Convergence Analysis of Distributed Set-Valued Information Systems

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***Abstract*—This paper focuses on the convergence of information in distributed systems of agents communicating over a network. The information on which the convergence is sought is not represented by real numbers, as often in the literature, rather by sets. The dynamics of the evolution of information across the network is accordingly described by set-valued iterative maps. While the study of convergence of set-valued iterative maps is highly complex in general, this paper focuses on Boolean maps, which are comprised of arbitrary combinations of unions, intersections, and complements of sets. For these important class of systems, we provide tools to study both global and local convergence. A distributed geographic information system, leading to successful information reconstruction from partial and corrupted data, is used to illustrate the applications of the proposed methods.**

***Index Terms*—Binary encoding, boolean dynamic systems, consensus algorithms, convergence, cooperative systems, distributed information systems, set-valued dynamic maps.**

I. Introduction

RECENT years have witnessed a constant increase of interests in applications involving many distributed agents that interact in order to achieve a common goal. Most of the problems attacked so far in the literature can be formulated as consensus problems over continuous domains, where local agents exchange data that consist of scalars (such as a temperature or the concentration of a chemical) or vectors (e.g., positions or velocities). Models used differ mainly in the type of rule each agent uses to combine its own information with the one received from its neighbors in the communication graph. In the simplest case, the evolution of the network of agents can be described by a linear iterative rule

$$x(t+1) = Ax(t) + Bu(t)$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is the system's state, with the i-th component x_i belonging to the i-th agent, $u \in \mathbb{R}^m$ is an input vector, and A is a weight matrix. More precisely, matrix A is designed so as to comply with available communication topology and ensure the convergence of the network to a unique decision, i.e., $x(\infty) \to \alpha 1$, with α depending on the initial system's state. Moreover, the input vector u can be used to model a known bias [1] or even an unknown disturbance signal [2]. Falling into this linear framework are most of the key papers on consensus [1], [3], [4]. By using more general nonlinear dynamical systems, other important schemes for achieving consensus on more complex functions of state variables can be accommodated for. For instance, the distributed algorithm based on the centroidal Voronoi tessellation proposed by [5] allows a collection of mobile agents to be deployed within a given environment while maximizing the network's sensing ability.

However, new emerging issues in the field of distributed control entail defining consensus algorithms on different representations of the state of information (see e.g., [6]). As a first example, consider the problem of averaging a set of initial measures taken by a collection of distributed sensors with limited communication bandwidth, which can be solved via a consensus system where agents' state information is represented by *symbols* obtained through a logarithmic quantizer [7]. As a second example, consider the problem of estimating the value of a logical decision task depending on binary input events by a set of agents with limited visibility on the events, e.g., the detection of malicious users in a networked computer system by interaction of local observation monitors [8]. A solution to the problem can be obtained through use of the so-called *logical consensus* approach, according to which agents share binary estimates of the events, combine them according to a suitable logical iterative function, and finally reach an agreement on their values. An interesting problem that is related to consensus is that of studying how to disseminate information through a network where nodes can only elaborate and share data over finite fields [9].

Furthermore, other applications involve problems of increasing complexity where the state of information takes value in possibly infinite domains. For instance, consider the problem of clock synchronization in distributed loosely-coupled systems via message exchange, where each node has a confidence interval on the true value of time, although the true value may be outside this interval for some sources. Marzullo's algorithm [10], on which the ubiquitous Network Time Protocol (NTP) [11] is based, is an agreement algorithm which estimates the smallest interval consistent with the largest number of sources. The problem of simultaneous localization and mapping by a

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$F_1 =$	$\mathcal{C}(T_{b_1}^{a_1}(X_3))$	X_3	X_3
$F_2 =$	$T_{b_2}^{a_2}(X_3) \cap \mathcal{C}(T_{b_3}^{a_3}(X_1))$	$X_2 \cap X_1$	$X_2 \cap \mathcal{C}(X_1)$
$F_3 =$	$T_{b_4}^{a_4}(X_1) \cup T_{b_5}^{a_5}(X_2)$	$\mathcal{C}(X_1) \cup X_2$	$X_1 \cap X_3$

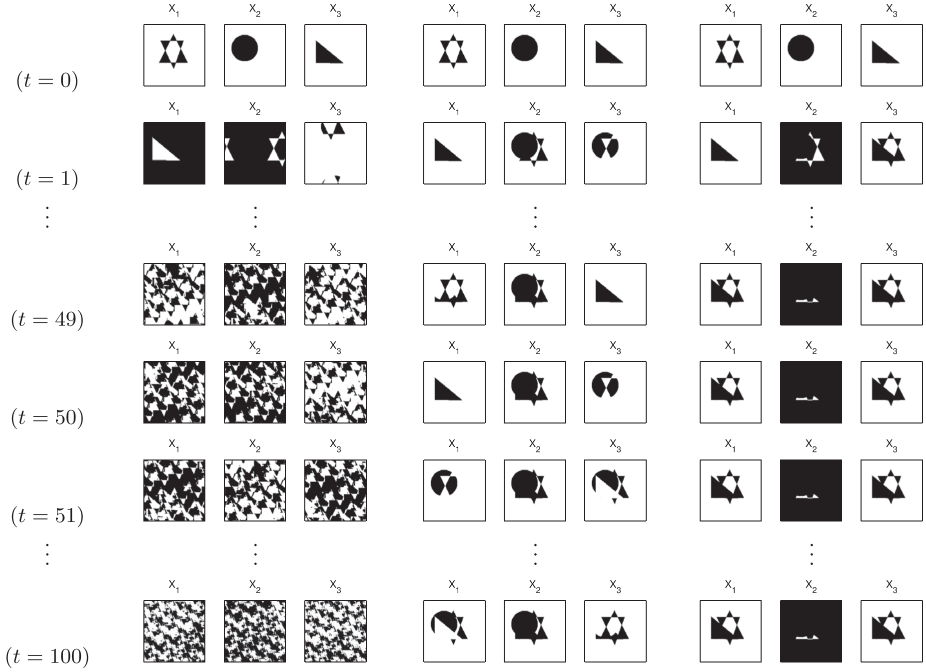

Fig. 1. Simulation runs of three systems with $n = 3$ agents running different set-valued maps to update their states. From left to right, starting from the same initial conditions, the three systems have a chaotic behavior, enter into a cycles, and reach an equilibrium point, respectively. In the formulas the terms $\mathcal{C}(X_i)$ is the complement of X_i, $T_k^h(X_i)$ is a translation of X_i by the vector $(h, k)^T$ (mod 1), and the coefficients are $a_1 = 2$, $b_1 = 7$, $a_2 = 66$, $b_2 = 12$, $a_3 = 1$, $b_3 = 99$, $a_4 = 77$, $b_4 = 11$, $a_5 = 56$, $b_5 = 154$.

set of mobile robotic agents is another example, where the traditional approach of modeling each agent's uncertainty on the positions of visually extracted features as additive or multiplicative signals is possible but not natural. As it was shown in [12], the problem can be solved by a consensus approach where agents exchange data represented by confidence *regions* containing the features' real positions. Finally, the detection of misbehavior in "societies" of robots comprised of agents that are supposed to obey to cooperation rules, depending on presence and absence of objects in their neighborhood, requires that every robot compute a local estimate of the neighborhood occupancy map of its neighbors and combine it with others' estimates via a *set-valued* consensus algorithm [13].

All these problems, and indeed many others, require that the information state of a network of n agents is a collection $X = (X_1, \ldots, X_n)^T$ of elements X_i, belonging to the power set[1] $\mathcal{P}(\mathbb{X})$ of a discrete or continuous, finite or infinite set $\mathbb{X}$, which is iteratively updated according to a set-valued map $F = (F_1, \ldots, F_n)^T$, with $F_i : \mathcal{P}(\mathbb{X})^n \to \mathcal{P}(\mathbb{X})$, i.e.,

$$X(t+1) = F\left(X(t)\right). \tag{1}$$

The evolution of such set-valued iterative systems can be extremely rich and complex in the general case. Consider e.g., the three relatively simple set-valued systems described in Fig. 1, where agents exchange and update their states $X_i \subseteq Q$, with $Q = [0, 1] \times [0, 1]$. The figure shows that, starting from the same initial conditions, set-valued iterative maps can display very different and interesting behaviors, ranging from chaotic sequences to cycles and equilibria with or without consensus. Although the study of these systems appears to be a formidable task in its full generality, in many applications of practical interest the set of rules used in the iterative map are limited to specific classes, which render analysis more tractable. Of particular relevance are certainly maps involving only Boolean operations, such as the set union $\cup$, the intersection $\cap$, and the complement $\mathcal{C}(\cdot)$. Fortunately, it is possible to provide a reasonably simple study and characterization of such systems.

The main intent of the paper is to show that information convergence in every instance of a Boolean iterative system can be studied in fundamentally the same way. This is achieved by extending the notions of convergence, local convergence, and contraction, already given in the binary domain [14], [15], to algebras of sets, taken with the union, intersection, and complement operations. The work presented here builds upon earlier results by the authors [16], where global convergence of Set-Valued Boolean Dynamic Systems (SVBDS)[2] was studied, and it provides also results on local convergence in terms of properties of binary matrices for which analysis [14], [15] and synthesis [8] results are available. These new results are mainly, but not only, contained in the following theorems: Theorem 4.1

[1]Recall that given a set $\mathbb{X}$, the power set $\mathcal{P}(\mathbb{X})$ is the set of all subsets of $\mathbb{X}$, including the empty set and $\mathbb{X}$ itself.

[2]The term "set-valued" is associated also with maps going from $\mathbb{X}$ to its power set $\mathcal{P}(\mathbb{X})$ (see e.g., [17]); however we adopt it here to indicate that the system is described by maps going from sets to sets.

of Section IV, Lemma 5.1, Theorem 5.2, Theorem 5.2, and Theorem 5.4 of Section V. By doing this, we believe that the present work is a step toward the definition of a unified framework for the convergence analysis of systems involving iterative maps based on Boolean algebras, and for the design of Boolean iterative systems producing consensus in such domains.

The paper is organized as follows. Section II recalls the definition of a Boolean algebra and summarizes known results on the convergence of maps defined over the simplest Boolean algebra, i.e., the one involving a binary domain. These results are extended in the following sections. Section III studies the global behavior of SVBDS. By using the binary encoding of a SVBDS that is presented in Section IV, conditions ensuring the global convergence of a SVBDS and local attractiveness of its equilibria are presented in Section V. In Section VI, the presented theory is applied to a problem of robust estimation of geographic information.

II. Boolean Dynamic Systems

In this section and in the remainder of the paper, we focus on a class of dynamic systems, namely Boolean Dynamic Systems (BDS), to define which we need to recall the following well-known notion (see e.g., [18]):

Definition 2.1: A Boolean Algebra is a sextuple $(\tilde{\mathbb{B}}, \wedge, \vee, \neg, 0, 1)$, consisting of a domain set $\tilde{\mathbb{B}}$, equipped with two binary operations $\wedge$ (called "meet" or "and") and $\vee$ (called "join" or "or"), a unary operation $\neg$ (called "complement" or "not"), and two elements 0 (null) and 1 (unity) belonging to $\tilde{\mathbb{B}}$, s.t. the following axioms hold, for all elements $a, b, c \in \tilde{\mathbb{B}}$:

1) $a \vee (b \vee c) = (a \vee b) \vee c, \quad a \wedge (b \wedge c) = (a \wedge b) \wedge c$ (associativity);
2) $a \vee b = b \vee a$, $a \wedge b = b \wedge a$ (commutativity);
3) $a \vee (a \wedge b) = a$, $a \wedge (a \vee b) = a$ (absorption);
4) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ (distributivity);
5) $a \vee \neg a = 1$, $a \wedge \neg a = 0$ (complementarity). ♦

From the first three pairs of axioms above, it follows that, for any two elements $a, b \in \tilde{\mathbb{B}}$, it holds that $a = a \wedge b$ if, and only if, $a \vee b = b$, which introduces a *partial order* relation $\leq$ among the elements of the domain. In particular, we will say that $a \leq b$, if, and only if, one of the two above equivalent conditions hold. Moreover, 0 and 1 are the smallest and greatest elements, respectively. Then, given any two elements $a, b \in \tilde{\mathbb{B}}$, the meet $a \wedge b$ and the join $a \vee b$ coincide with their infimum or supremum, respectively, w.r.t. $\leq$.

An element $a \in \tilde{\mathbb{B}}$ is referred to as a *scalar*. Consider the set $\tilde{\mathbb{B}}^n$ of Boolean *vectors* and the set $\tilde{\mathbb{B}}^{n\times n}$ of square Boolean *matrices*. We can give the following definitions that generalize the notions presented in e.g., [19] and [14], which are valid only when $\tilde{\mathbb{B}}$ is the binary domain $\{0, 1\}$:

Definition 2.2: Given two *vectors* $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$, and two square matrices $A = \{a_{i,j}\}$ and $B = \{b_{i,j}\}$, define the scalar product as $w \cdot v \stackrel{\text{def}}{=} \bigvee_{i=1}^{n} v_i \wedge w_i \in \tilde{\mathbb{B}}$, the product Av as the vector whose i-th element is the scalar product between the i-th row of A and the vector v, and the product AB as the matrix whose (i, j)-th element is the scalar product between the i-th row of A and the j-th column of B. In other words products between a matrix and a vector and between two matrices are computed in the usual way, substituting $+$ with $\vee$ and $\cdot$ with $\wedge$. ♦

We denote with 0 the null scalar, vector, or matrix, according to the context. The above described partial order relation $\leq$ between any two elements of $\tilde{\mathbb{B}}$ can be extended to Boolean vectors and matrices by assuming component-wise evaluation.

Definition 2.3 (Boolean Dynamic Systems (BDS)): Given a Boolean algebra, a BDS is an iterative system of the form in (1), whose state X is a vector in $\tilde{\mathbb{B}}^n$ and where F is a map combining the elements of its input argument to produce a new state vector, by using only the meet $\wedge$, the join $\vee$, and the complement $\neg$ operations of the Boolean algebra itself.

Definition 2.4 (Linear BDS): A BDS is *linear* if there exists a set-valued matrix $A \in \tilde{\mathbb{B}}^{n\times n}$ s.t., for all $X \in \tilde{\mathbb{B}}^n$, $F(X) = AX$.

For the following study, we need to give the following definitions:

Definition 2.5 (Canonical Basis): The set of the vectors $e_1, e_2, \ldots, e_n$, with $e_j \in \tilde{\mathbb{B}}^n$ contains 1 in the j-th element and zeros elsewhere, is called the canonical basis of $\tilde{\mathbb{B}}^n$.

Definition 2.6 (Eigenvalues and Eigenvectors): A scalar $\lambda \in \tilde{\mathbb{B}}$ is an *eigenvalue* of a Boolean matrix $A \in \tilde{\mathbb{B}}^{n\times n}$ if there exists a vector $x \in \tilde{\mathbb{B}}^n$, called *eigenvector*, s.t. $Ax = \lambda x$.

Definition 2.7 (Incidence Matrix): The incidence matrix of a Boolean map F is a Boolean binary matrix $B(F) = \{b_{i,j}\}$, with $b_{i,j} \in \tilde{\mathbb{B}}$ and where $b_{i,j} = 1$ if, and only if, the i-th component of $F(x)$ depends on the j-th component of the input vector x.

It is worth noting that, in the case of a distributed BDS, similarly to what happens in the example of Section I, every agent of the system share the value of its Boolean state by sending a message to its neighbors, and the map F specifies how these states need to be combined together in order to accomplish a global computation task. The incidence matrix $B(F)$ allows analyzing how the information flows from one agent to another, and, as it is shown below, plays an important role in the convergence study of the corresponding BDS.

III. Set-Valued Boolean Dynamic Systems—Global Convergence

As it is known from Stone's Representation Theorem [20], every Boolean algebra is isomorphic (i.e., it possesses the same structural properties) to a field of sets, which is, given a generic set $\mathbb{X}$, a subset $\Sigma(\mathbb{X}) \subset \mathcal{P}(\mathbb{X})$ that is closed under finite set unions, intersections, and complementations. For this reason, we focus on the following class of systems:

Definition 3.1 (Set-Valued Boolean Dynamic Systems (SVBDS)): A SVBDS is a BDS whose Boolean algebra is given by the sextuple $(\Sigma(\mathbb{X}), \cup, \cap, \mathcal{C}(\cdot), \emptyset, \mathbb{X})$, where $\mathbb{X}$ is a possibly infinite set called the unity, $\emptyset$ is the empty set, and the operators $\cup$, $\cap$, $\mathcal{C}(\cdot)$ are the set union, intersection, and complement, respectively. ♦

Remark 1: This class of SVBDM includes set-valued maps also involving the set difference $\setminus$ and the symmetric difference S operations between any two sets $X_i, X_j \in \Sigma(\mathbb{X})$ or between a set X_i and a constant set $A \in \Sigma(\mathbb{X})$. To show this, one can recall that both operations can be rewritten in terms of the Boolean algebra's basic operations: the former can be expressed as $X \setminus Y = \{x \in X \text{ s.t. } x \notin Y\} = \{X \cap \mathcal{C}(Y)\}$, and the latter as

$$\begin{aligned} S &: \mathcal{P}(\mathbb{X}) \times \Sigma(\mathbb{X}) \to \Sigma(\mathbb{X}) \\ &(x, y) \mapsto (\mathcal{C}(x) \cap y) \cup (x \cap \mathcal{C}(y)) . \end{aligned} \tag{2}$$

Moreover, if F involves operations with k constant sets, $A_1, \ldots, A_k$, one can define an augmented state vector $\tilde{X} = (X_1, \ldots, X_n, A_1, \ldots, A_k)^T$ and consider a system defined through the following dynamic map involving only Boolean operations on $\tilde{X}$:

$$\begin{aligned} \tilde{F}(\tilde{X}) = (&F_1(X_1, \ldots, X_n, A_1, \ldots, A_k), \ldots, \\ &F_n(X_1, \ldots, X_n, A_1, \ldots, A_k), A_1, \ldots, A_k)^T . \end{aligned}$$

♦

In the remainder of this section, we study under which conditions these systems converge to a unique equilibrium. The incidence matrix of a set-valued Boolean map F, as defined in Definition 2.7, specializes to a matrix $B(F) \in \{\emptyset, \mathbb{X}\}^{n \times n}$. Following the steps of [14] for binary systems, we provide the following:

Definition 3.2 (Boolean Vector Distance): Given any two set-valued vectors $X, Y \in \Sigma(\mathbb{X})^n$, the Boolean vector distance between the two vectors is described by the following application:

$$\begin{aligned} &\mathcal{D} : \Sigma(\mathbb{X})^n \times \Sigma(\mathbb{X})^n \to \Sigma(\mathbb{X})^n \\ &(X, Y) \mapsto (S(X_1, Y_1), \ldots, S(X_n, Y_n)) \end{aligned}$$

where X_i, Y_i are the i-th components of the vectors X and Y, respectively, and S is the symmetric difference in (2). ♦

Note that $\mathcal{D}$ satisfies the same formal properties of a metric:

1) $\mathcal{D}(X, Y) = \mathcal{D}(Y, X)$ for all X, Y (symmetry);
2) $\mathcal{D}(X, Y) = \emptyset$ if, and only if $X = Y$ (identity);
3) $\mathcal{D}(X, Y) \subseteq \mathcal{D}(X, Z) \cup \mathcal{D}(Z, Y)$ (sub-additivity);

where the inclusion relation $\subseteq$ is the partial order relation introduced after Definition 2.1, as specialized to the Boolean Algebra of the sets.

The following Proposition 3.1–3.5 are based on results that were first presented in the conference paper [16].

Proposition 3.1: Given any set-valued map F, it holds, for every $X, Y \in \Sigma(\mathbb{X})^n$

$$\mathcal{D}(F(X), F(Y)) \subseteq B(F)\mathcal{D}(X, Y). \tag{3}$$

Proof: Consider a chain of adjacent vector states, i.e., a sequence where any two successive states differ in exactly one component, connecting X to Y. By using the sub-additivity axiom, the i-th component of $\mathcal{D}(F(X), F(Y))$ can be decomposed as follows:

$$\begin{aligned} &S(F_i(X_1, \ldots, X_n), F_i(Y_1, \ldots, Y_n)) \subseteq \\ &\quad \subseteq S(F_i(X_1, \ldots, X_n), F_i(Y_1, X_2, \ldots, X_n)) \cup \\ &\quad \cup S(F_i(Y_1, X_2, \ldots, X_n), F_i(Y_1, Y_2, X_3, \ldots, X_n)) \cup \\ &\qquad\qquad \cdots \\ &\quad \cup S(F_i(Y_1, \ldots, Y_{n-1}, X_n), F_i(Y_1, \ldots, Y_{n-1}, Y_n)) . \end{aligned} \tag{4}$$

Let $\tilde{X}^j = (X_1, \ldots, X_{j-1}, X_j, Y_{j+1}, \ldots, Y_n)^T$ and $\tilde{Y}^j = (X_1, \ldots, X_{j-1}, Y_j, Y_{j+1}, \ldots, Y_n)^T$ be the j-th pair of consecutive state vectors in the chain. We want to prove that

$$S\left(F_i(\tilde{X}^j), F_i(\tilde{Y}^j)\right) \subseteq b_{i,j} S(X_j, Y_j) \tag{5}$$

where $b_{i,j}$ are the elements of $B(F)$. If $b_{i,j} = \emptyset$, i.e., the map F_i is independent of the j-th component of the state, (5) is trivially satisfied, since its first member is the empty set. Let us thus focus on the case with $b_{i,j} = \mathbb{X}$. Given the state vector $\tilde{X}^j$, the one-input argument function

$$\begin{aligned} &\tilde{F}_i^j : \Sigma(\mathbb{X}) \to \Sigma(\mathbb{X}) \\ &Z \mapsto F_i(X_1, \ldots, X_{j-1}, Z, Y_{j+1}, \ldots, Y_n) \end{aligned}$$

allows rewriting (5) as

$$S\left(\tilde{F}_i^j(X_j), \tilde{F}_i^j\left(S(X_j, S_j)\right)\right) \subseteq S_j \tag{6}$$

where $S_j = S(X_j, Y_j)$. Note that $\tilde{F}_i^j$ may consist only of one of the four applications: $A \cap Z$, $A \cap \mathcal{C}(Z)$, $A \cup Z$, $A \cup \mathcal{C}(Z)$, where $A \in \Sigma(\mathbb{X})$ is a set depending on the components of $\tilde{X}^j$, except for the j-th one. Direct computation on the above four cases shows that (6) holds for all X_j and all S_j, as it is shown in the following. Indeed, we have

$$\begin{aligned} &S(A \cap X_j, A \cap S(X_j, S_j)) = \\ &\quad = (\mathcal{C}(A) \cup \mathcal{C}(X_j)) \cap A \cap S(X_j, S_j) \cup \\ &\qquad \cup A \cap X_j \cap (\mathcal{C}(A) \cup \mathcal{C}(S(X_j, S_j))) = \\ &\quad = A \cap (\mathcal{C}(X_j) \cap S(X_j, S_j) \cup X_j \cap \mathcal{C}(S(X_j, S_j))) = \\ &\quad = A \cap (\mathcal{C}(X_j) \cup X_j) \cap S_j = A \cap S_j \subseteq S_j \\ &S(A \cap \mathcal{C}(X_j), A \cap \mathcal{C}(S(X_j, S_j))) \\ &\quad = (\mathcal{C}(A) \cup X_j) \cap A \cap \mathcal{C}(S(X_j, S_j)) \cup \\ &\qquad \cup A \cap \mathcal{C}(X_j) \cap (\mathcal{C}(A) \cup S(X_j, S_j)) = \\ &\quad = A \cap (X_j \cup \mathcal{C}(X_j)) \cap S_j = A \cap S_j \subseteq S_j \\ &S(A \cup X_j, A \cup S(X_j, S_j)) = \\ &\quad = \mathcal{C}(A) \cap \mathcal{C}(X_j) \cap (A \cup S(X_j, S_j)) \cup \\ &\qquad \cup (A \cup X_j) \cap \mathcal{C}(A) \cap \mathcal{C}(S(X_j, S_j)) = \\ &\quad = \mathcal{C}(A) \cap (\mathcal{C}(X_j) \cap S(X_j, S_j) \cup X_j \cap \mathcal{C}(S(X_j, S_j))) = \\ &\quad = \mathcal{C}(A) \cap (\mathcal{C}(X_j) \cup X_j) \cap S_j = \mathcal{C}(A) \cap S_j \subseteq S_j \\ &S(A \cup \mathcal{C}(X_j), A \cup \mathcal{C}(S(X_j, S_j))) = \\ &\quad = \mathcal{C}(A) \cap X_j \cap (A \cup \mathcal{C}(S(X_j, S_j))) \cup \\ &\qquad \cup (A \cup \mathcal{C}(X_j)) \cap \mathcal{C}(A) \cap S(X_j, S_j) = \\ &\quad = \mathcal{C}(A) \cap (X_j \cup \mathcal{C}(X_j)) \cap S_j = \mathcal{C}(A) \cap S_j \subseteq S_j \end{aligned} \tag{7}$$

where we made use of (2) of De Morgan's rules, $\mathcal{C}(M \cap N) = \mathcal{C}(M) \cup \mathcal{C}(N)$ and $\mathcal{C}(M \cup N) = \mathcal{C}(M) \cap \mathcal{C}(N)$ (see, e.g., [18]), and of the properties $M \cap \mathcal{C}(M) = \emptyset$, $M \cup \mathcal{C}(M) = \mathbb{X}$, for any sets $M, N \in \Sigma(\mathbb{X})$. Moreover, the second member of (4) is upper bounded by $b_{i,1}S(X_1, Y_1) \cup b_{i,2}S(X_2, Y_2) \cup \cdots \cup b_{i,n} S(X_n, Y_n) = B_i\mathcal{D}(X, Y)$, where $B_i = (b_{i,1}, \ldots, b_{i,n})$. The thesis immediately follows by repeating the process for all i. ■

Proposition 3.2: A Boolean matrix $M \in \{\emptyset, \mathbb{X}\}^{n\times n}$, with $M = \{m_{i,j}\}$, satisfies the Boolean inequality

$$\mathcal{D}\left(F(X), F(Y)\right) \subseteq M\mathcal{D}(X, Y) \tag{8}$$

for all vectors $X, Y \in \Sigma(\mathbb{X})^n$, if, and only if, $B(F) \subseteq M$.

Proof: The sufficiency can be shown by observing that, if $M = B(F) \cup \Delta M$, where ΔM is a nonempty matrix, the right-hand side of the inequality in (8) can be lower bounded as follows:

$$\begin{aligned} M\mathcal{D}(X, Y) &= B(F)\mathcal{D}(X, Y) \cup \Delta M\mathcal{D}(X, Y) \supseteq \\ &\supseteq B(F)\mathcal{D}(X, Y) \supseteq D\left(F(X), F(Y)\right) \end{aligned}$$

where the result of Proposition 3.1 has been used.

To prove the necessity, we show that, if

$$M \not\supseteq B(F) = \{b_{i,j}\} \tag{9}$$

then there exist two vectors X, Y s.t.

$$M\mathcal{D}(X, Y) \not\supseteq \mathcal{D}\left(F(X), F(Y)\right). \tag{10}$$

Since both M and $B(F)$ belong to $\{\emptyset, \mathbb{X}\}^{n\times n}$, the inequality in (9) implies that there exist i, j s.t. $m_{i,j} = \emptyset$ and $b_{i,j} = \mathbb{X}$. Since $b_{i,j} = \mathbb{X}$, there must exist two vectors

$$\begin{aligned} X &= (C_1, \ldots, C_{j-1}, X_j, C_{j+1}, \ldots, C_n)^T \\ Y &= (C_1, \ldots, C_{j-1}, Y_j, C_{j+1}, \ldots, C_n)^T \end{aligned}$$

where all C_s and X_j, Y_j are sets in $\Sigma(\mathbb{X})$, with $X_j \neq Y_j$, s.t. $F_i(X) \neq F_i(Y)$. This last condition implies that the right-hand side of (10) must be nonempty. By expanding the i-th row of left-hand side of (10), we obtain

$$\left(\cup_{s=1, s\neq j}^{n} m_{i,s}S(C_s, C_s)\right) \cup m_{i,j}S(X_j, Y_j) = \emptyset$$

since each $S(C_s, C_s) = \emptyset$ and $m_{i,j} = \emptyset$. Thus, we have $\emptyset = M\mathcal{D}(X, Y) \not\supseteq D(F(X), F(Y)) \neq \emptyset$. ■

Corollary 3.1: For any two set-valued maps $F, G : \Sigma(\mathbb{X})^n \to \Sigma(\mathbb{X})^n$, the incidence matrix of the function composition $F(G(X))$ satisfies the Boolean inequality $B(F(G)) \subseteq B(F)B(G)$.

Proof: The proof trivially follows from above. Indeed, if $(F \circ G)_i$ depends on X_j, then there exists k s.t. F_i depends on X_k and G_k depends on X_j. Hence, $B(F)_{i,k} \cap B(G)_{k,j} = \mathbb{X}$ which in turn implies that $(B(F)B(G))_{i,j} = \mathbb{X}$. ■

Moreover, recalling the notions introduced in Definition 2.6, we can provide the following:

Definition 3.3 (Boolean Spectrum): The Boolean spectrum $\sigma(\cdot)$ of a Boolean matrix $A \in \Sigma(\mathbb{X})^{n\times n}$ is set of the eigenvalues of A.

A first result about the spectrum of a Boolean map is the following:

Proposition 3.3: A Boolean matrix $A \in \Sigma(\mathbb{X})^{n\times n}$, $A = \{a_{i,j}\}$, admits the Boolean eigenvalue $\lambda = \emptyset$ if, and only if, it has at least one column for which the union of all its elements is less than $\mathbb{X}$, i.e., there exists $j \in \{1, \ldots, n\}$ s.t. $\cup_{i=1}^{n} a_{i,j} \subset \mathbb{X}$.

Proof—(Sufficiency): Suppose that j satisfies the condition $\cup_{i=1}^{n} a_{i,j} \subset \mathbb{X}$. We want to prove that $\lambda = \emptyset$ is a Boolean eigenvalue of A, i.e., there exists $X \neq \emptyset$ s.t. $AX = \emptyset X = \emptyset$. Consider a vector whose components are empty sets except for the j-th one. Then, we have $AX = A_jX_j$, where A_i is the i-th column of A, which we want to be the vector of empty sets. This last equation can be explicitly written as $a_{i,1} \cap X_j = \emptyset$, $a_{i,2} \cap X_j = \emptyset$, $a_{i,n} \cap X_j = \emptyset$, which hold if, and only if, it also happens that $(a_{1,j} \cap X_j) \cup (a_{2,j} \cap X_j) \cup \cdots \cup (a_{n,j} \cap X_j) = \emptyset$, and, by the distributivity property, that $(a_{1,j} \cup a_{2,j} \cup \cdots \cup a_{n,j}) \cap X_j = \emptyset$, and $\bigcup_{i=1}^{n} a_{i,j} \cap X_j = \emptyset$, for which the two sets are disjoint. Moreover, the value $\bar{X}_j = \mathbb{X} \setminus (\bigcup_{i=1}^{n} a_{i,j}) \neq \emptyset$ satisfies this condition and, due to the hypothesis that $\cup_{i=1}^{n} a_{i,j} \subset \mathbb{X}$, is different from $\emptyset$, which implies that $X = (\emptyset, \ldots, \emptyset, \bar{X}_j, \emptyset, \ldots, \emptyset)^T$ is an eigenvector of A.

(Necessity): Suppose that $\lambda = \emptyset$ is an eigenvalue of A. This implies that there exists $X \neq \emptyset$ s.t. $AX = \emptyset$. This means $\bigcup_{i=1}^{n} a_{i,j} \cap X_j = \emptyset$, for all j. This condition is trivially satisfied for every null component of X. For every other component of X that is different than $\emptyset$, the component itself must be disjoint to the union of the sets composing the corresponding column of A. This implies that their union can not cover the entire set $\mathbb{X}$, which gives the thesis. ■

Remark 2: Note that, if λ if an eigenvalue of a Boolean A with associated eigenvector X, then every matrix $A' = P^TAP$, where P is a permutation matrix,[3] has the same eigenvalue with eigenvector P^TX. To show this, left-multiply the equation $AX = \lambda X$ by P^T and recall that $P^TP = I$.[4] This gives $(P^TAP)(P^TX) = \lambda(P^TX)$, which proves the statement. ♦

The spectrum of a Boolean matrix may have a structure that is impossible in $\mathbb{R}^n$. The following example shows that the same eigenvector may be associated with different eigenvalues, or that the spectrum can be represented by the entire power set $\Sigma(\mathbb{X})$:

Example 3.1: Consider the entire real interval $\mathbb{X} = (-\infty, \infty)$ and the two following matrices:

$$A_1 = \begin{pmatrix} \emptyset & \{13\} \\ (17, 28] & \mathbb{X} \end{pmatrix}; \quad A_2 = \begin{pmatrix} [3, 5) & \mathbb{X} \\ \mathbb{X} & \{4\} \end{pmatrix}.$$

A_1 admits the eigenvalue $\lambda = \emptyset$ by Proposition 3.3, being the union of its first column's elements is less than $\mathbb{X}$, with associated eigenvectors $V_\lambda = (X, \emptyset)^T$, where X is any set in $(-\infty, 17] \cup (28, \infty)$. Moreover, A_2 does not admit the eigenvalue $\lambda = \emptyset$ by Proposition 3.3, while any scalar $\lambda \subseteq \mathbb{X} \setminus \emptyset$ is an eigenvalue of A_2, with associated eigenvector $V_\lambda = (X, X)^T$, with $X \subseteq \lambda$. ♦

A complete characterization of the Boolean spectrum of a generic map is complex (see, e.g., the work in [21]). However, for a subclass of these maps, the two following results can be stated:

[3] P is a permutation matrix in the classical sense, but where every 0 and 1 are replaced with $\emptyset$ and $\mathbb{X}$, respectively.

[4] I is the identify matrix with $\mathbb{X}$ on its diagonal and $\emptyset$ elsewhere.

Proposition 3.4: A matrix $A \in \{\emptyset, \mathbb{X}\}^{n\times n}$ admits the Boolean eigenvalue $\lambda = \mathbb{X}$ if, and only if, there exist no permutation bringing A in strictly lower or upper triangular form.

Proof (Sufficiency): Supposing the existence of a permutation matrix P s.t. $A' \stackrel{\text{def}}{=} P^T AP$ is strictly lower triangular, a vector $X \neq \emptyset$ s.t. $A'X = \mathbb{X}X = X$ does not exist. This trivially holds due to the form of matrix A'. Direct computation gives

$$\begin{aligned} \emptyset &= X_1 \\ a'_{2,1} \cap X_1 &= X_2 \\ a'_{3,1} \cap X_1 \cup a'_{3,2} \cap X_2 &= X_3 \\ &\vdots \\ a'_{n,1} \cap X_1 \cup \cdots \cup a'_{n,n-1} \cap X_{n-1} &= X_n. \end{aligned}$$

The value $\lambda = \mathbb{X}$ is not an eigenvalue of A since the only vector solving the system is $X = \emptyset$.

(Necessity): We need to prove that, if $\lambda = \mathbb{X}$ is not an eigenvalue of A, then there exists a permutation that brings A in strictly lower triangular form. Note that $\mathbb{X}$ is an eigenvalue of A if, and only if, A has a fixed point. So, let us start imposing that the vector $w = (\mathbb{X}, \ldots, \mathbb{X})^T$ is not a fixed point. Then, if A has not an empty row, the scalar product between every row of A and w is $\mathbb{X}$, and therefore w would be a fixed point. Then suppose that the i-th row of A is made of empty sets. We can now apply to A a permutation that exchanges the i-th row with the first one, and then exchanges the i-th with the first column. In this way we obtain a matrix where the first row is empty. By induction, suppose that there exists a permutation matrix P s.t. $P^T AP$ has the for

$$\begin{pmatrix} \emptyset & \cdots & \emptyset & \emptyset & \cdots & \emptyset \\ a'_{2,1} & \cdots & \emptyset & \emptyset & \cdots & \emptyset \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a'_{i,1} & \cdots & a'_{i,i-1} & \emptyset & \cdots & \emptyset \\ a'_{i+1,1} & & \cdots & & & a'_{i+1,n} \\ \vdots & & \ddots & & & \vdots \\ a'_{n,1} & & \cdots & & & a'_{n,n} \end{pmatrix}$$

and consider the vector $v = (\emptyset, \ldots, \emptyset, \mathbb{X}, \ldots, \mathbb{X})^T$, where the first i rows are null. v is not a fixed point of $P^T AP$ only if there exists $j > i$ s.t. the j-th row of $P^T AP$ has the form $(a'_{j,1}, \ldots, a'_{j,i}, \emptyset, \ldots, \emptyset)$. We can now apply to $P^T AP$ a permutation that exchanges the j-th row with the i-th one, and then exchanges the j-th with the i-th column. The inductive step is complete since we obtain the following matrix:

$$\begin{pmatrix} \emptyset & \cdots & \emptyset & \emptyset & \cdots & \emptyset \\ a'_{2,1} & \cdots & \emptyset & \emptyset & \cdots & \emptyset \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a'_{i+1,1} & \cdots & a'_{i+1,i} & \emptyset & \cdots & \emptyset \\ a'_{i+2,1} & & \cdots & & & a'_{i+2,n} \\ \vdots & & \ddots & & & \vdots \\ a'_{n,1} & & \cdots & & & a'_{n,n} \end{pmatrix}$$

which concludes the proof. ■

Proposition 3.5: If $A \in \{\emptyset, \mathbb{X}\}^{n\times n}$ is s.t. $\mathbb{X} \notin \sigma(A)$, then $\sigma(A) = \Sigma(\mathbb{X}) \setminus \mathbb{X}$.

Proof: By Proposition 3.4, we can assume that A is strictly lower triangular. Therefore, for every scalar $\lambda \subset \mathbb{X}$, the non-null vector $v = (\emptyset, \ldots, \emptyset, \mathcal{C}(\lambda))^T$ is an eigenvector of A associated with the eigenvalue λ, since it holds $\emptyset = Av = \lambda v = \emptyset$. ■

Corollary 3.2: For a matrix $A \in \{\emptyset, \mathbb{X}\}^{n\times n}$, the spectrum $\sigma(A)$ is given by either $\mathbb{X}$ or $\Sigma(\mathbb{X}) \setminus \mathbb{X}$.

Proof: The proof straightforwardly follows from Propositions 3.4 and 3.5. ■

Definition 3.4 (Contractive Map): A set-valued Boolean map $F : \Sigma(\mathbb{X})^n \to \Sigma(\mathbb{X})^n$ is said to be *contractive* w.r.t. the vector distance $\mathcal{D}$, if there exists a matrix $M \in \{\emptyset, \mathbb{X}\}^{n\times n}$, s.t. $\mathbb{X} \notin \sigma(M)$

$$\mathcal{D}\left(F(X), F(Y)\right) \subseteq M\mathcal{D}(X, Y), \text{ for all } X, Y \in \Sigma(\mathbb{X})^n.$$

♦

Remark 3: While Proposition 1 implies that, for every generic set-valued F, the distances $\mathcal{D}(X, Y)$ and $\mathcal{D}(F(X), F(Y))$ are always comparable through the incidence matrix $B(F)$, it is not ensured for F to satisfy the contractivity requirement of Definition 3.4, which is one of the reasons why contractivity is only sufficient for the global convergence of the corresponding SVBDS. ♦

Proposition 3.6: A set-valued Boolean map F is contractive if, and only if, $\mathbb{X} \notin \sigma(B(F))$.

Proof: The sufficiency is trivially satisfied by choosing $M = B(F)$. The necessity can be proved as follows. Let M be the Boolean matrix of Definition 3.4. Since $\mathbb{X} \notin \sigma(M)$, by Proposition 3.4, there exists a permutation P s.t. $P^T MP$ is strictly triangular. Moreover, by Proposition 3.5, we have that $B(F) \subseteq M$, and thus that also the matrix $P^T B(F)P$ is strictly triangular. Finally, by Proposition 3.5, we have that $\sigma(B(F)) = \Sigma(\mathbb{X}) \setminus \mathbb{X}$, which gives the thesis. ■

Definition 3.5 (Global Convergence): A SVBDS $X(t+1) = F(X(t))$, with $F : \Sigma(\mathbb{X})^n \to \Sigma(\mathbb{X})^n$, is *globally convergent*, if there exist $q \in \mathbb{N}$ and a unique $\xi \in \Sigma(\mathbb{X})^n$ s.t. for all $X \in \Sigma(\mathbb{X})^n$, it holds $F^t(X) = \xi$ for $t \geq q$. ♦

A result characterizing the global convergence of a SVBDS is the following:

Theorem 3.1: If the map F is contractive w.r.t. the vector distance $\mathcal{D}$, the SVBDS $X(t+1) = F(X(t))$ is globally convergent.

Proof: Since F is contractive, by Proposition 3.6, $\mathbb{X} \notin \sigma(B(F))$, and $B(F)$ is strictly lower or upper triangular, up to a transformation $P^T B(F)P$, with P a permutation matrix. This implies the existence of a non-negative integer $q \leq n$ s.t., $(B(F))^q = \emptyset$. By Corollary 3.1, we have $B(F^q) = B(F \cdots F) \subseteq B(F) \cdots B(F) = (B(F))^q = \emptyset$, which means that the application F^q is independent of X, i.e., there exists $\xi \in \Sigma(\mathbb{X})^n$ s.t., for all $X \in \Sigma(\mathbb{X})^n$, we have $F^q(X) = \xi$. Moreover, it holds that $F^{q+1}(\xi) = F^q(F(\xi)) = \xi$, being F^q is constant, and that $F^{q+1}(\xi) = F(F^q(\xi)) = F(\xi)$, thus $F(\xi) = \xi$, i.e., ξ is a fixed point of F. Suppose by absurdity that ξ is not unique, i.e., there exists $\eta \in \Sigma(\mathbb{X})^n$, $\eta \neq \xi$ s.t. $F(\eta) = \eta$. By Proposition 3.1, we have $D(\xi, \eta) = \mathcal{D}(F(\xi), F(\eta)) \subseteq B(F)\mathcal{D}(\xi, \eta)$, which repeated q times gives $D(\xi,\eta) \subseteq (B(F))^q \mathcal{D}(\xi,\eta) = \emptyset$, since $(B(F))^q = \emptyset$. Being $\mathcal{D}(\xi,\eta) = \emptyset$ we have the contradiction $\xi = \eta$. ■

Example 3.2: Consider a discrete-time dynamic system $X(t+1) = F(X(t))$, where $X = (X_1, X_2, X_3)^T \in \Sigma(\mathbb{X})^3$, F and thus its incidence matrix are

$$F(X)=\begin{pmatrix} X_1 \cup (X_2 \cap X_3) \\ X_1 \cup \mathcal{C}(X_2) \\ \mathcal{C}(X_1) \cap \mathcal{C}(X_2) \cap \mathcal{C}(X_3) \end{pmatrix}, B(F)=\begin{pmatrix} \mathbb{X} & \mathbb{X} & \mathbb{X} \\ \mathbb{X} & \mathbb{X} & \emptyset \\ \mathbb{X} & \mathbb{X} & \mathbb{X} \end{pmatrix}. \tag{11}$$

By Proposition 3.4, $\sigma(B(F))$ contains the eigenvalue $\mathbb{X}$, and thus, by Proposition 3.6, F is not contractive. ♦

We can finally prove the following result establishing a condition for global consensus convergence for a class of SVBDS:

Proposition 3.7 (Consensus of Linear SVBDS): A linear SVBDS of the form $X(t+1) = AX(t)$ possesses at least a set-valued equilibrium point that is a consensus state if, and only if

$$\bigcap_{i=1}^{n} a_{i,1} \cup a_{i,2} \cup \cdots \cup a_{i,n} \neq \emptyset.$$

Proof: The point $1_n\xi$ is a consensus equilibrium state if, and only if, $A1_n\xi = 1_n\xi$, i.e., for all $i = 1,\ldots,n$, $(a_{i,1} \cup a_{i,2} \cup \cdots \cup a_{i,n})\xi = \xi$. This holds if, and only if, the intersection of all matrix rows is not the emptyset. ■

IV. Binary Encoding of Set-Valued Boolean Dynamic Systems

While the results in Section III are very promising, a complete characterization of the spectrum of a general Boolean matrix is still far. Such an analysis is much simpler in the case of binary dynamic systems, which are BDS based on the simplest Boolean algebra where $\tilde{\mathbb{B}} = \mathbb{B} = \{0,1\}$ is the binary domain, $\wedge$ is the logical product ("and") $\cdot$, $\vee$ is the logical sum ("or") $+$, and $\neg$ is the "not" operator. Local convergence for binary dynamic systems has been addressed by introducing the notion of a discrete derivative [15]. A possible generalization of this notion for SVBDS is represented by the so-called Boolean derivative, proposed in [22]. However, this formulation of "derivative" gives rise to matrices containing not only the empty set and the unity, for which results characterizing their spectrum cannot be easily obtained.

For this reason, in the remainder of the paper, we pursue a different approach, which applies to SVBDS and allows their local convergence to be fully characterized. We show how a SVBDS can be translated into a binary dynamic system $x(t+1) = f(x(t))$, where $x \in \mathbb{B}^\kappa$ is binary state vector, and $f : \mathbb{B}^\kappa \to \mathbb{B}^\kappa$ is a binary iterative map, and κ is a suitable dimension [22]. We say that the above binary system encodes a SVBDS (see Definition 3.1), in the sense that every execution of the original system can be obtained by simulating the binary one. To this purpose, we first need to give the following definitions:

Definition 4.1 (Induced Unity Partition): Given a set-valued vector $\bar{X} = (\bar{X}_1,\ldots,\bar{X}_n)^T$, with each $\bar{X}_i \in \Sigma(\mathbb{X})$, we call the *induced unity partition* the following collection of sets:

$$\begin{aligned}
\bar{Z}_1(\bar{X}) &= \bar{X}_1 \cap \bar{X}_2 \cap \cdots \cap \bar{X}_{n-1} \cap \bar{X}_n \\
\bar{Z}_2(\bar{X}) &= \bar{X}_1 \cap \bar{X}_2 \cap \cdots \cap \bar{X}_{n-1} \cap \mathcal{C}(\bar{X}_n) \\
\bar{Z}_3(\bar{X}) &= \bar{X}_1 \cap \bar{X}_2 \cap \cdots \cap \mathcal{C}(\bar{X}_{n-1}) \cap \bar{X}_n \\
\bar{Z}_4(\bar{X}) &= \bar{\mathcal{C}}(X_1) \cap \bar{X}_2 \cap \cdots \cap \bar{X}_{n-1} \cap \bar{X}_n \\
\bar{Z}_5(\bar{X}) &= \bar{\mathcal{C}}(X_1) \cap \bar{X}_2 \cap \cdots \cap \bar{X}_{n-1} \cap \bar{\mathcal{C}}(X_n) \\
&\vdots \\
\bar{Z}_{\kappa-1}(\bar{X}) &= \mathcal{C}(\bar{X}_1) \cap \cdots \cap \mathcal{C}(\bar{X}_{n-1}) \cap \bar{X}_n \\
\bar{Z}_{\kappa}(\bar{X}) &= \mathcal{C}(\bar{X}_1) \cap \cdots \cap \mathcal{C}(\bar{X}_{n-1}) \cap \mathcal{C}(\bar{X}_n)
\end{aligned}$$

where $\kappa = 2^n$. ♦

It is straightforward to verify that the above sets form a partition of $\mathbb{X}$, i.e., $\bar{Z}_i \cap \bar{Z}_j = \emptyset$, and $\bar{Z}_1 \cup \cdots \cup \bar{Z}_n = \mathbb{X}$. Let $\mathcal{Z}(\hat{X})$ be the smallest collection of sets closed under set union and including $\bar{Z}_i$, $i = 1,\ldots,\kappa$.

Definition 4.2 (Binary Encoding): Given a unity partition $\bar{Z}_1,\ldots,\bar{Z}_\kappa$, a binary encoding is represented by a left-invertible application $\mathcal{L} : \mathcal{Z}(\hat{X}) \to \mathbb{B}^\kappa$, called the *encoder map*, that associates any set $X_i \in \mathcal{Z}(\hat{X})$ with a κ-dimensional binary vector x_i whose h-th component is 1 if, and only if, X_i has non-null overlapping with the set $\bar{Z}_h$, i.e., $x_i = \mathcal{L}(X_i)$ where

$$\begin{aligned}
\mathcal{L} :\quad & \mathcal{Z}(\hat{X}) \to \mathbb{B}^\kappa \\
& X_i \mapsto x_i = \left(x_1^i,\ldots,x_\kappa^i\right)^T, x_h^i = \begin{cases} 0 & \text{if } X_i \cap \bar{Z}_h = \emptyset \\ 1 & \text{otherwise.} \end{cases}
\end{aligned}$$

The left-inverse relation $\mathcal{L}^\dagger : \mathbb{B}^\kappa \to \mathcal{Z}(\hat{X})$, referred to as the *decoder map*, returns the set X_i originally associated with a binary vector x_i, i.e., $X_i = \mathcal{L}^\dagger(x_i)$ where

$$\begin{aligned}
\mathcal{L}^\dagger :\quad & \mathbb{B}^\kappa \to \mathcal{Z}(\hat{X}) \\
& x_i \mapsto X_i = \bigcup_{h=1,\ldots,\kappa,x_h^i=1} \bar{Z}_h.
\end{aligned}$$

♦

Definition 4.3: Given two Boolean Algebras with domain sets $\tilde{\mathbb{B}}_1$ and $\tilde{\mathbb{B}}_2$, respectively, two functions $\Phi_1 : \tilde{\mathbb{B}}_1^n \to \tilde{\mathbb{B}}_1$ and $\Phi_2 : \tilde{\mathbb{B}}_2^n \to \tilde{\mathbb{B}}_2$ are *formally identical*, if Φ_1 can be obtained from Φ_2 by replacing the operations of the second algebra involved in Φ_2 with the corresponding one in the first algebra, and vice-versa.

In the remainder of this section, by proving that all sets that can be obtained from arbitrary combination of unions, intersections, and complements of the sets $\bar{X}_i$, can be described as suitable unions of the sets $\bar{Z}_1,\ldots,\bar{Z}_\kappa$, we provide a method to find a binary encoding of the set-valued Boolean map F. This fact is formalized in the following main result:

Theorem 4.1 (Binary Encoding of SVBDS): Given a generic set-valued Boolean map $F : \Sigma(\mathbb{X})^n \to \Sigma(\mathbb{X})^n$ and an initial set-valued vector state $X^0 \in \Sigma(\mathbb{X})^n$, the evolution $X(t)$, for $t = 0, 1, \ldots$, of the SVBDS

$$X(t+1) = F\left(X(t)\right)$$
$$X(0) = X^0$$

can be computed as

$$X(t) = \mathcal{L}^{\dagger}\left(x(t)\right)$$

where $\mathcal{L}$ is the binary encoding associated with the unity partition corresponding the sets $X_1^0, \ldots X_n^0$, and $x(t) \in \mathbb{B}^{n\times\kappa}$, for $t = 0, 1, \ldots$, is the evolution of the binary dynamic system

$$(x_{i,1}(t+1), \ldots, x_{i,\kappa}(t+1))$$
$$= \left(f_i\left(x_{1,1}(t), \ldots, x_{n,1}(t)\right), \ldots, f_i\left(x_{1,\kappa}(t), \ldots, x_{n,\kappa}(t)\right)\right)$$
$$(x_{i,1}(0), \ldots, x_{i,\kappa}(0)) = \mathcal{L}\left(X_i^0\right)^T$$

for $i = 1, \ldots, n$, where each function $f_i : \mathbb{B}^\kappa \to \mathbb{B}$ is formally identical to the corresponding function F_i.

Proof: As a first step we need to prove that the intersection, the union, and the complement of any two sets $\bar{X}_i$ and $\bar{X}_j$ can be computed by operating bitwise logical product, sum, and complement, respectively, on the binary vectors $\bar{x}_i = \mathcal{L}(\bar{X}_i)$ and $\bar{x}_j = \mathcal{L}(\bar{X}_j)$, and then projecting the results back through the left-inverse map $\mathcal{L}^{\dagger}$. First consider the following set intersection:

$$\bar{X}_i \cap \bar{X}_j = \mathcal{L}^{\dagger}(\bar{x}_i) \cap \mathcal{L}^{\dagger}(\bar{x}_j) =$$
$$= \left(\bigcup_{h=1,\bar{x}_h^i=1}^{\kappa} Z_h\right) \cap \left(\bigcup_{l=1,\bar{x}_l^j=1}^{\kappa} Z_l\right)$$

which can be expanded, by distributing the set intersection, as the union of the sets given by the intersection of one Z_h with one Z_l. As all Z_i are disjoint, only those Z_i appearing in both the original sets, $\bar{X}_i$ and $\bar{X}_j$, remain in the intersection. Therefore, we can write

$$\bar{X}_i \cap \bar{X}_j = \bigcup_{h=1,\left(\bar{x}_h^i=1\right)\wedge\left(\bar{x}_h^j=1\right)}^{\kappa} Z_h =$$
$$= \bigcup_{h=1,\bar{x}_h=1}^{\kappa} Z_h = \mathcal{L}^{\dagger}(\bar{x})$$

where $\bar{x} \in \mathbb{B}^\kappa$ is obtained through the following *bit-wise operation* on the vectors $\bar{x}_i$ and $\bar{x}_j$:

$$\bar{x} = \bar{x}_i\bar{x}_j = (\bar{x}_{i,1}\bar{x}_{j,1}, \ldots, \bar{x}_{i,\kappa}\bar{x}_{j,\kappa})$$

which proves the correspondence relation

$$\bar{X}_i \cap \bar{X}_j \underset{\mathcal{L}^{\dagger}}{\overset{\mathcal{L}}{\rightleftharpoons}} \bar{x}_i\bar{x}_j. \tag{12}$$

Moreover, consider the following set union:

$$\bar{X}_i \cup \bar{X}_j = \mathcal{L}^{\dagger}(\bar{x}_i) \cup \mathcal{L}^{\dagger}(\bar{x}_j) =$$
$$= \left(\bigcup_{h=1,\bar{x}_h^i=1}^{\kappa} Z_h\right) \cup \left(\bigcup_{l=1,\bar{x}_l^j=1}^{\kappa} Z_l\right) =$$
$$= \bigcup_{h=1,\left(\bar{x}_h^i=1\vee\bar{x}_h^j=1\right)}^{\kappa} Z_h = \bigcup_{h=1,\bar{x}_h=1}^{\kappa} Z_h =$$
$$= \mathcal{L}^{\dagger}(\bar{x})$$

with

$$\bar{x} = \bar{x}_i + \bar{x}_j = (\bar{x}_{i,1} + \bar{x}_{j,1}, \ldots, \bar{x}_{i,\kappa} + \bar{x}_{j,\kappa})$$

which proves the correspondence relation

$$\bar{X}_i \cup \bar{X}_j \underset{\mathcal{L}^{\dagger}}{\overset{\mathcal{L}}{\rightleftharpoons}} \bar{x}_i + \bar{x}_j. \tag{14}$$

Finally, consider the following complementation of two sets:

$$\mathcal{C}(\bar{X}_i) = \mathcal{C}\left(\mathcal{L}^{\dagger}(\bar{x}_i)\right) = \mathcal{C}\left(\bigcup_{h=1,\bar{x}_h^i=1}^{\kappa} Z_h\right) =$$
$$= \bigcap_{h=1,\bar{x}_h^i=1}^{\kappa} \mathcal{C}(Z_h).$$

By definition $\mathcal{C}(Z_h)$ is the set of points not belonging to Z_h, that can be obtained as the union of all the other partition sets

$$\bar{Z}_h = \mathcal{C}(Z_h) = \bigcup_{h'=1,h'\neq h}^{\kappa} Z_h' =$$
$$= Z_1 \cup Z_2 \cup \cdots \cup Z_{h-1} \cup Z_{h+1} \cup \cdots \cup Z_\kappa =$$
$$= \mathcal{L}^{\dagger}(z_1) \cup \mathcal{L}^{\dagger}(z_2) \cup \cdots \cup \mathcal{L}^{\dagger}(z_{h-1}) \cup$$
$$\cup \mathcal{L}^{\dagger}(z_{h+1}) \cup \cdots \cup \mathcal{L}^{\dagger}(z_\kappa) =$$
$$= \bigcup_{l=1,\alpha_{l,h}=1}^{\kappa} Z_l$$

with $\alpha_h = z_1 + z_2 + \cdots + z_{h-1} + z_{h+1} + \cdots + z_\kappa$. Easy computation gives a logical vector $\alpha_h = (1, \ldots, 1, 0, 1, \ldots, 1)^T$ containing all entries to 1 except for the h-th one. Finally, intersection of all $\bar{Z}_h$ yields

$$\mathcal{C}(\bar{X}_i) = \bigcap_{h=1,\bar{x}_h^i=1}^{\kappa} \bar{Z}_h =$$
$$= \bigcap_{h=1,\alpha_h=1}^{\kappa} Z_h, \text{ with } \alpha = \alpha_1\alpha_2, \ldots, \alpha_r$$

where r is the number of $\bar{x}_i$'s non-null components. As all these components are assigned with a logical vector α_l containing a null element at position l, and as all these components are considered, the sets that remain in the intersection are those not belonging to $\bar{X}_i$, or in other words, for which $\bar{x}^i_h = 0$. Hence, we have

$$\mathcal{C}(\bar{X}_i) = \bigcup_{h=1,\bar{x}^i_h=0}^{\kappa} Z_h = \bigcup_{h=1,\bar{x}^i_h=1}^{\kappa} Z_h = \\ = \mathcal{L}^\dagger(y_i)$$

with

$$y_i = \neg\bar{x}_i = (\neg\bar{x}_{i,1},\dots,\neg\bar{x}_{i,\kappa})$$

which proves the correspondence relation

$$\mathcal{C}(\bar{X}_i) \underset{\mathcal{L}^\dagger}{\overset{\mathcal{L}}{\rightleftharpoons}} \neg\bar{x}_i. \tag{14}$$

The proof of the theorem can now be given as follows. First assume the choice $\bar{X} = (\bar{X}_1,\dots,\bar{X}_n) = X^0$. The generic function F_i, involved in the application F, generates a new set $\bar{X}'_i$, by combining, according to a suitable order, pairs of the elements of $\bar{X}$ or of their combinations, via set intersection, union, and complement. Based on the discussion above, it follows that $\bar{X}'_i$ is a set that can be obtained by operating, with the same order, all the corresponding bit-wise logical operations on suitable logical vectors. More precisely, having denoted with f_i the binary function formally identical to F_i and given the binary vectors $\bar{x}_i = \mathcal{L}(\bar{X}_i)$, we can write $\bar{X}'_i = \mathcal{L}^\dagger(\bar{x}'_i)$, where

$$\bar{x}'_i = (\bar{x}'_{i,1},\dots,\bar{x}'_{i,\kappa}) = \\ = (f_i(\bar{x}_{1,1},\dots,\bar{x}_{n,1}),\dots,f_i(\bar{x}_{1,\kappa},\dots,\bar{x}_{n,\kappa}))$$

which proves the correspondence relation

$$F_i(\bar{X}) \underset{\mathcal{L}^\dagger}{\overset{\mathcal{L}}{\rightleftharpoons}} (f_i(\bar{x}_{1,1},\dots,\bar{x}_{n,1}),\dots,f_i(\bar{x}_{1,\kappa},\dots,\bar{x}_{n,\kappa}))\,.$$

Repeating the same reasoning on the new state $\bar{X}' = (\bar{X}'_1, \dots,\bar{X}'_n)^T$ of the SVBDS yields to the state $\bar{X}''$, which can also be expressed as a suitable union of the sets $\bar{Z}_i$. This fact also shows that the same binary encoding can be used at every time step t, which concludes the proof. ■

Remark 4: Note that, although the binary encoding $\mathcal{L}$ depends on the initial state $X(0)$, the above theorem implies that the same encoding can be used during the entire evolution of the original SVBDS.

Remark 5: Reordering the binary state variables based on their second index reveals that the evolution of a SVBDS can be computed through that of the following κ decoupled binary systems:

$$(x_{1,j}(t+1),\dots,x_{n,j}(t+1)) = \phi\,(x_{1,j}(t),\dots,x_{n,j}(t))$$

for $j = 1,\dots,\kappa$, where

$$\phi : \mathbb{B}^n \to \mathbb{B}^n \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \mapsto \begin{pmatrix} f_1(y_1,\dots,y_n) \\ \vdots \\ f_n(y_1,\dots,y_n) \end{pmatrix}. \tag{15}$$

While each of these systems are initialized with a different state, they share the same binary dynamic map ϕ.

Example 4.1: Consider again the system of Example 3.2, where the unity is $\mathbb{X} = [0,\infty)$ and system's initial state is $X(0) = ([1,5],[3,7],[2,3)\cup[4,5]\cup[6,\infty))^T$. The first two values of the system's state can be obtained according to (11), which yields

$$X(1) = \begin{pmatrix} [1,5]\cup[6,7] \\ [0,5]\cup(7,\infty) \\ [0,1) \end{pmatrix}, X(2) = \begin{pmatrix} [0,5]\cup[6,7] \\ [1,7] \\ (5,6) \end{pmatrix}. \tag{16}$$

The same results can be obtained by using the binary encoding of the system. We first need to consider the collection of sets

$$\begin{aligned} Z_1 &= X_1 \cap X_2 \cap X_3 = [4,5] \\ Z_2 &= X_1 \cap X_2 \cap \mathcal{C}(X_3) = [3,4) \\ Z_3 &= X_1 \cap \mathcal{C}(X_2) \cap X_3 = [2,3) \\ Z_4 &= X_1 \cap \mathcal{C}(X_2) \cap \mathcal{C}(X_3) = [1,2) \\ Z_5 &= \mathcal{C}(X_1) \cap X_2 \cap X_3 = [6,7] \\ Z_6 &= \mathcal{C}(X_1) \cap X_2 \cap \mathcal{C}(X_3) = (5,6) \\ Z_7 &= \mathcal{C}(X_1) \cap \mathcal{C}(X_2) \cap X_3 = (7,\infty) \\ Z_8 &= \mathcal{C}(X_1) \cap \mathcal{C}(X_2) \cap \mathcal{C}(X_3) = [0,1) \end{aligned}$$

and then associate each state X_i with a binary vector $x_i \in \mathbb{B}^8$. Based on Theorem 4.1, the original system can be simulated by the binary dynamic system $x(t+1) = f(x(t))$, where $x = (x_1^T, x_2^T, x_3^T)^T$ and

$$\begin{aligned} f(x) = (&x_{1,1} + x_{2,1}x_{3,1},\dots,x_{1,8} + x_{2,8}x_{3,8}, \\ &x_{1,1}\neg x_{2,1},\dots,x_{1,8}\neg x_{2,8}, \\ &\neg x_{1,1}\neg x_{2,1}\neg x_{3,1},\dots,\neg x_{1,8}\neg x_{2,8}\neg x_{3,8}) \end{aligned} \tag{17}$$

with initial state $x(0) = (x_1^T(0), x_2^T(0), x_3(0)^T)^T$, where $x_1(0) = \mathcal{L}(X_1(0)) = (1,1,1,1,0,0,0,0)$, $x_2(0) = \mathcal{L}(X_2(0)) = (1,1,0,0,1,1,0,0)$, and $x_3(0) = \mathcal{L}(X_3(0)) = (1,0,1,0,1,0,1,0)$. The first two values of the binary state are $x(1) = (x_1(1), x_2(1), x_3(1))^T = f(x(0))$ and $x(2) = (x_1(2), x_2(2), x_3(2))^T = f(x(1))$, with

$$\begin{aligned} x_1(1) &= (1,1,1,1,1,0,0,0), & x_1(2) &= (1,1,1,1,1,0,0,1) \\ x_2(1) &= (1,1,1,1,0,0,1,1), & x_2(2) &= (1,1,1,1,1,1,0,0) \\ x_3(1) &= (0,0,0,0,0,0,0,1), & x_3(2) &= (0,0,0,0,0,1,0,0) \end{aligned}$$

which corresponds to the original system's states

$$X(1) = \mathcal{L}^\dagger\left(x(1)\right) = \begin{pmatrix} Z_1 \cup Z_2 \cup Z_3 \cup Z_4 \cup Z_5 \\ Z_1 \cup Z_2 \cup Z_3 \cup Z_4, \cup Z_7 \cup Z_8 \\ Z_8 \end{pmatrix}$$

$$X(2) = \mathcal{L}^\dagger\left(x(2)\right) = \begin{pmatrix} Z_1 \cup Z_2 \cup Z_3 \cup Z_4 \cup Z_5 \cup Z_8 \\ Z_1 \cup Z_2 \cup Z_3 \cup Z_4 \cup Z_5 \cup Z_6 \\ Z_6 \end{pmatrix}$$

being clearly equal to the values obtained in (16). ♦

V. Convergence Revisited and Completed

We first show how the encoding technique presented above gives rise to the same global convergence conditions of Section III. First recall from [14] the following notions and result:

Definition 5.1 (Spectral Radius): Given a binary matrix $A \in \mathbb{B}^{m\times m}$, its spectral radius $\rho(A)$ is given by its biggest eigenvalue in the sense of the partial order relation $\leq$.

Definition 5.2 (Binary Vector Distance): Given two binary vectors $x, y \in \mathbb{B}^m$, the binary vector distance between the two vectors is represented by the application

$$\begin{aligned} d: \quad & \mathbb{B}^m \times \mathbb{B}^m \to \mathbb{B}^m \\ & (x,y) \mapsto (x_1 \otimes y_1, \ldots, x_m \otimes y_m) \end{aligned}$$

where $\otimes$ is the exclusive disjunction

$$\begin{aligned} \otimes: \quad & \mathbb{B} \times \mathbb{B} \to \mathbb{B} \\ & (x_i, y_i) \mapsto (\neg x_i y_i) + (x_i \neg y_i). \end{aligned}$$

Note that the Boolean vector distance of Definition 3.2 specializes to the binary vector distance if the domain $\tilde{\mathbb{B}}$ is binary.

Theorem 5.1: A map $f: \mathbb{B}^m \to \mathbb{B}^m$ is contractive w.r.t. the binary vector distance d if, and only if, the following equivalent conditions hold: 1) $\rho(B(f)) = 0$, 2) there exists a permutation matrix P s.t. $P^T B(f) P$ is strictly lower or upper triangular; 3) $B(f)^q = 0$, with $0 \leq q \leq m$. Moreover, if f is contractive, there exists a positive integer $q \leq m$ s.t. f^q, the composition of f with itself q times, is a constant map, i.e., it is independent of the input vector. ♦

Consider a SVBDS characterized by a set-valued map $F: \Sigma(\mathbb{X})^n \to \Sigma(\mathbb{X})^n$ and its corresponding binary dynamic system characterized by the function $f: \mathbb{B}^m \to \mathbb{B}^m$, with $m = n\kappa$. We first want to show that the properties of global contractivity of F can be investigated in the binary domain by studying the same properties of f. To this purpose we can prove the following:

Lemma 5.1: Having denoted with $B(f)$ the incidence matrix of f, the following equivalence holds $\{B(F)\}_{i,j} = \mathbb{X}$ if, and only if, $\{B(f)\}_{2^{n(i-1)}+1:2^{n(i-1)},2^{n(j-1)}+1:2^{n(j-1)}} = I$, where I is the identity matrix and the notation $M_{i:j,k:l}$ indicates a submatrix of M obtained by extracting its rows from i to j and its columns from k to l.

Proof: First rewrite the map $F(X_1, \ldots, X_n) = (F_1(X^1_{i_1}, \ldots, X^1_{i_{k_1}}), \ldots, F_n(X^n_{i_1}, \ldots, X^n_{i_{k_n}}))^T$, where $X^j_{i_l}$ are the variables on which the j-th component of the image of F actually depends. By the encoding map, we have that $B(f) \in \mathbb{B}^{n\kappa \times n\kappa}$ equals

$$\begin{pmatrix} 0 & \ldots & 0 & \overbrace{I}^{i_1^1} & 0 & \cdots & \overbrace{I}^{i_2^1} & \cdots & \overbrace{I}^{i_{k_1}^1} & \cdots & 0 \\ \vdots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \vdots \\ 0 & \cdots & \underbrace{I}_{i_1^n} & 0 & \cdots & \underbrace{I}_{i_2^n} & 0 & \cdots & \underbrace{I}_{i_{k_n}^n} & \cdots & 0 \end{pmatrix}$$

where 0 and I are here the zero and identity matrices, respectively. The thesis easily follows since the matrix $B(F)$, by replacing 0 with $\emptyset$ and I with $\mathbb{X}$, has exactly the same form:

$$\begin{pmatrix} 0 & \ldots & 0 & \overbrace{\mathbb{X}}^{\mathbb{X}_1^1} & 0 & \cdots & \overbrace{\mathbb{X}}^{\mathbb{X}_2^1} & \cdots & \overbrace{\mathbb{X}}^{\mathbb{X}_{k_1}^1} & \cdots & 0 \\ \vdots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \vdots \\ 0 & \cdots & \underbrace{\mathbb{X}}_{\mathbb{X}_1^n} & 0 & \cdots & \underbrace{\mathbb{X}}_{\mathbb{X}_2^n} & 0 & \cdots & \underbrace{\mathbb{X}}_{\mathbb{X}_{k_n}^n} & \cdots & 0 \end{pmatrix}.$$

■

Theorem 5.2 (Global Convergence): The dynamic map $F: \Sigma(\mathbb{X})^n \to \Sigma(\mathbb{X})^n$ of a SVBDS is contractive if, and only if, its encoding $\mathcal{L}(F): \mathbb{B}^{n\kappa} \to \mathbb{B}^{n\kappa}$ is contractive.

Proof: By Theorems 5.1 and 3.1, it is sufficient to prove that $\mathbb{X} \notin \sigma(B(F))$ if, and only if, $\rho(B(f)) = 0$. We have that $\mathbb{X} \notin \sigma(B(F))$ if, and only if, there exists a permutation matrix P s.t. $P^T B(F) P$ is strictly lower or upper triangular, and Theorem 5.1 assures that $\rho(B(f)) = 0$ if, and only if, there exists a permutation matrix p s.t. $p^T B(f) p$ is strictly lower triangular. By Lemma 5.1, it holds $\{B(F)\}_{ij} = \mathbb{X}$ if, and only if, $\{B(f)\}_{2^{n(i-1)}+1:2^{n(i-1)},2^{n(j-1)}+1:2^{n(j-1)}} = I$, which immediately implies that $\mathbb{X} \notin \sigma(B(F))$ if, and only if, $\rho(B(f)) = 0$. ■

Remark 6: It is worth remarking that, for some initial conditions $X(0)$, the induced unity partition $\bar{Z}_i$, for $i = 1, \ldots, \kappa$, can be degenerate, i.e., some sets in the collection can be emptysets. For this specific initial condition the original SVBDS may converge notwithstanding the fact that state components of the encoded binary system associated with empty sets may not converge. However, other initial conditions certainly exist s.t. the same state components are not associated to empty sets, hence the necessity that the encoded binary system global convergence for the same property for the SVBDS.

Remark 7: $\rho(B(f)) = 0$ if, and only if, $\rho(\tilde{B}(F)) = 0$, where $\tilde{B}(F)$ is the matrix obtained substituting 1 to $\mathbb{X}$ and 0 to $\emptyset$. This can be easily seen by using the equivalent formulation in terms of permutation matrices given by Theorem 5.1. ♦

Example 5.1 (Cont'd): Consider again the system of Example 3.2. Following the derivation of the associated logical system (17), the incidence matrix of the corresponding binary dynamic system is $B(f) = \{C_{i,j}\}$, with $C_{i,j} \in \mathbb{B}^{8\times 8}$ and $C_{i,j} = I_8$ for $i \neq 2$, $j \neq 3$ and $C_{2,3} = 0_8$, where 0_8 and I_8 are the null and identity matricies of dimension 8, respectively. According to Theorem 5.2, the system of the Example 3.2 is contractive if, and only if, its binary dynamic system is contractive. Based on Theorem 5.1, this system is not contractive, since $B(f)$ cannot be put in a strictly triangular form by a permutation matrix (there should be a zero row). Then, based on Theorem 5.2, we can conclude that the system of Example 3.2 is not contractive, as we obtained in Section III by using Theorem 3.1. ♦

We now move on to attack the study of local convergence of a SVBDS, for which we recall from [15] the following definitions and results on the local convergence of a binary map f about an *equilibrium point* x s.t. $f(x) = x$.

Definition 5.3 (Von-Neumann Neighborhood (VNN)): The VNN of a point $x \in \mathbb{B}^m$ is the set $V(x)$ of all points differing from x in at most one component, i.e., $V(x) = \{x, \tilde{x}^1, \ldots, \tilde{x}^m\}$, where $\tilde{x}^j = (x_1, \ldots, x_{j-1}, \neg x_j, x_{j+1}, \ldots, x_m)^T$.

Definition 5.4 (Discrete Derivative): The discrete derivative of a binary map $f : \mathbb{B}^m \to \mathbb{B}^m$ at a point $x \in \mathbb{B}^m$ is a binary matrix $f'(x) = \{f'_{i,j}\}$, s.t. $f'_{i,j} = 1$ if, and only if, a variation in the j-th component of x produces a variation in the i-th component of $f(x)$, i.e., $f'_{i,j}(x) = f_i(x) \otimes f_i(\tilde{x}^j)$.

Definition 5.5: An equilibrium point $x^* \in \mathbb{B}^m$ is *attractive* in its VNN $V(x^*)$ if, for all $y \in V(x^*)$, the following relations hold: 1) $f(y) \in V(x^*)$ and 2) $\exists \bar{m} \in \mathbb{N}$ s.t., $f^{\bar{m}}(y) = x^*$.

Definition 5.6: A binary map f is said to be *locally convergent* at an equilibrium point x^* if x^* is attractive in its VNN.

Theorem 5.3: An equilibrium point $x^* \in \mathbb{B}^m$ is attractive in its VNN if, and only if, the following two relations hold: 1) $\rho(f'(x^*)) = 0$, and 2) $f'(x^*)$ contains at most one element to 1 in each column. ♦

We can now focus on a generic SVBDS with an equilibrium point given by $X^* = (X_1^*, \ldots, X_n^*)^T \in \Sigma(\mathbb{X})^n$. The following definitions and results extend the corresponding ones in [15]:

Definition 5.7 (Complemented Neighborhood (CN)): The CN $\tilde{V}(X^*)$ of a point $X^* \in \Sigma(\mathbb{X})^n$ is the set of points differing from it in at most one complemented component, i.e., $\tilde{V}(X) = \{X, \tilde{X}^1, \ldots, \tilde{X}^n\}$, with $\tilde{X}^j = (X_1^*, \ldots, X_{j-1}^*, \mathcal{C}(X_j^*), X_{j+1}^*, \ldots, X_n^*)^T$. ♦

Note that, if $X^* \in \Sigma(\mathbb{X})^n$, it also holds that $\tilde{V}(X^*) \in \mathcal{Z}(X^*)$.

Definition 5.8: An equilibrium point X^* of $F : \Sigma(\mathbb{X})^n \to \Sigma(\mathbb{X})^n$ is *attractive* in its CN if the following two relations hold: 1) $F(\tilde{V}(X^*)) \subset \tilde{V}(X^*)$ and 2) $F^n(Y) = X$, for all $Y \in \tilde{V}(X^*)$.

We can prove the following result:

Theorem 5.4 (Local Convergence of SVBDS): An equilibrium point X^* of the generic set-valued map $F : \Sigma(\mathbb{X})^n \to \Sigma(\mathbb{X})^n$ is attractive in its CN $V(X^*)$ if, and only if, the binary equilibria $y_j^* = (x_{1,j}^*, \ldots, x_{n,j}^*)^T$, for $j = 1, \ldots, \kappa$, where $(x_{1,1}^*, \ldots, x_{1,\kappa}^*, \ldots, x_{n,1}^*, \ldots, x_{n,\kappa}^*) = \mathcal{L}(X^*)$, are all attractive in their VNN $V(y_j^*)$ for the binary map ϕ in (15).

Proof: By comparing Definition 5.3 and Definition 5.7, we first obtain that $F(\tilde{V}(X^*)) \in \tilde{V}(X^*)$ if, and only if, $\phi(V(y_j^*)) \in V(y_j^*)$, for all y_j^*. Moreover, we also obtain that $F^n(Y) = X^*$ for all $Y \in \tilde{V}(X^*)$ if, and only if, $\phi^n(y) = y_j^*$ for all $y \in V(y_j^*)$ and all y_j. Based on this, by Theorem 5.2, the thesis is implied. ■

Example 5.2 (Cont'd): Consider again the system of Example 3.2, with a generic initial condition given by $X(0) = (A, B, C)^T$, where A, B, and C are time-invariant sets in $\Sigma(\mathbb{X})$. As discussed above, the system is not contractive (the spectrum of its incidence matrix $B(F)$ contains $\mathbb{X}$, or equivalently the spectral radius of the incidence matrix $B(f)$ of its encoding f is 1). Moreover, it is easy to verify that the state vector obtained with $A = B = \mathbb{X}$ and $C = \emptyset$ is an equilibrium of the system. The partition sets of the binary encoding are $Z_1 = Z_3 = \cdots = Z_8 = \emptyset$ and $Z_2 = \mathbb{X}$. After reordering the unique non-empty set, i.e., $\kappa = 1$, we obtain the partition set described by $Z_{1'} = \mathbb{X}$. The encoded binary vector state is $\bar{x} = \mathcal{L}(\bar{X}) = (x_1^T, x_2^T, x_3^T)^T$, with $x_1 = x_2 = 1$ and $x_3 = 0$, which is attractive in its VNN for the encoded binary map $f(x) = (x_{1,1} + x_{2,1}x_{3,1}, x_{1,1}\bar{x}_{2,1}, \bar{x}_{1,1}\bar{x}_{2,1}\bar{x}_{3,1})^T$. Therefore, by Theorem 5.4, the equilibrium point $\bar{X}$ is attractive in its CN for the original system. ♦

VI. Application to Distributed Chart Estimation

Consider a mosaicking application involving reconstruction of a geographical chart, by using n balloon stations deployed over the area. Let $\mathcal{Q}$ be the set of points on the Earth surface with latitude and longitude comprised within 30° N and 75° N, and 30° W and 50° E, respectively, roughly corresponding to the European continent. By acquiring a spotlight-type image of the underneath surface, each station $\mathcal{A}_i$ is able to produce a local *estimated chart* $I_i(0) \subseteq \mathcal{Q}$, composed of a collection of connected sets representing the estimated emerged lands. Moreover, let $V_i(0) \subseteq \mathcal{Q}$ be a region representing the field-of-view of $\mathcal{A}_i$, i.e., the set of points that can be "seen" by $\mathcal{A}_i$. For simplicity we model each $V_i(0)$ as a circle centered at the projection of $\mathcal{A}_i$'s position on the Earth surface and having radius given by a constant d. Within its field-of-view, each $\mathcal{A}_i$ may incorrectly include portions of sea or neglect parts of existing lands in $I_i(0)$.

We assume a minimum *measurement multiplicity constraint* requiring that each point in a subset $\mathcal{Q}' \subseteq \mathcal{Q}$ lays within the intersection of at least $r > 0$ field-of-views; we further assume a *bounded detection error constraint* requiring that, in every set of r stations satisfying the measurement multiplicity constraint, at most γ of these stations may perform an incorrect detection. By assuming that each station is able to share data via communication with other neighboring stations, we seek a solution enabling an end-user on the ground, willing e.g., to use the chart information for navigation purpose, to efficiently and promptly poll its nearest station so as to retrieve a unique and *consistent* chart of the continent's surface.

A first solution can be found by following a *centralized* approach. In this solution a central processor with high computation and memory capacities must receive the estimated charts and visibility regions from all the stations, combine them into a global geographical chart, and then send this chart back to all stations. To cope with incorrect land detection, a well-known result from fault-tolerance theory can be used [23], requiring that, for every point $q \in \mathcal{Q}'$, the central processor uses the estimated charts received from at least $r' = 2\gamma + 1$ different stations including q in their field-of-view (thus it must hold the condition $r \geq r'$). Among these r' estimated charts, if γ is the maximum number of them that are possibly containing detection errors at least $\gamma + 1$ charts—the majority—contain correct information for that point. According to this approach, the central process can reconstruct an estimated chart I^* by using the following formula:

$$I^* = \bigcup_{q \in \mathcal{Q}} \left(\bigcup_{H \in S_{\gamma+1}(K_q)} \left(\bigcap_{h \in H} I_h(0) \right) \right) \tag{18}$$

where $S_\alpha(A)$ returns the set of all sets of cardinality α composed of elements in A, and

$$K_q = \{i \in \{1, \ldots, n\} | q \in V_i(0)\}.$$

Intuitively, the formula can be explained as follows. For every point $q \in \mathcal{Q}$, it is necessary to generate all possible agent index r'-tuples and, for each tuple, to intersect the initially estimated charts $I_h(0)$ of the involved agents. Note that each $I_h(0)$ may include estimated lands that lie outside the initial confidence region $V_h(0)$; however these estimated lands are dropped out from I^* if they are not confirmed by a sufficient number of agents. With the same reasoning, a region of global visibility can be defined as follows, representing the region for which the centralized process has received sufficient information to perform high accuracy land detection:

$$V^* = \bigcup_{q \in \mathcal{Q}} \left(\bigcup_{H \in S_{\gamma+1}(K_q)} \left(\bigcap_{h \in H} V_h(0) \right) \right). \tag{19}$$

While it effectively solves the problem, this centralized solution is unsatisfactory for at least three reasons: The first is *non-scalability*, since the amount of data to be elaborated by the processor requires computation and memory capacities increasing super-linearly with the number n of stations; secondly, the approach requires an explicit management of *message routing* in order to allow every station to reach and be reached from the central processor; third, it leads to the implementation of a system that has a *single-point of failure*.

By bearing in mind the centralized solution as an indicator of achievable performance, we seek a solution that is fully distributed, i.e., no central processor is used, and that requires no message routing, namely all stations must reach a consensus on the continent chart by exchanging messages only with their one-hop neighbors. We assume a minimum *communication connectivity constraint* requiring that, for every point $q \in V_i(0)$, each $\mathcal{A}_i$ has at least $2\gamma + 1$ communication neighbors whose field-of-view comprises q. Let the *set-valued variable* $X_i \subseteq \mathcal{Q} \times \mathcal{Q}$ be the state of $\mathcal{A}_i$, and C_i the index set of its communication neighbors. A possible distributed solution can be obtained by using a SVBDS, where $\mathcal{A}_i$'s state is initialized with the value

$$X_i(0) = (I_i(0), V_i(0))$$

and then iteratively updated according to the rule

$$\begin{cases} I_i(t+1) = \bigcup_{H \in S_{\gamma+1}(C_i)} \bigcap_{h \in H} \left(V_h(t) \cap I_h(t)\right) \\ V_i(t+1) = \bigcup_{H \in S_{\gamma+1}(C_i)} \bigcap_{h \in H} V_h(t). \end{cases} \tag{20}$$

We need to prove that, by means of the update rule in (20), each state X_i converges to the state (I^*, V^*). First note that, having defined the set $K = \{1, \ldots, n\}$, (18) and (19) can be rewritten as

$$\begin{cases} I^* = \bigcup_{H \in S_{\gamma+1}(K)} \bigcap_{h \in H} \left(V_h(0) \cap I_h(0)\right) \\ V^* = \bigcup_{H \in S_{\gamma+1}(K)} \bigcap_{h \in H} V_h(0). \end{cases}$$

It is straightforward to verify that the state $X^* = \mathbf{1}_n(I^*, V^*)$ is an equilibrium for the above SVBDS. While the system is not globally convergent to X^*, it is possible to show that such an equilibrium is attractive in a region that is large enough to tolerate up to γ incorrect land detections. Let us consider the general case in which three assumptions hold: 1) $I^* \neq \emptyset$ and

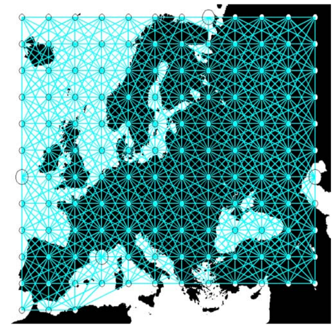

Fig. 2. Deployment and connectivity of 135 stations over the European continent (from top to down, $\mathcal{A}_8$, $\mathcal{A}_{73}$, $\mathcal{A}_{84}$, and $\mathcal{A}_{112}$ are represented with bigger circles).

$V^* \neq \emptyset$, indicating that some land exists and is in the field-of-view of at least $2\gamma + 1$ stations; 2) a portion of sea, $\mathcal{C}(I^*)$, is included in the global visibility region V^*; and 3) $V^* \subset \mathcal{Q}$, indicating that a portion of the European continent is not in the field-of-view of at least $2\gamma + 1$ stations. The unity partition sets described in Section IV are given by

$$Z_1 = I_1(0) \cap \cdots \cap I_n(0) \cap V_1(0) \cap \cdots \cap V_n(0) = I^* \cap V^* = I^*$$

$$\vdots$$

$$\begin{aligned} Z_\kappa &= \mathcal{C}\left(I_1(0)\right) \cap \cdots \cap \mathcal{C}\left(I_n(0)\right) \cap \mathcal{C}\left(V_1(0)\right) \cap \cdots \cap \mathcal{C}\left(V_n(0)\right) = \\ &= \mathcal{C}(I^*) \cap \mathcal{C}(V^*) = \mathcal{C}(V^*) \end{aligned}$$

with $\kappa = 2^{2n}$. In the above equations the property $I^* \subseteq V^*$, which can be deduced by simple reasoning on the definitions of I^* and V^*, has been used. It is possible to provide physical interpretations for the sets Z_1, representing the emerged lands that can be detected by using the information available from all stations, and Z_κ, representing the region not included in the field-of-view of at least $2\gamma + 1$ stations. The original SVBDS can be simulated by a binary dynamic system, where the (i, j)-th update rule is

$$\begin{cases} \eta_{i,j}(t+1) = \sum_{H \in S_{\gamma+1}(C_i)} \prod_{h \in H} \left(v_{h,j}(t)\eta_{h,j}(t)\right) \\ v_{i,j}(t+1) = \sum_{H \in S_{\gamma+1}(C_i)} \prod_{h \in H} v_{h,j}(t) \end{cases} \tag{21}$$

for $i = 1, \ldots, n$, $j = 1, \ldots, \kappa$, and the initial state is obtained from the relations

$$x_i(0) = \left(\eta_{i,1}(0), \ldots, \eta_{i,\kappa}(0), v_{i,1}(0), \ldots, v_{i,\kappa}(0)\right) = \mathcal{L}\left(X_i(0)\right)$$

for $i = 1, \ldots, n$. The corresponding equilibrium point is $x^* = (\eta_1^*, v_1^*, \ldots, \eta_n^*, v_n^*)$, with $(\eta_i^*, v_i^*) = \mathcal{L}((I^*, V^*)) = (1, 0, \ldots, 0, 1, 0, \ldots, 0)$. After reordering the state variables according to Remark (5), i.e., $(\eta_{1,1}^*, v_{1,1}^*, \ldots, \eta_{1,\kappa}^*, v_{1,\kappa}^*, \ldots, \eta_{n,1}^*, v_{n,1}^*, \ldots, \eta_{n,\kappa}^*, v_{n,\kappa}^*)^T$, and thus obtaining the corresponding map ϕ as in (15), one can find that the spectral radii of the discrete derivatives of ϕ, evaluated at the points $y_1^* = (1, \ldots, 1)^T$ and $y_2^* = \cdots = y_\kappa^* = (0, \ldots, 0)^T$, are all null. This implies by Theorem 5.3 that each y_j^* is attractive in its VNN $V(y_j^*)$, and hence, by Theorem 5.4, that X^* is attractive for the original SVBDS at least in its CN. Furthermore, consider the region Γ composed of the states that differ in at most γ components from x^*. It is easy to verify that the value of the map in (21)

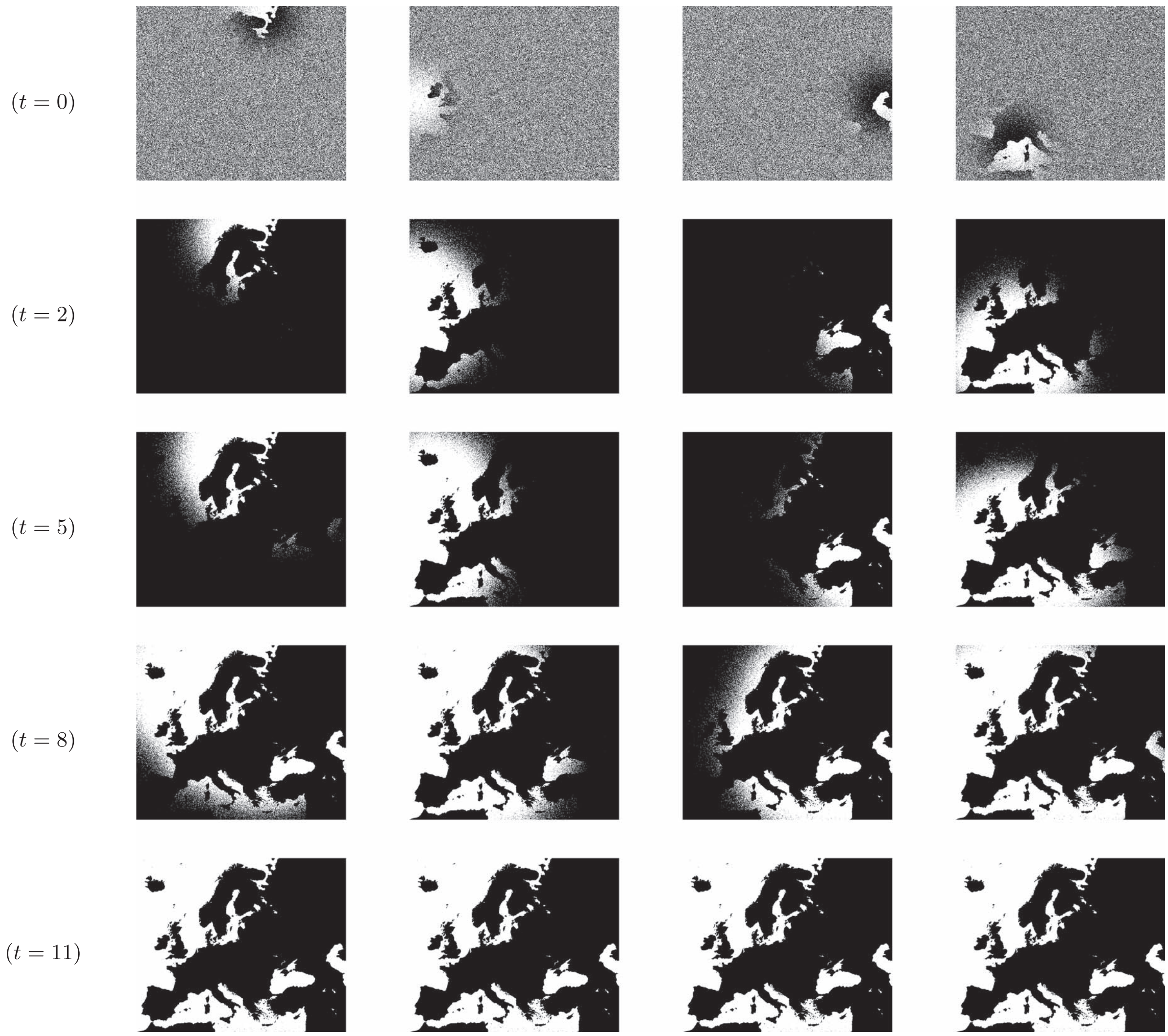

Fig. 3. Simulation run with 135 balloon stations and maximum number of faults per pixel given by $\gamma = 3$. Only the evolutions of the charts estimated by 4 stations is reported for space reasons: from left to right, $\mathcal{A}_8$ is placed approximately at latitude 66° N and longitude 41° E, $\mathcal{A}_{73}$ at 48° N and 8° W, $\mathcal{A}_{84}$ at 45° N and 32° E, and $\mathcal{A}_{112}$ at 39° N, and 6° E. The network of stations effectively consents on a global map of the European continent with reduced noise.

remains constant for all states in Γ, i.e., for all $\tilde{x} \in \Gamma$, it holds $f(\tilde{x}) = f(x^*) = x^*$. This fact tells us that Γ is included in the region of attractiveness of x^*. By projecting back Γ to the original system domain, we can prove that X^* is attractive at least in the set $\mathcal{L}^\dagger(\Gamma)$, which is large enough to tolerate γ incorrect land detections.

Remark 8: Based on the above discussion, the system in (20) is guaranteed to converge in at most 2^n steps. Moreover, it has been established that, if the set-valued map F corresponding to the update rule in (20) is *commutative*, *associative*, and *idempotent* w.r.t. any pair of its input arguments, the system's convergence time is upper bounded by the diameter of the communication graph [13]. In this respect, note first that the second function of (20) is idempotent w.r.t. any pair of its inputs. Note also that the function is commutative and associative w.r.t. any pair of input argument set $H \in S_{\gamma+1}(C_i)$. To show this, define, for every index set H, a new fictitious input argument as $U_H = \bigcap_{h \in H} V_h(t)$. The set-valued function can be written as $V_i(t+1) = \bigcup_{H \in S_{\gamma+1}(C_i)} U_H$, which clearly satisfies the above two properties w.r.t. the new input arguments U_H. By drawing a similar discussion about the first function in (20), we can conclude that the communication graph's diameter is an upper bound for the system's convergence time. The view that is thus finally reconstructed and shared among the agents is unique and consistent with all available measures. However, the required land detection accuracy is guaranteed only within the region Q'.

Remark 9: The binary encoding of (21) is only needed for analysis purposes. Every station update its state by making computations in the original set-valued domain, for which it is only required to know who its neighbors are.

A more general case can be considered, where, due to noise increasing with the distance and to local atmospheric conditions, such as the presence of stratus clouds, each $\mathcal{A}_i$ can incorrectly detect points within its field-of-view with probability ε. Each region $V_i(0)$ can thus be interpreted as the initial *region of ε-confidence* of $\mathcal{A}_i$, i.e., the set of points where the probability of land detection error is not greater than ε. Furthermore, we require, for every point in $\mathcal{Q}$, that the probability E of land detection errors in the global chart is bounded as $E \leq \bar{E} < 1$.

For each $\mathcal{A}_i$, the probability of having more than γ land detection errors in $V_i(1)$ is

$$p(\varepsilon) = 1 - \sum_{k=0}^{\gamma} \binom{r}{k} \varepsilon^k (1-\varepsilon)^{r-k}.$$

With the same reasoning, the probability of having more than γ land detection errors in $V_i(2)$ is $p \circ p(\varepsilon)$, and in $V_i(t)$ is $s_\varepsilon(t) = p\circ, \ldots, \circ p(\varepsilon)$, i.e., the composition of p with itself t times. Therefore, we need to chose a set of sensors with the probability ε satisfying the constraint

$$s_\varepsilon(t) < \bar{E} \text{ for all } t \geq 1. \tag{22}$$

It is possible to show that, for $\varepsilon < 1/2$ and for all γ, $p(\varepsilon) < \varepsilon$ and the function $s_\varepsilon(t)$ monotonically decreases for $t \geq 1$. Hence, for an admissible error probability $\bar{E} < 1/2$, the condition in (22) is implied by $\varepsilon < \bar{E}$.

Let us finally consider a simulative example including $n = 135$ stations with the hypothesis of $\gamma = 3$, $r = 7$, $\varepsilon = 0.05$. By placing the stations on a grid with mesh size δ, we satisfy the measurement multiplicity constraint with $r = \lfloor \pi d(d-1)/\delta \rfloor + 1$, as known from number theory [24]. Fig. 2 is a depiction of the stations' deployment and the available communication graph. The diameter of the communication graph, i.e., the maximum distance between any two nodes on the graph, is 11. Let us consider a case in which $E = 0.02$. Fig. 3 shows how the network of stations iteratively update their estimated charts I_i by running an instance of the consensus algorithm described in (20). The first row reports the initial estimated charts of four stations obtained from processing of the images taken by their onboard vision systems, while the last row reveals that the stations have successfully converged to the centralized estimated chart I^*. In the figures the visibility sets V_i are not drawn for legibility purpose. Their borders roughly corresponds to the transition zones from the regions with clear land contours to the ones containing only noise. We can observe that, during the estimation process, each V_i expands as the land contours become clear and finally include the entire chart.

VII. Conclusion

This paper focused on the convergence towards consensus on information in distributed systems, where agents share data that is not represented by real numbers, rather by logical values or sets. We showed that both types of information convergence problems can indeed be attacked in a unified way in the framework of Boolean distributed information systems. Based on a notions of contractivity and local convergence for Boolean dynamical systems, a necessary and sufficient condition ensuring the global and local convergence toward an equilibrium point is presented. Application of achieved results to some examples was finally shown. Future works will address the convergence of more general set-valued maps.

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