Invited Paper

# Linear and nonlinear approximations for periodically driven bistable systems

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## ABSTRACT

We analyze periodically driven bistable systems by two different approaches. The first approach is a linearization of the stochastic Langevin equation of our system by the response on small external force. The second one is based on the Gaussian approximation of the kinetic equations for the cumulants. We obtain with the first approach the signal power amplification and output signal-to-noise ratio for a model piece-wise linear bistable potential and compare with the results of linear response approximation. By using the second approach to a bistable quartic potential, we obtain the set of nonlinear differential equations for the first and the second cumulants.

**Keywords**: Stochastic resonance, Stochastic linearization, Linear response theory, Gaussian approximation

## 1. INTRODUCTION

The stochastic resonance observing in a bistable system under external sinusoidal excitation was investigated in a great number of papers (see, for example, reviews[1]). This phenomenon discovered in 1981 in studies of strange periodicity of Earth's ice ages[2] was further detected in physical experiments with two-state Schmitt trigger electronic circuit,[3] bidirectional ring laser,[4] and later with tunnel diode.[5] Theory of the stochastic resonance was mostly developed in 1989 due to a series of papers. The authors of the paper[6] used the approximate master equations for two stable states populations, the solution of Fokker-Planck equation based on analogy with the quantum perturbation theory was approximately obtained in the work,[7] and in the article[8] the technique of eigenvalues and eigenfunctions for kinetic operator of Fokker-Planck equation was applied. At last, authors of the work[9] suggested the linear response approximation method based on the results of linear response theory[10] to analysis of signal power amplification and output signal-to-noise ratio. The predictions of stochastic resonance theory for bistable quartic potential were checked by authors[11] through analog simulations.

In this paper to investigate the stochastic resonance phenomenon we propose two different methods: the linearization of stochastic Langevin equation by the system response on small external excitation, and the Gaussian approximation in the framework of well-known cumulant-neglect closure method. We compare the results obtained for the system with piece-wise linear bistable potential with the results giving the linear response approximation method.

## 2. THE STOCHASTIC LINEARIZATION METHOD

To explain the method of stochastic linearization by the response on small external additive excitation $s(t)$ we consider the simple non-autonomous nonlinear dynamical system describing by the following stochastic differential equation

$$\dot{x} = f(x) + \xi(t) + s(t), \tag{1}$$

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where $\xi(t)$ is the Gaussian white noise with zero mean and intensity $2D$, and $f(x)$ is arbitrary differentiable function. We should find the solution of Eq. (1) in the following form: $x(t) = x_0(t) + x_1(t)$, where $x_0(t)$ is unperturbed motion of the system, i.e.

$$\dot{x}_0 = f(x_0) + \xi(t)\,, \tag{2}$$

and $x_1(t)$ is the system response on external excitation $s(t)$. Substituting superposition in Eq. (1) and taking into account Eq. (2), we arrive at

$$\dot{x}_1 = f(x_0 + x_1) - f(x_0) + s(t)\,. \tag{3}$$

Since $x_1(t) \ll x_0(t)$ owing to a small value of external signal $s(t)$ Eq. (3) can be approximately rewritten as

$$\dot{x}_1 \simeq f'(x_0)\, x_1 + s(t)\,. \tag{4}$$

The solution of linear differential equation (4) for the external excitation $s(t)$ acting from $t = -\infty$ is

$$x_1(t) \simeq \int_{-\infty}^{t} \exp\left\{ \int_{\tau}^{t} f'(x_0(\theta))\, d\theta \right\} s(\tau) d\tau. \tag{5}$$

Averaging of Eq. (5) in the case of deterministic $s(t)$ with consideration for stationary Markovian process $x_0(t)$ gives

$$\langle x_1(t) \rangle \simeq \int_0^{\infty} \chi(\tau) s(t-\tau) d\tau, \tag{6}$$

where

$$\chi(\tau) = \left\langle \exp\left\{ \int_0^{\tau} f'(x_0(\theta))\, d\theta \right\} \right\rangle \tag{7}$$

is the linear response function of system (1).

For sinusoidal excitation $s(t) = A \sin(\Omega t + \varphi)$ from Eq. (6) we have

$$\langle x_1(t) \rangle \simeq A \cdot \mathtt{Im}\left[ e^{i(\Omega t + \varphi)} \tilde{\chi}(i\Omega) \right], \tag{8}$$

where $\tilde{\chi}(p)$ is the Laplace transform of response function (7) and $\tilde{\chi}(i\Omega)$ is so-called linear susceptibility of the system. Thus, to calculate the linear response function in the approximation scheme used we must find the functional average (7) or its Laplace transform $\tilde{\chi}(p)$.

Hereafter we shall use so-called moment-producing function of an additive functional of Markovian process $z(t)$ defined as the following conditional functional average[12]

$$R_\lambda(z, t | z_0, t_0) = \left\langle \delta(z - z(t)) \exp\left\{ -\lambda \int_{t_0}^{t} H(z(\theta), \theta)\, d\theta \right\} \middle| z(t_0) = z_0 \right\rangle, \tag{9}$$

where $H(z,t)$ is arbitrary function. The moment-producing function (9) satisfies the following equation

$$\frac{\partial R_\lambda}{\partial t} = \left[ \hat{L} - \lambda H(z,t) \right] R_\lambda\,, \tag{10}$$

where $\hat{L}$ is the kinetic operator of Markovian process $z(t)$. The initial condition for Eq. (10) reads (see Eq. (9))

$$R_\lambda(z, t_0 | z_0, t_0) = \delta(z - z_0)\,. \tag{11}$$

For stationary Markovian process $x_0(t)$ the linear response function (7) can be expressed in terms of the producing function

$$R(y, \tau | y_0) = \left\langle \delta(y - x_0(\tau)) \exp\left\{ \int_0^{\tau} f'(x_0(\theta))\, d\theta \right\} \middle| x_0(0) = y_0 \right\rangle,$$

obeying for the system under consideration the differential equation

$$\frac{\partial R}{\partial \tau} = -f\left(y\right)\frac{\partial R}{\partial y} + D\frac{\partial^2 R}{\partial y^2}\,, \tag{12}$$

as

$$\chi(\tau) = \int_{-\infty}^{\infty} G(y,\tau)dy\,, \qquad G(y,\tau) = \int_{-\infty}^{\infty} R\left(y,\tau|\,y_0\right)W_\infty(y_0)dy_0\,, \tag{13}$$

where $W_\infty(x)$ is the stationary probability distribution of unperturbed system. Going to the Laplace transform $\tilde{R}\left(y,p|\,y_0\right)$ of the producing function $R\left(y,\tau|\,y_0\right)$ in Eq. (12) and taking into account the initial condition (11) we get

$$D\tilde{R}'' - f\left(y\right)\tilde{R}' - p\tilde{R} = -\delta\left(y - y_0\right). \tag{14}$$

From Eq. (14) it is easily to obtain the following equation for Laplace transform $\tilde{G}\left(y,p\right)$ of the function $G(y,\tau)$ entering in Eq. (13)

$$D\tilde{G}'' - f\left(y\right)\tilde{G}' - p\tilde{G} = -W_\infty\left(y\right). \tag{15}$$

We must solve Eq. (15) with special boundary conditions. As a result, the Laplace transform of linear response function, in accordance with Eq. (13), can be found as

$$\tilde{\chi}(p) = \int_{-\infty}^{\infty} \tilde{G}(y,p)dy\,. \tag{16}$$

## 3. CALCULATIONS OF LINEAR RESPONSE FUNCTION FOR BISTABLE SYSTEM

To demonstrate the possibilities of above-mentioned linearization method we consider the overdamped Brownian diffusion in a bistable potential $U\left(x\right)$ in the presence of small external harmonic force. This motion is governed by the following Langevin equation for the coordinate $x\left(t\right)$ of Brownian particle

$$\dot{x} = -U'\left(x\right) + \xi\left(t\right) + A\sin\left(\Omega t + \varphi\right). \tag{17}$$

We shall make calculations of the linear response function $\chi(\tau)$ of the system (17) for model piece-wise linear potential

$$U(x) = \begin{cases} U_0\left(1 - |x|/L\right), & |x| < L\,, \\ +\infty\,, & |x| > L\,, \end{cases} \tag{18}$$

shown at Fig. 1. The model potential (18) has two reflecting boundaries at points $x = \pm L$ and the triangular

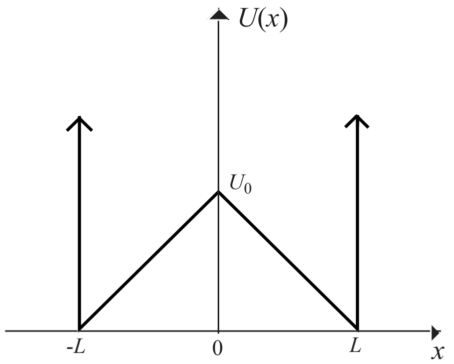


**Figure 1.** Piece-wise linear bistable potential.

barrier separating two stable states. For bistable system (17) with potential (18): $f\left(x\right) = -U'\left(x\right)$, and the integral in expression (7) for linear response function can be interpreted in physical sense. Namely, in accordance with Eq. (18), $U''(x) = -2U_0\delta(x)/L$, and, as a result, we have

$$\chi(\tau) = \left\langle \exp\left\{-\int_0^\tau U''\left(x_0(\theta)\right) d\theta\right\}\right\rangle = \left\langle \exp\left\{\frac{2U_0 N(\tau)}{L}\right\}\right\rangle, \tag{19}$$

where $N(\tau)$ is the crossing points number of axis $x = 0$ by the random process $x_0(t)$ on the interval $(0, \tau)$, i.e. the transitions number of unperturbed system from one stable state to other. In such a case the linear response function (19) is the moment-producing function of this random value.

By symmetry of the potential (18) the stationary probability distribution $W_\infty(x)$ of unperturbed system (17) is an even function

$$W_\infty(x) = \frac{\beta e^{\beta|x|/L}}{2L\left(e^\beta - 1\right)} \qquad (|x| < L), \tag{20}$$

where $\beta = U_0/D$ is the dimensionless height of potential barrier. In accordance with Eq. (15) the same property is inherent of the function $\tilde{G}(x,p)$: $\tilde{G}(-x,p) = \tilde{G}(x,p)$. Solving Eq. (15) in the region $0 < y < L$, where $f(y) = -U'(y) = U_0/L$, and taking into account Eq. (20), we obtain

$$\tilde{G}(y,p) = c_1 e^{\lambda_1 y} + c_2 e^{\lambda_2 y} + \frac{W_\infty(y)}{p}, \tag{21}$$

where

$$\lambda_{1,2} = \frac{\beta \pm \sqrt{\beta^2 + 4pL^2/D}}{2L}.$$

To find unknown constants $c_1$ and $c_2$ we use the condition at right reflecting boundary $x = L$: $\tilde{G}'(L,p) = 0$ and the condition: $\tilde{G}'(0,p) = 0$, readily following from the evenness of function $\tilde{G}(y,p)$. As a result, after some rearrangements we arrive at

$$\begin{aligned} c_1\gamma e^{-\mu} + c_2\mu e^{-\gamma} + \frac{\beta W_\infty(0)}{p} &= 0, \\ c_1\gamma + c_2\mu + \frac{\beta W_\infty(0)}{p} &= 0, \end{aligned} \tag{22}$$

where: $\gamma = \lambda_1 L$, $\mu = \lambda_2 L$. Substitution of Eq. (21) in Eq. (16) and integration over $y$ give

$$\tilde{\chi}(p) = 2\int_0^L \tilde{G}(y,p)dy = \frac{1}{p} + \frac{2Lc_1}{\gamma}\left(e^\gamma - 1\right) + \frac{2Lc_2}{\mu}\left(e^\mu - 1\right). \tag{23}$$

Calculating $c_1$ and $c_2$ from Eqs. (22) and substituting in Eq. (23), we finally obtain the following expression for the Laplace transform of linear response function of the bistable system (17) in the framework of stochastic linearization method

$$\tilde{\chi}(p) = \frac{1}{p} - \frac{\beta^2}{p^2\tau_0} - \frac{2\beta^3}{p^3\tau_0^2}\cdot\frac{\mu\sinh^2(\gamma/2) - \gamma\sinh^2(\mu/2)}{\cosh\gamma - \cosh\mu}, \tag{24}$$

where the dimensionless parameter $\tau_0 = L^2/D$ means the time of free diffusion on the distance $L$.

Based on Eqs. (8) and (24) it is easily to derive the signal power amplification (SPA)

$$\eta = \frac{\langle x_1\rangle^2}{A^2/2} = |\tilde{\chi}(i\Omega)|^2 \tag{25}$$

and output signal-to-noise ratio (SNR)

$$SNR = \frac{\langle x_1\rangle^2}{S_0(\Omega)} = \frac{A^2\,|\tilde{\chi}(i\Omega)|^2}{2S_0(\Omega)}, \tag{26}$$

where $S_0(\Omega)$ is the power spectrum of unperturbed system (17) previously obtained by authors.[13]

Further we should compare the results (24)–(26) with the results giving by linear response theory.[10] In accordance with fluctuation–dissipation theorem (FDT) for the system (17) with thermal noise source $\xi(t)$ the linear response function $\chi_{LRA}(\tau)$ is inextricably connected[9] with the correlation function $K_0[\tau]$ of equilibrium thermal fluctuation $x_0(t)$ of unperturbed system (17)

$$\chi_{LRA}(\tau) = -\frac{1}{D} \cdot \frac{dK_0[\tau]}{d\tau}. \tag{27}$$

Making the Laplace transform in Eq. (27) we arrive at

$$\tilde{\chi}_{LRA}(p) = \frac{\langle x_0^2 \rangle_\infty}{D} - \frac{p\tilde{K}_0[p]}{D}, \tag{28}$$

where $\langle x_0^2 \rangle_\infty$ is the variance of Brownian particle coordinate. Substituting the exact relation (22) for the Laplace transform $\tilde{K}_0[p]$ of the correlation function $K_0[\tau]$ from the paper[13] in Eq. (28), we have

$$\tilde{\chi}_{LRA}(p) = \frac{1}{p} - \frac{\beta}{p\left(1-e^{-\beta}\right)\left(\gamma e^{\mu} - \mu e^{\gamma}\right)} \left\{ e^{\gamma} - e^{\mu} + 4\beta \left[ \frac{\sinh^2(\mu/2)}{\mu} - \frac{\sinh^2(\gamma/2)}{\gamma} \right] \right\}. \tag{29}$$

As one would expect, the result (29) of linear response approximation theory differs from the result (24) obtained by the method of stochastic linearization.

## 4. STOCHASTIC LINERIZATION VERSUS LINEAR RESPONSE APPROXIMATION

The dependences of signal power amplification (25) and output signal-to-noise ratio (26) on white noise intensity $D$ for different values of signal frequency $\Omega$ obtained in framework of linear response approximation for model bistable potential (Fig. 1) are shown in Fig. 2 and Fig. 3 respectively. The height of triangular potential barrier

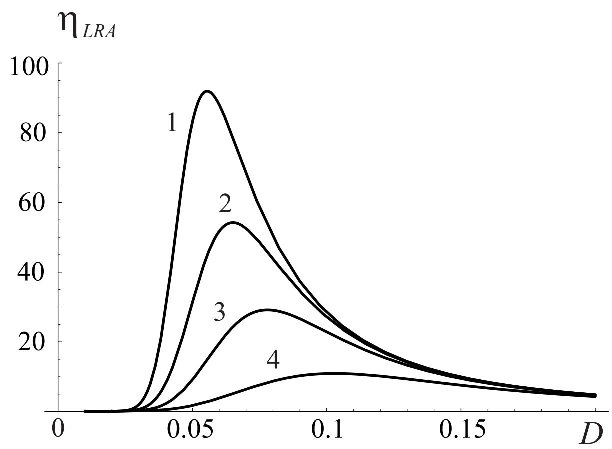


**Figure 2.** Linear response approximation: Signal power amplification $\eta_{LRT}$ versus white noise intensity $D$ for different values of external signal frequency: curve 1 – $\Omega = 0.01$, curve 2 – $\Omega = 0.02$, curve 3 – $\Omega = 0.04$, curve 4 – $\Omega = 0.1$. The parameters are $U_0 = 0.25$, $A = 0.1$, $L = 1$.

was adopted with the aim of comparison the same as for the quartic potential $U(x) = -x^2/2 + x^4/4$ abundant in literature. For such a potential it has been possible to obtain only approximate results for $\eta$ and $SNR$ because of unknown formula for the correlation function of unperturbed fluctuations. Usually, for these purposes the first term of correlation function expansion regarding the minimal eigenvalue of conjugate kinetic operator is used.[9]

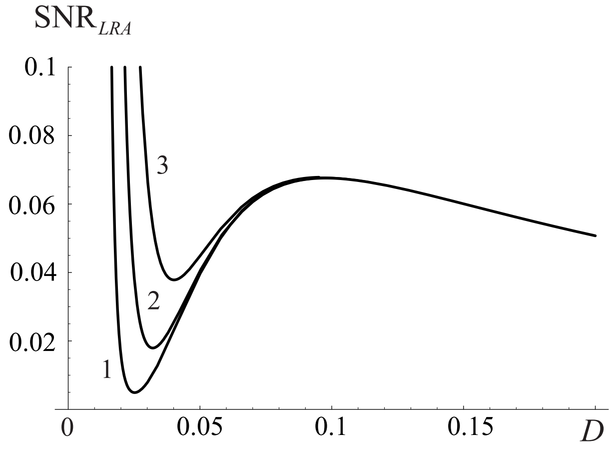


**Figure 3.** Linear response approximation: Signal-to-noise ratio $SNR_{LRT}$ versus white noise intensity $D$ for different values of external signal frequency: curve 1 – $\Omega = 0.01$, curve 2 – $\Omega = 0.04$, curve 3 – $\Omega = 0.1$. The parameters are $U_0 = 0.25$, $A = 0.1$, $L = 1$.

We also took into account the condition on the amplitude of external sinusoidal signal: $A < U_0/L$, which is necessary for stochastic resonance realization.

As is evident from Fig. 2, the plot of signal power amplification has the inherent maximum achieving at some white noise intensity. This phenomenon or so-called stochastic resonance can be explained as follows. The mean rate of system transitions from one stable state to other increases with increasing white noise intensity $D$ and synchronizes with rocking frequency of a potential by external force. In such a situation we obtain the signal amplification. On further increasing $D$ the transitions become fast and are independent on rocking rate. As a result, the synchronization breaks down, and the signal power amplification decreases.

As Fig. 2 suggests, the value of intensity corresponding to a maximal amplification decreases with decreasing external force frequency, while the height of peak increases. A correlation with approximate results in the case of quartic bistable potential for external signal frequency $\Omega = 0.01$ shows that the maximal amplification factor in our case is approximately seven times greater than the value found in the work.[9] This feature can be explained as follows. For considering model bistable potential the diffusion area is restricted by two reflecting boundaries (see Fig. 1), and, as a result, a transitions from one stable state to other become more effective because the most part of Brownian particles are involved in the barrier crossing.

We can also observe the maximum in the behavior of output signal-to-noise ratio in Fig. 3, but, in contrary with the plot of signal power amplification, this maximum having the same value for different frequencies of external field is more gentle and achieved at the white noise intensity $D = 0.09$. We call attention to the minimum in the dependence of SNR on white noise intensity $D$ detected in a large body of experiments.[1,4] This minimum is achieved at small intensity of white noise when the transitions across a barrier become a rare events, and thus it must be associated with intrawell motion of Brownian particles. In such a regime the exponentially fast narrowing of power spectrum width of unperturbed system with decreasing white noise intensity $D$ takes place.[13] As a result, the power spectrum $S_0(\Omega)$ at external field frequency decreases steeply and SNR (26) increases. It must be emphasized that the minimum becomes more deep with decreasing the frequency of external field.

The signal power amplification for different values of white noise intensity monotonically decreases with increasing frequency $\Omega$, as it is shown in Fig. 4. Actually, at fixed rate of diffusion, Brownian particles have no time to cross the barrier in synchronism with the increasing rocking frequency of a potential, and, as a result, the stochastic resonance breaks down.

Let us compare the results of linear response approximation with the results (24)–(26) obtained by stochastic linearization method. The plots of signal power amplification $\eta_{SL}$ and the output signal-to-noise ratio $SNR_{SL}$

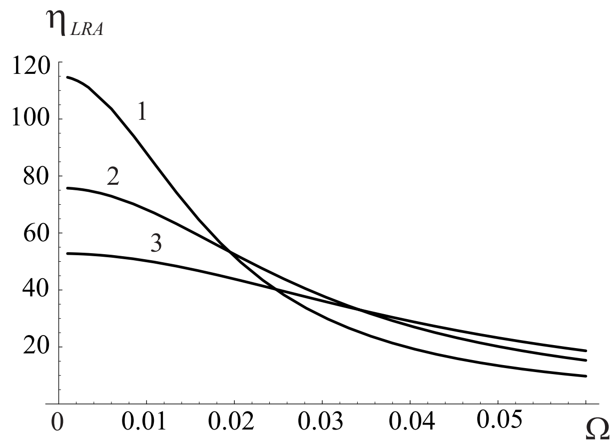


**Figure 4.** Linear response approximation: Signal power amplification $\eta_{LRT}$ versus the signal frequency $\Omega$ for different values of white noise intensity: curve 1 – $D = 0.06$, curve 2 – $D = 0.07$, curve 3 – $D = 0.08$. The parameters are $U_0 = 0.25$, $A = 0.1$, $L = 1$.

is presented in Fig. 5 and Fig. 6 respectively. Contrary to Fig. 2 and Fig. 3 the curves presented in Fig. 5

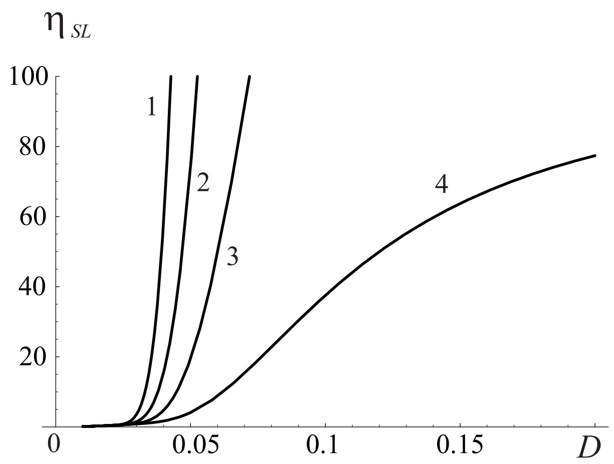


**Figure 5.** Stochastic linearization: Signal power amplification $\eta_{SL}$ versus white noise intensity $D$ for different values of external signal frequency: curve 1 – $\Omega = 0.01$, curve 2 – $\Omega = 0.02$, curve 3 – $\Omega = 0.04$, curve 4 – $\Omega = 0.1$. The parameters are $U_0 = 0.25$, $A = 0.1$, $L = 1$.

Fig. 6 have not a maximum which is indicator of stochastic resonance phenomenon. But as it is seen from Fig. 5 the amplification of external sinusoidal signal takes place, and we can also observe a minimum in SNR behavior in Fig. 6. To clear up the reason of absence of the maximum in plots at Figs. 5 and 6 we should estimate the error of stochastic linearization method. The condition of applicability of stochastic linearization method reads $x_1(t) \ll x_0(t)$, and in averaged version as

$$\varepsilon\left(D\right) = \frac{A\left|\tilde{\chi}(i\Omega)\right|}{\sqrt{\langle x_0^2\rangle_\infty}} \ll 1\,, \tag{30}$$

because the fluctuations $x_0(t)$ of unperturbed system has zero mean. It is quite difficult to obtain in the explicit

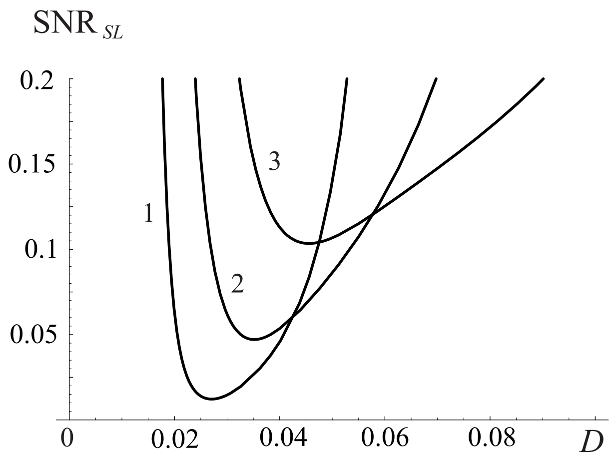


**Figure 6.** Stochastic linearization: Signal-to-noise ratio $SNR_{SL}$ versus white noise intensity $D$ for different values of external signal frequency: curve 1 – $\Omega = 0.01$, curve 2 – $\Omega = 0.04$, curve 3 – $\Omega = 0.1$. The parameters are $U_0 = 0.25$, $A = 0.1$, $L = 1$.

form the inequality for white noise intensity $D$ from Eq. (30). Because of this, the plots of the error $\varepsilon(D)$ for different values of external force frequency $\Omega$ are shown in Fig. 7. As it is seen from Fig. 7, the above-mentioned

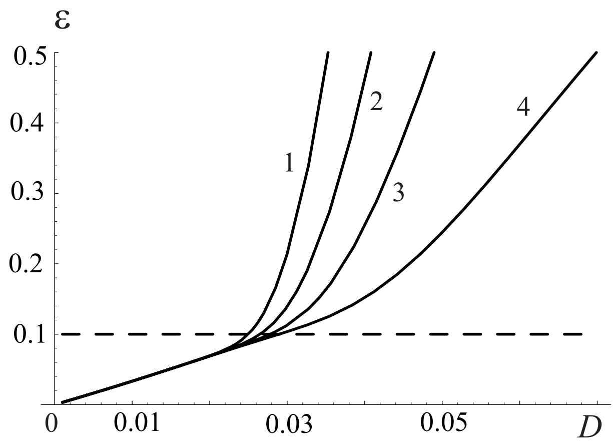


**Figure 7.** The error of stochastic linearization method $\varepsilon$ versus intensity of white noise $D$ for different values of signal frequency: curve 1 – $\Omega = 0.01$, curve 2 – $\Omega = 0.02$, curve 3 – $\Omega = 0.04$, curve 4 – $\Omega = 0.1$. The parameters are $U_0 = 0.25$, $A = 0.1$, $L = 1$.

method of Langevin equation linearization by response on small external force is only valid at $D \leq 0.03$. At the same time, the resonant maximum in the plots of signal power amplification in Fig. 2 lies in the range $0.05 \leq D \leq 0.11$, and is achieved at white noise intensity $D \simeq 0.09$ for output signal-to-noise ratio (see Fig. 3).

## 5. GAUSSIAN APPROXIMATION FOR BISTABLE SYSTEM

Now we should consider the general equations for the cumulants $\kappa_s$ of arbitrary Markovian process obtained by Malakhov[14]

$$\frac{d\kappa_s}{dt} = \sum_{m=1}^{s} C_s^m \left\langle x^{[s-m]}, K_m(x,t) \right\rangle , \tag{31}$$

where $K_m(x,t)$ are the kinetic coefficients and $\langle \ldots, \ldots \rangle$ is the cumulant bracket. For continuous Markovian process $x(t)$ describing by Fokker-Planck equation from Eq. (31) we have

$$\frac{d\kappa_s}{dt} = s \left\langle x^{[s-1]}, K_1(x,t) \right\rangle + \frac{s(s-1)}{2} \left\langle x^{[s-2]}, K_2(x,t) \right\rangle , \tag{32}$$

because of $K_m(x,t) = 0$ for $m \geq 3$. To analyze Eqs. (32) it can usually used the cumulant-neglect closure procedures.[15] The simplest one is the Gaussian approximation when only two first equations should be remained in set of equations (32), and all cumulants $\kappa_s = 0$ for $s \geq 3$. In accordance with the results reported in the paper,[16] we arrive at

$$\begin{aligned} \frac{d\kappa_1}{dt} &= \langle K_1(x,t) \rangle_G , \\ \frac{d\kappa_2}{dt} &= 2 \langle K_1'(x,t) \rangle_G \cdot \kappa_2 + \langle K_2(x,t) \rangle_G , \end{aligned} \tag{33}$$

where average $\langle \ldots \rangle_G$ must be calculated using Gaussian probability distribution.

For bistable system (17) we have

$$K_1(x,t) = -U'(x) + s(t) , \qquad K_2(x,t) = 2D . \tag{34}$$

Substitution of Eq. (34) in Eqs. (33) gives

$$\begin{aligned} \frac{d\kappa_1}{dt} &= -\langle U'(x) \rangle_G + s(t) , \\ \frac{d\kappa_2}{dt} &= -2\kappa_2 \cdot \langle U''(x) \rangle_G + 2D . \end{aligned} \tag{35}$$

For the smooth quartic bistable potential $U(x) = -x^2/2 + x^4/4$ from Eq. (35) we have

$$\begin{aligned} \frac{d\kappa_1}{dt} &= -\left\langle x^3 \right\rangle_G + \kappa_1 + s(t) , \\ \frac{d\kappa_2}{dt} &= -6\kappa_2 \cdot \left\langle x^2 \right\rangle + 2\kappa_2 + 2D . \end{aligned} \tag{36}$$

Using the connection between moments and cumulants

$$\left\langle x^2 \right\rangle = \kappa_2 + \kappa_1^2, \qquad \left\langle x^3 \right\rangle = \kappa_3 + 3\kappa_2\kappa_1 + \kappa_1^3,$$

we arrive at the following set of equations in the Gaussian approximation

$$\begin{aligned} \frac{d\kappa_1}{dt} &= -3\kappa_2\kappa_1 - \kappa_1^3 + \kappa_1 + s(t) , \\ \frac{d\kappa_2}{dt} &= -6\kappa_2^2 - 6\kappa_2\kappa_1^2 + 2\kappa_2 + 2D . \end{aligned} \tag{37}$$

We must numerically solve the nonlinear system (37) for small sinusoidal signal $s(t) = A\sin(\Omega t + \varphi)$ and then remain only the first harmonic term in the expression for $\kappa_1(t)$. Of course, this approximation is not so good because the real probability distribution has two peaks and deviates from Gaussian law.

## 6. CONCLUSIONS

To calculate the linear susceptibility of a system we suggested the method of stochastic linearization by the response on small external signal with the apparatus of moment-producing functions of an additive functional of Markovian process. The calculations for the model bistable system driving by sinusoidal force have analogy with calculations of the power spectrum of unperturbed system made recently by authors.[13] In the framework of linear response approximation we found the significantly large signal amplification in the system with piece-wise linear bistable potential restricting by two reflecting boundaries in comparison with the smooth quartic bistable potential. It was shown that the value of resonant maximum and its position in the behavior of signal-to-noise ratio versus white noise intensity are independent on the external force frequency. The correlation between the results of stochastic linearization method and of linear response approximation showed that the first method is only valid in the range of relatively small white noise intensities. As a result, the dependencies of the signal power amplification and output signal-to-noise ratio depict the right tendency, but have not the characteristic maximum indicating the stochastic resonance phenomenon. We also obtained the nonlinear differential equations for first and second cumulants of Brownian motion in the smooth quartic potential prepared for forthcoming numerical calculations.

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## REFERENCES

1. M.I. Dykman, D.G. Luchinsky, R. Mannella, P.V.E. McClintock, N.D. Stein, and N.G. Stocks, Nuovo Cim. **17D**, 661 (1995); L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni, Rev. Mod. Phys. **70**, 223 (1998); V.S. Anishchenko, A.B. Neiman, F. Moss, L. Schimansky-Geier, Sov. Phys. Usp. **42**, 7 (1999).
2. R. Benzi, A. Sutera, and A. Vulpiani, J. Phys. A: Math. Gen. **14**, L453 (1981); C. Nicolis, Tellus **34**, 1 (1982); R. Benzi, G. Parisi, A. Sutera, and A. Vulpiani, Tellus **34**, 10 (1982).
3. S. Fauve and F. Heslot, Phys. Lett. A **97**, 5 (1983); V.I. Melnikov, Phys. Rev. E **48**, 2481 (1993).
4. B. McNamara, K. Wiesenfeld, and R. Roy, Phys. Rev. Lett. **60**, 2626 (1988); G. Vemuri and R. Roy, Phys. Rev. A **39**, 4668 (1989).
5. R.N. Mantegna and B. Spagnolo, Phys. Rev. E **49**, R1792 (1994); R.N. Mantegna, B. Spagnolo, and M. Trapanese, Phys. Rev. E **63**, 011101 (2000).
6. B. McNamara and K. Wiesenfeld, Phys. Rev. A **39**, 4854 (1989).
7. C. Presilla, F. Marchesoni, and L. Gammaitoni, Phys. Rev. A **40**, 2105 (1989).
8. R.F. Fox, Phys. Rev. A **39**, 4148 (1989).
9. P. Jung and P. Hänggi, Phys. Rev. A **44**, 8032 (1991).
10. P. Hänggi and H. Thomas, Phys. Rep. **88**, 207 (1982).
11. T. Zhou and F. Moss, Phys. Rev. A **41**, 4255 (1990).
12. D.A. Darling and A.J.F. Siegert, IRE Trans. Inform. Theory **3**, 32 (1957).
13. A.A. Dubkov, V.N. Ganin, and B. Spagnolo, Acta Physica Polonica B **35**, 1447 (2004).
14. A.N. Malakhov, Radiophys. and Quant. Electr. **19**, 48 (1976).
15. P. Hänggi and P. Talkner, J. Stat. Phys. **22**, 65 (1980); S.F. Wojtkiewicz, B.F. Spencer, Jr. and L.A. Bergman, Int. J. Non-Linear Mech. **31**, 657 (1996).
16. A.N. Malakhov, Radiophys. and Quant. Electr. **17**, 1329 (1974).