On low-frequency vibrations of a composite string with contrast properties for energy scavenging fabric devices

ON LOW-FREQUENCY VIBRATIONS OF A COMPOSITE STRING WITH CONTRAST PROPERTIES FOR ENERGY SCAVENGING FABRIC DEVICES

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Abstract. Free vibrations of a two-component string with high-contrast material parameters are considered at different boundary conditions to illustrate the very low-frequency energy harvesting capability of fabric devices. It is revealed that, only for the case of mixed boundary conditions, low-frequency (locally) almost rigid-body vibrations are admissible, provided that material parameter ratios lie in some well defined interval. A low-frequency perturbation procedure is carried out to determine the eigenfrequencies as well as the eigenforms. The analysis is extended to a piecewise inhomogeneous string and to a string supported on an elastic foundation. It is shown that both situations may still admit low-frequency vibrations, under certain restrictions on the material properties. This is particularly remarkable, given that the situation of elastic support normally possesses two nonzero cutoff frequencies. The results may be especially relevant for energy scavenging fabric devices, where very low-frequency (< 10 Hz) mechanical vibrations of textile fibers are harvested through friction.

1. Introduction

Low-frequency mechanical vibrations of composite structures have been the object of extensive studies, see the classic textbooks [4], [12] and, for instance, [19] for a modern account. In recent years, a revival of interest in the subject has been taking place owing to the appearance of new applications connected to the development of multi-phase or multi-layered structures with high-contrast in the geometrical and mechanical properties. Alongside multi-layered composite structures with high-contrast material parameters, which are currently widely used in various fields of civil and mechanical engineering, see [5, 2], another promising application area is related to the rapidly developing field of meta-materials. Meta-materials are engineered materials endowed with unique properties, often stemming from the interplay of periodically arranged phases exhibiting extremely high contrast [13].

Key words and phrases. low-frequency vibration; energy scavenging; contrast properties; strings.
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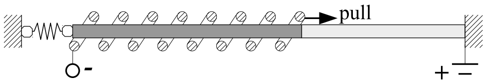

FIGURE 1. Schematics of a textile energy harvester, cfr.[17, Fig.2a]. A micro-wire is wound around the vibrating string and frictional energy is harvested through pulling or vibrating. Note that friction causes (mechanical) tension to vary along the string.

The same principle of phase periodicity is adopted to design and construct waveguides with tailored filtering properties [3, 16, 10].

Mechanical vibrations arise naturally in a variety of environments and they can be harvested to power self-sustaining micro- and especially nano-devices. Despite high-frequency vibrations being very attractive in light of their high energy content, much greater interest lies in the exploitation of low-frequency vibrations for their ubiquitous character (body movements such as footsteps or heartbeat, wind or thermal generated vibrations, air flow and noise) [18].

Recently, textile fabric devices have been proposed as a mean of scavenging very low-frequency (< 10 Hz) mechanical energy through the coupling of a vibrating string wound around by electrically coupled micro-wires [17, 15]. A device schematics is given in Fig.1, although other arrangements are equally possible. In this paper, we focus attention on low-frequency vibrations of a two-component piecewise-constant finite string, in an attempt to better elucidate the energy harvesting capability of the passive element in an energy scavenging device. For the best performance, the string is endowed with high-contrast in the material and/or in the geometrical properties. Continuity conditions are assumed between the components. The analysis is carried out for three types of boundary conditions, namely free-free, fixed-fixed and fixed-free (mixed) end conditions. It is shown that the low-frequency behavior is possible only for the case of mixed boundary conditions, which appears especially attractive for energy harvesting purposes. A low-frequency perturbation approach is adopted to obtain the lowest eigenfrequency and the corresponding eigenform, whose character appears almost rigid-body like. The analysis of the case of variable material parameters confirms that the low-frequency regime is accessible only in a fixed-free setup, although the almost rigid-body behavior is now restricted to the strong component. Finally, vibrations of a high-contrast two-component piecewise homogeneous string supported on a Winkler elastic foundation are considered [14]. In this case, the asymptotic approximation is carried out for frequencies standing in the vicinity of the cut-off frequency of the stronger component, which still can be made very small under some conditions on the ratios of the geometrical and mechanical properties.

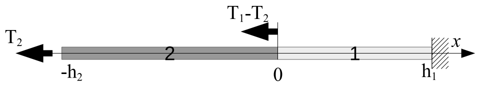

FIGURE 2. A two-component fixed-free string. The tension jump $T_1 - T_2$ approximate the effect of friction with the micro-wire (see Fig.1); a more refined model considering continuous variation of the tension is discussed in Sec.3B

It is worth mentioning that the eigenfrequencies of a composite string correspond to the lowest cutoff frequencies for a plate (or a shell) from the standpoint of 3D elasticity [6]. The same analogy can be extended to incorporate the effects of pre-stress [11] and anisotropy [8].

The paper is organized as follows. In Section 2, three types of boundary conditions for the ends of the string are considered and it is revealed that the low-frequency regime is possible only in the case of free-fixed boundary conditions. A restriction on the material parameters entailing such behavior is also obtained. In Section 3, a perturbative approach is first conducted on piecewise homogeneous string and then extended to the case of variable material properties. It is shown that, although the almost rigid-body eigen-behaviour retrieved in the former case is lost in the latter, low-frequency vibrations are still admissible under suitable conditions. The effect of an elastic support is considered in Section 4 and it brings a cutoff frequency which may be greatly decreased. Finally, conclusions are drawn in Section 5.

2. Frequency equation for a two-component string

Let us consider a finite linear string made of two-components, named 1 and 2, with high material and geometric contrast parameters. The length of the components is h_1 and h_2. Let the x axis be taken to lie along the string with the origin coinciding with the interface between the two components (see Fig.2).

The governing equation of the string in harmonic motion is

$$\frac{\mathrm{d}^2 u_i}{\mathrm{d}x^2} + \frac{\omega^2}{c_i^2} u_i = 0, \quad i = 1, 2, \tag{2-1}$$

where u_i is the transverse displacement in the relevant component of the string, T_i and $c_i = \sqrt{T_i/\rho_i}$ the corresponding string tension and wave speed, ρ_i the linear mass density and $\omega > 0$ the vibration frequency [19, Chap.2]. The conditions enforcing displacement and traction continuity at the interface between the components are given by

$$u_1(0) = u_2(0), \quad T_1 \frac{\mathrm{d}u_1}{\mathrm{d}x}(0) = T_2 \frac{\mathrm{d}u_2}{\mathrm{d}x}(0). \tag{2-2}$$

Let introduce the following notation for the ratios of the material parameters in the two components of the string:

$$T = \frac{T_1}{T_2}, \quad h = \frac{h_1}{h_2}, \quad \rho = \frac{\rho_1}{\rho_2}, \quad c = \frac{c_1}{c_2}, \tag{2-3}$$

together with the non-dimensional frequency parameters

$$\lambda_i = \frac{\omega}{c_i} h_i > 0, \quad i = 1, 2. \tag{2-4}$$

It is observed that the parameters λ_1 and λ_2 are related through the connection

$$\lambda_2 = \frac{1}{h}\sqrt{\frac{T}{\rho}}\lambda_1. \tag{2-5}$$

The general solution of the constant coefficient linear ODEs (2-1) is given by

$$\begin{aligned} u_1(x) &= A\cos\left(\lambda_1 \frac{x}{h_1}\right) + B\sin\left(\lambda_1 \frac{x}{h_1}\right), \quad 0 \le x \le h_1, \\ u_2(x) &= C\cos\left(\lambda_2 \frac{x}{h_2}\right) + D\sin\left(\lambda_2 \frac{x}{h_2}\right), \quad -h_2 \le x \le 0, \end{aligned} \tag{2-6}$$

where A, B, C, D are arbitrary constants.

For a one-parameter asymptotic analysis, the small positive quantity ε is introduced as follows:

$$\varepsilon = \frac{T}{h} \ll 1. \tag{2-7}$$

Besides, let

$$\eta = \frac{T}{c} = \sqrt{T\rho}, \tag{2-8}$$

whence Eq.(2-5) gives the connection

$$\lambda_2 = \frac{\varepsilon}{\eta}\lambda_1. \tag{2-9}$$

We shall consider three types of end conditions for the string and, in each case, investigate the possibility for low-frequency vibrations.

2A. **Traction free end conditions.** The boundary conditions for a string with traction free ends can be written in the form

$$\frac{\mathrm{d}u_1}{\mathrm{d}x}(h_1) = 0 \quad \text{and} \quad \frac{\mathrm{d}u_2}{\mathrm{d}x}(-h_2) = 0. \tag{2-10}$$

Substituting the general solution (2-6) into the boundary conditions (2-10) and into the continuity relations (2-2) we arrive at a homogeneous system of algebraic equations which is linear in the integration constants. As well known, such system possesses non-trivial solution provided that the determinant of the associated coefficient matrix is equal to zero, namely

$$\begin{vmatrix} -\sin\lambda_1 & \cos\lambda_1 & 0 & 0 \\ 0 & 0 & \sin\lambda_2 & \cos\lambda_2 \\ 1 & 0 & -1 & 0 \\ 0 & \eta & 0 & -1 \end{vmatrix} = 0. \tag{2-11}$$

Such requirement leads to the frequency equation

$$\eta\tan\lambda_1 + \tan\lambda_2 = 0, \tag{2-12}$$

which, clearly, cannot sustain low-frequency vibrations (i.e. vibrations at $\lambda_1, \lambda_2 \ll 1$) unless materials with exotic properties, like negative density, are considered (see, for example, [13] and references therein).

2B. **Fixed end conditions.** The situation of a string with fixed ends is now considered. In this case, the boundary conditions (2-10) are substituted by

$$u_1(h_1) = 0 \quad \text{and} \quad u_2(-h_2) = 0. \tag{2-13}$$

Following the usual procedure, we arrive at the frequency equation

$$\tan\lambda_1 + \eta\tan\lambda_2 = 0, \tag{2-14}$$

which closely resembles Eq.(2-12). Hence, low-frequency vibrations in a string with fixed ends cannot be achieved. However, an interesting remark appears in [19], where it is observed that, in the case of a fixed-fixed two-segment string, the eigenfrequency is a decreasing function of the density ratio ρ.

2C. **Fixed-free ends.** Let us consider yet another type of boundary conditions, namely the fixed-free end conditions, wherein

$$u_1(h_1) = 0 \quad \text{and} \quad \frac{\mathrm{d}u_2}{\mathrm{d}x}(-h_2) = 0. \tag{2-15}$$

The frequency equation may be written as

$$\tan\lambda_1\tan\lambda_2 = \eta.$$

In the low-frequency regime, characterized by $\lambda_1 \ll 1$ and $\lambda_2 \ll 1$, the frequency equation is approximated by

$$\lambda_1\lambda_2 = \eta,$$

or, employing the connection (2-9), by the following condition on λ_1:

$$\lambda_1 = \frac{\eta}{\sqrt{\varepsilon}} \ll 1. \tag{2-16}$$

Clearly, Eq.(2-9) demands

$$\lambda_2 = \sqrt{\varepsilon} \ll 1. \tag{2-17}$$

According to the definitions (2-7,2-8), the (order) inequalities (2-16) and (2-17) amount to

$$\rho h \ll 1 \quad \text{and} \quad \frac{T}{h} \ll 1, \tag{2-18}$$

respectively. In particular, the first inequality may be rewritten in term of masses, i.e.

$$m_1 = \rho_1 h_1 \ll \rho_2 h_2 = m_2,$$

that is one string component mass needs be much smaller than the other's (note that 1 and 2 are exchangeable). Together, the inequalities (2-18) require

$$\eta^2 \ll \varepsilon \ll 1$$

which, by the definitions (2-7,2-8), gives a single condition on the geometric/mechanical parameters allowing for low-frequency vibrations, namely

$$T \ll h \ll \frac{1}{\rho}. \tag{2-19}$$

Low-frequency vibrations may arise, for example, in a string with soft and light part 1, while part 2 is stiff and heavy, i.e. $T_1 \ll T_2$ and $\rho_1 \ll \rho_2$. Besides, the corresponding string component lengths, h_1 and h_2, need be chosen of the same order of magnitude, i.e. $h_1/h_2 \sim 1$.

3. Asymptotic analysis of low-frequency vibrations in a composite string with free-fixed ends

The study of the frequency equation carried out in Sec.2 leads to the conclusion that low-frequency vibrations are only possible for a string with free-fixed end conditions, provided some restriction on the material parameter ratios is met. For a more refined analysis, a low-frequency asymptotic approximation is now employed. A constant coefficient boundary-value problem is considered first and then results are generalized to the variable coefficients situation.

3A. **Piecewise homogeneous string.** For a two-component string with homogeneous material parameters, the equations of motion (2-1) hold together with the boundary conditions (2-10) and the continuity relations (2-2) at the interface. We restrict our attention to low-frequency vibrations at $\eta \sim \varepsilon$, whence $\lambda_1 \sim \lambda_2 \sim \sqrt{\varepsilon}$. To this aim, let

$$\eta = \alpha\varepsilon, \quad \text{where} \quad \alpha = O(1). \tag{3-1}$$

Hence, the connection (2-9) between λ_1 and λ_2 now reads

$$\lambda_2 = \alpha^{-1}\lambda_1. \tag{3-2}$$

Let us introduce non-dimensional spatial variables in each component of the string

$$\xi_1 = \frac{x}{h_1} \in [0,1] \quad \text{and} \quad \xi_2 = \frac{x}{h_2} \in [-1,0].$$

Then, our boundary-value problem may be re-written in terms of the dimensionless variables

$$\frac{\mathrm{d}^2 u_i^*}{\mathrm{d}\xi_i^2} + \lambda_i^2 u_i^* = 0, \quad i = 1,2, \tag{3-3}$$

together with the fixed-free end conditions

$$u_1^*(1) = 0, \quad \frac{\mathrm{d}u_2^*}{\mathrm{d}\xi_2}(-1) = 0$$

and the continuity conditions at the interface $\xi_1 = \xi_2 = 0$

$$u_1^*(0) = u_2^*(0), \quad \varepsilon \frac{\mathrm{d}u_1^*}{\mathrm{d}\xi_1}(0) = \frac{\mathrm{d}u_2^*}{\mathrm{d}\xi_2}(0).$$

Here, it is let $u_i^*(\xi_i) = u_i(x/h_i), i = 1, 2$. Assuming for u_i^* a regular asymptotic expansion in the small parameter ε, we write

$$u_i^* = u_i^{(0)} + \varepsilon u_i^{(1)} + \varepsilon^2 u_i^{(2)} + O(\varepsilon^3), \quad i = 1, 2, \tag{3-4}$$

while, in the low-frequency regime, it is

$$\lambda_1^2 = \varepsilon(\Lambda_0 + \varepsilon\Lambda_1 + \varepsilon^2\Lambda_2 + O(\varepsilon^3)). \tag{3-5}$$

Clearly, λ_2 follows from the connection (3-2).

3A.1. *Leading order problem.* At the leading order, the equations of motion (3-3) give

$$u_1^{(0)} = A_0\xi_1 + B_0,$$
$$u_2^{(0)} = C_0\xi_2 + D_0,$$

where A_0, B_0, C_0 and D_0 are integration constants. Using the boundary and continuity conditions, we arrive at

$$u_1^{(0)} = D_0(1 - \xi_1), \tag{3-6a}$$
$$u_2^{(0)} = D_0, \tag{3-6b}$$

which shows a local rigid-body behavior (rotation for 1 and translation for 2). Given that frequency cannot be derived at this stage, we need proceed to the next order.

3A.2. *First order problem.* At the first order, compatibility gives the leading order frequency term

$$\Lambda_0 = \alpha^2,$$

whereupon

$$\lambda_1^2 = \alpha^2\varepsilon\left(1 + O(\varepsilon)\right), \quad \lambda_2^2 = \varepsilon\left(1 + O(\varepsilon)\right).$$

Expressions for first order correction to displacements, $u_1^{(1)}$ and $u_2^{(1)}$, take up the form:

$$u_1^{(1)} = \frac{1}{6}\alpha^2 D_0(1 - \xi_1)\left(2 - \xi_1\right)\xi_1 + D_1(1 - \xi_1),$$
$$u_2^{(1)} = -D_0\xi_2\left(\frac{\xi_2}{2} + 1\right) + D_1,$$

where D_1 is yet another integration constant.

3A.3. *Second order.* At the second order, compatibility yields the first order correction to the frequency

$$\Lambda_1 = -\frac{\alpha^2(1+\alpha^2)}{3},$$

and we arrive at the expansion

$$\lambda_1^2 = \varepsilon\alpha^2\left(1 - \frac{1+\alpha^2}{3}\varepsilon + O(\varepsilon^2)\right).$$

3B. **Piecewise inhomogenenous string.** It is now assumed that the material properties of each component of the string are no longer constant along the length, namely $T_i = T_i(x) > 0, \rho_i = \rho_i(x) > 0,\ i = 1, 2$. The equation for harmonic transverse vibrations are [19]

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(T_i(x)\frac{\mathrm{d}u_i}{\mathrm{d}x}\right) + \rho_i(x)\omega^2 u_i = 0, \quad i = 1, 2, \tag{3-7}$$

while the fixed-free boundary conditions (2-15), together with the continuity relations (2-2), hold. It is expedient to introduce the non-dimensional quantities

$$T_i^*(\xi_i) = \frac{T_i(x)}{T_i(0)}, \quad \rho_i^*(\xi_i) = \frac{\rho_i(x)}{\rho_i(0)}, \quad i = 1, 2 \tag{3-8}$$

as well as the ratios

$$T = \frac{T_1(0)}{T_2(0)}, \quad \rho = \frac{\rho_1(0)}{\rho_2(0)}. \tag{3-9}$$

In terms of the dimensionless co-ordinates ξ_i, Eq.(3-7) becomes

$$\frac{\mathrm{d}}{\mathrm{d}\xi_i}\left(T_i^*(\xi_i)\frac{\mathrm{d}u_i^*}{\mathrm{d}\xi_i}\right) + \lambda_i^2\rho_i^*(\xi_i)u_i^* = 0, \tag{3-10}$$

where

$$\lambda_i^2 = \frac{\omega^2 h_i^2 \rho_i(0)}{T_i(0)}, \quad i = 1, 2 \tag{3-11}$$

and the connection (3-2) still holds. The boundary conditions give

$$u_1^*(1) = 0, \quad \frac{\mathrm{d}u_2^*}{\mathrm{d}\xi_2}(-1) = 0,$$

while continuity at the interface reads

$$u_1^*(0) = u_2^*(0), \quad \varepsilon\frac{\mathrm{d}u_1^*}{\mathrm{d}\xi_1}(0) = \frac{\mathrm{d}u_2^*}{\mathrm{d}\xi_2}(0),$$

where the small parameter ε is introduced as at Eq.(2-7). Solution of this boundary value problem is taken in the form of the asymptotic expansions (3-4) and (3-5) for u_i^* and λ_1, respectively.

At the leading order we obtain

$$u_1^{(0)}(\xi_1) = D_0\left(1 - \frac{\phi_1(\xi_1)}{\phi_1(1)}\right), \tag{3-12a}$$

$$u_2^{(0)}(\xi_2) = D_0, \tag{3-12b}$$

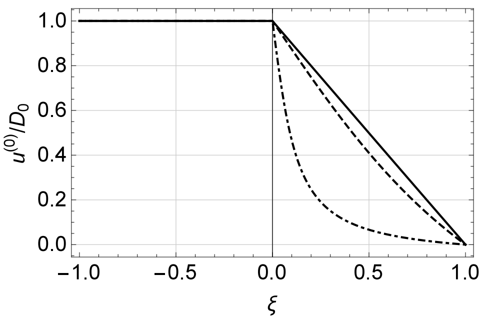

FIGURE 3. Leading order eigenforms for $T_1^*(\xi_1) = 1 + A^2\xi_1^2$ and $T_2^*(\xi_2) = 1$ at $A = 0.1$ (solid), $A = 1$ (dashed) and $A = 10$ (dot-dashed)

where D_0 is an integration constant and we have let

$$\phi_i(\xi_i) = \int_0^{\xi_i} \frac{\mathrm{d}t}{T_i^*(t)}, \quad i = 1, 2. \tag{3-13}$$

In order to better illustrate the leading order expressions for the first eigenform (3-12), a string with quadratic and constant behavior for the dimensionless tensions T_1^* and T_2^* is considered, i.e. $T_1^*(\xi_1) = 1 + A^2\xi_1^2$ and $T_2^*(\xi_2) = 1$. Then, it is

$$\phi_1(\xi_1) = \frac{1}{|A|}\arctan(|A|\xi_1) \quad \text{and} \quad \phi_2(\xi_2) = \xi_2.$$

In Fig.3, the leading order expressions for the first eigenform, given by Eqs.(3-12), are plotted at three values of the parameter A. As expected, the locally rigid-body behavior is retrieved for small values of A. However, it is perhaps less obvious that large values of A lead to a step function. Besides, we further observe that the transformation $u_i^*(\xi_i) = U_i^*(z_i)$, with the mapping

$$\xi_i \mapsto z_i \mid z_i(\xi_i) = \frac{\phi_i(\xi_i)}{\phi_i((-1)^{i+1})},$$

can be used to turn the variable coefficient problem (3-10) into Lioville's normal form [5, 19].

Bringing the analysis one step further, we obtain the displacement first order correction

$$u_1^{(1)}(\xi_1) = D_0\Lambda_0\left(\frac{1}{\phi_1(1)}\int_0^{\xi_1}\frac{\int_0^{\sigma_2}\rho_1^*(\sigma_1)\phi_1(\sigma_1)\,\mathrm{d}\sigma_1}{T_1^*(\sigma_2)}\,\mathrm{d}\sigma_2 - \Phi_1(\xi_1)\right) + A_1^{(1)}\phi_1(\xi_1) + B_1^{(1)}$$
$$u_2^{(1)}(\xi_2) = -D_0\Lambda_0\frac{1}{\alpha^2}\Phi_2(\xi_2) + C_1^{(1)}\phi_2(\xi_2) + D_1^{(1)}$$

being

$$\Phi_i(\xi) = \int_0^{\xi}\frac{\int_0^{\sigma_2}\rho_i^*(\sigma_1)\,\mathrm{d}\sigma_1}{T_i^*(\sigma_2)}\,\mathrm{d}\sigma_2, \quad i = 1, 2.$$

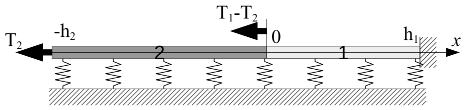

FIGURE 4. A two-component free-fixed string supported by a Winkler elastic foundation. The foundation is introduced to account for an embedding elastic matrix

Here, compatibility gives the leading order term in the frequency expansion

$$\Lambda_0 = \frac{\alpha^2}{\phi_1(1) m_2^*},$$

having let the dimensionless mass of the 2 component

$$m_2^* = \int_{-1}^{0} \rho_2^*(\sigma_1) \mathrm{d}\sigma_1.$$

Clearly, the expression for λ_2 may be readily obtained from (3-2).

4. Piecewise homogenenous string on a Winkler foundation

In this Section, near-zero frequency vibrations of a two-component piecewise homogenenous string on a Winkler foundation are considered (Fig.4).

The equations of motion for a string on a Winkler foundation are [4]

$$\frac{\mathrm{d}^2 u_i}{\mathrm{d}x^2} + \left(\frac{\omega^2}{c_i^2} - \frac{\kappa}{T_i} \right) u_i = 0, \quad i = 1, 2, \tag{4-1}$$

where κ is the Winkler foundation modulus (whose physical dimensions are force over length squared). These equations clearly show that local cutoff frequencies exist [9]

$$\omega_{\mathrm{cutoff}}{}_i^2 = \frac{\kappa}{\rho_i},$$

such that harmonic vibrations are possible only whenever $\omega > \max\left(\omega_{\mathrm{cutoff}1}, \omega_{\mathrm{cutoff}2}\right)$. Eqs.(4-1) are most conveniently put in dimensionless form

$$\frac{\mathrm{d}^2 u_i}{\mathrm{d}\xi_i^2} + \gamma_i^2 u_i = 0, \quad i = 1, 2, \tag{4-2}$$

where it is let

$$\gamma_i^2 = \lambda_i^2 - \beta_i^2 \quad \text{and} \quad \beta_i^2 = \frac{h_i^2}{T_i} \kappa, \quad i = 1, 2. \tag{4-3}$$

It is remarked that the λ_i are defined according to Eq.(2-4) and therefore the connection (2-9) still holds. Obviously, we demand $\lambda_i > \beta_i$ for global vibrations to take place, which shows that β_i are the dimensionless local cutoff frequencies. Furthermore, the following connection stands between β_1 and β_2:

$$\beta_2^2 = \frac{\varepsilon}{h} \beta_1^2. \tag{4-4}$$

The general solution of the ODEs (4-2) is given by

$$u_1(\xi_1) = A\cos(\gamma_1\xi_1) + B\sin(\gamma_1\xi_1), \quad 0 \le \xi_1 \le 1, \tag{4-5a}$$

$$u_2(\xi_2) = C\cos(\gamma_2\xi_2) + D\sin(\gamma_2\xi_2), \quad -1 \le \xi_2 \le 0, \tag{4-5b}$$

where A, B, C, D are arbitrary constants. As in the case of a string with homogeneous parameters, we shall consider three types of boundary conditions, namely free-free, fixed-fixed and fixed-free end conditions. The frequency equation for harmonic vibrations of a string is, for the case of fixed ends,

$$\varepsilon\frac{\gamma_1}{\gamma_2}\tan\gamma_1 + \tan\gamma_2 = 0,$$

while, for the case of free ends, it is

$$\varepsilon\frac{\gamma_1}{\gamma_2}\tan\gamma_2 + \tan\gamma_1 = 0.$$

Clearly, both equations do not allow for low-frequency vibrations. Conversely, the frequency equation for the fixed-free case reads

$$\varepsilon\frac{\gamma_1}{\gamma_2} = \tan\gamma_1\tan\gamma_2, \tag{4-6}$$

which may admit low-frequency vibrations. Indeed, assuming γ_1 and γ_2 small (which amounts to considering near-cutoff vibrations, see also [1]), we get

$$\gamma_2^2 = \varepsilon + O(\varepsilon^3),$$

whence the condition for γ_2 to be small is given by

$$\varepsilon = T/h \ll 1. \tag{4-7}$$

The (squared) scaled frequency λ_2^2 is readily obtained from Eq.(4-3) through shifting by the local cutoff frequency β_2^2, i.e.

$$\lambda_2^2 = \beta_2^2 + \varepsilon + O(\varepsilon^3)$$

which, in light of (4-4), is enough for low-frequency vibrations of the 2-component (assuming that β_1 is of order unity or smaller and h of order unity or larger). In order to achieve global low-frequency vibrations, we demand λ_1 to be small as well and, using the connections (2-9), this requires

$$\lambda_1^2 = \frac{\eta^2}{\varepsilon}\left(\frac{\beta_1^2}{h} + 1 + O(\varepsilon^2)\right) \ll 1, \tag{4-8}$$

which amounts to the condition

$$(\beta_1^2 + h)\rho \ll 1. \tag{4-9}$$

Similarly to what was chosen in Sec.3, this condition may be fulfilled taking, for instance,

$$\eta = \eta_0\varepsilon. \tag{4-10}$$

It rests to be seen whether γ_1 is also small as it was initially assumed. To this aim, using the definition (4-3), we further demand

$$0 < \beta_1^2(\rho - 1) + \rho h \ll 1. \tag{4-11}$$

Together, Eqs.(4-9) and (4-11) imply

$$\beta_1^2 < (\beta_1^2 + h)\rho \ll 1,$$

whereupon β_1 needs also be a small quantity. For instance, we could set

$$\beta_1 = \varepsilon\beta_0. \tag{4-12}$$

Leading order asymptotic expansions for the displacement may be equally well derived. To this end, we first write the eigenforms u_1 and u_2 through introducing the general solution (4-5) into the fixed-free boundary conditions (2-15) and into the continuity relations (2-2)

$$u_1(\xi_1) = D\left(\sin(\gamma_1\xi_1) - \tan(\gamma_1)\cos(\gamma_1\xi_1)\right),$$
$$u_2(\xi_2) = D\left(\varepsilon\frac{\gamma_1}{\gamma_2}\sin(\gamma_2\xi_2) - \tan(\gamma_1)\cos(\gamma_2\xi_2)\right),$$

where it is understood that γ_1 and γ_2 are related through the frequency equation (4-6). Then, we introduce the smallness assumptions (4-10,4-12) together with the expansion (4-8) for λ_1 and proceed to expand in the small parameter ε. Thus, the asymptotic expansion for the eigenform is obtained

$$u_1 = D(1 - \xi_1) + O(\varepsilon),$$
$$u_2 = D + O(\varepsilon).$$

It is perhaps surprising to observe that, under the smallness assumptions, the presence of the Winkler foundation does not alter the leading order shape of the eigenforms, cfr. Eqs.(3-6), and a local rigid body-motion is again retrieved.

5. Conclusions

Very low-frequency vibrations in a two-component high-contrast string have been investigated for the case of fixed-fixed, free-free and free-fixed boundary conditions, in an attempt to enhance the energy scavenging capability of the soft element in a fabric device. It is shown that low-frequency vibrations are achievable only for the case of fixed-free end conditions, which seems especially apt at harvesting low-power energy sources. Besides, conditions on the material and geometrical property ratios were given in order to sustain near-zero frequency vibrations. Piecewise constant as well as variable material parameters are considered. In the former case, the exact solution is obtainable and an almost rigid-body motion is found. The almost rigid-body behavior is especially welcome as it warrants little wear in the system. Conversely, the latter situation can only be addressed in an approximate way, through a two-scale approach, and it is shown that, although the almost rigid body behavior is generally lost, low-frequency vibrations can still be sustained, provided that suitable conditions on the material parameters hold. The question whether low-frequency vibrations may be still admitted in a soft element supported by a Winkler foundation was then addressed, because energy harvesters may be embedded in an elastic matrix. In this case, the soft string is assumed piecewise constant and an exact solution is obtained. As expected, two cutoff frequencies appear which, however, may be brought close to zero under suitable smallness assumptions on the material parameter ratios. A somewhat surprising result is obtained, for the local rigid-body character of the leading order expressions for the eigenforms, already met in the unsupported case, is again retrieved. It is finally observed that our results apply equally well to the analysis of low-frequency vibrations in composite plates and shells treated within the framework of 3D elasticity. For example, in case of a two-layered elastic plate, the eigenfrequencies λ_1, λ_2 would be associated with the lowest cutoff frequencies, see, for instance, [7] and [12].

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