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Abstract: In this paper a physical model for the anomalous temperature time evolution (decay) observed in complex thermodynamical system in presence of uniform heat source is provided. Measures involving temperatures T with power-law variation in time shows a different evolution of the temperature time rate $\dot{T}$ (t) with respect to the temperature time-dependence T (t). Indeed the temperature evolution is a power-law increasing function whereas the temperature time rate is a power-law decreasing function of time. Such a behavior may be captured by a physical model that allows for a fast thermal energy diffusion close to the insulated location but must offer more resistance to the thermal energy flux as soon as the distance increases. In this paper this idea has been exploited showing that such thermodynamical system is represented by an heterogeneous one-dimensional distributed mass one with power-law spatial scaling of its physical properties. The model yields, exactly a power-law evolution (decay) of the temperature field in terms of a real exponent as that is related to the power-law spatial scaling of the thermodynamical property of the system. The obtained relation yields a physical ground to the formulation of fractional-order generalization of the Fourier diffusion equation.

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Palermo, April 22, 2015

Dear Prof. Jeong,

enclosed please find the rev.1 of the manuscript entitled "An exact thermodynamical model of power-law temperature time scaling" for possible publication in the journal: Annals of Physics. All the considerations and comments provided by reviewers have been accounted for in the enclosed rev.1.

I hereby declare that this is an original paper that has not been submitted elsewhere.

Best Regards

Prof. Massimiliano Zingales

Manuscript #: AOP 70792

Title: An exact thermodynamical model of power-law temperature time-scaling

Corresponding Author: Prof. Massimiliano Zingales

AUTHOR REBUTTALS

The author is very grateful for the positive comments provided by the present reviewer. The provided suggestions have been included in rev.1 of the paper. In more detail:

R1) The discrete thermodynamic system described in eqs.(3,4) yields an approximation of power-law temporal evolution of the temperature defined by beta as the thermal conductivity c(V) and the specific heat coefficient C(V) varies along masses mj with the relation defined by alpha. The eqs.(3) and (4) do not explicitely contain the parameters alpha and beta. Is that an assumption? or is it a statement ?

A1) The balance equations, in eq.(3) for the generic mass and in eq.(4) for the boundary mass m1, have been written in a generic form involving the non-homogeneity of the conductor in the thermal conductivity χ_j and in the specific heat c_j. Their specific variation, as power-laws of alpha, along the thermodynamic system provided in eqs.(5 a,b) corresponds to a power-law decay of the temperature with beta. It is a challenging problem whether other functional class of parameter variation may correspond to other measured time-varying evolution of the temperature fields. Therefore I decided not to specify the functional class of parameter variation in the generic balance equations.

R2) Eq. (5a) and (5b) Why different denominators are different, it should be explained.

A2) Thanks for the observation, I kept the same denominator and re-performed derivations in the revised manuscript.

R3) Text after Eq.(5a) and (5b) "the real exponent alpha belongs to the interval -1<=alpha<=1". This interval should be -1<alpha<1

A3) Thanks again, I used same denominator in eq.(5a, 5b) of the revised manuscript and therefore I can keep the left extrema for the exponent alpha that may achieve alpha=-1 in rev.1

R4) The ref.[35] should be updated M. Zingales, a fractional-order non-local thermal energy transport in rigid conductors, *Comm., Nlin. Sc. Num. Sim.*

A4) Reference updated in rev.1

R5) The following paper can be cited M. Zingales, G. Failla, "The finite element method for fractional non-local thermal energy transfer in non-homogeneous rigid conductors", *Comm., Nlin. Sc. Num. Sim:* doi: 10.1016/j.cnsns.2015.04.23

A5) Reference added in rev.1

R6) "Measures involving temperatures T with power-law variation in time as T^b where beta is real number show high time rate at the beginning of the experiment followed by a slowing phase". The high time rate at the beginning of the experiment followed by a slowing phase mean that two different beta b1>0 and b2<2 should be considered on two different regions [0,ts] and [ts, infinity]. As a result we should have two different equations for these regions. May be second region should contain the integral (or derivative) I_{t_s+} (or D_{t_s+}) instead of I_{0+} (or D_{0+}) , i.e. it should contain integration from t_s to infinity, see eqs.(61) and eq.(62).

A6) The author is very grateful for the nice observation. Indeed the phrase is misleading and it has been changed in rev.1. About the general comment involving fractional-order operators in t_s it may be a challenging idea introducing the kernel of the operator in the form: t^{beta2} * H(t-ts)+ t^{beta2} * H(ts-t) where we introduced an Heaveside function H[]. However in presence of different regions in time evolution of the measured temperature a best fitting procedure involving linear combinations of the power-laws will provide a precise description of the measured data.

R7) Eq.(59) contains Gamma function with (1-beta) therefore beta cannot be a real number in general. Beta can not be equal to integer positive values, At least beta should be form open interval [0,1(instead of [0,1].

A7) Correction made in rev.1

An exact thermodynamical model of power-law temperature time scaling

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Abstract

In this paper a physical model for the anomalous temperature time evolution (decay) observed in complex thermodynamical system in presence of uniform heat source is provided. Measures involving temperatures T with power-law variation in time as $T(t) \propto t^{\beta}$ with $\beta \in \mathbb{R}$ shows a different evolution of the temperature time rate $\dot{T}(t)$ with respect to the temperature time-dependence $T(t)$. Indeed the temperature evolution is a power-law increasing function whereas the temperature time rate is a power-law decreasing function of time.

Such a behavior may be captured by a physical model that allows for a fast thermal energy diffusion close to the insulated location but must offer more resistance to the thermal energy flux as soon as the distance increases. In this paper this idea has been exploited showing that such thermodynamical system is represented by an heterogeneous one-dimensional distributed mass one with power-law spatial scaling of its physical properties. The model yields, exactly a power-law evolution (decay) of the temperature field in terms of a real exponent as $T \propto t^{\beta}$ (or $T \propto t^{-\beta}$ that is related to the power-law spatial scaling of the thermodynamical property of the system. The obtained relation yields a physical ground to the formulation of fractional-order generalization of the Fourier diffusion equation.

Keywords: Temperature evolution, Fractional derivatives, Anomalous conduction,

1. Introduction

Thermal energy transfer due to phonon-phonon diffusion is very often observed in engineering and physical sciences leading to prediction of heat fluxes and temperature fields by means of the well-known Fourier transport equation. In this latter relation the instantaneous value of the thermal energy flux at any location is related to the local spatial gradient of the temperature ([1]). The evolution of the temperature field provided by the Fourier transport equation is an exponential growth or decay and it proves to be accurate in several applications. However as soon as thermal energy flux is investigated in complex, multiphase and multiscale conductors and/or in presence of high frequency phenomena, then marked deviations of the exponential-type temperature evolution from experimental data may be observed [2],[3]. Under these circumstances, several generalization of the Fourier equation have been proposed in scientific literature from the mid of the last century [4]. Inertial correction to pure diffusive heat transport, including ballistic phononic transport, shows interesting features of the temperature field as the propagation of second-sound thermal waves observed in superfluids [5]-[6] and the pathological non-monotonic behavior of the entropy state function ([7],[8],[9]).

Fourier/Cattaneo models of thermal energy transport are not suitable, however, to describe the power-law rising of the temperature in recent challenging applications. Indeed an anomalous evolution of the temperature field has been observed in ultrafast phenomena as the laser pulsatile radiation in biological tissues[10] as well as in thermal energy transport in nanostructured materials [[11]].

In this context generalization of the Fourier transport equation has been proposed replacing classical differential operators with their real-order (fractional) counterparts $\frac{d^j f}{dt^j} = \left(D^j f\right)(t) \rightarrow \frac{d^\beta f}{dt^\beta} = (D^\beta f)(t)$ with $\beta \in \mathbb{R}$ [12].

This approach has been used in several context of physics and engineering yielding the so-called fractional-order Fourier transport equation ([13],[14],[15],[16]) or the non-local fractional-order thermodynamics ([17],[18],[19],[20],[21]).

Despite the wide success beyond the introduction of fractional-order Fourier equations it has been only presented on phenomenological basis and no thermodynamical systems have been provided, at the best of the author knowledge, with power-law time scaling of the temperature.

This paper aims to fill this gap introducing an heterogeneous conductor that yields, exactly, the anomalous time evolution of the temperature field as $T(t) \propto t^{\beta}$ and $\beta \in [0,1]$. It is shown that, as the thermodynamical properties of the system vary as power-law of the distance from the insulated border, a relation among the thermal properties of the conductor and the exponent of the decay of the temperature field is obtained.

A similar feature was first encountered in the field of classical mechanics where the anomalous material creep/relaxation has been modeled with a proper mechanical ladder yielding the exact description of material hereditariness ([22][23],[24],[25],[26],[27]). In other studies the presence of anomalous evolution of pressure and/or mass flux has been related to the transport across fractal porous materials ([28],[29],[30]). In other studies the physical representation of the spatial interactions involved in the use of fractional-order calculus have been investigated (see e.g. [31],[32],[33]) also in the context of long-range thermal energy fluxes [34],[35],[36])

The paper is organized as follow: In the next section a discrete thermodynamical system will be shown to approximate the power-law evolution of the temperature field. In sec.3 the non-homogeneous continuous thermodynamical system representation is reported and it is shown to describe, exactly, the power-law evolution of the temperature with exponent related to the heterogeneity of the material decay; Some conclusions are reported in sec.4 whereas details on fractional-order operators generaling classical operators have been reported in Appendix.

2. A discrete mass representation of the anomalous temperature evolution

The idea described in this paper stems out from a physical consideration about the temperature rising observed in complex thermodynamical systems as biological tissues, nanostructured materials as well as in presence of multiphasic materials in which different kinds of thermal energy carriers exist. Power-law temperature evolution is a high rate phenomenon during the first time instants that is followed by a progressive decay of the temperature time rate. Such an observation suggests that a thermodynamical model capable to follow this phenomena shall allow for a fast thermal energy diffusion close to the thermal energy source and it must be less compliant as the distance from the source increase.

The challenge to describe the temperature evolution by means of such consideration is exploited in this section introducing a proper thermodynamical system that approximate the power-law evolution of the temperature time scaling.

To this aim let us consider the thermodynamical system in fig.(2 a) representing $n+1$ masses $m_j = A_j \Delta z$ with $j = 1,2,...,n+1$, located at abscissas $z_j = j\Delta z$ and $\Delta z = l/(n+1)$ with $l = (n+1)\Delta z$ the overall length of the system. The masses are separated by adiabatic walls from the external environment so that thermal energy exchange may occur only along the z direction and connected each other by a perfect conductor. The thermodynamic state variables describing the system are assumed as the macroscopic temperatures $T_j(t)$ of the masses m_j for $j = 1,2,...,n+1$.

Energy balance of the j^{th} mass m_j of the system involves the rate of the internal energy U_j of the mass and the energy flux along the conductors m_j, namely, $q_j(t)$ and $q_{j-1}(t)$ that may be written as:

$$\frac{dU_j(t)}{dt} = m_j \frac{du_j(t)}{dt} = m_j C_j^{(V)} \frac{dT_j(t)}{dt} = A_{j-1} q_{j-1}(t) - A_j q_j(t) \tag{1}$$

where we denoted $C_j^{(V)}$ the specific heat at constant volume of the mass m_j that is obtained as: $C_j^{(V)} = \left(\frac{\partial u_j}{\partial T} \right)_{T_0}$ assumed uniform for the considered temperature interval and $u_j(t)$ is the internal energy function density of the mass m_j.

In the following it is assumed that only diffusive phonon-phonon interaction yields thermal energy transport so that the j^{th} flux may be expressed as:

$$q_j(t) = -\chi_j^{(T)} \frac{T_{j+1}(t) - T_j(t)}{z_{j+1} - z_j} = -\chi_j^{(T)} \frac{T_{j+1}(t) - T_j(t)}{\Delta z} \tag{2}$$

where we denoted $\chi_j^{(T)}$ the thermal conductivity of the $j-th$ conductor. Substitution of eq.(2) into eq.(1) yields the balance of the thermal energy as an the explicit differential equation system in the temperatures $T_j(t)$.

$$\rho \Delta z C_j^{(V)} \dot{T}_j(t) = \frac{1}{\Delta z}\left[\chi_{j+1}^{(T)} T_{j+1}(t) - \left(\chi_j^{(V)} + \chi_{j+1}^{(V)}\right) T_j(t) + \chi_{j-1}^{(T)} T_{j-1}(t)\right] \tag{3}$$

where we assumed $A_{j-1} = A_j = A$ for $j = 1, 2, ..., n+1$ and that mass m_j occupies the volume $A\Delta z$ so that, introducing the mass density ρ it may be expressed as: $m_j = \rho A \Delta z$ (see fig.2).

The energy balance equations reported in eq.(3) involve masses m_j with $j = 2, 3, ..., n$ as the temperature of the m_{n+1} mass of the system has been set to the value $T_{n+1} = 0$ without loss of generality. Energy balance of mass m_1 of the thermodynamical system in fig.(2) involves an external thermal energy flux, denoted in the following $\bar{q}(t)$ yielding:

$$C_1^{(V)} \Delta z \rho \dot{T}_1(t) + \chi_1^{(T)} \frac{T_2(t) - T_1(t)}{\Delta z} = \bar{q}(t) \tag{4}$$

In order to achieve an approximate description of the first mass temperature evolution as $T_1 \propto t^{\beta}$ of the system described by eqs.(3,4) a non-homogeneous thermodynamical conductor shall be introduced.

In this regard we assume that the thermal conductivity $\chi_j^{(V)}$ and the specific heat

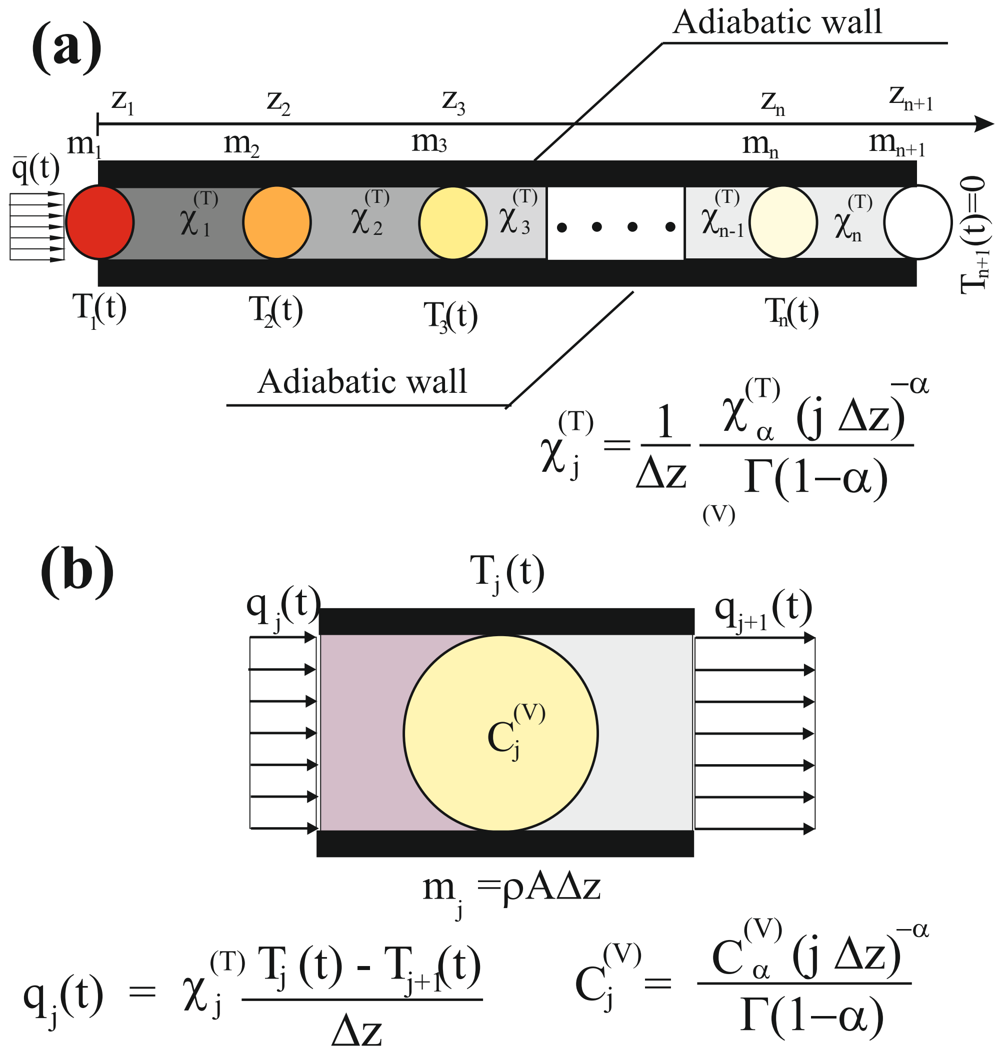

Figure 1: Thermodynamical model of anomalous temperature rising: a) The concentrated mass system; b) Thermal energy balance of the $j^{th}-$mass

coefficient $C_j^{(V)}$ varies along masses m_j with the relations:

$$C_j^{(V)} = \frac{C_\alpha^{(V)} (j\Delta z)^{-\alpha}}{\Gamma(1-\alpha)} \tag{5a}$$

$$\chi_j^{(T)} = \frac{\chi_\alpha^{(T)} (j\Delta z)^{-\alpha}}{\Gamma(1-\alpha)} \tag{5b}$$

where $\Gamma(\bullet)$ is the Euler-Gamma function and the real exponent α belongs to the interval $-1 \leq \alpha < 1$ for diffusion-type considerations that will be reported in the next section.

Coefficients $C_\alpha^{(V)}$ and $\chi_\alpha^{(T)}$ are specific heat and thermal conductivity with anomalous physical dimensions as:

$$\left[C_\alpha^{(V)}\right] = \frac{L^{2+\alpha}T^2}{K} \qquad ; \qquad \left[\chi_\alpha^{(T)}\right] = \frac{ML^{1+\alpha}}{KT^3} \tag{6}$$

Under these circumstances the mass temperature evolution $T_j(t)$ $(j = 1, 2, ..., , n)$ is ruled by the solution of the differential equation system:

$$\begin{cases} p_\alpha \dot{T}_1(t) - r_\alpha \left[T_2(t) - T_1(t)\right] = \bar{q}(t) \\ p_\alpha j^{-\alpha} \dot{T}_j(t) = r_\alpha \left[(j+1)^{-\alpha} T_{j+1}(t) - [(j+1)^{-\alpha} + j^{-\alpha}] T_j(t) + (j-1)^{-\alpha} T_{j-1}(t)\right] \end{cases} \tag{7}$$

where we introduced the coefficients:

$$p_\alpha = \frac{\rho C_\alpha^{(V)} \Delta z}{\Gamma(1-\alpha)} \qquad ; \qquad r_\alpha = \frac{\chi_\alpha^{(T)}}{\Gamma(1-\alpha)\Delta z} \tag{8}$$

that may be converted to a compact notation as we introduce the n-dimensional temperature vector $\mathbf{T}(t) \in \mathbb{R}^n$ that gathers the mass temperatures $T_j(t)$ as:

$$p_\alpha \mathbf{A} \dot{\boldsymbol{T}}(t) + r_\alpha \mathbf{B} \boldsymbol{T}(t) = \mathbf{v} \bar{q}(t) \tag{9}$$

where: In Eq. (9):

$$\mathbf{T}^T = \begin{bmatrix} T_1(t) & T_2(t) & \dots & T_n(t) \end{bmatrix}; \ \mathbf{v}^T = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \end{bmatrix} \tag{10}$$

where the apex T means transpose. The coefficient matrices $\mathbf{A}$ and $\mathbf{B}$ reads:

$$\mathbf{A} = \begin{bmatrix} 1^{-\alpha} & 0 & 0 & \dots & 0 \\ 0 & 2^{-\alpha} & 0 & \dots & 0 \\ 0 & 0 & 3^{-\alpha} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & n^{-\alpha} \end{bmatrix}. \tag{11}$$

$$\mathbf{B} = \begin{bmatrix} 1^{-\alpha} & -1^{-\alpha} & 0 & \dots & 0 \\ -1^{-\alpha} & 1^{-\alpha}+2^{-\alpha} & -2^{-\alpha} & \dots & 0 \\ 0 & -2^{-\alpha} & 2^{-\alpha}+3^{-\alpha} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & (n-1)^{-\alpha}+n^{-\alpha} \end{bmatrix} \tag{12}$$

Matrices **A** and **B** are symmetric and positive definite so that temperature evolution reported in Eq. (9) may be obtained by using standard tools of linear analysis introducing an eigenvector decomposition of the differential system in eq.(7) with the linear mapping:

$$\mathbf{A}^{1/2}\mathbf{T}(t) = \mathbf{x}(t) \tag{13}$$

that substituted in eq.(9) yields, after premultiplication by $\mathbf{A}^{-1/2}$, a differential equation for the unknown vector $\mathbf{x}$ obtained as:

$$p_\alpha \dot{\mathbf{x}} + r_\alpha \mathbf{D}\mathbf{x} = \tilde{\mathbf{v}}\bar{q}(t) \tag{14}$$

where $\tilde{\mathbf{v}} = \mathbf{A}^{-1/2}\mathbf{v}$ and $\mathbf{D}$ is the matrix $\mathbf{D} = \mathbf{A}^{-1/2}\mathbf{B}\mathbf{A}^{-1/2}$ given as:

$$\mathbf{D} = \begin{bmatrix} 1 & -\left(\frac{2}{1}\right)^{\frac{\alpha}{2}} & 0 & \dots & 0 & 0 \\ -\left(\frac{2}{1}\right)^{\frac{\alpha}{2}} & 1+\left(\frac{2}{1}\right)^{\alpha} & -\left(\frac{3}{2}\right)^{\frac{\alpha}{2}} & \dots & 0 & 0 \\ 0 & -\left(\frac{3}{2}\right)^{\frac{\alpha}{2}} & 1+\left(\frac{3}{2}\right)^{\alpha} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1+\left(\frac{n-1}{n-2}\right)^{\alpha} & -\left(\frac{n}{n-1}\right)^{\frac{\alpha}{2}} \\ 0 & 0 & 0 & \dots & -\left(\frac{n}{n-1}\right)^{\frac{\alpha}{2}} & 1+\left(\frac{n}{n-1}\right)^{\alpha} \end{bmatrix} \tag{15}$$

that is $\mathbf{D}$ is symmetric and positive definite and it may be obtained straightforwardly as n and α have been chosen. Once the differential equation of the temperature evolution has been set the eigenvector decoupling of the system yields a nice picture of the temperature evolution.

Indeed let $\mathbf{\Phi}$ be the eigenvector matrix whose columns are the orthonormal eigenvectors of $\mathbf{D}$ that is:

$$\mathbf{\Phi}^T\mathbf{D}\mathbf{\Phi} = \mathbf{\Lambda}; \qquad \mathbf{\Phi}^T\mathbf{\Phi} = \mathbf{I} \tag{16}$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{\Lambda}$ is the diagonal matrix collecting the eigenvalues $\lambda_j > 0$ of $\mathbf{D}$. In the following we order λ_j in such a way that $\lambda_1 > \lambda_2 > \cdots > \lambda_n$. As we indicate $\mathbf{y}(t)$ the modal coordinate vector, defined as:

$$\mathbf{x}(t) = \mathbf{\Phi}\,\mathbf{y}(t); \qquad \mathbf{y}(t) = \mathbf{\Phi}^T\,\mathbf{x}(t) \tag{17}$$

and we substitute in Eq. (14) a decoupled set of differential equation is obtained in the form:

$$p_\alpha\,\dot{\mathbf{y}} + r_\alpha\,\mathbf{\Lambda}\mathbf{y} = \bar{\mathbf{v}}\bar{q}(t) \tag{18}$$

where $\bar{\mathbf{v}} = \mathbf{\Phi}^T\tilde{\mathbf{v}} = \mathbf{\Phi}^T\mathbf{A}^{-1/2}\mathbf{v} = \mathbf{\Phi}^T\mathbf{v}$.

The j^{th}-equation of Eq. (18) reads:

$$\dot{y}_j + \gamma_j\,y_j = \frac{\phi_{1,j}}{p_\alpha}\bar{q}(t); \qquad j = 1, 2, 3, \dots n \tag{19}$$

where $\gamma_j = \lambda_j r_\alpha / p_\alpha \geq 0$ and $\phi_{1,j}$ is the j^{th} element of the first row of the matrix $\mathbf{\Phi}$ and it corresponds to a decoupled set of memory dependent conductors that may be obtained from the integral description of the transport equation ([13],[14]) as the kernel function is assumed in the form: $K_j(t-\tau) = \delta(\tau) + \gamma_j \delta^{'}(\tau)$ with $\delta(\bullet)$ the Delta function. Additionally, it is worth noticing that the choice of exponential-type kernel function in the memory equation of integral transport models yields, instead the Fourier-Cattaneo model of thermal energy transport.

As a general consideration it must be observed that the presence of decaying eigenvalues $\gamma_1 \geq \gamma_2 \geq ... \geq \gamma_n$ led to conclude that the more important contributions to the initial, faster, temperature time rate is provided by the fist generalized conductors, whereas the final decreasing time rate is contributed by the latter, memory-dependent, conductors.

Indeed the solution of Eq. (19) is provided in the form:

$$y_j(t) = y_j(0)\, e^{-\gamma_j t} + \frac{\phi_{1,j}}{p_\alpha} \int_0^t e^{-\gamma_j (t-\tau)} \bar{q}(\tau)\, d\tau \tag{20}$$

where $y_j(0)$ is the j^{th} component of the vector $\mathbf{y}(0)$ related to the vector of initial conditions $\boldsymbol{T}(0)$ as:

$$\mathbf{y}(0) = \mathbf{\Phi}^T \mathbf{A}^{1/2} \mathbf{T}(0). \tag{21}$$

Solution of the differential equation system in Eq. (9) may be obtained as the modal vector $\mathbf{y}(t)$ has been evaluated by solving Eq. (20) with the aid of Eqs. (13) and (17) as:

$$\mathbf{T}(t) = \mathbf{A}^{-1/2} \mathbf{\Phi} \mathbf{y}(t). \tag{22}$$

As we are interested to a relation among the heat flux applied to the first mass m_1 and the increment of the state variable, namely $T_1(t)$ we must evaluate the first element of vector $\boldsymbol{\gamma}(t)$ obtained as:

$$T(t) = \mathbf{v}^T \mathbf{T}(t). \tag{23}$$

The expression of the eigenvalues and eigenvectors for of the system is obtained in closed-form for the specific case $\alpha = 0$. In this case the thermodynamical constants of the system reads $C_j^{(V)} = C_0$ and $\chi_j^{(\alpha)} = \chi_0$ yielding the coefficients of the differential equation system, namely, p_0 and r_0 as:

$$p_0 = C_0^{(V)} \Delta z; \quad r_0 = \frac{\chi_0^{(T)}}{\Delta z} \tag{24}$$

The thermal balance equation system is then written in compact form, similarly to the Eq. (9), as:

$$\mathbf{A}\dot{\boldsymbol{T}} + \frac{1}{\triangle \tau_0}\mathbf{B}\boldsymbol{T} = \mathbf{v}\frac{\bar{q}(t)}{C_0^{(V)}\Delta z} \tag{25}$$

where $\triangle \tau_0 = \dfrac{\chi_0^{(T)}}{C_0^{(V)} \triangle z^2}$ is a system-dependent relaxation time and the coefficient matrices $\mathbf{A} = \mathbf{I}$ and matrix $\mathbf{B}$ read:

$$\mathbf{B} = \begin{bmatrix} 1 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 2 \end{bmatrix} \tag{26}$$

The eigenvalues λ_j and the normalized eigenvectors $\boldsymbol{\phi}_j$ of particular tridiagonal matrix $\mathbf{B}$ may be found in [8],[9] and they are reported below:

$$\lambda_j = 2 - 2\cos\left(\frac{2j-1}{2n+1}\pi\right), \ j = 1, 2 \dots, n \tag{27}$$

$$\phi_{k,j} = \sqrt{\frac{4}{2n+1}}\cos\left[\frac{(2j-1)(2k-1)}{2(2n+1)}\pi\right], \ j,k = 1, 2 \dots, n. \tag{28}$$

Using the Eq. (28) can be easily calculate the modal matrix $\mathbf{\Phi}$, obtaining the following equation in the modal space:

$$C_0^{(V)} \triangle z \dot{\mathbf{y}} + \frac{\chi_0^{(T)}}{\triangle z}\mathbf{\Lambda}\mathbf{y} = \bar{\mathbf{v}}\bar{q}(t) \tag{29}$$

where $\bar{\mathbf{v}} = \mathbf{\Phi}^T \mathbf{v}$ and the j^{th}-equation of the system (29), corresponding to the thermal balance equilibrium equation of the j^{th}- conductor:

$$\dot{y}_j + \frac{\lambda_j}{\triangle\tau_0} y_j = \frac{\phi_{1,j}}{C_0^{(V)}\triangle z}\bar{q}(t), \quad j = 1, 2, \ldots, n. \tag{30}$$

yielding the solution of j^{th}-equation as:

$$y_j(t) = y_j(0)\, e^{-\frac{\lambda_j}{\triangle\tau_0}t} + \frac{\phi_{1,j}}{\triangle z C_0^{(V)}} \int_0^t e^{-\frac{\lambda_j}{\triangle\tau_0}(t-\tau)}\bar{q}(\tau)\, d\tau \tag{31}$$

and the temperature evolution of the first mass of the system reads:

$$\begin{aligned} T(t) &= \mathbf{v}^T \mathbf{\Phi}\, \mathbf{y}(t) = \\ &= \sum_{j=1}^{n} \left[\phi_{1,j} y_j(0)\, e^{-\frac{\lambda_j}{\triangle\tau_0}t} + \frac{\phi_{1,j}^2}{\triangle z C_0^{(V)}} \int_0^t e^{-\frac{\lambda_j}{\triangle\tau_0}(t-\tau)}\bar{q}(\tau)\, d\tau \right]. \end{aligned} \tag{32}$$

The temperature evolution of the first mass m_1 of the system is represented in fig.(2), named $T_0(t)$, for an uniform thermal energy flux, that is $\bar{q}(t) = \bar{q}_0 = 1$. It may be observed that, as soon as a specified value of the exponent α is chosen, then a power-law evolution of the temperature $T_1(t) \propto t^{\beta}$, with β a proper real number may be observed (fig.2).

The temperature evolutions have been obtained assuming the overall lenght of the conductor $l = 30$, $C_\alpha^{(V)} = 1$ and $\chi_\alpha^{(T)} = 1$. The three curves have involved number of masses $n = 4000 \quad , n = 1000 \quad , n = 2500$ for $\alpha = 0.2$, $\alpha = 0.0$ and $\alpha = 0.2$, respectively. Several numerical analysis, not reported for brevity, showed that as soon as the number n of masses increases and the length of the conductor l increases a better approximation of a power-law up to a specified time instant t_α is achieved. Value of t_α depends on the number of mass considered and, independently on the length of the domain occupied by the system. Indeed, as soon as the number n increases for fixed value of Δz or the value of Δz decreases for fixed number of masses n, larger values of the time t_α describing a curve fitted by a power-law t^{β} is obtained. In virtue of this consideration it may be inferred that the choice of an unbounded, distributed mass thermodynamical system may induce the exact representation of the power-law

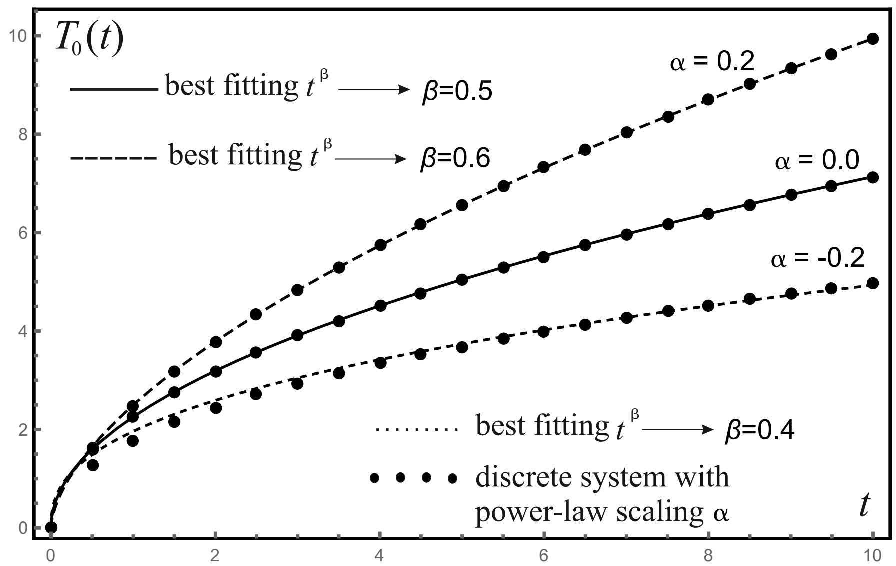

Figure 2: Temperature evolution for different values of the decaying exponent of the thermodynamical properties of the system: Dots solution of the discrete mass system; Lines best fitting process with t^{β}

temperature evolution and a precise relation among the scaling parameter α and the exponent decay β may be found as it is shown in the next section.

3. Exact thermodynamical representation of temperature power-law evolution

Thermodynamical analysis of the discrete mass system reported in previous section revealed that, under some conditions about the macroscopic thermodynamical properties of the masses and the conductors, an approximate anomalous power-law time dependence of the temperature as t^{β} is obtained.

In this section it is shown that the exact representation of the power-law time dependence of the temperature is obtained as limiting case of the discretized mass system presented in previous paper letting, at the same time $n \to \infty$, $\Delta z \to 0$ and $l \to \infty$. To this aim let us introduce the temperature field so that $T_j(t) \to T(z_j, t)$, and the thermal energy flux field as $q_j(t) \to q(z_j, t)$. In this respect, the balance equation reported in eq.(1) has the expression:

$$\rho C^{(V)}(z)\frac{\partial T(z,t)}{\partial t} = -\frac{\partial q(z,t)}{\partial z} \tag{33}$$

that describes the balance among the rate of the thermal energy $\dot{U} = \rho \frac{\partial u}{\partial t}$ and the differences among the outgoing $q(z+dz,t)$ and incoming $q(z,t)$ thermal energies at location z in unit time. Introducing the Fourier transport equation, obtained for $\Delta z \to 0$ as:

$$q(z,t) = -\chi^{(T)}(z)\frac{\partial T(z,t)}{\partial z} \tag{34}$$

in eq.(33) the diffusion temperature equation is obtained as:

$$\rho C^{(V)}(z)\frac{\partial T(z,t)}{\partial t} = \frac{\partial}{\partial z}\left[\chi^{(T)}(z)\frac{\partial T(z,t)}{\partial z}\right] \tag{35}$$

where the thermodynamical properties of the distributed mass system are described as continuous functions of the abscissa z as the continuous counterparts of eqs.(3 a,b) $C_j^{(V)} \to C^{(V)}(z_j)$ and $\chi_j^{(T)} \to \chi^{(T)}(z_j)$ that read:

$$C^{(V)}(z) = \frac{C_\alpha^{(V)} z^{-\alpha}}{\Gamma(1-\alpha)} \qquad ; \qquad \chi^{(T)}(z) = \frac{\chi_\alpha^{(T)} z^{-\alpha}}{\Gamma(1-\alpha)} \tag{36}$$

The boundary conditions associated to the temperature diffusion relation in eq.(33) are obtained as the continuous conditions on the first m_1 and the latter m_{n+1} mass of the discrete system of the previous section as:

$$\bar{q}(t) = \lim_{z\to 0} -\chi^{(T)}(z)\frac{\partial T(z,t)}{\partial z} \qquad ; \qquad \lim_{z\to\infty} T(z,t) = 0 \tag{37}$$

According to the observations reported in previous section two cases are considered: *i)* the case of heat flux across an homogeneous thermodynamical system yielding a temperature evolution as $T_0(t) \propto t^{1/2}$ and *ii)* the case of the thermal energy transfer across a non-homogeneous conductor that corresponds to an exact description of the temperature evolution as $T_0(t) \propto t^\beta$ with β related to the scaling exponent of the thermodynamical properties α.

3.1. Temperature evolution in an uniform conductor: The power-law $T(t) \propto t^{1/2}$

Let us assume, in the following, that the mass density of the system is selected, without loss of generality as $\rho = 1$ and $\alpha = 0$. In this context the temperature equation,obtained by eq.(35) reads:

$$C_0^{(V)} \frac{\partial T(z,t)}{\partial t} = \chi_0^{(T)} \frac{\partial^2 T(z,t)}{\partial z^2} \tag{38}$$

with associated boundary conditions:

$$\bar{q}(t) = -\lim_{z \to 0} \chi_0^{(T)} \frac{\partial T(z,t)}{\partial z} \tag{39a}$$

$$\lim_{z \to \infty} T(z,t) = 0 \tag{39b}$$

The solution of the temperature equation may be obtained of the temperature field in the continuous domain is formulated in Laplace domain as:

$$C_0^{(V)} s\widehat{T}(z,s) = \chi_0^{(T)} = \frac{d^2\widehat{T}(z,s)}{dz^2} \tag{40}$$

Solution of eq.(40) may be obtained as a linear combination of exponential functions as:

$$\widehat{T}(z,s) = B_1 \exp\left(-\sqrt{\frac{C_0^{(V)} s}{\chi_0^{(T)}}}\, z\right) + B_2 \exp\left(\sqrt{\frac{C_0^{(V)} s}{\chi_0^{(T)}}}\, z\right) \tag{41}$$

Position of the boundary condition in eqs. (39a) and (39b) yields

$$B_1 = \frac{\widehat{\bar{q}}(s)}{\sqrt{\chi_0^{(T)} C_0^{(V)}}} s^{-1/2} \qquad ; \qquad B_2 = 0 \tag{42}$$

yielding the temperature field $\widehat{T}(z,s)$ along the thermodynamic distributed system as:

$$\widehat{T}(z,s) = \frac{\widehat{\bar{q}}(s)}{\sqrt{\chi_0^{(T)} C_0^{(V)}}} s^{-1/2} \exp\left(-\sqrt{\frac{C_0^{(V)} s}{\chi_0^{(T)}}} z\right) \tag{43}$$

yielding, after inverse Laplace transform, a Riemann-Liouville fractional-order integral ([22]) among the ingoing thermal energy flux $\bar{q}(t)$ and measured temperature $T_0(t) =$

$T(0,t)$ as:

$$T_0(t) = \frac{1}{\sqrt{C_0^{(V)}\chi_0^{(T)}}}\left(\mathrm{I}_{0^+}^{\frac{1}{2}}\bar{q}\right)(t) = \frac{1}{R_{1/2}}\left(\mathrm{I}_{0^+}^{\frac{1}{2}}\bar{q}\right)(t) \tag{44}$$

The fractional-order relation in eq.(44) yields a power-law evolution of the temperature as we assume an uniform flux $\bar{q}(t) = < t >^0$ with $< \bullet >^0$ the singularity function of order 0 coalescing with the well-known Heaveside theta function. Indeed, in this case:

$$T_0(t) = \frac{1}{R_{1/2}}\left(\mathrm{I}_{0^+}^{\frac{1}{2}}\bar{q}\right)(t) = \frac{1}{\sqrt{\chi_0^{(T)}C_0^{(V)}}}t^{1/2} \propto t^{1/2} \tag{45}$$

That is as far as we control the ingoing flux at $z = 0$, the measured temperature at the same location depends on the histories of the flux field with a fractional integral order $\beta = 1/2$. As a consequence the temperature evolution for uniform heat flux, at the same location, involves exactly a power-law time dependence with exponent $\beta = 1/2$ as already observed in the discrete thermodynamical system reported in previous section (see fig.2). It may be concluded that the distributed thermodynamical system considered in this case, obtained as an asymptotic expansion of the corresponding discretized counterpart shown in fig.(1 a), is the exact thermodynamical model of anomalous power-law temperature evolution $t^{1/2}$. The generalization to real powers of t as t^β will be discussed in the next section.

3.2. The temperature evolution $T(t) \propto t^\beta$ $(0 \leq \beta < 1)$

In this section the generalization of the exact result obtained previously to arbitrary values of the exponent β is reported. In this regard it is shown that a linear one-to-one relation exists among the scaling exponent of the thermodynamical property of the system α and the exponent β.

The temperature field $T(z,t)$ may be obtained introducing Laplace transform of eq.(35) yielding an ordinary differential equation in Laplace domain as:

$$\frac{d}{dz}\left[\chi^{(T)}(z)\frac{d\widehat{T}(z,s)}{dz}\right] = sC_\alpha^{(V)}(z)\widehat{T}(z,s) \tag{46}$$

that may be cast, after some straightforward manipulation as:

$$\frac{d^2\widehat{T}(s,z)}{dz^2}+\frac{\left[\chi^{(T)}(z)\right]'}{\chi^{(T)}(z)}\frac{d\widehat{T}(s,z)}{dz}-\frac{C^{(V)}(z)}{\chi^{(T)}(z)}s\widehat{T}(s,z)=0 \tag{47}$$

Substitution for the thermal conductivity coefficient $\chi^{(T)}(z)$ and the specific heat $C^{(V)}(z)$ the corresponding power-laws reported in eqs.(36) the differential equation ruling the temperature field reads:

$$\frac{d^2\widehat{T}(z,s)}{dz^2}-\frac{\alpha}{z}\frac{d\widehat{T}(z,s)}{dz}-\tau_\alpha s\widehat{T}(z,s)=0 \tag{48}$$

with:

$$\tau_\alpha=\frac{C_\alpha^{(V)}}{\chi_\alpha^{(T)}} \tag{49}$$

is an anomalous decaying time depending on the anomalous thermal properties of the mass system.

The governing equation of the temperature field may be reverted into a Bessel equation of the second kind introducing an auxiliary function: $\overline{T}(z,s)$ related to the unknown function $\hat{T}(z,s)$ by means of the non-linear mapping $\hat{T}(z,s)=z^\alpha\overline{T}(z,s)$. In this setting the first and second-order derivatives involved in eq.(48) read, respectively:

$$\frac{d\widehat{T}(z,s)}{dz}=\alpha z^{\alpha-1}\overline{T}(z,s)+z^\alpha\frac{d\overline{T}(z,s)}{dz} \tag{50a}$$

$$\begin{aligned}\frac{d^2\widehat{T}(z,s)}{dz^2}&=\frac{d}{dz}\left[\alpha z^{\alpha-1}\overline{T}(z,s)+z^\alpha\frac{d\overline{T}(z,s)}{dz}\right]=\\&=\alpha(\alpha-1)z^{\alpha-2}\overline{T}(z,s)+2\alpha z^{\alpha-1}\frac{d\overline{T}(z,s)}{dz}+\\&+z^\alpha\frac{d^2\overline{T}(z,s)}{dz^2}\end{aligned} \tag{50b}$$

and substitutions into eq.(48) yield a modified Bessel equation for function $\overline{T}(z,s)$ as:

$$z^2\frac{d^2\overline{T}(z,s)}{dz^2}+\alpha z\frac{d\overline{T}(z,s)}{dz}-\left(z^2\tau_\alpha s+\alpha\right)\overline{T}(z,s)=0 \tag{51}$$

Eq.(51) may be solved in terms of the first and the second modified Bessel functions denoted, respectively $Y_\beta\left(z\sqrt{\tau_\alpha s}\right)$ and $K_\beta\left(z\sqrt{\tau_\alpha s}\right)$ defined as:

$$Y_\beta(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{\beta+2k}}{k!\Gamma(k+\beta+1)} \tag{52a}$$

$$K_\beta(z) = \frac{\pi}{2sin(2\pi\beta)}\left[Y_{-\beta}(z) - Y_\beta(z)\right] \tag{52b}$$

yielding a solution of the modified Bessel function in the form:

$$\widehat{T}(z,s) = z^\beta\left(B_1 Y_\beta\left(z\sqrt{\tau\alpha s}\right) + B_2 K_\beta\left(z\sqrt{\tau_\alpha s}\right)\right) \tag{53}$$

where we introduced the α-dependent relaxation time τ_α and the exponent β that is related to the scaling exponent α as:

$$\beta = \frac{(1+\alpha)}{2} \tag{54}$$

Integration constants B_1 and B_2 in eq.(53) are defined as we impose the relevant boundary conditions that are defined in Laplace domain as:

$$\lim_{z\to 0} -\chi^{(T)}(z)\frac{\partial \widehat{T}(z,s)}{\partial z} = \widehat{q}(s) \tag{55a}$$

$$\lim_{z\to\infty} \widehat{T}(z,s) = 0 \tag{55b}$$

yielding the integration constants:

$$B_1 = 0 \quad ; \quad B_2 = \frac{2^\beta\Gamma(\beta)}{\chi_\alpha^{(T)}\Gamma(1-\beta)}(s\tau_\alpha)^{-\beta/2}\widehat{q}(s) \tag{56}$$

and the temperature field of the distributed mass systems in the form:

$$\widehat{T}(z,s) = \frac{2^\beta\Gamma(2(1+\beta))(\tau_\alpha s)^{-\beta/2}}{\chi_\alpha^{(T)}\Gamma(1-\beta)} K_\beta\left(z\sqrt{s\tau_\alpha}\right)\widehat{q}(s) \tag{57}$$

As soon as the temperature $T(z,t)$ has been obtained in the whole distributed system the explicit relation among the measured temperature at $z=0$ as a consequence of the ingoing flux of thermal energy is obtained as:

$$\widehat{T}_0(s) = \lim_{z \to 0} \widehat{T}(z,s) = \frac{s^{-\beta}}{R_\beta \Gamma(\beta)} \widehat{\bar{q}}(s) \tag{58}$$

where the anomalous thermal diffusivity coefficient, R_β is expressed in terms of the physical properties of the conductor as:

$$R_\beta = \frac{\Gamma(2\beta)\tau_\alpha^\beta}{\chi_\alpha^{(T)} 2^{1-2\beta} \Gamma(\beta)\Gamma(1-\beta)} \tag{59}$$

The relation in eq.(58) yields, under the assumption of uniform thermal energy flux as $\bar{q}(t) = 1$, the time-varying temperature function $T_0(t)$ as the inverse Laplace transform:

$$T_0(t) = \frac{t^\beta}{R_\beta} \propto t^\beta \tag{60}$$

thatis the power-law temperature time scaling observed in fig.(2) for the discretized mass system considered in the analysis with $\beta \in [0,1)$.

Under the assumption that the thermal energy flux is a time-dependent function, the inverse Laplace transform of eq.(58) yields

$$T_0(t) = \frac{1}{R_\beta} \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} \bar{q}(\tau) d\tau = \frac{1}{R_\beta} \left(\mathrm{I}_{0^+}^\beta \bar{q} \right)(t) \tag{61}$$

that is a Riemann-Liouville fractional-order integral (see Appendix for details) of order $\beta \in [0,1)$.

Eq.(61) represents the generalization of the Fourier-Cattaneo transport equation in terms of fractional-order integrals obtained with an exact thermodynamical model of thermal energy transport. It may be observed that a generalization of the Fourier-Cattaneo equation involving fractional-order derivatives of the heat fluxes ([13][14]) may be obtained in the framework of the proposed model is possible as we include in the analysis values of the decaying α in the interval $-3 \leq \alpha \leq -1$ (see e.g. [37]).

The inverse relation of eq.(61) may be obtained introducing the $\beta-$order fractional derivative of both sides of eq.(61) yielding:

$$\bar{q}(t) = R_\beta \left(\mathrm{D}^\beta_{0^+} T_0 \right)(t) \tag{62}$$

that is a fractional-order generalization of the transport equation analogous to fractional-order generalization of the Darcy filtration equation as reported in ([37]).

The considerations reported in this section show that, as a fractional-order heat transport equation is considered, three different physical models of thermal energy transport may be involved: *i*); A Subdiffusive phonon-phonon transmission, ii); a Normal phononic energy transport and *iii*); a Superdiffusive phononic energy transmission.

Indeed, in case *ii*), the Normal phononic energy transfer ($\beta = 1/2$) occurs in an homogeneous media without any difference during the phononic path. If instead a Subdiffusive thermal process (case *i*) is observed, then the phonons carrying thermal energy moves freely close to the thermal energy source and they are, instead more confined along their path in the conductor ($\beta \leq 1/2$). A similar, but opposite, effect (case *iii*) is involved in Superdiffusive thermal energy path ($\beta \geq 1/2$) where the motion of the phonons is more confined in the neighborhood of the thermal energy source and it is less confined as the distance increases.

The aformentioned connection among the physics of the anomalous temperature rising and classical thermodynamics has not been pointed out, at the best of the authors knowledge, and it is useful to realize material conductors with sub/superdiffusive properties at the nanoscale.

4. Conclusions

The anomalous temperature rising in the form of a time-varying power law as $T(t) \propto t^\beta$ with $\beta \in [0,1)$ can not be predicted with the use of the well-known equations of the thermal energy transfer as the Fourier/Cattaneo equation and proper generalization of the transpot equations in in terms of fractional-order calculus has been proposed. However, beside the phenomenological replacement of the Cattaneo/Fourier equations with their fractional-order counterpat no physical model has been provided to describe anomalous time scaling at the best of the author's knowledge.

In this study it is shown that, based on the consideration that the power-law rising show a fast time rate at the beginning followed by decreasing time rate a novel thermodynamical model yielding the anomalous temperature evolution $T \propto t^{\beta}$ is proposed. The idea stems out from the observation that a thermodynamical system with a functionally graded thermodynamical parameters undergoes non-homogeneous temperature evolution. As a proper functional class of the parameters is considered in terms of power-laws of the distance from the insulated border, then a time evolution of the temperature in the form t^{β} is obtained. A specific relation on the scaling of the thermodynamical properties and the exponent β has been explicitly obtained.

It is shown that a discrete mass system corresponds to an approximation of the power-law t^{β} and, at the limit, the exact expression $T_0(t) \propto t^{\beta}$ is obtained. In this context the presence of a non-constant thermal energy flux yields a fractional-order generalization of the Fourier transport equation is obtained providing a physical ground to the use of fractional-order thermodynamics.

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Appendix: Remarks on fractional calculus

In this appendix the essential features of fractional calculus will be shortly discussed. Let us consider a real-valued, Lebesgue integrable function $f(x), x \in \mathbb{R}$ such that $f(x) \in L^1$.The left and right Riemann-Liouville (RL) fractional-order integrals are defined as:

$$\begin{aligned} \left(I_{+}^{\alpha} f\right)(x) &= \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} \frac{f(y)}{(x-y)^{1-\alpha}} dy \\ \left(I_{-}^{\alpha} f\right)(x) &= \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} \frac{f(y)}{(y-x)^{1-\alpha}} dy \end{aligned} \tag{63}$$

with $\alpha \in [0,1]$ and $\Gamma(\bullet)$ is the Euler-Gamma function. The left and right fractional derivatives are defined as:

$$\begin{aligned} \left(D_{+}^{\alpha} f\right)(x) &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{-\infty}^{x} \frac{f(y)}{(x-y)^{1-\alpha}} dy \\ \left(D_{-}^{\alpha} f\right)(x) &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{x}^{\infty} \frac{f(y)}{(y-x)^{1-\alpha}} dy \end{aligned} \tag{64}$$

As we assume that function $f(x) \in C^1$with $C^1$the class of continuous functions with continuous first derivative, then the left and right RL fractional derivatives coalesces with the Marchaud (M) fractional operator that is defined as:

$$\left(\mathbf{D}_{+}^{\alpha} f\right)(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^{x} \frac{f(x)-f(y)}{(x-y)^{1+\alpha}} dy = \left(D_{+}^{\alpha} f\right)(x) \tag{65}$$

for the left M fractional derivative, whereas, the right M fractional derivative is related to the right RL fractional derivative as:

$$\left(\mathbf{D}_{-}^{\alpha} f\right)(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_{x}^{\infty} \frac{f(x) - f(y)}{(y-x)^{1+\alpha}} dy = \left(D_{-}^{\alpha} f\right)(x) \tag{66}$$

The definition of RL and M fractional derivatives operating on functions defined on bounded intervals $[a,b] \subset \mathbb{R}$ involves integral terms as well as algebraic contributions as:

$$\left(D_{a^{+}}^{\alpha} f\right)(x) = \frac{f(a)}{\Gamma(1-\alpha)(x-a)^{\alpha}} + \frac{1}{\Gamma(1-\alpha)} \int_{a}^{x} \frac{f(y)'}{(x-y)^{\alpha}} d\xi \tag{67}$$

$$\left(D_{b^{-}}^{\alpha} f\right)(x) = \frac{f(b)}{\Gamma(1-\alpha)(b-x)^{\alpha}} - \frac{1}{\Gamma(1-\alpha)} \int_{x}^{b} \frac{f(y)'}{(y-x)^{\alpha}} d\xi \tag{68}$$

where $f(y)' = \frac{df}{dy}$, showing divergence at the boundaries of the considered domains, unless function $f(x) \to 0$ faster than x^{α}as $x \to 0$.