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# Quantum correlations of identical particles subject to classical environmental noise

Andrea Beggi · Fabrizio Buscemi · Paolo Bordone

Date: XXXXXX

**Abstract** In this work we propose a measure for the *quantum discord* of indistinguishable particles, based on the definition of *entanglement of particles* given in [H. M. Wiseman *et al.*, *Phys. Rev. Lett* **91**, 097902 (2003)]. This *discord of particles* is then used to evaluate the quantum correlations in a system of two identical bosons (fermions), where the particles perform a quantum random walk described by the Hubbard Hamiltonian in a 1D lattice. The dynamics of the particles is either unperturbed or subject to a classical environmental noise – such as random telegraph, pink or brown noise. The observed results are consistent with those for the *entanglement of particles*, and we observe that on-site interaction between particles have an important protective effect on correlations against the decoherence of the system.

**Keywords** Indistinguishable particles · Entanglement · Quantum Discord · RTN ·  $1/f$  noise ·  $1/f^2$  noise

**PACS** 03.65.Ud · 03.65.Yz · 03.67.-a · 03.67.Mn · 05.40.-a · 71.10.Fd · 03.75.Lm

## 1 Introduction

Since its first introduction[1–4], quantum *entanglement* has become one of the most intriguing and characteristic traits of quantum mechanics. In the last two decades, it has turned into a fundamental resource for quantum information theory and quantum computing[5, 6], since it can be used to implement protocols and calculations that would be impossible in a classical context. However, recent developments in the field of quantum information showed that the entanglement is not able to capture all the quantum correlations contained in a system, and therefore some separable (i.e. not entangled) states still possess a certain amount of correlations that can be used to perform non-classical computing tasks[7, 8]. Thus, in order to take into account these correlations, many quantifiers were proposed in recent years[9, 10], among which the most used is certainly *quantum discord*[11, 12]. Quantum discord represents a measure of the (quantum) correlations of a system that are destroyed by a measurement on a subparty of the system, and in general it does not coincide with entanglement, nor entanglement is necessarily contained within discord[13].

Although there are many possible quantifiers and witnesses for entanglement in multipartite systems[14], the entanglement in bipartite systems of distinguishable particles has a well defined formulation[15–17]. The same consideration does not hold for systems of identical particles, where many definitions have been introduced in recent years, raising debates about the reliability of the proposed quantifiers[18–32]. The main difficulties appearing in the estimation of entanglement for indistinguishable particles arise from the exchange symmetry, which requires the (anti-)symmetrization of the wavefunction: as a consequence, pure states cannot be factorized anymore, even if the particles are not entangled. This requires the introduction of a specific criterion that can distinguish between the non-separability due to genuine entanglement and the “spurious” correlations due to exchange symmetry, which cannot be used to violate Bell’s inequality and therefore are not a resource for quantum-information processing[23, 24].

Among the different proposed approaches for calculating the entanglement of identical particles, the Wiseman and Vaccaro criterion[26] can overcome some problems shown by other methods [27]. It has been recently

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A. Beggi, F. Buscemi, P. Bordone  
Dipartimento di Scienze Fisiche, Informatiche e Matematiche, Università di Modena e Reggio Emilia  
Via Campi 213/A, I-41125, Modena, Italy  
P. Bordone  
Centro S3, CNR-Istituto di Nanoscienze Modena  
Via Campi 213/A, I-41125, Modena, Italy  
E-mail: andrea.beggi@unimore.com  
fabrizio.buscemi@unimore.it

extended to multipartite systems[33] and it was applied for studying the correlations in quantum walks of identical non-interacting particles[34] and the entanglement between sites in Hubbard spin chains[35], where it also acts as an order parameter which is able to capture quantum phase transitions[36]. Indeed, it was also suggested that this kind of entanglement could be measurable in many experimental scenarios [27], thus making it an interesting quantifier for quantum correlations.

At present, however, the evaluation of quantum discord of identical particles is still an almost unexplored topic, with few exceptions [29, 37]. It is well known, however, that quantum discord possesses some features that make it more promising than entanglement in accounting for correlations - such as e.g. a higher robustness under decoherence, the general absence of sudden death phenomena and the ability to identify quantum phase transitions that are missed by entanglement [10].

In the last years, the interest in entanglement among identical particles is increased, since it is crucial for the understanding of many physical phenomena which involve highly correlated indistinguishable subsystems, such as photons in nonlinear waveguides[38–40], ultracold atoms trapped in optical lattices[41–44] or electrons in solid state systems[45, 46]. These systems constitute a possible prototype for implementing quantum computing devices, and indeed some experimental realizations of photonic chips have been achieved recently[47–49], whose architecture rely on photonic quantum walks. The most basic Hamiltonian that can describe a continuous-time Quantum walk (QW)[50, 51] in those physical systems is the Hubbard Hamiltonian[52–54], which has also been used as a benchmark for entanglement criteria of indistinguishable particles[20, 27]. Besides this, QWs of identical particles are of peculiar interest since the indistinguishability of the walkers is responsible for the building-up of genuinely quantum correlations, even in the absence of interactions between particles (e.g. photons) [34, 55–57].

In real physical systems QWs are subject to environmental noise, whose effect is to destroy quantum correlations among the walkers[58]. Therefore, the study of decoherence is of vital importance for the realization of devices that are able to implement robust quantum information protocols, in order to preserve correlations against the action of the environment. A possible model to represent an external noise source appearing in many nanodevices is a random bistable fluctuator with switching frequency  $f_0$ , which produces the so-called *random telegraph noise* (RTN)[59, 60]. A collection of bistable fluctuators with different switching rates  $f$  can also be used to model[61] *colored noises*, such as  $1/f$  (“pink”) or  $1/f^2$  (“brown”) noises, which are very common in solid-state physics[62–65]. There are many works in quantum information literature focused on RTN[66–72] or  $1/f$  noises[68, 73–79], but few studies are available for  $1/f^2$  noise [61, 80–82]. The exploration of these kind of noises is of utmost importance for practical applications, since the system can exhibit peculiar phenomena (like sudden death and revival of correlations, or memory effects) as a function of the noise spectrum[83–85], even for classical noise sources[61, 66, 86–90] such as RTN fluctuators<sup>1</sup>.

For what concerns QWs, many studies are devoted to noise in discrete-time quantum walks, but for continuous-time quantum walks there is a limited amount of studies[91, 92], which are concerning e.g. static noise [93], RTN [66, 94] or other mechanisms of decoherence, such as phonon thermal baths (see [95] and cited references), unitary noise[96], measurements[97] and lattice defects[98]. Among those, very few are focused on the time-evolution of quantum correlations. It seems, therefore, of interest to realize a general overview of the effects of noise on continuous-time QWs correlations, keeping also into account the effects due to indistinguishability of the walkers.

The aim of this paper is double: first of all, we perform an extensive study on quantum correlations in continuous-time QWs of identical particles (fermions and bosons), whose dynamics is either unperturbed or subject to a classical noisy environment – namely a single RTN fluctuator or a collection of bistable fluctuators mimicking colored noises ( $1/f$  and  $1/f^2$ ), in order to quantify the role of classical noise in the decoherence of the system. Secondly, in order to fully characterize quantum correlations, we introduce a measure for the quantum discord of identical particles, and we confront it with the corresponding value of entanglement, to prove that it is a good quantifier of quantum correlations. The dynamics of quantum correlations are then explored as a function of many physical parameters, such as the strength of interactions among the walkers, the number of open decohering channels and the frequency of the noise sources.

The paper is organized as follows. In Sect. 2 we first review the entanglement criteria for identical particles introduced in the literature and their known problems; after this, we illustrate the concept of *entanglement of particles* as introduced by Wiseman and Vaccaro[26], then we extend this approach in order to introduce the *quantum discord of particles*. Then, in Sect. 3 we introduce and characterize the physical models that describe the quantum walks of our fermionic and bosonic particles – which are both based upon the Hubbard Hamiltonian – and the mechanisms that we exploit to generate RTN and colored noises. Sections 4-5 are devoted to the discussion of the results of the numerical simulations: the first one is concerning bosons, the second one

<sup>1</sup> We refer to a noise as *classical* when the system is coupled with the environment but the environment is unaffected by the system, i.e. no back-action-induced correlations can be transferred from the system to the environment. Classical environments, however, can have “effective” back-action on a physical system, as recently shown[61, 66, 86–90], thus determining a revival of quantum correlations.

is about fermions. Finally, in Sect. 6 some conclusions are drawn, and some further perspectives of research are suggested.

## 2 Bipartite entanglement and discord of identical particles

In this section, we shortly review the criteria used in the literature to estimate the entanglement in bipartite systems of two indistinguishable particles, then we illustrate the quantifier that we adopted in our work, which we extend to the evaluation of quantum discord. Specifically, our approach relies on the concept of *entanglement of particles*, introduced by Wiseman and Vaccaro [26, 27] in order to give an operational definition of the entanglement of two identical particles which does not violate the superselection rule of the local particle number.

One of the first proposed quantifiers for the entanglement of identical particles was the Schliemann criterion [18, 19], which is based upon the Slater decomposition of the (anti-)symmetrized state and evaluates the quantum correlations as a function of the minimum Slater rank of the considered state (over all its possible decompositions). Unfortunately, this approach behaves incorrectly under local and nonlocal mode transformations when the entanglement is evaluated between the modes of a system [99]. The same problem affects the entanglement measures proposed in Refs. [100, 101]. To overcome these limits, another criterion was proposed by Zanardi [20]: in his approach, the space of the modes of the systems is mapped into qubit states, and the entanglement between particles is evaluated as the entanglement between modes. It has been recently demonstrated [25] that this criterion is equivalent to those that identify pure separable states as states where both parties possess a complete set of defined properties [23, 24]. The approach used by Zanardi, however, can lead to the overestimation of entanglement due to violation of the local number of particles superselection rule, and this is the reason why it has been generalized by Wiseman and Vaccaro [26]. In their proposal, the so-called *entanglement of particles* is represented by the maximum amount of nonclassical correlations that can be extracted (“accessible entanglement”) from the system with local operations, and then can be encoded into conventional *quantum registers* (i.e. a set of distinguishable qubits). Within this definition, the problems of the aforementioned criteria can be overcome. Other approaches investigated in the literature define entanglement via non-classical correlations between subsets of observables [102–105], but do not possess the same easiness of computability of this criterion.

To go into further detail, the Wiseman and Vaccaro criterion addresses the non-classical correlations in the form of entanglement between two distant parties (namely Alice and Bob) of a quantum system in the mixed state  $\rho$ , each accessing a given set of modes. Any subsystem is assumed to have a standard quantum register, that is a set of distinguishable qubits, in addition to the indistinguishable particles described by  $\rho$ . The *entanglement of particles*  $E_P$  is given by the maximum amount of entanglement that Alice and Bob can produce between their standard quantum registers by means of local operations on the modes that they have access to. For a two-particle system,  $E_P$  can simply be expressed as

$$E_P = P_{1,1} \mathcal{E}(\rho_{1,1}) \quad (1)$$

where  $\rho_{1,1} = \Pi_{1,1} \rho \Pi_{1,1}$  is the state obtained from  $\rho$  by means of the projectors  $\Pi_{1,1}$  onto the state having one particle in each subsystem.  $P_{1,1}$  denotes the probability of finding 1 in measurements of the local number of particles by both Alice and Bob, and  $\mathcal{E}$  represents a bipartite standard entanglement measure estimating the degree of non classical-correlation between the quantum registers of distinguishable qubits controlled by Alice and Bob. Specifically, here the latter is evaluated in terms of the entanglement of formation [16]:

$$\mathcal{E} = h \left( \frac{1 + \sqrt{1 - C^2}}{2} \right), \quad (2)$$

where  $h(x) = -x \ln_2 x - (1 - x) \ln_2 (1 - x)$  and  $C$  denotes the so-called Wootters concurrence

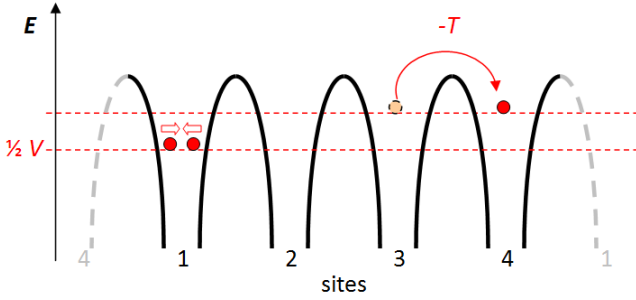
$$C = \max \{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\} \quad (3)$$

with  $\lambda_j$ 's indicating the eigenvalues of the matrix  $\eta$

$$\eta = \sqrt{\rho_{1,1}} (\sigma_y^A \otimes \sigma_y^B) \rho_{1,1}^* (\sigma_y^A \otimes \sigma_y^B) \sqrt{\rho_{1,1}} \quad (4)$$

arranged in decreasing order. In the above expression  $\rho_{1,1}^*$  denotes the complex conjugation of  $\rho_{1,1}$ , and  $\sigma_y^{A(B)}$  is the well-known Pauli matrix acting on the qubit state controlled by Alice(Bob).

The *entanglement of particles* given in Eq. (1) does not violate the superselection rule of the local particle number. Indeed, local operations can be performed onto the collapsed state  $\rho_{1,1}$  without any restrictions in order to transfer its entanglement  $\mathcal{E}$  to the standard quantum registers of each subsystem.



**Fig. 1** Bosonic Hubbard chain:  $V$  is the interaction strength between particles sharing the same site,  $-T$  is the energy gain for hopping (in figure  $T > 0$  and  $V < 0$ ).

Only recently, the estimation of quantum correlations (QCs) other than entanglement has begun to be explored in systems of identical particles by means of different approaches relying on the measurement-induced disturbances [106] or the use of witness operators [29]. By adopting the basic ideas of the approach by Wiseman and Vaccaro, here we introduce a *quantum discord of particles*  $D_P$ . Similarly to  $E_P$ , it represents the maximum amount of discord that can be extracted from the quantum registers of distinguishable qubits of Alice and Bob with no violation of the superselection rule of the local particle number. Thus,  $D_P$  can be expressed as:

$$D_P = P_{1,1} \mathcal{D}(\rho_{1,1}), \quad (5)$$

where  $\mathcal{D}$  quantifies the quantum correlations in the form of discord between the Alice and Bob standard quantum registers in the state  $\rho_{1,1}$ . It is given by [11, 12]:

$$\mathcal{D}(\rho_{1,1}) = \mathcal{T}(\rho_{1,1}) - \mathcal{J}(\rho_{1,1}), \quad (6)$$

that is the difference between the mutual information  $\mathcal{T}$  and the classical correlation  $\mathcal{J}$ . Specifically, the former can be expressed as:

$$\mathcal{T}(\rho_{1,1}) = S(\rho_{1,1}^A) + S(\rho_{1,1}^B) - S(\rho_{1,1}) \quad (7)$$

where  $\rho_{1,1}^{A(B)}$  is the partial trace of the bipartite system  $\rho_{1,1}$  and  $S(\rho_{1,1}) = -\text{Tr}[\rho_{1,1} \log_2(\rho_{1,1})]$  is the von Neumann entropy. On the other hand, the amount of classical correlation  $\mathcal{J}$  is given by

$$\mathcal{J} = \max_{\{\Pi_k^B\}} [S(\rho_{1,1}^A) - S(\rho_{1,1}^A | \{\Pi_k^B\})], \quad (8)$$

where  $\{\Pi_k^B\}$  are projective measurements on subsystem  $B$  and  $S(\rho^A | \{\Pi_k^B\}) = \sum_k p_k \rho_k^A$ , where

$$\rho_k^A = \text{Tr}_B[\Pi_k^B \rho \Pi_k^B] / \text{Tr}[\Pi_k^B \rho \Pi_k^B] \quad (9)$$

is the density operator describing  $A$  conditioned by the measurement outcome  $k$  on  $B$ .

It is worth noting that, from the definitions themselves, both  $E_P$  and  $D_P$  depend upon the partition of the system, that is upon which modes Alice and Bob control.

Considering the definition of Eq. (5), together with the invariance of  $P_{1,1}$  and  $\mathcal{D}(\rho_{1,1})$  under local transformations (which conserve the local particle number), it is easy to prove that the *discord of particles* possesses all the required properties for a quantum discord quantifier [10, 12], as shown in Appendix A. This means that, like the conventional *quantum discord*, our quantifier should be able to capture some features of quantum correlations among particles that are missed by the *entanglement of particles*.

### 3 Physical models

The Hubbard model represents a valid mean to describe a number of different systems in solid-state physics ranging from ultracold atoms trapped in optical lattices to high temperature superconductivity [52, 107]. Since its physical relevance, it is of interest to examine the dynamics of quantum correlations in simplified Hubbard models of interacting bosons and fermions subject to a classical external noise. Specifically, we focus on Bose- and Fermi-Hubbard models affected by environmental RTN, pink  $1/f$ , and brown  $1/f^2$  noises.

### 3.1 Two-boson Bose-Hubbard model

Here, we consider two spinless bosons interacting among each other and hopping among four sites of a lattice. This system is the bosonic version of the Hubbard plaquette, which is used in solid state as a building block for highly-correlated many-body systems[108–111]. In agreement with previous works [61, 66, 67, 90], in order to mimic the effect of the noise on the quantum system the hopping amplitudes among the sites are assumed to follow a time-dependent stochastic behavior. Such an assumption yields the Hamiltonian [27, 40]

$$\mathcal{H}_{BH}(t) = -T \sum_{i=1}^4 q_i(t) (b_i^\dagger b_{i+1} + b_{i+1}^\dagger b_i) + \frac{V}{2} \sum_{i=1}^4 \hat{n}_i (\hat{n}_i - 1), \quad (10)$$

where  $b_i^\dagger, b_i$  are the bosonic creation and annihilation operators for a particle at site  $i$  (satisfying the commutation rules  $[b_i^\dagger, b_j] = \delta_{ij}$ ) in the Fock space,  $\hat{n}_i = b_i^\dagger b_i$  is the corresponding number operator, and we have imposed periodic boundary conditions (PBC), namely  $i + 1 = 1$  for  $i = 4$ .  $V$  denotes the on-site interaction energy,  $T$  indicates the tunneling amplitude between the neighbor sites in absence of noise, and  $q_i(t)$  is a time-dependent random parameter related to the kind of noise.

For the case of random telegraph noise (RTN),  $q_i(t) = \eta_i(t)$  where  $\eta_i(t)$  describes a single fluctuator randomly flipping between the values  $-1$  and  $1$  at rate  $\gamma_0$ . In our approach, we assume that the sources of noise affecting the tunneling among neighbor sites are independent of each other, that is  $q_0(t) \neq q_1(t) \neq q_2(t) \neq q_3(t)$ . On the other hand, for the pink and Brownian noise,  $q_i(t) = \frac{1}{N_f} \sum_{j=1}^{N_f} \eta_{ij}(t)$  is given by an averaged linear superposition of  $N_f$  bistable fluctuators  $\eta_{ij}(t)$ , each one with a proper switching rate  $\gamma_j$  taken from the range  $[\gamma_{\text{inf}}, \gamma_{\text{sup}}]$  according to the distribution

$$p(\gamma) = \begin{cases} \frac{1}{\gamma} \frac{1}{\ln(\gamma_{\text{sup}}/\gamma_{\text{inf}})} & \text{for } 1/f \text{ noise} \\ \frac{1}{\gamma^2} \frac{1}{1/\gamma_{\text{inf}} - 1/\gamma_{\text{sup}}} & \text{for } 1/f^2 \text{ noise.} \end{cases} \quad (11)$$

As shown elsewhere [61], the greater is  $N_f$ , the closer is the power spectrum of the stochastic process described by  $\sum_{j=1}^{N_f} \eta_{ij}(t)$  to the  $1/f^\alpha$ -like behavior (where  $\alpha=1,2$ ), in a frequency interval  $[f_1, f_2]$  with  $\gamma_{\text{inf}} \ll f_1, f_2 \ll \gamma_{\text{sup}}$ . Here, we average the effect of the  $N_f$  bistable fluctuators in order to keep the hopping amplitude  $T_i(t) = T \cdot q_i(t)$  in the interval  $[-|T|, |T|]$ , since - as we will see briefly - the absolute value of  $T$  determines the characteristic speed of the evolution of the system.

For both kinds of noise, given the random nature of the Hamiltonian of Eq. (10), the time evolution of the two-boson system is obtained by averaging the stochastic dynamics of the quantum state over different noise configurations. Specifically, we adopt a numerical approach able to generate, for each noise parameter  $q_i(t)$ , a given number  $M$  of histories (i.e., different temporal sequences of RTN or  $1/f^\alpha$  noise signals) which are inserted in  $\mathcal{H}_{BH}(t)$  to evaluate  $M$  unitary time evolutions of the system. Once estimated these, the two-boson density matrix  $\rho(t)$  at time  $t$  can be evaluated as:

$$\rho(t) = \frac{1}{M} \sum_{k=1}^M U_k(t) \rho(0) U_k^\dagger(t), \quad (12)$$

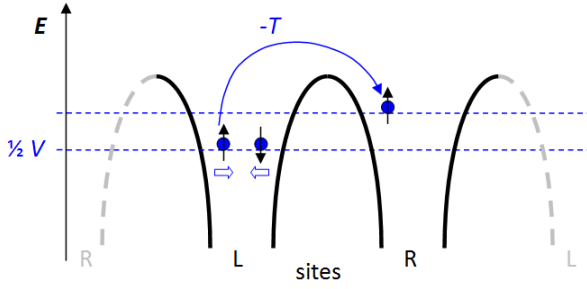
where  $U_k(t)$  is the time-evolution operator corresponding to the  $k$ -th history, and the initial state of the system  $\rho(0)$  is

$$\rho_B(0) = |\Psi_B\rangle \langle \Psi_B|, \quad (13)$$

where

$$|\Psi_B\rangle = \frac{1}{\sqrt{2}} \left( b_3^\dagger b_1^\dagger + b_4^\dagger b_2^\dagger \right) |0\rangle \quad (14)$$

and  $|0\rangle$  indicating the vacuum state containing zero particles in each mode. Then, for the bipartition of the system  $A = \{1, 2\}$  and  $B = \{3, 4\}$ ,  $|\Psi_B\rangle$  is a two-boson maximally entangled state. Here, we address the decohering effects suffered by the two-boson entanglement and discord between the two-site bipartitions  $A$  and  $B$ .



**Fig. 2** Fermionic Hubbard dimer:  $V$  is the interaction strength between electrons sharing the same site,  $-T$  is the energy gain for hopping (in figure  $T > 0$  and  $V < 0$ ).

### 3.2 Two-fermion Fermi-Hubbard model

Here, we illustrate a Hubbard dimer of two electrons with spin degrees of freedom  $\sigma = \uparrow, \downarrow$  hopping between two spatial sites  $L$  and  $R$  in the presence of noise. This simple model allows for the description of a large number of systems[112, 113] – among which there is the hydrogen molecule – since it is strictly connected to the antiferromagnetic Heisenberg model[20] in the high interaction limit.

The two-fermion dynamics is ruled by the stochastic Hamiltonian:

$$\mathcal{H}_{HD}(t) = -T \sum_{\sigma=\uparrow,\downarrow} q_{\sigma}(t)(c_{L\sigma}^{\dagger}c_{R\sigma} + c_{R\sigma}^{\dagger}c_{L\sigma}) + \frac{V}{2} \sum_{i=L,R} \hat{n}_{i\uparrow}\hat{n}_{i\downarrow}, \quad (15)$$

where  $c_{i\sigma}^{\dagger}, c_{i\sigma}$  are the creation and annihilation operators of a fermion at site  $i$  with spin  $\sigma$  satisfying the anticommutation rules  $\{c_{i\sigma}^{\dagger}, c_{i'\sigma'}^{\dagger}\} = \delta_{i,i'}\delta_{\sigma,\sigma'}$ .  $T$  still denotes the hopping amplitude between the spatial sites while the  $V$  term mimics a sort of Coulomb interaction between electrons on the same site. The RTN, pink, and brown noises are reproduced by means of the random term  $q_{\sigma}(t)$  which is assumed to be dependent upon spin. Unlike the Bose-Hubbard model, the electron hopping among spatial sites is only affected by two different sources of noise, and spin-flip transitions are prohibited.

The numerical procedure described in the Sec. 3.1 is again adopted to evaluate the time-evolution of the two-fermion state

$$\rho_F(0) = |\Psi_F\rangle \langle\Psi_F|, \quad (16)$$

where

$$|\Psi_F\rangle = \frac{1}{2} \left( c_{L\uparrow}^{\dagger}c_{L\downarrow}^{\dagger} + c_{R\uparrow}^{\dagger}c_{R\downarrow}^{\dagger} \right) |0\rangle. \quad (17)$$

This is a maximally-entangled state when the bipartition of the system  $A = \{L \uparrow, R \uparrow\}$  and  $B = \{L \downarrow, R \downarrow\}$  is considered. Indeed, our aim is to evaluate how the noise affects the quantum correlations between the spins up and down of the two electrons.

### 3.3 General remarks on the Hubbard Hamiltonian

With a simple factorization, the Bose-Hubbard Hamiltonian of Eq. (10) can be rewritten as:

$$\mathcal{H}_{BH}(t) = T \left[ - \sum_{i=1}^4 q_i(t)(b_i^{\dagger}b_{i+1} + b_{i+1}^{\dagger}b_i) + \frac{v}{2} \sum_{i=0}^3 \hat{n}_i(\hat{n}_i - 1) \right], \quad (18)$$

where  $v = V/T$  is the relative strength of on-site interaction with respect to the kinetic term. As we can see, the physical meaning of  $T$  can be reduced to a time-scale factor, i.e. the higher is  $T$ , the faster is the dynamics of the system (so the evolution of the correlations has to be evaluated against the adimensional time  $\tau = |T| \cdot t$ ). Therefore, the only parameter which is able to change significantly the dynamics of the system is  $v$ , and analogous conclusions can be drawn for the Fermi-Hubbard Hamiltonian. One could expect that the sign

of  $V$ , that distinguishes between attractive ( $V < 0$ ) and repulsive ( $V > 0$ ) interactions among particles, should alter dramatically the dynamics of the system. This, however, seems in contrast with the literature, where it is shown that the evolution of the system is invariant with respect to the sign of  $V$  [40]. Indeed, after a closer look, it turns out that the dynamics of quantum correlations (QC) is independent from the sign of  $V$  only when the system is in a state of the Fock space with exact occupation numbers for each site (the case studied in Ref. [40]), while for linear combinations of states the dynamics can be affected by sign of interactions in a non negligible way, as we will show briefly.

This point will be rigorously detailed in a forthcoming work.

#### 4 Numerical simulations: Bose-Hubbard model

In this Section we report the results of our numerical simulations. Our algorithm generates the time-dependent Hamiltonian  $\mathcal{H}(t)$  at time steps of  $\delta t$ , updating the values of the hopping amplitudes according to the random noise (if present), then it calculates the evolution  $\rho(t) = U(t)\rho(0)U^\dagger(t)$  of the initial state  $\rho(0)$  through the time-ordered evolution operator

$$U(t) = \mathcal{T} \left\{ \exp \left[ -i \int_0^t \mathcal{H}(\tau) d\tau \right] \right\} \quad (19)$$

at desired time  $t = n \cdot \delta t$ . The operator  $U(t)$  is discretized in time and written as[114]

$$\mathcal{T} \left\{ \exp \left[ -i \sum_{j=0}^n \mathcal{H}(j \cdot \delta t) \delta t \right] \right\} \simeq \prod_{j=0}^n \exp [-i\mathcal{H}(j \cdot \delta t) \delta t] \quad (20)$$

resorting to the Trotter factorization in the limit of  $\delta t \rightarrow 0$ , and each term  $\exp [-i\mathcal{H}(j \cdot \delta t) \delta t]$  is obtained through the diagonalization of  $\mathcal{H}(j \cdot \delta t)$ . The evolution is calculated in the joint Hilbert space of the two particles (symmetrization or anti-symmetrization of the wavefunction can be performed equally at the beginning of the evolution or at each time-step). The values of  $\rho(t)$  are then averaged over a number of histories which is large enough to ensure convergence, and the average density matrix  $\langle \rho(t) \rangle$  is used to evaluate entanglement  $E_P$  and discord  $D_P$  of particles at time  $t$ , as explained in Sect. 2. The numerical optimization of quantum discord  $\mathcal{D}(\rho_{1,1})$  is performed with the *exhaustive enumeration* algorithm (aka brute-force search), in order to reach a global solution.

##### 4.1 Noiseless system

The numerical simulations for the noiseless system give the following results. We start with the maximally entangled state  $|\Psi_B\rangle$  of Eq.(14), and the evolution of the correlations are practically identical for Entanglement  $E_P$  (Fig. 3 a) and Quantum Discord  $D_P$  (Fig. 3 b), thus showing that our quantifier is qualitatively able to capture the quantum correlations of the noiseless system.

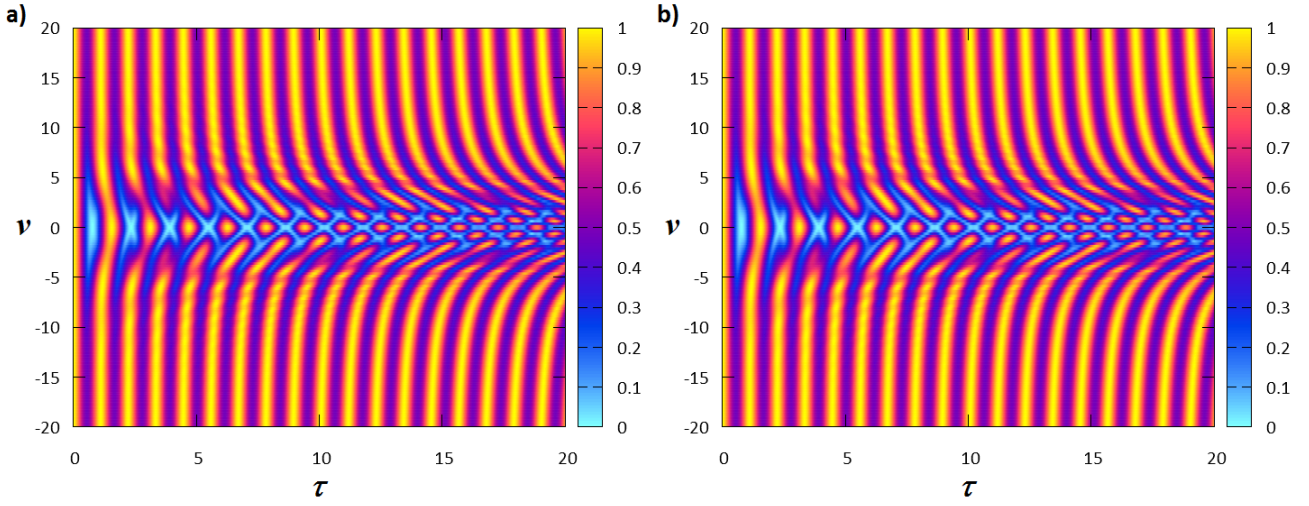
Both entanglement and discord show a periodic behavior in time (sometimes with beatings - see Fig. 4). This is perfectly reasonable since the Hilbert space of the system is finite and the noiseless Hamiltonian is constant in time, therefore the dynamics can repeat itself after a certain period of time (provided that the involved eigenvalues have rational quotients). For longer chains, where the effect of the PBC shows up at later times, the period of the dynamics would be obviously larger.

For what concerns the effects of the potential energy  $V$ , we see that at larger values of  $v$  the oscillations of the quantum correlations are smaller in amplitude and characterized by a higher average value (see Fig. 4), thus showing a sort of “protective” effect of the interactions with respect to both entanglement and discord. As we can see from both Fig. 3 a) and b), QC are independent from the sign of  $V$ , as already observed for both bosonic[40] and fermionic systems[115]. Interestingly, however, this effects is not universal but it depends on the chosen initial state. Indeed, for a different initial state, such as  $|\Xi_B\rangle = \frac{1}{\sqrt{2}}(b_4^\dagger b_1^\dagger + b_4^\dagger b_2^\dagger)|0\rangle$ , the correlations show an asymmetry<sup>2</sup> with respect to  $V = 0$  (see Fig. 6).

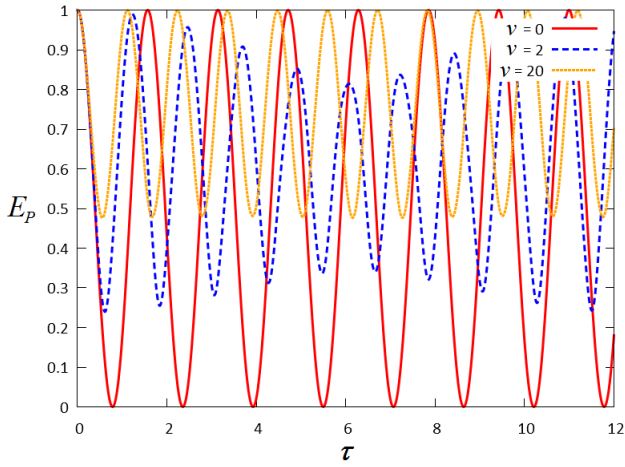
In this second case, even if the initial entanglement is zero (since the state can be factorized), correlations do appear in any case, and not only because of the effect of interactions (which can entangle the particles), but also because of periodic boundary conditions: indeed, even when  $v = 0$  (see Fig. 7), the interference effects can build up nonclassical correlations between particles, as already observed in the literature [34, 55, 57]. Also, no periodicity is apparent for the observed cases, except for  $V = 0$ , but we observe again a larger average value of QC for higher values of  $v$ .

<sup>2</sup> The same asymmetry is observed for the diagonal elements of  $\rho'_B(t) = |\Xi_B\rangle\langle\Xi_B|$ , while it is absent in  $\rho_B(t) = |\Psi_B\rangle\langle\Psi_B|$ , but we do not report this result here for reasons of space: however, we mention this phenomenon since it is reproduced also for larger chains (e.g. with 30 sites) and for couples of particles initialized far from the edges of the chain, thus showing that this effect is genuine and not due to PBC.

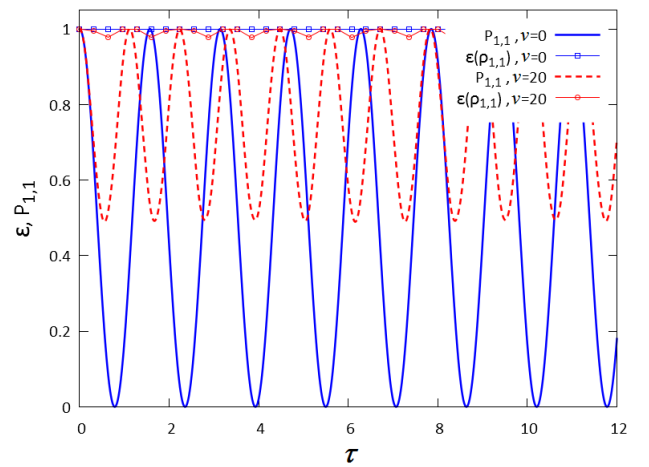




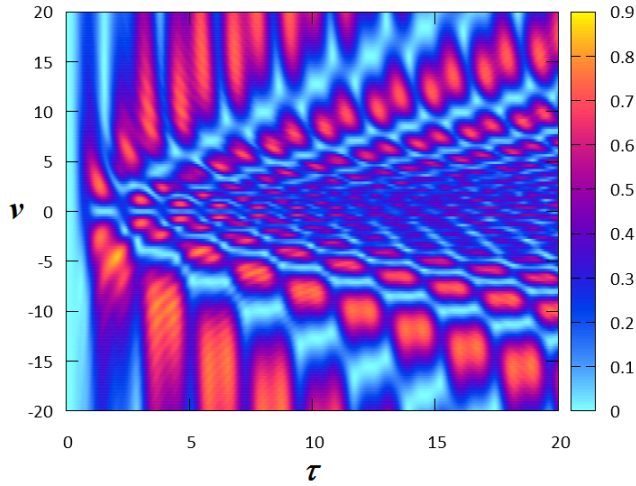
**Fig. 3** Entanglement of particles and Discord of particles for the initial state  $|\Psi_B\rangle$ . **a)** Entanglement  $E_P$  **b)** Discord  $D_P$ .



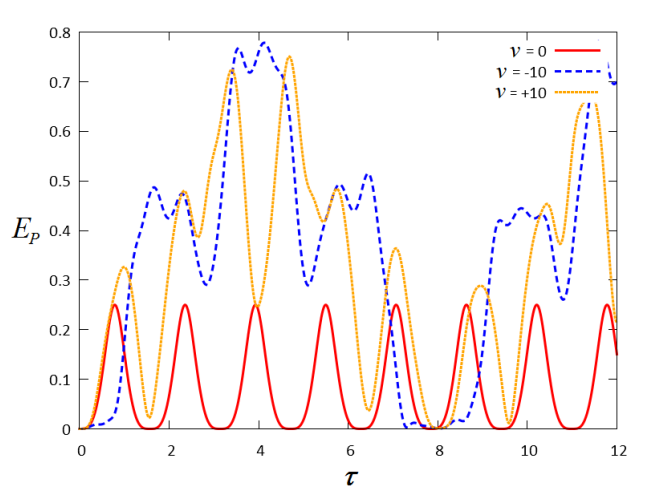
**Fig. 4** Entanglement of particles for the initial state  $|\Psi_B\rangle$  at different relative strength  $v$  of interactions.



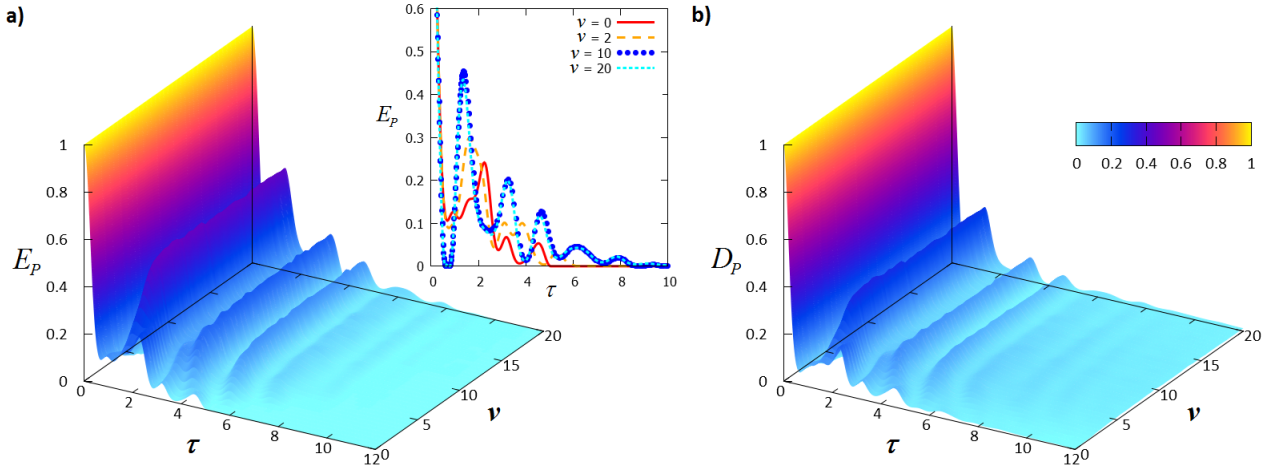
**Fig. 5** Entanglement of formation  $\mathcal{E}(\rho_{1,1})$  and probability  $P_{1,1}$  of finding the particles in different partitions for the initial state  $|\Psi_B\rangle$  at different relative strength  $v$  of interactions.



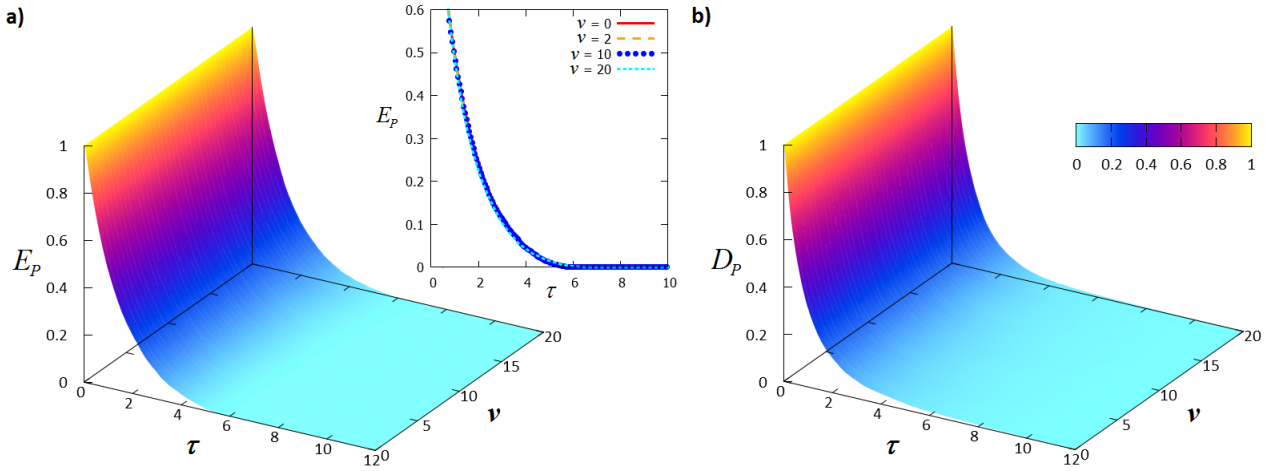
**Fig. 6** Entanglement of particles  $E_P$  for the initial state  $|\Xi_B\rangle$  (see text for definition). The results for Discord of particles  $D_P$  are not reported since they are identical.



**Fig. 7** Entanglement of particles for the initial state  $|\Xi_B\rangle$  (see text for definition) at different relative strength  $v$  of interactions.



**Fig. 8** Entanglement  $E_P$  and Discord of particles  $D_P$  with a single RTN fluctuator in the strong coupling regime ( $\gamma_0\tau_s = 0.1$ ). **Inset:**  $E_P$  for different values of the interaction strength  $v$ .



**Fig. 9** Entanglement  $E_P$  and Discord of particles  $D_P$  with a single RTN fluctuator in the weak coupling regime ( $\gamma_0\tau_s = 10$ ). **Inset:**  $E_P$  for different values of the interaction strength  $v$ .

However, since we are mainly interested in the evolution of a maximally entangled state, from now on we will focus only on the effects of the noise on the correlations of the state  $|\Psi_B\rangle$ . As we will see, the QC for  $|\Psi_B\rangle$  are always symmetrical with respect to  $v$  (but this result should be considered with care since - as we just saw - this is not a general property for the system).

Looking back at Eq. (1), we recall that entanglement of particles  $E_P$  is the product of two contributions: the first is the probability of finding one particle in each partition, namely  $P_{1,1}$ , and the second is the entanglement of modes for the standard quantum registers, namely  $\mathcal{E}(\rho_{1,1})$ . The role of these contributions is compared in Fig. 5. As we can see, the entanglement of modes  $\mathcal{E}$  has very low dependence on the relative strength  $v$  of the interactions. On the contrary, the probability  $P_{1,1}$  depends strongly on  $v$ : for non-interacting particles it oscillates between 0 and 1, while for strongly interacting particles ( $v = 20$ ) it is bounded between 0.5 and 1, thus producing the strong increase in the average value of  $E_P$  that we observe in Fig. 4 for high values of  $v$ . This means that the role of strong interactions is that of keeping preferentially both particles in their original partition, or at least discouraging two particles from occupying the same partition of the system. From a physical point of view, this can be connected to the band-structure of the Hubbard Hamiltonian[40]. Indeed, as observed also in Mott-Hubbard transitions[116], the increase in  $v$  separates the bandstructure into two subbands: a main one, for states in which the particles are on first-neighbor or second-neighbor sites (e.g.  $|1, 2\rangle$  and  $|1, 3\rangle$ ), and a miniband, for states where the particles share the same site (e.g.  $|1, 1\rangle$ ). Given our initial state  $|\Psi_B\rangle$ , the separation in energy of the subbands lowers the probability of certain transitions for the particles, namely those in which they occupy the same site (and therefore they are in the same partition), thus increasing the relative probability of occupying different partitions. Obviously, the same discussion holds for the discord of particles  $D_P$ .

## 4.2 Single RTN fluctuator

When the hopping amplitudes of the particles are perturbed each one by a single RTN fluctuator with switching rate  $\gamma_0$ , two regimes of behavior can be identified for the system: the strong coupling regime  $\gamma_0\tau_s < 1$ , in which the switching dynamics of the fluctuators are slower than the characteristic time  $\tau_s = |T|^{-1}$  of the system (thus allowing a back-action of the environment over the system), and the weak coupling regime  $\gamma_0\tau_s > 1$ , where the previous condition is reversed, and the effect of the environment is perceived by the system as an average over many noise cycles.

All the simulations for the initial state  $|\Psi_B\rangle$  show that the results are independent from the sign of  $V$ , therefore we report the values of QC only for  $v > 0$ . For what concerns the strong coupling regime, as we see from Fig. 8, the qualitative behavior for entanglement and discord is the same, and the system shows a sort of memory effect, with sudden deaths and revivals of correlations before reaching a complete decoherence condition. However, discord seems to be more sensitive to the noisy environment, since it assumes typically lower values with respect to entanglement after the switching on of the noise (except for long times, when a crossing of the two curves can occur before they both go to zero). This result is not surprising [13], since it is known that the two quantifiers do capture different kinds of quantum correlations, and therefore a direct quantitative comparison between them is not possible. Again, higher values of  $v$  have a protective effects on correlations, that is they disappear at longer times. However, for  $v \geq 10$ , a sort of limit behavior is reached, and higher values of  $v$  do not produce any modification in QC (see inset of Fig. 8).

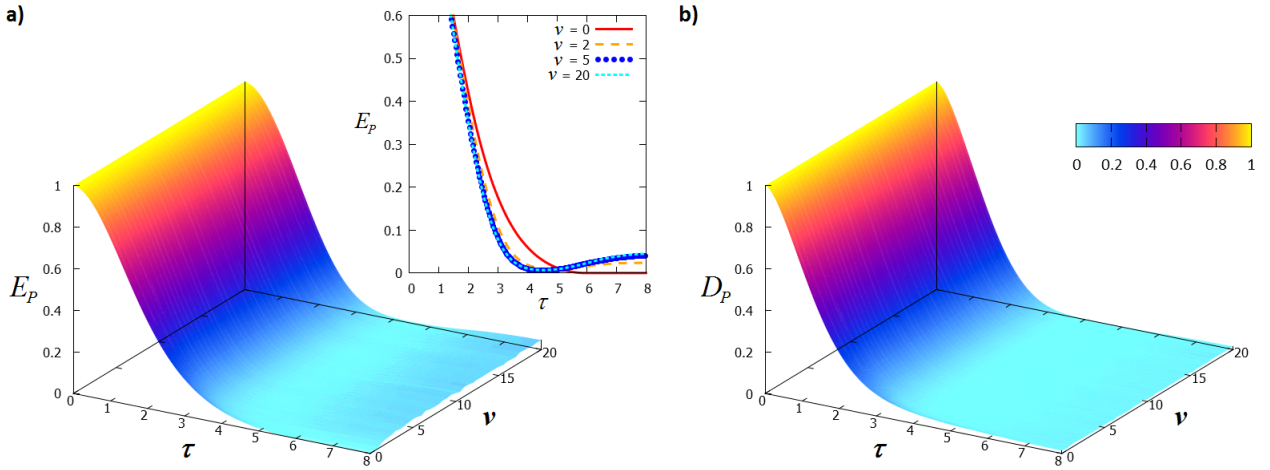
A different behavior of QC is observed in the weak coupling regime (see Fig. 9), where discord is still lower than entanglement (except for long times, where a crossing can again occur), but the variation of  $v$  does not produce any change in the time evolution of discord or entanglement (see inset in Fig. 9). Under this coupling regime, the system does not show memory effects, and the decay of QC is monotone, but not necessarily faster than that of the strong coupling regime. These behaviors are very similar to those observed in a couple of qubits subject to RTN in the Markovian and non-Markovian regime [90], except for the decay times of correlations, which are apparently faster for qubits in the Markovian regime.

## 4.3 Colored noises

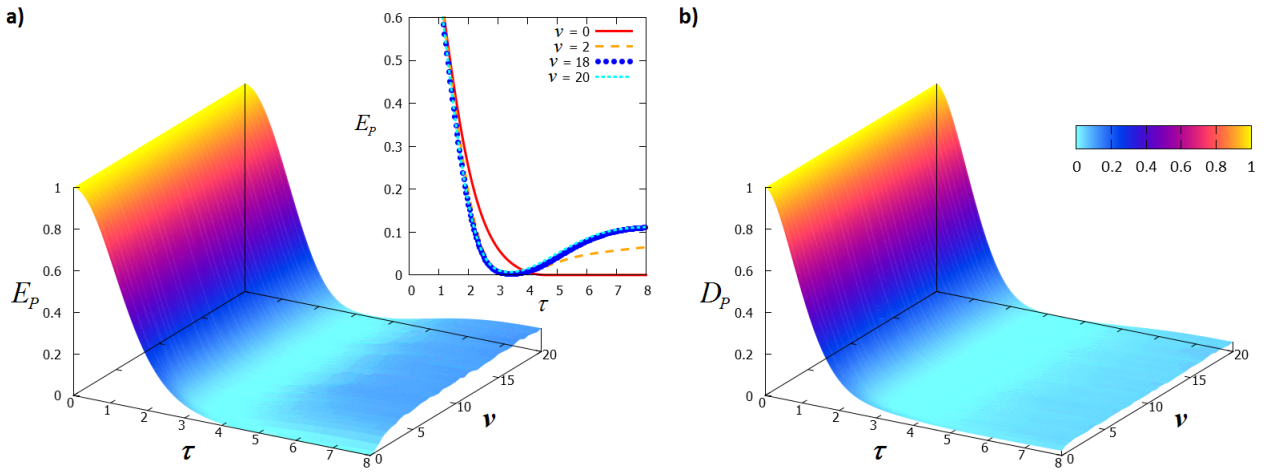
If the hopping amplitudes are perturbed by a collection of  $N_f$  bistable fluctuators (each of them mimicking a decohering channel), whose switching frequencies  $\gamma = f$  are distributed with the power law  $f^{-\alpha}$ , then the system is subject to pink ( $\alpha = 1$ ) or brown ( $\alpha = 2$ ) colored noise. To reproduce these kinds of noise, we generated randomly the rates  $\gamma$  of our fluctuators in the interval of frequencies  $\gamma \in |T| \cdot [1.25 \cdot 10^{-4}, 1.25 \cdot 10^2]$ , using the appropriate power law for each noise. Again, since the simulations for the initial state  $|\Psi_B\rangle$  show that the results are independent from the sign of  $V$ , we report the evolution of QC only for  $v > 0$ . We notice an interesting dependence of both entanglement and discord from the absolute values of the interaction  $v$ , both for  $1/f$  (Figs. 10 a and b) and  $1/f^2$  noises (Figs. 11 a and b). In detail, while QC for  $v = 0$  show their first decay at long times, with  $v > 0$  they decay faster but show a marked revival (see the insets of Figs. 10 and 11). These phenomena are present in both pink and brown noise scenario, but they are more marked for brown noise (Fig. 12), where QC at long times reach an almost constant value (see Fig. 12). This revival is stronger for higher values of  $v$ , when an asymptotic behavior is reached – e.g., in the case of  $E_P$  the saturation is given by  $v \gtrsim 5$  for  $1/f$  noise,  $v \gtrsim 18$  for  $1/f^2$  noise (see insets of Figs. 10 and 11). Again, interactions show to have a protective effect on QC. However, we also notice that quantum discord  $D_P$ , although it decays at first generally faster than  $E_P$  and has weaker revivals, at  $v = 0$  manages to overcome entanglement: in the pink noise scenario it happens before they both go to zero, while in the brown noise scenario discord never fully vanishes (see Fig. 12), while entanglement goes rapidly to zero (and shows a negligible revival). We comment further on this behavior of entanglement and discord in Appendix B.

The decohering effects strongly depends upon the number  $N_f$  of fluctuators, and if the sum  $q(t)$  of the fluctuators is not renormalized, the decohering effect is stronger for a higher number  $N_f$  of open noise channels (also because this produces a larger average value of  $T(t)$ , and therefore a faster dynamics for the system), as it is shown in [61]. However, since in our simulations we renormalize the noise  $q(t)$  to the number of fluctuators, the effect we observe here is reversed: in Fig. 13 we see that for higher values of  $N_f$  a longer time is needed to destroy the QC. This is reasonable since the fluctuators are generated randomly from the distribution  $1/f^\alpha$ , and therefore for higher values of  $N_f$  there is a larger number of fluctuators with low switching rates, and this situation keeps almost constant the value of  $q(t)$  for longer times, thus inducing a weaker decohering effect on the system.

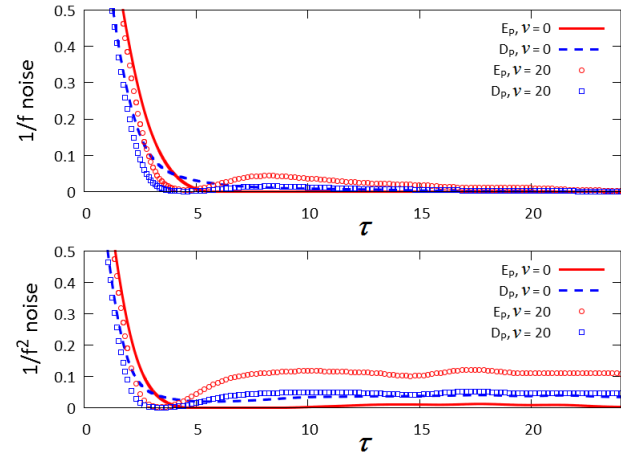
We notice then that QC are destroyed a bit faster by brown noise, and this is due to the fact that the distribution  $1/f^2$  produces a higher number of low switching-rate fluctuators with respect to the  $1/f$  distribution, therefore the system is more subject to back-action from the environment, as we observed for RTN in the strong coupling regime (see the previous section 4.2). This is the reason for which the first decay of QC is faster for



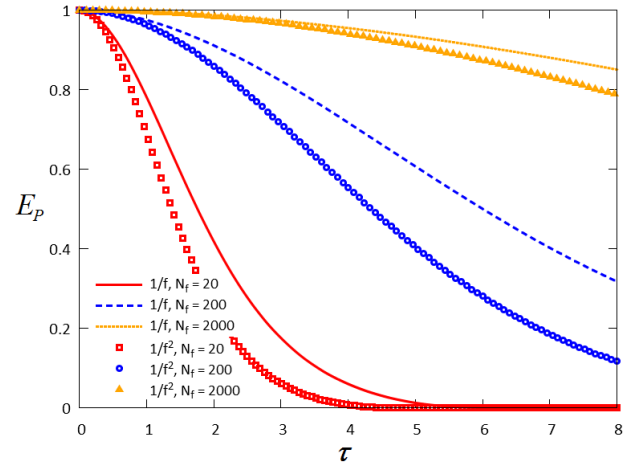
**Fig. 10** Entanglement  $E_P$  and Discord of particles  $D_P$  for  $1/f$  (pink) noise, with  $N_f = 20$  fluctuators. **Inset:** Evolution of  $E_P$  for different strengths  $\nu$  of the interactions.



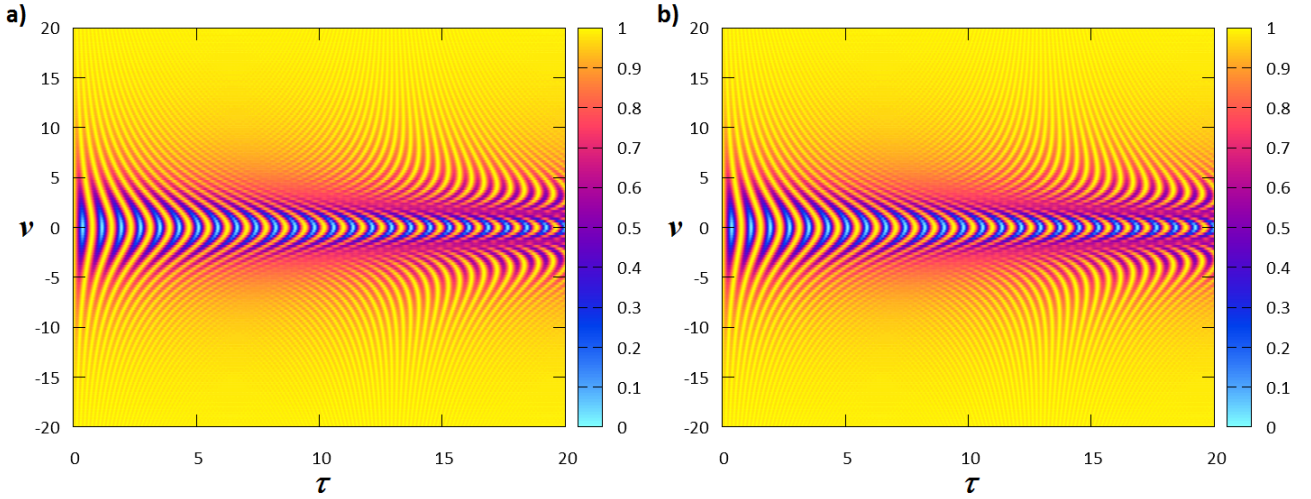
**Fig. 11** Entanglement  $E_P$  and Discord of particles  $D_P$  for  $1/f^2$  (brown) noise, with  $N_f = 20$  fluctuators. **Inset:** Evolution of  $E_P$  for different strengths  $\nu$  of the interactions.



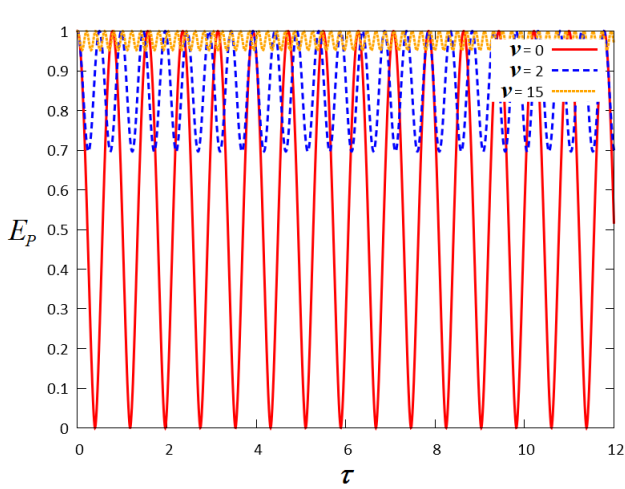
**Fig. 12** Entanglement  $E_P$  (red) and Discord  $D_P$  (blue) for  $1/f$  (top panel) and  $1/f^2$  (bottom panel) noise, with  $N_f = 20$  fluctuators, at different values of the interaction strength  $\nu$  and for long time scales.



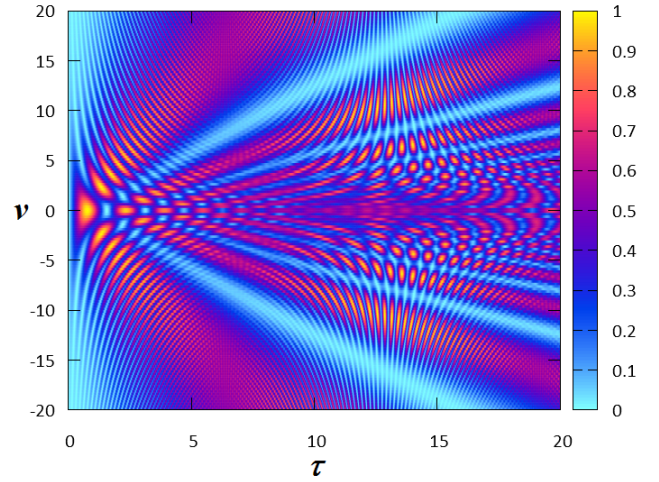
**Fig. 13** Entanglement of particles for  $1/f$  (lines) and  $1/f^2$  (dots) noise at zero interaction strength, for different values in the number of fluctuators  $N_f$ .



**Fig. 14** Entanglement of particles and Discord of particles for the initial state  $|\Psi_F\rangle$ . a) Entanglement  $E_P$  b) Discord  $D_P$ .



**Fig. 15** Entanglement of particles for the initial state  $|\Psi_F\rangle$  at different relative strength  $v$  of interactions.



**Fig. 16** Entanglement of particles  $E_P$  for the initial state  $|\Xi_F\rangle$  (see text for definition). The results for Discord of particles  $D_P$  are not reported since they are identical.

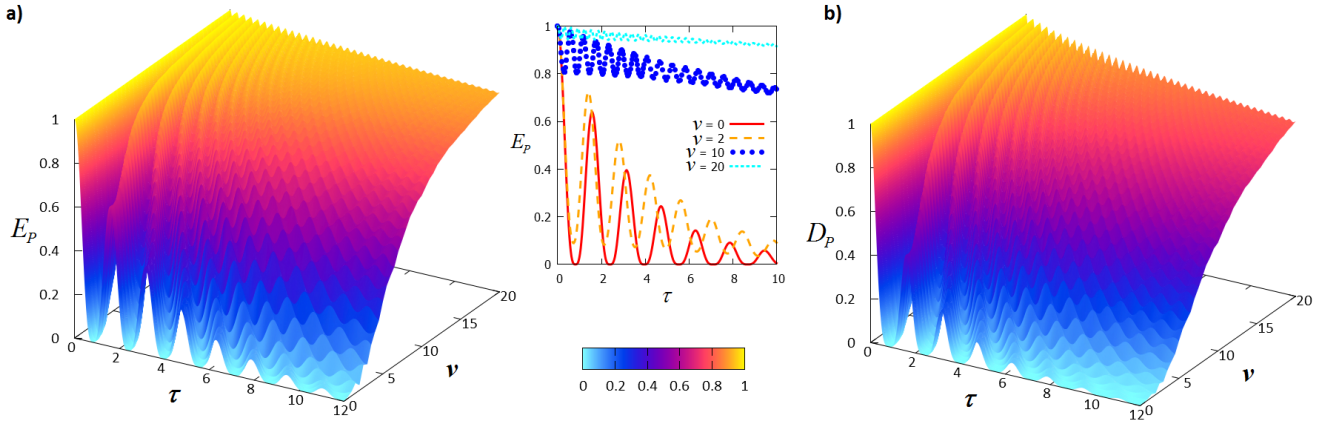
brown noise (see for comparison the insets of Figs. 8 and 9), but then we observe also a marked revival, which is absent in the pink noise case. However, the difference in the decay times is not so pronounced, due to the fact that in both noise distributions there is also the effect of the fast-switching fluctuators (which was absent in the RTN scenario for the strong coupling regime), therefore the evolution of QC shows a hybrid character between strong and weak coupling. With respect to the 2-qubit case, described in Ref. [61], the effects of pink noise at  $v = 0$  are very similar, while for brown noise the dynamics of QC at  $v = 0$  are quite different, since in our system  $1/f^2$  noise does not give rise to entanglement sudden deaths and sudden births, as it happens instead for qubits. Here indeed we observe the saturation of correlations at long times, for the brown noise scenario (see Fig. 12).

## 5 Numerical simulations: Fermi-Hubbard model

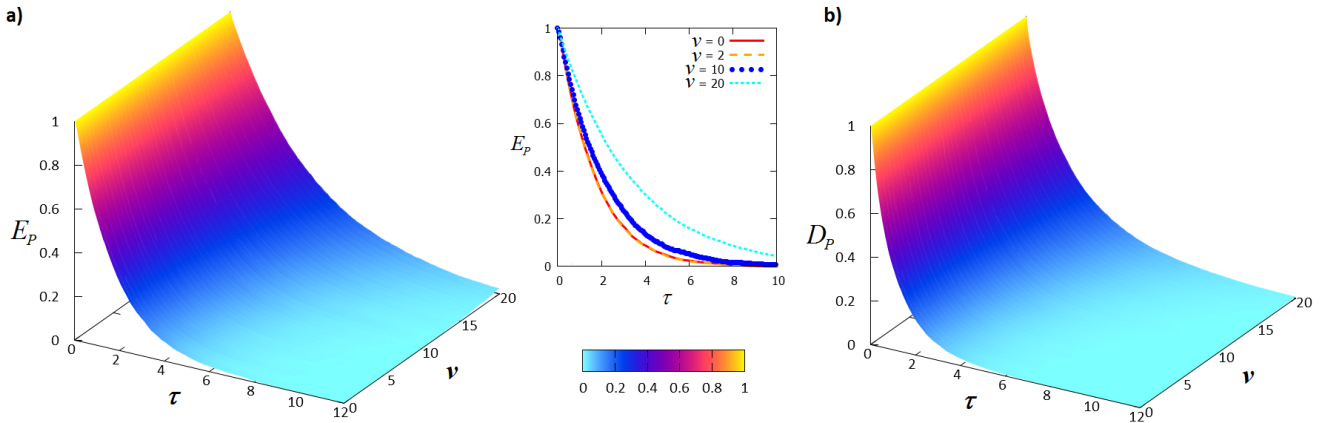
### 5.1 Noiseless system

Here we describe the results of the numerical simulations for the noiseless Fermi-dimer system. We start with the maximally entangled state  $|\Psi_F\rangle$  of Eq. (17), and the evolution of the correlations are practically identical for Entanglement  $E_P$  (Fig. 14 a) and Quantum Discord  $D_P$  (Fig. 14 b), as we saw for the bosonic case. Also in this case the behavior of the system is perfectly symmetrical for positive and negative  $v$ .

To compare the situation with the boson case, we see that for  $|v| > 5$  the entanglement approaches a sort of limit behavior ( $|v| \gtrsim 15$ ), with small oscillations around its maximum value (see Fig. 15). Therefore we deduce that the effect of interactions in preserving the correlations is stronger for the fermionic system.



**Fig. 17** Entanglement  $E_P$  (a) and Discord of particles  $D_P$  (b) for  $|\Psi_F\rangle$  with a single RTN fluctuator in the strong coupling regime ( $\gamma_0\tau_s = 0.1$ ). **Inset:**  $E_P$  for different values of the interaction strength  $v$ .



**Fig. 18** Entanglement  $E_P$  (a) and Discord of particles  $D_P$  (b) for  $|\Psi_F\rangle$  with a single RTN fluctuator in the weak coupling regime ( $\gamma_0\tau_s = 10$ ). **Inset:**  $E_P$  for different values of the interaction strength  $v$ .

Due to the formal analogies between Eq. (10) and (15), we wanted to check whether the Hamiltonian of Eq. (15) allows to create a fermionic state with behavior analogous to  $|\Xi_B\rangle$ . Therefore, we initialized the system in the state  $|\Xi_F\rangle = \frac{1}{\sqrt{2}}(c_{L\uparrow}^\dagger + c_{R\uparrow}^\dagger)c_{R\downarrow}^\dagger|0\rangle$ , which is the only possibility left<sup>3</sup> for having one particle in each partition of the system (notice that, during the evolution, the particles cannot change their spin). Even in this case, however, the evolution is symmetrical with respect to the sign of  $v$  (see Fig. 16): this is a consequence of anti-symmetrization and/or inhibition of spin-flip, whose effect is to suppress many of the allowed transitions for the bosonic system<sup>4</sup>.

It is worth noting that the suppression of spin-flip transitions implies that at any time each subsystem contains exactly one particle ( $P_{1,1} = 1$ ), and therefore  $E_P = \mathcal{E}(\rho_{1,1})$ ,  $D_P = \mathcal{D}(\rho_{1,1})$ . This means that in general QC are stronger in the Fermi dimer than in the bosonic system.

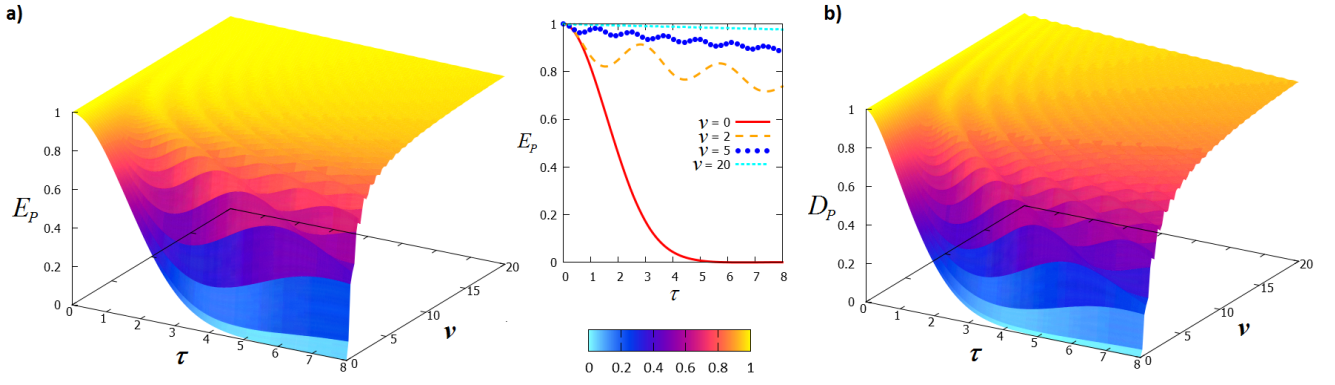
## 5.2 Single RTN fluctuator

Next, we study the effect of a single RTN fluctuator (with rate  $\gamma_0$ ) on the Fermi dimer, both in the strong ( $\gamma_0\tau_s < 1$ ) and weak ( $\gamma_0\tau_s > 1$ ) coupling regime. Again the behavior is perfectly symmetrical, so we show only the evolution of QC for  $v > 0$ .

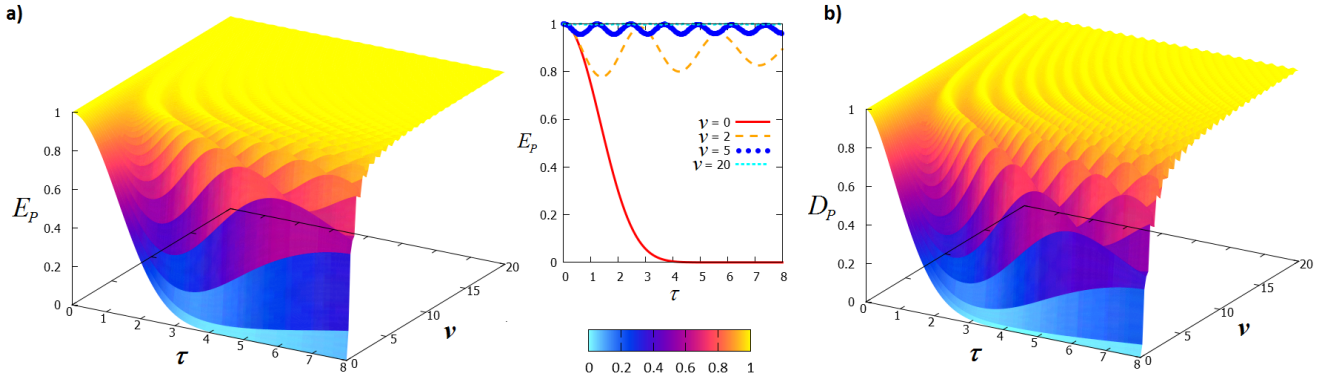
As can be seen from Figure 17, in the strong-coupling regime both entanglement and discord are subject to a series of oscillations. When  $v = 0$ , we clearly observe sudden deaths and revivals, but when  $v > 0$  the protective effect of  $V$  prevents QC from going to zero for a longer time, and at higher  $v$  the effect is stronger, so that decoherence is significantly reduced. As in the bosonic case, the decay of discord is faster than that of entanglement.

<sup>3</sup> Except for symmetrical states that are equivalent to  $|\Xi_F\rangle$  – within a relabeling of spins and/or sites – and a relative phase between the two terms. Notice that  $|\mathcal{T}_F\rangle = \frac{1}{\sqrt{2}}(c_{L\uparrow}^\dagger c_{R\downarrow}^\dagger + c_{R\uparrow}^\dagger c_{L\downarrow}^\dagger)|0\rangle$  is equivalent to  $|\Psi_F\rangle$  in terms of entanglement.

<sup>4</sup> Notice also that the *Pauli exclusion principle* here is given automatically by spin-flip suppression, therefore states like  $|j, j\rangle$  would have remained unoccupied even if the particles initialized in the state  $|\Psi_F\rangle$  were bosons.



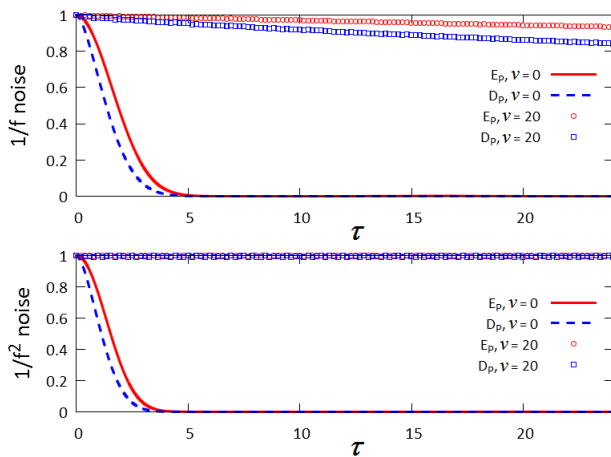
**Fig. 19** Entanglement  $E_P$  (a) and Discord of particles  $D_P$  (b) for  $1/f$  (pink) noise, with  $N_f = 20$  fluctuators. **Inset:** Evolution of  $E_P$  for different strengths  $v$  of the interactions.



**Fig. 20** Entanglement  $E_P$  (a) and Discord of particles  $D_P$  (b) for  $1/f^2$  (brown) noise, with  $N_f = 20$  fluctuators. **Inset:** Evolution of  $E_P$  for different strengths  $v$  of the interactions.

For what concerns the strong-coupling regime – that is represented in 18 – the oscillatory behavior is absent: QC show a monotonic decay, which is again faster for quantum discord, and the protective effect of interactions is still present but almost negligible with respect to the previous case (see Figure 18, Inset). We conclude that the protective action of interactions over QC is effective only for low frequency noises and, in this condition, it is much more effective than for bosons.

Both regimes are very similar to those observed in the literature for quantum walks of classical (distinguishable) particles, subject to a dynamical noise respectively in non-Markovian and Markovian regime.



**Fig. 21** Entanglement  $E_P$  (red) and Discord  $D_P$  (blue) for  $1/f$  (top panel) and  $1/f^2$  (bottom panel) noise, with  $N_f = 20$  fluctuators, at different values of the interaction strength  $v$  and for long time scales.

### 5.3 Colored noises

Moving to the colored noise scenario, we introduce again a perturbation in the hopping amplitudes of electrons due to a collection of  $N_f$  bistable fluctuators, whose switching frequencies  $\gamma$  are distributed with the power law of pink or brown noise and are randomly generated in the interval  $\gamma \in |T| \cdot [1.25 \cdot 10^{-4}, 1.25 \cdot 10^2]$ .

We observe again a situation which is quite different with respect to the bosonic case, since the effect of interactions over QC is very strong in this scenario, both for pink and brown noise. In both cases, the decay of correlations is fast for  $v = 0$  and very slow for  $v > 0$  (see Figs. 19, 20 and 21), practically negligible at high  $v$  in the brown noise scenario (see Fig. 21). Again, we observe that discord has the same qualitative behavior of entanglement but a faster decay. Moreover, as for bosons, the decay of QC at  $v = 0$  is slightly faster in the brown noise scenario, but the differences between the two noisy environments are very small. Both for pink and brown noise, the decay of QC for  $v > 0$  is oscillatory (see Insets of Figs. 19 and 20) – a difference with respect to the bosonic case – but the amplitudes get smaller and the oscillations are faster at higher  $v$ .

For what concerns the number of fluctuators  $N_f$ , we observe the same effects seen for bosons, namely a slower decay of QC for a higher number of fluctuators, which is again due to the renormalization of noise.

## 6 Conclusions

In this paper we propose an expression for the *quantum discord*  $D_P$  of a couple of indistinguishable particles through the generalization of the notion of *entanglement of (indistinguishable) particles*  $E_P$  introduced by Wiseman and Vaccaro [26]. We used this quantifier of quantum correlations for studying two 1D lattice model systems, both described by the Hubbard Hamiltonian, in which a couple of interacting bosons or fermions tunnels between the sites of a chain subject to classical environmental noise (RTN, pink  $1/f$ , brown  $1/f^2$ ). We confronted the results for  $D_P$  with the corresponding values of  $E_P$ , showing that in all the studied cases there are no marked qualitative differences between the two quantifiers: the peculiarity of this behavior (which is expected for pure and not for mixed states) in our opinion is due to the dynamics of the noise (since it was observed also for qubits subject to classical environments [61, 90]), but at the same time it is a strong evidence of the reliability of the proposed measure  $D_P$  as a quantifier for quantum correlations of indistinguishable particles. In detail, for pure states  $D_P$  always coincides with entanglement of formation, while for noisy systems  $D_P$  is generally lower than  $E_P$ : a situation which is not common but it has already been observed in the literature, both in bipartite and multipartite systems of qubits [13, 117] and in particular under the effect of classical noise [61, 90]. On the other hand, in one of the studied cases we observed that the entanglement goes to zero while the quantum discord does not vanish, as it is usually observed in the literature. We expect that a different behavior of  $E_P$  and  $D_P$  will be more evident when the model is changed (adding e.g. external fields or more complex interactions) or when the proposed quantifier will be used to describe other kinds of systems composed by identical particles: both cases, however, are beyond the scope of the present work, but can be considered as promising developments of this research.

As far as the dynamic of the systems is concerned, we observed – both in the bosonic Hubbard plaquette and in the fermionic Hubbard dimer – that in general the effect of the on-site interactions  $V$  over quantum correlations does not depend on its sign: a symmetry already described in the literature [40, 115] and attributed to the invariance of the bandstructure with respect to  $\text{sgn}(V)$ . However, for the bosonic system, an exception was found in the state  $|\Xi_B\rangle = \frac{1}{\sqrt{2}}(b_4^\dagger b_1^\dagger + b_4^\dagger b_2^\dagger)|0\rangle$ , where the switching from repulsive to attractive interactions modified deeply the evolution of both entanglement and discord. This phenomenon requires a further analysis, since it has never been observed before, and it can offer new interesting possibilities for the controlled manipulation of quantum correlations.

We also notice that the interactions  $V$  are able to inhibit the decoherence of the system by preserving QC (i.e., both entanglement and discord): the stronger is  $V$ , the higher is the degree of protection, and this effect is much more marked in the fermionic system. To a further level of detail, we notice that the protective action of the on-site interaction  $V$  over correlations is much more effective for low-frequency noises than for high-frequency ones. This effect is clearly visible for RNT noise in the weak and strong coupling regimes, but also partially for brown and pink noises, since the former scenario is dominated by low-frequency fluctuators much more than the latter, and therefore it shows a higher “effective back-action” of the environment over the system. This is the reason why we observe a higher resistance to decoherence (fermions) and a revival of quantum correlations (bosons) for the systems subject to brown noise, while these phenomena are weaker or absent in the pink noise scenario.

The main difference between the bosonic and the fermionic system is that the latter is able to preserve QC in a more efficient way than the former, even with a lower value of the relative interaction strength  $v$ . On the other hand, these discrepancies can be attributed to some basic distinctive elements, such as the differences in geometry of the two models (two vs. four sites) and the lower number of allowed transitions in the fermionic system, due both to spin-flip suppression and anti-symmetrization.



These results evidence many connections with those obtained in systems of quantum bits or classical particles subject to RTN or  $1/f^\alpha$  noises[61, 66, 90], thus showing that some previous results concerning model system of qubits can act as a guideline to interpret experimental observations on quantum random walk, also suggesting new possible directions of investigation.

Further perspective of research on quantum correlations could be the extension of the interaction to particles occupying neighboring sites[118] and possibly the introduction of a constant external field: two aspects which could be used to engineer or preserve quantum correlations at a deeper and more efficient level. Also the effects of static noise (resulting in lattice disorder) could be explored in the case of identical particles – due to their role in phenomena like Anderson localization or the transition from quantum to classical random walks[119–122]– in order to explore differences, if present, with the distinguishable particle case[66].

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## A Properties of Quantum Discord

Quantum discord  $\mathcal{D}_B(\rho_{AB})$  gives the amount of correlations between the subsystem  $A$  and  $B$  that are destroyed by a measurement on  $B$ , which can be interpreted as a measure of quantum correlations between the two subsystems. The properties required for a good quantifier of *quantum discord* are the following ones[10, 12]:

- (i). discord is non-negative;
- (ii). discord is not symmetric, i.e. in general  $\mathcal{D}_B(\rho_{AB}) = S(\rho_{1,1}^B) + S(\rho_{1,1}^A | \{\Pi_k^B\}) - S(\rho_{1,1}) \neq \mathcal{D}_A(\rho_{AB})$  depends upon the subsystem on which the measurements  $\{\Pi_k\}$  are performed;
- (iii). discord is invariant under local-unitary transformations, i.e.  $\mathcal{D}_S(\rho_{AB}) = \mathcal{D}_S((U_B^\dagger \otimes U_A^\dagger)\rho_{AB}(U_A \otimes U_B))$  for any couple of unitary transformations  $U_A, U_B$ ;
- (iv). discord is non-decreasing under local operations;
- (v). discord is a monotone of entanglement for pure states, i.e.  $\mathcal{D}_A(\rho_{AB}) = S(\rho_A) = S(\rho_B)$ ;
- (vi). discord vanishes if the state  $\rho_{AB}$  is classical-quantum with respect to the measured subparty  $A$ , i.e.  $\rho_{AB} = \sum_i p_i \Pi_i^A \otimes \rho_i^B$  (we can consider classical states as a subclass of classical-quantum states);
- (vii). discord is bounded from above, as  $\mathcal{D}_B(\rho_{AB}) \leq S(\rho_B)$ .

As we will show briefly, our quantifier for discord of particles  $D_P$  possesses all the above properties, and therefore it can be considered as a measure for quantum correlations of identical particles.

For a system of  $N$  indistinguishable particles, shared among two subparties  $A$  and  $B$ ,  $D_P = D_P^{(S)}$  assumes the following form

$$D_P^{(S)} = \sum_{k=0}^N P_{k,N-k} \mathcal{D}_S(\rho_{k,N-k}), \quad (21)$$

where  $S$  is the measured subsystem and

$$\rho_{k,N-k} = \Pi_{k,N-k} \rho \Pi_{k,N-k} \quad (22)$$

is the projection of the system state  $\rho$  over the subspace in which  $A$  possesses exactly  $k$  particles and  $B$  the remaining  $N - k$  ones, while  $P_{k,N-k} = \text{Tr}_{AB}[\rho_{k,N-k}]$  is the corresponding probability. In detail, if  $A$  controls the modes  $\{n_{A1}, n_{A2}, \dots\}$  and  $B$  controls the modes  $\{n_{B1}, n_{B2}, \dots\}$ , we can write the projector  $\Pi_{k,N-k}$  as:

$$\begin{aligned} \Pi_{k,N-k} &= \sum_{\substack{\sum_i n_{Ai}=k \\ \sum_i n_{Bi}=N-k}} |\{n_{Ai}\}\{n_{Bi}\}\rangle \langle \{n_{Ai}\}\{n_{Bi}\}|, \\ &= \sum_{\sum_i n_{Ai}=k} |\{n_{Ai}\}\rangle \langle \{n_{Ai}\}| \otimes \\ &\quad \sum_{\sum_i n_{Bi}=N-k} |\{n_{Bi}\}\rangle \langle \{n_{Bi}\}|, \end{aligned} \quad (23)$$

where we used the property  $\forall |\{n_{Ai}\}\{n_{Bi}\}\rangle \quad \sum_i n_{Ai} = k \iff \sum_i n_{Bi} = N - k$ . Of course, the sum of all the projectors gives the identity:

$$\sum_{k=0}^N \Pi_{k,N-k} = I. \quad (24)$$

Now the reduced matrices for the two subsystems can be written, e.g., as:

$$\begin{aligned} \rho_{k,N-k}^B &= \frac{\text{Tr}_A[\rho_{k,N-k}]}{\text{Tr}_{AB}[\rho_{k,N-k}]} \\ &= \frac{\text{Tr}_A[\rho_{k,N-k}]}{P_{k,N-k}} \\ &= \frac{1}{P_{k,N-k}} \sum_{\{n_{Ai}\}} \langle \{n_{Ai}\} | \rho_{k,N-k} | \{n_{Ai}\} \rangle \\ &= \frac{1}{P_{k,N-k}} \sum_{\sum_i n_{Ai}=k} \langle \{n_{Ai}\} | \rho_{AB} | \{n_{Ai}\} \rangle. \end{aligned} \quad (25)$$

Therefore, it is easy to see that:

$$\begin{aligned}
\rho_B &= \text{Tr}_A[\rho_{AB}] \\
&= \sum_{\{n_{Ai}\}} \langle \{n_{Ai}\} | \rho_{AB} | \{n_{Ai}\} \rangle \\
&= \sum_k \sum_{\Sigma_i n_{Ai}=k} \langle \{n_{Ai}\} | \rho_{AB} | \{n_{Ai}\} \rangle \\
&= \sum_k P_{k,N-k} \rho_{k,N-k}^B.
\end{aligned} \tag{26}$$

Property (i) is immediately verified, since  $D_P$  is a convex combination of non-negative quantities. The same discussion holds for property (ii), since  $D_P^{(S)}$  is also a convex combination of non-symmetrical quantifiers with respect to  $S$ .

Let us recall now a property of the local operations  $L_A, L_B$  (acting on  $A$  or  $B$ ) stated in Ref [26]: they do not change the local number of particles, therefore they commute with the operation of measuring the local number of particles and conserve the probability  $P_{k,N-k}$ :

$$\Pi_{k,N-k} L_A \rho L_A^\dagger \Pi_{k,N-k} = L_A \Pi_{k,N-k} \rho \Pi_{k,N-k} L_A^\dagger \tag{27}$$

$$\begin{aligned}
P_{k,N-k} &= \text{Tr}_{AB}[\rho_{k,N-k}] \\
&= \text{Tr}_{AB}[L_A \rho_{k,N-k} L_A^\dagger]
\end{aligned} \tag{28}$$

Using this property, it is immediate to prove properties (iii) and (iv) since  $D_P$  is a convex combination (with constant coefficients  $P_{k,N-k}$ ) of non decreasing quantities  $\mathcal{D}_S(L_A \rho_{k,N-k} L_A^\dagger) \geq \mathcal{D}_S(\rho_{k,N-k})$ , which are also invariant if  $L_A$  is unitary.

If we consider a pure state, recalling that the projective operations  $\Pi_{k,N-k}$  do not change its nature, we use the properties of  $\mathcal{D}_A(\rho_{AB})$  and we get:

$$\begin{aligned}
D_P^{(A)} &= \sum_{k=0}^N P_{k,N-k} S(\text{Tr}_B[\rho_{k,N-k}]) \\
&= \sum_{k=0}^N P_{k,N-k} S(\rho_{k,N-k}^A),
\end{aligned} \tag{29}$$

which proves property (v) (notice that for these states  $D_P = E_P$  if we use the von Neumann entropy as a measure of entanglement).

Then, when considering a classical-quantum state  $\rho_{AB} = \sum_i p_i \Pi_i^A \otimes \rho_i^B$ , its projection  $\Pi_{k,N-k} \rho_{AB} \Pi_{k,N-k}$  is still a classical-quantum state, and therefore  $D_P^{(A)}$  results in a linear combination of vanishing quantities, giving thus property (vi).

Finally, it is easy to prove that  $D_P$  is bounded from above (vii) since it is a convex combination of quantities that are bounded from above:

$$\begin{aligned}
D_P^{(B)} &= \sum_{k=0}^N P_{k,N-k} \mathcal{D}_B(\rho_{k,N-k}) \\
&\leq \sum_{k=0}^N P_{k,N-k} S(\rho_{k,N-k}^B) \leq S(\rho_B),
\end{aligned} \tag{30}$$

where the last passage comes from the concavity property  $S(\sum_i p_i \rho_i) \geq \sum_i p_i S(\rho_i)$  of Von Neumann entropy.

## B Purity and Decoherence

In all the analyzed cases, our *discord of particles* is very similar to entanglement of particles and, moreover, it is typically a bit smaller than it. This behavior is uncommon, since usually discord tends to be higher than entanglement even under external detrimental agents (such as, e.g., high temperatures or strong magnetic fields[123]). A typical condition in which there's no significant difference between entanglement and discord is when the considered state is almost pure, but this is definitely not the case for our system, whose states are heavily mixed by noise. The mixing of the state can be quantified with the purity,

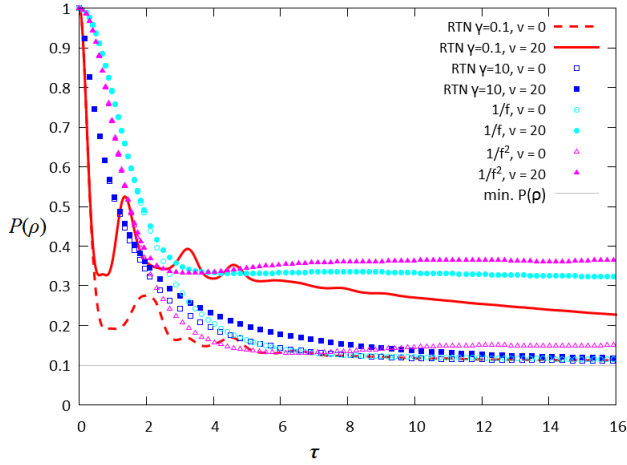
$$P(\rho_{AB}) = \text{Tr}[\rho_{AB}^2] \tag{31}$$

which is 1 for a pure state and  $1/d$  ( $d = \dim(\rho_{AB})$ ) for a maximally mixed state, but also through the *decoherence*, which is quantified by the von Neumann entropy

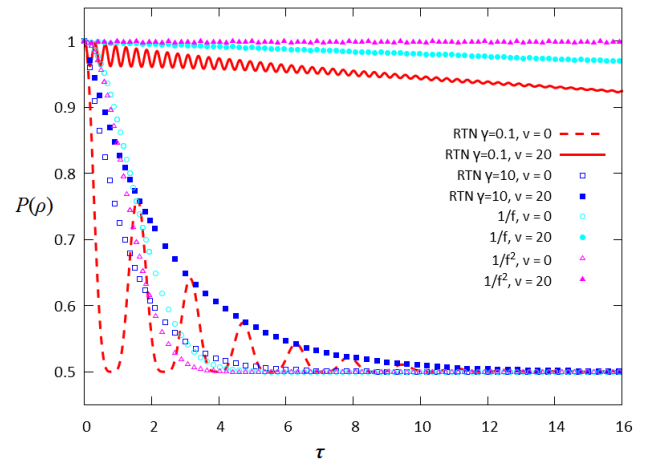
$$S_D(\rho_{AB}) = -\frac{1}{\ln(d)} \rho_{AB} \ln \rho_{AB}. \tag{32}$$

To be more precise, we should mention that the decoherence of this system should not be interpreted as a measure of the entanglement between the system and the environment, since our environment is not included in the quantum evolution, but it is modeled in a classical way. As a result, there isn't a true flow of information between the system and the environment, but noise is still capable of mixing the density matrix, thus determining a loss of coherence in  $\rho_{AB}$  which is quantified by Eq. (32). We computed both quantities for our simulations, but in the following we report only purity for reasons of clarity and brevity, since the two quantifiers agree qualitatively in all considered cases.

As it can be seen in Fig. 22, the action of classical noise over the bosonic system drives it towards a heavily mixed state. Mixing is almost maximal for  $V = 0$  - where purity goes to  $1/d = 0.1$  and decoherence (not shown) goes to 1 - while at high strengths of the interaction the loss of coherence is limited, in agreement with our observations on the role of  $V$  in preserving quantum correlations. A similar behaviour is observed for the fermionic system, shown in Fig. 23, but here even at  $V = 0$  mixing is never maximal (purity downfalls and saturates at 0.5, which is larger than  $1/d \simeq 0.17$ ), coherently with the fact that we observed a higher resistance of



**Fig. 22** Purity of the state  $|\Psi_B\rangle$  under different classical noises and interaction strengths  $v$



**Fig. 23** Purity of the state  $|\Psi_F\rangle$  under different classical noises and interaction strengths  $v$

correlations to noise in this kind of system. Moreover, we notice that at high  $V/J$  there is almost no mixing - as well as there is almost no loss in quantum correlations - for slow RTN and  $1/f^\alpha$  noises (see Figs. 17 and 19-21 for comparison).

Starting the simulation with a state which is not maximally entangled does not change the dynamics of correlations: e.g., the state  $|\Psi'_B\rangle = \frac{1}{2} (b_1^\dagger b_3^\dagger + \sqrt{3} b_2^\dagger b_4^\dagger) |0\rangle$  still shows an evolution in which discord decays faster than entanglement under the action of noise (even if, at larger times, they cross before they go both to zero).

Moreover, the same relationship between entanglement and discord (i.e., a very similar evolution in time and the hierarchy  $D < E$ ), was observed for quantum bits subject to classical RTN [90] and colored noise [61]: we therefore conclude that this behavior of entanglement and discord can be related to the peculiar features of the noise generated by randomly switching bistable fluctuators and is neither a consequence of the choice of the initial state nor an effect of low mixing.

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