
g(X_T) - \bar{v}(t, x) &= g(X_T) - \bar{v}(S_k, X_{S_k}) + \sum_{i=1}^{k} [\bar{v}(S_i, X_{S_i}) - \bar{v}(S_i, X_{S_i-})] \\
&\quad + \sum_{i=2}^{k} [\bar{v}(S_i, X_{S_{i-1}}) - \bar{v}(S_{i-1}, X_{S_{i-1}})] + \bar{v}(S_1, x) - \bar{v}(t, x).
\end{aligned}
$$

The first sum can be written as

$$
\sum_{i=1}^{k} [\bar{v}(S_i, X_{S_i}) - \bar{v}(S_i, X_{S_i-})] = \int_t^T \int_E [\bar{v}(s, y) - \bar{v}(s, X_{s-})] \, p(ds\,dy),
$$

while the other can be estimated from above by repeated applications of (5.12), taking into account that X is constant in the intervals $(t, S_1]$, $(S_{i-1}, S_i]$ $(2 \leq i \leq k)$ and $(S_k, T]$:

$$
\begin{aligned}
\bar{v}(S_i, X_{S_{i-1}}) - \bar{v}(S_{i-1}, X_{S_{i-1}}) &\leq -\int_{S_{i-1}}^{S_i} \left(\mathcal{L}_E^{\alpha(s, X_{S_{i-1}})} \bar{v}(s, X_{S_{i-1}}) + f(s, X_{S_{i-1}}, \alpha(s, X_{S_{i-1}})) \right) ds \\
&= -\int_{S_{i-1}}^{S_i} \left(\mathcal{L}_E^{\alpha(s, X_s)} \bar{v}(s, X_s) + f(s, X_s, \alpha(s, X_s)) \right) ds
\end{aligned}
$$

for $2 \leq i \leq k$ and similar formulae for the intervals $(t, S_1]$, and $(S_k, T]$. We end up with

$$
\begin{aligned}
g(X_T) - \bar{v}(t, x) &\leq \int_t^T \int_E [\bar{v}(s, y) - \bar{v}(s, X_{s-})] \, p(ds\,dy) \\
&\quad - \int_t^T \left(\mathcal{L}_E^{\alpha(s, X_s)} \bar{v}(s, X_s) + f(s, X_s, \alpha(s, X_s)) \right) ds.
\end{aligned}
$$

Recalling that the compensator of the measure p under $\mathbb{P}_\alpha^{t,x}$ is $1_{[t,\infty)}(s)\lambda(X_{s-}, \alpha(s, X_{s-}), dy)\, ds$ we have, taking expectation,

$$
\mathbb{E}_\alpha^{t,x} \int_t^T \int_E [\bar{v}(s, y) - \bar{v}(s, X_{s-})] \, p(ds\,dy) = \mathbb{E}_\alpha^{t,x} \int_t^T \mathcal{L}_E^{\alpha(s, X_s)} \bar{v}(s, X_s) \, ds,
$$

which implies, by the previous inequality, $\mathbb{E}_\alpha^{t,x}[g(X_T)] - \bar{v}(t, x) \leq -\mathbb{E}_\alpha^{t,x} \int_t^T f(s, X_s, \alpha(s, X_s)) \, ds$ and so $\bar{v}(t, x) \geq J(t, x, \alpha)$. Since $\alpha \in \mathcal{A}_{ad}$ was arbitrary we conclude that $\bar{v}(t, x) \geq V(t, x)$.

$\square$

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